

CREDIT SPREAD MODELING EFFECTS ON COUNTERPARTY RISK VALUATION ADJUSTMENTS: A SPANISH CASE STUDY

Abstract:

We analyze the effects of the financial crisis in credit valuation adjustments (CVA's). Following the arbitrage-free valuation framework presented in Brigo et al. (2009), we consider a model with stochastic Gaussian interest rates and CIR++ default intensities. Departing from previous literature, we are able to calibrate default intensities profiting from Gaussian mapping techniques presented in Brigo and Alfonsi (2004), and reproduce the historically observed instantaneous covariances of CDS prices. To test the calibration procedure, we track the Spanish financial sector, who has behaved in a singular manner through the crisis, regarded among the safest in Europe at the beginning, and in need of a partial bailout a few years later. We calculate adjustments involving the two major Spanish banks and a generic European counterpart in these two situations for both interest rate and credit derivatives.

Keywords: Counterparty Risk, Arbitrage-Free Credit Valuation Adjustment, Credit Default Swaps, Credit Spread Volatility

EFFECTOS DE MODELIZACIÓN DE SPREADS DE CRÉDITO EN AJUSTES DE VALORACIÓN POR RIESGO DE CONTRAPARTE: UN CASO ESPAÑOL

Resumen:

En este trabajo se analizan los efectos de la crisis financiera en los ajustes por valoración de riesgo de crédito. Siguiendo el marco de valoración libre de riesgo presentado en Brigo et al. (2009), se considera un modelo híbrido estocástico con tipos de interés gaussianos e intensidades de quiebra CIR++. A diferencia de literatura anterior, se calibran las intensidades de quiebra aprovechando las técnicas de mapeo gaussiano mostradas en Brigo y Alfonsi (2004), reproduciendo las covarianzas instantáneas históricas observadas de precios de permutas de incumplimiento crediticio. Este procedimiento de calibración se prueba sobre el sector financiero español, que ha seguido un comportamiento singular durante la crisis reciente, pasando de ser considerado de los más sólidos de Europa a necesitar un rescate parcial pocos años después. Se calculan ajustes involucrando a los dos mayores bancos españoles y a una contraparte europea genérica en ambas situaciones para derivados de tipos de interés y de crédito.

Palabras clave: Riesgo de contraparte, ajuste de valoración de crédito libre de riesgo, permutas de incumplimiento crediticio, volatilidad del spread de crédito

Materia: Riesgo del crédito

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1 Introduction

1.1 Context

The financial crisis that started in 2007 has caused a paradigm shift in the business of banking. Every stakeholder in the industry, from regulators to investment banks, from rating agencies to hedge funds, has been obliged to make a halt and reconsider the very basics of their daily tasks.

In a similar fashion to the stock market crash of October 1987, when the volatility smile first appeared in equity option prices¹, this crisis is challenging traditional financial engineering in several ways. Typical non arbitrage relationships between spot and forward interest rates do not hold anymore due to the appearance of basis spreads among tenors. Interest rates are entering negativity for certain products. Traditional safe assets, such as OECD sovereign bonds, are becoming dubious when not dangerously risky.

Another area in need of revision is the treatment of Counterparty Credit Risk (CCR) in market activities, that is, the risk that the counterparty defaults before the final settlement of a transaction's cash flows. If the portfolio has a positive value for the bank at the time of default, an economic loss will occur. Notice that CCR has a bilateral nature in derivatives, since the market value of the portfolio can be positive or negative depending on time-varying market factors. In this setup, Credit Valuation Adjustment (CVA) is the difference between the risk-free portfolio value and the true portfolio value that takes into account the counterparty's default. In short, CVA is the market value of CCR.

The review on CCR has been threefold. Previously addressed under International Accounting Standards (IAS) 39, its importance was further stressed in January 2013, when the International Financial Reporting Standard (IFRS) 13 "Fair Value Measurement" entered into force. Largely based on the accounting standard applied in the United States, it intends to harmonize the definition of fair value, which is characterized as an exit price, that is, the one that would be received or paid in an orderly transaction between market participants. In this context, CCR plays a major role in the computation of fair value.

¹See, for example, [Hull]

From a regulatory perspective, the Basel Committee on Banking Supervision recognized in 2009 that capital for CCR had proved to be inadequate. The then ongoing regulatory framework, Basel II, addressed CCR as a default and credit migration risk, not fully accounting for market value losses short of default. However, as the Basel Committee pointed, "roughly two-thirds of CCR losses were due to CVA losses and only about one-third were due to actual defaults". The identification of this and other related shortcomings led to a comprehensive reform on the calculation of capital for CCR which is being implemented by central banks.

A third aspect related to CCR which is being currently addressed has to do with pricing financial products. Counterparty risk has been gradually incorporated in valuation procedures, altering the price to be charged for specific instruments in order to account for the default risk of the counterparty. This change in price, CVA, appears as an option on the residual value of the portfolio, with a random maturity given by the default time of the counterparty. Furthermore, if the investor wants to account for the possibility of him defaulting, a second change on price should be added, named Debt Valuation Adjustment (DVA). Both changes in price generate a source of risk that needs to be taken into account.

The ubiquity of these concepts may lead to different definitions of CVA: an accounting one for books and records, a front office CVA for pricing new deals and a regulatory CVA for defining capital requirements. An accountant will comfortably accept the presence of DVA in defining fair value as the natural contrary of CVA. A trader, though, will complain against considering an adjustment that takes into account the possibility of him defaulting, becoming virtually impossible to hedge (who would buy the insured insurance insuring the insured?) These misalignments can lead to inappropriate trading decisions, with apparently profitable trades not appearing that way to shareholders.

A recent survey² about current market practices on CCR showed a rapid evolution along the past two to three years motivated by changes in regulatory and accounting guidelines. Banks are focusing on building models for advanced capital treatments, including collateral optimization and funding. Effort is also being placed on quantifying Wrong Way Risk (WWR), that is, the risk that

²See [DeloitSolum]

the exposure to our counterparty gets higher when its credit quality worsens, that gets captured by jointly simulating credit spreads and underlying risk factors.

The same survey unveiled a clear divergence in approaches and methodologies across the market. While the use of risk-neutral default probabilities via credit spreads is becoming a standard practice in the quantification of CVA, DVA considerations and the extent to which it should be used to reduce CVA charges is a source of variation. Further ambiguities related to possible funding adjustments, outside the Basel III mandate but subject of increased focus, adds to the confusion, ensuring that CCR will remain a hot topic for a long time.

1.2 Literature on CCR

Literature on bilateral counterparty risk is extensive and relatively recent. Although some references can be traced back to the 1990s³, the review on CCR triggered by the 2008 credit events has contributed to generate a huge amount of works on this topic. A good (and entertaining) survey can be found at [Brigo11].

Since the early 2000s, Damiano Brigo himself has written several papers on counterparty credit risk. A typical work by Brigo is configured as follows:

1. Enunciation of a model-free bilateral counterparty risk valuation formula based on expectations and default indicator functions.
2. Focus on a particular type of product. At this stage, a concrete model is needed, typically a Gaussian two factor (G2++) model for interest rates, Cox-Ingersoll-Ross (CIR) without jumps for credit and Schwartz-Smith for commodities.
3. Numerical analysis, either computing sensitivities from a range of values for specific parameters or calibrating the model to real data.

Complexity on the first step has been gradually increased when valuing financial products. There exists a growing trend in the banking industry on modeling collateral treatment rather than

³See, for example, [DuffieHuang]

relying on simplifications for the sake of capital optimization. In this context, recent works, like [Brigoetal11] or [Brigoetal12], generalize the framework for arbitrage-free valuation of bilateral counterparty risk to the case where collateral is included, with possible re-hypotecation.

Another source of variation has to do with adding jumps when modeling credit. Brigo himself has written some papers applying SSRJD (Shifted Square Root Jump Diffusion) for default intensities, like [BrigoElBachir]. [LiptonSepp] even compute CVA for credit default swaps including jumps. However, the complexity of the numerical methods required to successfully manage this type of models has prevented the literature to tackle this topic in depth.

1.3 The Spanish case

This paper is focused on the Spanish financial sector for two reasons. First, few studies, if any, have modeled the dynamics of credit spreads in this market, and prefer to analyze those of American or British companies instead. However, the Spanish financial sector has behaved in a rather unorthodox manner when compared to other European counterparts. As we shall see below, at the beginning of the crisis, Spanish banks were regarded as one of the most solid entities in the continent, having escaped from the subprime mortgage meltdown at the other side of the Atlantic. However, as time went by, the situation reversed. International financial markets calmed down and the focus was put on the soundness of the Spanish recession and its effects on its financial entities. This quick twist, with two opposite situations in less than five years, gives us another reason for studying the Spanish case.

The Spanish financial sector is highly concentrated. As pointed in [VillOhan], at the end of 2008 there were 362 credit institutions operating in Spain, with 159 banks that represented 53.53 percent of total assets and 46 savings and loans (*cajas*), which accumulated an additional 38.40 percent. However, among the banks, Banco Santander controlled assets of over \$1.4 trillion and BBVA of around \$0.75 trillion. In comparison, the then third largest bank, Banco Popular, had assets of only around \$150 billion.

Traditionally, Spanish banks and *cajas* have held long-standing relations with industry, both in terms of controlling equity positions in companies and through large credits. For individuals,

Spanish banks offer their clients a wide variety of products including deposits, mortgages, credit cards or pension funds. Additionally, although there have been some insights on investment banking, the local monetary authority, the Banco de España, has prevented Spanish banks from playing with complicated structured investment vehicles. Securitization, despite increasing, involved instruments much less complicated than in the U.S. and banks kept most of the credit risk in their own balances.

The apparent universal character of Spanish banks masked the excessive concentration of their lending in the real estate sector. The economic recession that hit Spain at the end of 2008, when the National Statistics Institute first published negative figures of GDP, revealed the ongoing collapse of a real estate bubble and the subsequent meltdown of Spanish economy. Local unemployment rates doubling EU average and concerns about the possibility of a financial bailout, that finally took place in June 2012, rocketed local Treasury yields. Despite their international character and diversification, both Santander and BBVA were not immune to the situation in their country of origin. Figure 1 displays the evolution of the 5 year Credit Default Swap (CDS) spreads for some of the biggest Euro area banks between 2010 and 2013. Starting in comparable levels, as time goes by only Italian banks, Unicredito and Intesa San Paolo, exhibit spreads in line with Santander and BBVA, while CDS from German and French banks are perceived by the market as much less risky.

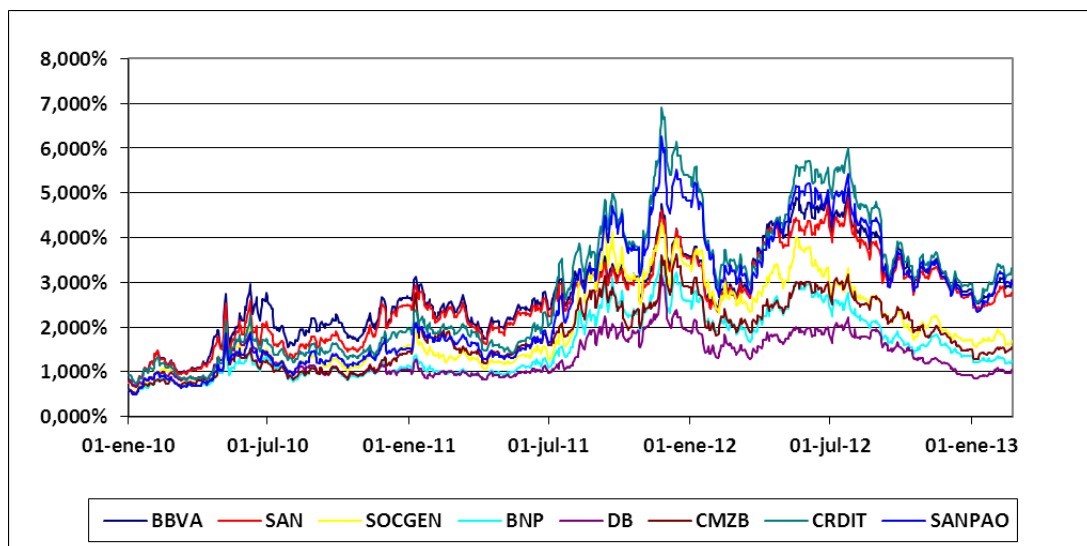


Figure 1: Evolution of CDS5y for main Euro area banks (2010-2013)

This asymmetry between peripheral and core European countries will influence the pricing of financial products if counterparty risk valuation adjustments are taken into account. As an example, imagine a firm engaged in a loan linked to a floating reference who is willing to get rid of interest rate risk by entering a payer swap (paying fixed and receiving floating). Formerly, in this swap, a solid firm would pay the same fixed amount than another on the verge of defaulting (the swap rate). If CVA is added, the bad firm will be charged with a prohibitively high spread over the swap rate compared with the good one. Conversely, if DVA is taken into account, the firm will be tempted to close the deal with a troubled bank, since the spread it will charge to her will be lower. Further paradoxical effects can appear if correlations are taken into account.

1.4 This paper

Our purpose here will be analyzing the effects of the financial crisis in counterparty risk valuation adjustments. Following the arbitrage-free valuation framework presented in [Brigoetal09], we will consider a model with stochastic Gaussian interest rates and CIR++ default intensities. Departing from previous literature, we will be able to calibrate default intensities profiting from Gaussian mapping techniques presented in [BrigoAlfonsi04], and reproduce the historically observed instantaneous covariances of CDS prices. We will calculate adjustments involving the two major Spanish banks, BBVA and Santander, and a generic European counterpart before and along the Spanish recession in both interest rate and credit derivatives. We shall allow for credit spread volatility, correlation between the default times of the investor and counterparty, and for correlation of each with interest rates, and will investigate the effects of incorporating counterparty risk valuation adjustments in pricing.

The paper is structured as follows: Section 2 summarizes the counterparty risk valuation framework from [Brigoetal09], establishing the appropriate notation. Section 3 describes the reduced form model setup of the paper with stochastic interest rates and intensities plus a copula on the exponential triggers. Section 4 presents the calibration procedure for interest rates along with estimation results. Section 5 deals with the calibration of default intensities, presenting the market of Credit Default Swaps and the application of the Gaussian mapping technique in our modeling framework to generate a closed-form expression for the price of a CDS, showing estimation re-

sults. Section 6 presents counterparty credit risk valuation adjustments in several scenarios for an interest rate swap and Section 7 does the same for a Credit Default Swap. Finally, Section 8 concludes.

2 Arbitrage-free valuation of counterparty risk

There exists no consensus in the banking industry about how to calculate counterparty risk credit valuation adjustments. For a long time, the debate focused on whether the entity should account for the possibility of its own default, including DVA in the valuation (symmetric CVA), or whether it should consider itself default-free (asymmetric CVA). Basel II defined counterparty credit risk as the one arising from the possibility that the counterparty to a transaction could default before the final settlement of the transaction's cash flows. No explicit mention about considering the own default was done. Nevertheless, new accountancy and regulatory rules have stressed the importance of fair value, taking asymmetric CVA off the table and inducing financial entities to include DVA in the quantification of counterparty risk.

However, there is still a source of divergence around bilateral counterparty risk. CVA reflects the economic loss to the investor in case the counterparty defaults. Conversely, DVA does the same for the counterparty in case the investor cannot fulfill his obligations. Once we are admitting that both participants in the transaction can default, we might need to reconsider the definitions outlined above. CVA would reflect the economic loss to the investor in case the counterparty defaults *before the investor*. The same applies to DVA, which would become the loss to the counterparty in case the investor defaults *before the counterparty*. This approach is called contingent CVA. We will explore valuation adjustments in these two situations, contingent and non-contingent, defining a general arbitrage-free valuation framework for both approximations.

There exists a third matter of discussion when computing these adjustments. Under the International Swaps and Derivatives Association (ISDA) 2009 protocol, in the event of default, the closeout amount "may take into account the creditworthiness of the Determining Party", suggesting that an institution may consider their own DVA in determining the amount to be settled. This is called the "risky closeout" paradigm, as opposed to the "risk-free closeout" one, where the value

of the transactions to be settled in the event of default is based on risk-free valuation. For simplicity, we shall follow the latter paradigm. Both approaches have their shortcomings, outlined in, for example, [GregGerm]. Currently, there is a hot debate around this issue which is beyond the scope of this paper.

Following the notation of [Brigoetal09], we will refer to the two names involved in the transaction and subject to default risk as *investor*, named I, and *counterparty*, named C. Valuation will be seen from the point of view of the investor I, so that cash flows received by I will be positive whereas cash flows paid by I (and received by C) will be negative.

We denote by τ_I and τ_C respectively the default times of the investor and counterparty. We place ourselves in a probability space $(\Omega; \mathcal{G}; \mathcal{G}_t; \mathbb{Q})$. The filtration \mathcal{G}_t models the flow of information of the whole market, including credit, and \mathbb{Q} is the risk neutral measure. This space is endowed also with a right-continuous and complete sub-filtration \mathcal{F} representing all the observable market quantities but the default events.

2.1 Contingent CVA

Let us call T the final maturity of the payoff which we need to evaluate and let us define the stopping time

$$\tau = \min\{\tau_I, \tau_C\}$$

If $\tau > T$, there is neither default of the investor, nor of his counterparty during the life of the contract and they both can fulfill the agreements of the contract. On the contrary, if $\tau \leq T$ then either the investor or his counterparty (or both) default. At τ , the Net Present Value (NPV) of the residual payoff until maturity is computed. We can distinguish two cases:

- $\tau = \tau_C$: If the NPV is negative for the investor, it is completely paid by the investor itself. If the NPV is positive for the investor, only a recovery fraction REC_C of the NPV is exchanged.

- $\tau = \tau_I$: If the NPV is positive for the defaulted investor, it is completely received by the defaulted investor itself. If the NPV is negative for the defaulted investor, only a recovery fraction REC_I of the NPV is exchanged.

We can define the following (mutually exclusive and exhaustive) events ordering the default times:

$$\begin{aligned}
A &= \{\tau_I \leq \tau_C \leq T\} & E &= \{T \leq \tau_I \leq \tau_C\} \\
B &= \{\tau_I \leq T \leq \tau_C\} & F &= \{T \leq \tau_C \leq \tau_I\} \\
C &= \{\tau_C \leq \tau_I \leq T\} \\
D &= \{\tau_C \leq T \leq \tau_I\}
\end{aligned}$$

Notice that A to D are the default events, while E and F are the non-default ones.

Let us call $\Pi^D(t; T)$ the discounted payoff of a generic defaultable claim at t and $\Pi(t; T)$ the discounted payoff for an equivalent claim with a default-free counterparty. We then have that at valuation time t , and conditional on the event $\{\tau > t\}$, the price of the payoff under bilateral counterparty risk is:

$$\begin{aligned}
\mathbb{E}_t[\Pi^D(t, T)] &= \mathbb{E}_t[\Pi(t, T)] + \\
&\quad \mathbb{E}_t[\text{LGD}_I 1_{\{A \cup B\}} D(t, \tau_I) (-NPV(\tau_I))^+] - \\
&\quad \mathbb{E}_t[\text{LGD}_C 1_{\{C \cup D\}} D(t, \tau_C) (NPV(\tau_C))^+]
\end{aligned}$$

Where \mathbb{E} is the risk-neutral expectation, REC_i is the recovery fraction with $i \in \{I, C\}$, and $\text{LGD}_i = 1 - REC_i$ is the loss given default.

Therefore, the value of a defaultable claim is the value of the corresponding default-free claim plus a long position in a put option plus a short position in a call option.

The second term and the third term being subtracted from the second one are called respectively Debit Valuation Adjustment (DVA) and Credit Valuation Adjustment.

2.2 Non-contingent CVA

In the non-contingent approach, each participant in the transaction considers itself default-free. Using the same notation than above, the adjustment calculated by the investor would be:

$$\mathbb{E}_t[\text{LGD}_C 1_{\{\tau_C < T\}} D(t, \tau_C) (\text{NPV}(\tau_C))^+]$$

While the one calculated by the counterparty would be:

$$\mathbb{E}_t[\text{LGD}_I 1_{\{\tau_I < T\}} D(t, \tau_I) (-\text{NPV}(\tau_I))^+]$$

The main drawback of this approximation is that one adjustment is not the opposite of the other as in the contingent case. Therefore, the two parties would not agree on the value of the counterparty risk adjustment to be added to the default free price unless one of them was default-free.

3 A dynamic model for default intensity and interest rates

To price CVA and DVA we must consider a model with stochastic intensity and interest rates. As exposed in [Schonbucher], there are basically two types of tractable approaches when trying to model credit and interest rates simultaneously:

1. The Gaussian setup. This framework suffers from the possibility of reaching negative credit spreads and interest rates with positive probability, but a high degree of analytical tractability is retained.

2. The Cox-Ingersoll-Ross (CIR) setup. This approach gives us the required properties of non-negativity, but it loses some analytical tractability.

Non-negative intensities are even more desirable than non-negative interest rates since credit traditionally shows higher levels of volatility. Thus, to ensure non-negative intensities while retaining as much analytical tractability as possible, we follow the hybrid approach of [Brigoetal09]: Gaussian setup for interest rates, and CIR for intensities.

3.1 Interest rate model

For interest rates, we will assume that the dynamics of the instantaneous short-rate process under the risk-neutral measure \mathbb{Q} will be given by a G2++:

$$r(t) = x(t) + z(t) + \varphi(t, \alpha) \quad (1)$$

where α is a set of parameters and the processes x and z are \mathcal{F}_t adapted and satisfy

$$\begin{cases} dx(t) = -ax(t)dt + \sigma dW_1(t), & x(0) = 0 \\ dz(t) = -bz(t)dt + \eta dW_2(t), & z(0) = 0 \end{cases} \quad (2)$$

where (W_1, W_2) is a two-dimensional Brownian motion with instantaneous correlation ρ_{12} , being $-1 \leq \rho_{12} \leq 1$, and a, b, σ and η are positive constants. These are the parameters defining $\alpha = [a, b, \sigma, \eta, \rho_{12}]$. The function $\varphi(t, \alpha)$ is set to match the initial zero coupon curve observed in the market.

3.2 Counterparty and Investor Credit Spread models

For the stochastic intensity models we will set

$$\lambda_t^i = y_t^i + \psi^i(t, \beta^i), \quad i \in \{I, C\} \quad (3)$$

The function ψ is a deterministic function and is set to match the initial CDS spread curve. y is assumed to be a Cox-Ingersoll-Ross process under the risk-neutral measure:

$$dy_t^i = \kappa^i(\mu^i - y_t^i)dt + \nu^i \sqrt{y_t^i} dW_3^i(t), \quad i \in \{I, C\} \quad (4)$$

where the parameter vector is $\beta^i = (\kappa^i, \mu^i, \nu^i, y_0^i)$ and each parameter is a positive deterministic constant. Notice that, in principle, y_0^i is a parameter at our disposal. y_t^i will be always positive as long as $2\kappa^i \mu^i > (\nu^i)^2$. As usual, W_3^i is a standard Brownian motion process under the risk neutral measure.

3.3 Spread correlation

Short interest-rate factors x and z are correlated with the intensity process y through their driving Brownian motions:

$$dW_j dW_3^i = \rho_{ji} dt, \quad j \in \{1, 2\}, \quad i \in \{I, C\}$$

In order to reduce the number of free parameters, we will proceed as in [Brigoetal09], and consider that

$$\rho_{1i} = \rho_{2i}, \quad i \in \{I, C\}$$

Further, we also allow for correlation between default intensities of the investor and the counterparty:

$$dW_3^I dW_3^C = \rho_{IC} dt$$

3.4 Default correlation

Cumulated intensity can be defined as:

$$\Lambda(t) = \int_0^t \lambda_s ds$$

such that $\mathbb{Q}\{\tau \geq t\} = \exp\{-\Lambda(t)\}$. We are in a Cox process setting, where:

$$\tau_i = \Lambda_i^{-1}(\xi_i), \quad i \in \{I, C\}$$

with ξ_I and ξ_C standard (unit-mean) exponential random variables. We impose their associated uniforms $U_i = 1 - \exp(-\xi_i)$, $i \in \{I, C\}$ to be correlated through a copula function. Thus,

$$\mathbb{Q}\{U_I < u_I, U_C < u_C\} = C(u_I, u_C)$$

We choose copula C to be Gaussian with correlation parameter ρ_{Cop} . Notice that this is a default correlation, connecting default times even if spreads were independent. As pointed in [Brigoetal09], where a Gaussian copula is used too, in general high default correlation creates more dependence between default times than a high correlation in spreads.

3.5 Monte Carlo techniques

Payoffs will be valued using Monte Carlo simulation.

The transition density for the G2++ model is known in closed form. As shown in, for example, [BrigoMercurio], let us consider the stochastic process

$$dx(t) = -kx(t)dt + \zeta dW(t), \quad x(0) = 0$$

Then, for $t \geq s$, $x(t)$ is normally distributed with mean $x(s) \exp\{-k(t-s)\}$ and variance $\frac{\zeta^2}{2k}[1 - \exp\{-2k(t-s)\}]$.

Regarding default intensities, we will use the Euler Implicit positivity-preserving scheme presented in [BrigoAlfonsi04]. If we consider the CIR process:

$$dy(t) = \kappa(\mu - y(t))dt + \nu\sqrt{y(t)}dW(t)$$

Then, for $t \geq s$, we have:

$$\sqrt{y(t)} = \frac{\nu(W_t - W_s) + \sqrt{\nu^2(W_t - W_s)^2 + 4\left[y_s + (\kappa\mu - 0.5\nu^2)(t-s)\right][1 + \kappa(t-s)]}}{2[1 + \kappa(t-s)]}$$

4 Calibration of interest rates

4.1 Calibration procedure

The parameters of the interest-rate model under the risk-neutral measure can be calibrated to the surface of at-the-money (ATM) swaption volatilities. A swaption is an option on an interest rate swap (IRS). There are basically two types of swaptions: payer and receiver.

A European payer swaption gives the right (but not the obligation) to enter a payer IRS (paying fixed, receiving floating) of a given length (tenor) at a given fixed rate (strike) and at a given future time (maturity). Conversely, a European receiver swaption gives the right to enter a receiver IRS (receiving fixed, paying floating).

Consider a swaption with strike S_K , maturity $T = t_0$ and swap payment times $\mathcal{T} = \{t_1, \dots, t_n\}$, $t_1 > T$. It is a common practice to value swaptions with a Black-like formula. In this setup, the price of a swaption is⁴:

$$ES^{Black}(0, T, \mathcal{T}, S_K, \omega; \sigma) = \omega \sum_{i=1}^n \tau_i P(0, T_i) \left[S(0) \Phi(\omega d_1) - S_K \Phi(\omega d_2) \right]$$

where $\omega = 1$ ($\omega = -1$) for a payer (receiver) swaption, $S(0)$ is the forward swap rate, $P(0, T)$ is the discount factor between 0 and T , τ_i the year fraction from t_{i-1} to t_i , Φ is the standard normal cdf, and

$$d_1 = \frac{\ln(S(0)/S_K) + \sigma^2 T/2}{\sigma \sqrt{T}} \quad d_2 = d_1 - \sigma \sqrt{T}$$

Swaption prices are typically displayed in a matrix, where each row is indexed by the swaption maturity T_α , whereas each column is indexed in terms of the underlying swap length, $T_\beta - T_\alpha$. The $x \times y$ -swaption is then the swaption in the table whose maturity is x years and whose underlying swap is y years long. Thus a 2×10 swaption is a swaption maturing in two years and giving then

⁴See [Hull]

the right to enter a ten-year swap. It is a common practice to quote Black-volatilities instead of prices.

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	66.9	64.5	57.9	53.3	47.1	42.7	39.2	36.6	34.3	32.5
2y	58.7	52.6	57.1	43.2	40.4	37.2	34.9	33.1	31.7	30.6
3y	50.6	44.7	40.6	37.6	35.6	33.4	31.8	30.5	29.5	28.6
4y	42.2	37.8	35.1	33.0	31.5	30.1	29.1	28.2	27.6	27.0
5y	36.0	33.1	31.2	29.7	28.6	27.8	27.1	26.5	26.1	25.8
7y	28.3	27.3	26.2	25.5	25.1	24.7	24.5	24.4	24.4	24.4
10y	23.1	23.0	22.7	22.6	22.7	22.8	23.0	23.3	23.6	23.7

Table 1: ATM swaption volatilities. 07-Feb-2013

[BrigoMercurio] find out the expression for European swaption prices under G2++, shown in Appendix A. Using this formula, swaption prices can be fitted to the market data surface by minimizing square errors, that is, we solve

$$\hat{\alpha} = \operatorname{argmin}_{\alpha} \sum_i [ES_{ATM}^{Black}(0, T_i, T_i; \sigma^{Mkt}) - ES_{ATM}(0, T_i, T_i; \alpha)]^2$$

Notice that we match swaption *prices* obtained from market quoted *volatilities*. Once we have obtained the parameters of $\alpha = [a, b, \sigma, \eta, \rho_{12}]$, we can use $\varphi(t, \alpha)$ to automatically fit the initial zero coupon curve:

$$\varphi(t, \alpha) = f(0, t) + \frac{\sigma^2}{2} B(a, 0, t)^2 + \frac{\eta^2}{2} B(b, 0, t)^2 + \rho_{12} \sigma \eta B(a, 0, t) B(b, 0, t)$$

where $f(0, t)$ is the instantaneous forward rate at time 0 for a maturity t implied by the initial zero coupon curve:

$$f(0, T) = -\frac{\partial \ln P(0, T)}{\partial T}$$

4.2 Estimation results

We intend to capture counterparty valuation adjustments in two different scenarios, depending on whether the market considered Spanish financial institutions to be more or less solvent than the average European company. Figure 2 displays the evolution of 5-year tenor CDS of BBVA, Santander and iTraxx Europe from 2007 to the beginning of 2013. We will pick two dates for our analysis, only four years apart: February 2009 and February 2013. The first can be regarded as the last moment before the situation reversed: although Lehman Brothers had expired only a few months ago, financial markets were starting to calm down, while Spain had just entered recession. The latter date, however, depicts a moment somewhat less stressed than June 2012, when Spain accepted the financial bailout, but still the market made a clear distinction between Spanish and European institutions. To capture the recent dynamics on each case, we will use two years of data from these two moments backwards.

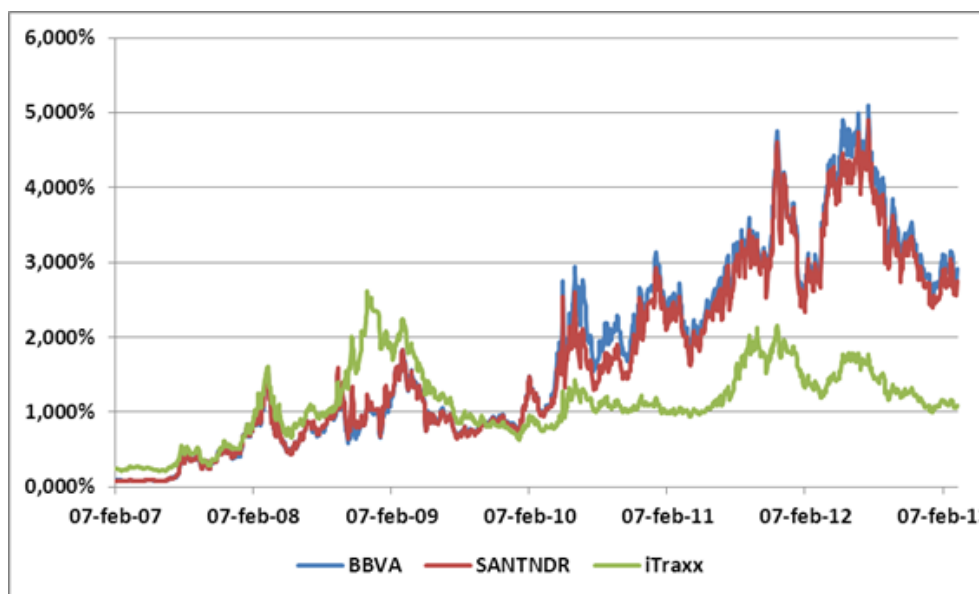


Figure 2: Evolution of CDS5y for BBVA, Santander and iTraxx Europe (2007-2013)

4.2.1 After the crisis

We calibrate the risk-neutral dynamics to the surface of ATM swaption volatilities on 07 February 2013. Minimization of the sum of the squares of the percentage differences between model and market swaption prices produces the following calibrated parameters: $a = 0.56160993$, $b = 0.011979556$, $\sigma = 0.005145749$, $\eta = 0.007824323$ and $\rho_{12} = -0.780480924$. Calibration results are summarized in Tables 2 and 3. In the first, we show the fitted swaption volatilities as implied by the G2++ model, whereas in the second we report the absolute differences. Apart from some exceptions in the shortest maturities, differences are rather low given that we are fitting seventy prices with only five parameters:

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	105.0	86.1	71.4	60.0	51.9	46.0	41.7	38.5	36.0	34.1
2y	72.4	61.6	52.7	46.4	41.8	38.3	35.7	33.7	32.1	30.7
3y	52.7	46.0	41.3	37.8	35.1	33.1	31.5	30.2	29.1	28.3
4y	40.0	36.8	34.3	32.3	30.8	29.5	28.5	27.6	27.0	26.5
5y	33.4	31.6	30.1	28.9	28.0	27.2	26.5	26.0	25.6	25.1
7y	26.8	26.3	25.7	25.3	24.8	24.6	24.3	24.0	24.2	24.3
10y	23.5	23.3	23.5	23.4	23.1	23.7	24.0	24.1	24.1	24.0

Table 2: G2++ calibrated swaption volatilities. 07-Feb-2013

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	38.1	21.6	13.6	6.7	4.8	3.3	2.5	2.0	1.7	1.6
2y	13.7	9.0	5.6	3.2	1.4	1.1	0.8	0.5	0.3	0.1
3y	2.1	1.3	0.7	0.2	-0.5	-0.4	-0.3	-0.3	-0.4	-0.3
4y	-2.2	-1.0	-0.8	-0.7	-0.7	-0.6	-0.6	-0.6	-0.5	-0.5
5y	-2.6	-1.6	-1.1	-0.8	-0.7	-0.6	-0.6	-0.5	-0.5	-0.7
7y	-1.5	-1.0	-0.5	-0.3	-0.3	-0.1	-0.2	-0.4	-0.2	-0.1
10y	0.4	0.3	0.7	0.7	0.4	0.9	0.9	0.8	0.5	0.3

Table 3: Swaption calibration results: absolute differences. 07-Feb-2013

4.2.2 Before the crisis

We calibrate the risk-neutral dynamics to the surface of ATM swaption volatilities on 06 February 2009, just before the Spanish economy started entering recession. Minimization of the sum of

the squares of the percentage differences between model and market swaption prices produces the following calibrated parameters: $a = 0.03027296$, $b = 0.040462377$, $\sigma = 0.008905182$, $\eta = 0.008788244$ and $\rho_{12} = -0.378510205$. Calibration results are summarized in Tables 4 and 5. In this case, there are no extreme differences as in 2013. Instead, we observe a sort of V-structure, with high and positive differences for short tenors and large maturities that decrease with tenor and become negative for short maturities.

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	41.7	35.1	31.2	28.6	26.5	24.9	23.6	22.5	21.5	20.8
2y	30.9	28.2	26.2	24.6	23.3	22.2	21.3	20.5	19.8	19.2
3y	26.4	24.8	23.4	22.3	21.3	20.5	19.8	19.2	18.6	18.3
4y	23.9	22.6	21.6	20.7	19.9	19.2	18.7	18.1	17.9	17.6
5y	21.8	20.9	20.1	19.4	18.7	18.3	17.7	17.5	17.3	16.9
7y	19.4	18.7	18.1	17.7	17.2	17.1	16.9	16.5	16.7	16.8
10y	17.6	16.9	17.1	16.9	16.5	16.9	17.1	17.1	17.0	16.8

Table 4: G2++ calibrated swaption volatilities. 06-Feb-2009

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	1.2	1.3	0.1	-1.0	-2.1	-2.9	-3.7	-4.5	-5.2	-5.9
2y	5.5	4.6	3.2	2.1	1.1	0.2	-0.7	-1.5	-2.4	-3.3
3y	5.7	4.7	3.4	2.4	1.5	0.6	-0.1	-0.8	-1.6	-2.0
4y	5.5	4.4	3.3	2.4	1.5	0.8	0.3	-0.4	-0.8	-1.3
5y	5.0	4.1	3.1	2.3	1.5	1.1	0.4	0.0	-0.3	-0.9
7y	4.2	3.4	2.6	2.2	1.5	1.3	0.9	0.2	0.1	-0.1
10y	3.2	2.5	2.4	2.0	1.3	1.3	1.2	0.7	0.2	-0.4

Table 5: Swaption calibration results: absolute differences. 06-Feb-2009

5 Calibration of default intensities

5.1 Introduction

Literature has not been conclusive about how to calibrate the dynamics of default intensities. In general terms, quotes for CDS options, specially single name, are considered illiquid and not reliable. Therefore, one needs to rely entirely on CDS prices. The typical calibration procedure can be schematized as follows:

1. Assume independency between interest rates and default intensities.
2. Guess a suitable β .
3. Set ρ to a desired value.

Notice that there is some contradiction between steps 1 and 3, since we are assuming zero correlation first to depart from that assumption at the end. However, literature has repeatedly shown⁵ that correlations have a small impact on CDS prices, so the consequences of this contradiction are negligible.

Regarding step 2, some works specify "reasonable" values for the parameters, as in [BrigoPallav]. Another usual approach, followed by, for example, [BrigoChourd] for a CIR++ for both interest rates and default intensities, consists on imposing some constraints in the calibration of CDS. In their case, they require β to be found that keep Ψ positive and increasing⁶, which is achieved by setting $2\kappa\mu > \nu^2$, and that minimize $\int_0^T \Psi(s, \beta)^2 ds$. This minimization amounts to contain the departure of λ from its time-homogeneous component y as much as possible. Unfortunately, this approach involves using no real information about the evolution of intensities.

We will propose a method that will allow us to calibrate intensities from historical market data. By using several approximations, we will be able to obtain a closed-form expression for both CDS prices and instantaneous covariances among them, that will be fitted to observed historical ones.

The remainder of this section is structured as follows: first, we will give a quick overview of Credit Default Swaps. Then, we will present separability, that is, the assumption of independency between interest rates and default intensities. As we shall see, this will allow us to easily price CDS contracts in our modeling framework. Next, we will depart from separability, and still, using some approximations, we will reach a closed form expression for the price of a CDS. Then, applying some stochastic calculus to this expression plus some approximations, we will come out with an expression for the instantaneous covariance between different CDS contracts that can be used to

⁵See, for example, [BrigoAlfonsi04]

⁶A definition of Ψ will be shown in equation (7)

calibrate the dynamics of default intensities. Finally, we apply this method to the same two periods described above for interest rate calibration and show some results.

5.2 Credit Default Swaps

A credit default swap is a contract ensuring protection against default. Two companies, named "A" and "B", agree on the following:

If a third reference company "C" defaults at time $\tau < T$, where T is the maturity of the contract, "B" pays to "A" a certain amount of cash LGD. This cash is a *protection* for "A" in case "C" defaults.

In exchange for this protection, company "A" agrees to pay periodically to "B" a fixed amount S . Payments occur at times $\mathcal{T} = \{T_1, \dots, T_n\}$, day-count-fractions are described as $\alpha_i = T_i - T_{i-1}$, $T_0 = 0$, fixed in advance at time 0 up to default time τ if this occurs before maturity T , or until maturity T otherwise.

Credit events are carefully defined in CDS contracts, and they typically include bankruptcy, failure to pay, restructuring, obligation acceleration, obligation default and repudiation. A detailed survey on configuration and settlement of CDS contracts can be found in [Gregory10].

Formally, we may write the CDS discounted value to "B" at time t as:

$$1_{\{\tau > t\}} \left(D(t, \tau) (\tau - T_{\beta(t)-1}) 1_{\{\tau < T_n\}} S + \sum_{i=\beta(t)}^n D(t, T_i) \alpha_i 1_{\{\tau > T_i\}} S - 1_{\{\tau < T\}} D(t, \tau) \text{LGD} \right)$$

where $t \in [T_{\beta(t)-1}, T_{\beta(t)})$, i.e. $T_{\beta(t)}$ is the first date of T_1, \dots, T_n following t . The stochastic discount factor at time t for maturity T is denoted by $D(t, T) = B(t)/B(T)$, where $B(t) = \exp(\int_0^t r_u du)$ denotes the bank-account numeraire, r being the instantaneous short interest rate.

We denote by $CDS(t, T, T, S, \text{LGD})$ the price of the above CDS. We can compute this price according to risk-neutral valuation as in [BielRut]:

$$\begin{aligned}
CDS(t, T, T, S, \text{LGD}) = & 1_{\{\tau > t\}} \mathbb{E} \left[D(t, \tau) (\tau - T_{\beta(t)-1}) 1_{\{\tau < T_n\}} S \right. \\
& \left. + \sum_{i=\beta(t)}^n D(t, T_i) \alpha_i 1_{\{\tau > T_i\}} S - 1_{\{\tau < T\}} D(t, \tau) \text{LGD} \middle| \mathcal{G}_t \right] \quad (5)
\end{aligned}$$

5.3 Separability

Let us assume independence between interest rates and default intensities, i.e., $\rho_{1i} = \rho_{2i} = 0$, $i = I, C$. In this case, we can *separate* some terms in expression (5) and get the following model independent formula:

$$\begin{aligned}
CDS(t, T, T, S, \text{LGD})^0 = & \text{LGD} \left[\int_{T_0}^{T_n} P(0, t) dt \mathbb{Q}\{\tau_i \geq t\} \right] \\
& + S \left[\int_{T_0}^{T_n} P(0, t) (t - T_{\beta(t)-1}) dt \mathbb{Q}\{\tau_i \geq t\} \right. \\
& \left. + \sum_{j=1}^n \alpha_j P(0, T_j) \mathbb{Q}\{\tau_i \geq T_j\} \right] \quad (6)
\end{aligned}$$

where we have assumed a constant LGD, $\mathbb{Q}\{\tau \geq T\}$ is the survival probability at T , and

$$d_t \mathbb{Q}\{\tau \geq t\} = \mathbb{Q}\{\tau \in [t, t + dt)\}$$

Let us define the following integrated quantities:

$$\Lambda(t) = \int_0^t \lambda_s ds, \quad Y(t) = \int_0^t y_s ds, \quad \Psi(t, \beta) = \int_0^t \psi(s, \beta) ds \quad (7)$$

In the market, we can observe CDS spread quotes for several maturities $T_n = 1y, 2y, \dots$, with T_i 's resetting quarterly. We can assume a piecewise constant structure for implied hazard rates and solve

$$\begin{aligned}
CDS_{0,1y}(S_{0,1y}, \text{LGD}; \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_{Mkt}^1) &= 0; \\
CDS_{0,2y}(S_{0,2y}, \text{LGD}; \lambda_{Mkt}^1; \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_{Mkt}^2) &= 0; \dots
\end{aligned}$$

iteratively, obtaining a function for the survival probabilities implied by the market:

$$\mathbb{Q}_{Mkt}\{\tau \geq t\} = \exp\{-\Lambda_{Mkt}(t)\}$$

Back to our model, the survival probabilities are given by:

$$\mathbb{Q}\{\tau \geq t\} = \mathbb{E}_0[\exp(-\Lambda(t))] = \mathbb{E}_0[\exp(-\Psi(t, \beta) - Y(t))]$$

If we match these probabilities to the ones observed in the market, we obtain:

$$\Psi(t, \beta) = \ln\left(\frac{\mathbb{E}_0[e^{-Y(t)}]}{\mathbb{Q}_{Mkt}\{\tau \geq t\}}\right) = \ln(P_{CIR}(0, t, y_0; \beta)) + \Lambda_{Mkt}(t)$$

where P_{CIR} is the closed form expression for bond prices in the time-homogeneous CIR model with initial condition y_0 and parameters β , given by:

$$P_{CIR}(t, T, y_t; \beta) = A_{CIR}(t, T)e^{B_{CIR}(t, T)y_t}$$

where

$$A_{CIR}(t, T) = \left(\frac{2h \exp\{(\kappa+h)(T-t)/2\}}{2h+(\kappa+h)(\exp\{(T-t)h\}-1)}\right)^{2\kappa\mu/\nu^2}$$

$$B_{CIR}(t, T) = \frac{2(\exp\{(T-t)h\}-1)}{2h+(\kappa+h)(\exp\{(T-t)h\}-1)}$$

$$h = \sqrt{\kappa^2 + 2\nu^2}$$

If β is chosen in order to have a positive function ψ , the model will be automatically calibrated to the market survival probabilities stripped from CDS data.

5.4 Departing from separability

The former chain of reasoning has relied on the possibility to disentangle survival probabilities from discount factors. If $\rho_{1i} = \rho_{2i} \neq 0$, $i = I, C$, then formula (6) no longer holds and to calibrate CDS data we need to solve

$$CDS(t, \mathcal{T}, T, S, \text{LGD}) = 0$$

numerically to compute prices of CDS and use them to calibrate the model, which can be a rather dramatic task. However, using Gaussian mapping techniques as in [BrigoAlfonsi04] we are able to approximate formula (5) with a closed form solution to ease the calibration process:

$$\begin{aligned} CDS(t, \mathcal{T}, T, S, \text{LGD}; x_t, z_t, y_t) = & S \int_t^T e^{-\int_t^u (\varphi(s, \alpha) + \psi(s, \beta)) ds} \left[\psi(u, \beta) \widehat{H}_2(u) \right. \\ & \left. + \widehat{H}_1(u) \right] (u - T_{\beta(u)-1}) du \\ & + S \sum_{i=1}^n \alpha_i e^{-\int_t^{T_i} (\varphi(s, \alpha) + \psi(s, \beta)) ds} \widehat{H}_2(T_i) \\ & - \text{LGD} \int_t^T e^{-\int_t^u (\varphi(s, \alpha) + \psi(s, \beta)) ds} \left[\psi(u, \beta) \widehat{H}_2(u) + \widehat{H}_1(u) \right] du \end{aligned} \quad (8)$$

Details about the auxiliary functions \widehat{H}_i , as well as the derivation of the formula, can be found in Appendix B.

5.5 Calibration procedure

We intend to obtain an expression for the instantaneous covariance between CDS spreads. To do so, we first write our state variables in matricial form, and name:

$$\bar{X}_t = \begin{bmatrix} x(t) \\ z(t) \\ y^I(t) \\ y^C(t) \end{bmatrix}; \quad W_t = \begin{bmatrix} W_1(t) \\ W_2(t) \\ W_3^I(t) \\ W_3^C(t) \end{bmatrix}$$

\bar{X}_t has the following dynamics:

$$d\bar{X}_t = \mathcal{O}(dt) + \left[\Sigma_1 + \Sigma_2 \sqrt{\bar{X}_t} \right] dW_t$$

Where $\mathcal{O}(dt)$ are terms in dt and

$$\Sigma_1 = \begin{bmatrix} \sigma & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \Sigma_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \nu^I & 0 \\ 0 & 0 & 0 & \nu^C \end{bmatrix}$$

By inverting equation (8), we can obtain a function S_T for the par value of a CDS spread, that is:

$$CDS(t, T, T, S_T(t, \bar{X}_t), \text{LGD}; \bar{X}_t) = 0$$

We can apply Ito's Lemma to find out the dynamics of the par spread of the CDS:

$$dS_T = \frac{\partial S_T}{\partial t} + \nabla^\top S_T(t, \bar{X}_t) d\bar{X}_t + \frac{1}{2} d\bar{X}_t^\top \nabla^2 S_T(t, \bar{X}_t) d\bar{X}_t$$

where $\nabla^\top S_T(t, \bar{X}_t)$ and $\nabla^2 S_T(t, \bar{X}_t)$ are the gradient and hessian matrix of S with respect to \bar{X} , respectively. We can isolate the terms in dt and get:

$$dS_T = \mathcal{O}(dt) + \nabla^\top S_T(t, \bar{X}_t) \left[\Sigma_1 + \Sigma_2 \sqrt{\bar{X}_t} \right] dW_t = \mathcal{O}(dt) + \Sigma_T(t, \bar{X}_t) dW_t$$

We realize that the diffusion of the CDS spread is stochastic itself, since it depends on \bar{X}_t . We propose to approximate it by taking:

$$\Sigma_T(t, \bar{X}_t) \approx \Sigma_T(t, E[\bar{X}_t]) \quad (9)$$

This assumption is similar to the one adopted in [AndPiterbarg] to proxy the dynamics of swap rates. Additionally, we will need to impose that processes y^I and y^C start at their long-term mean, that is, we state:

$$y_0^i = \mu^i, \quad i \in \{I, C\}$$

To check the validity of the approximation taken in equation (9), we do three exercises with a given set of parameters:

1. Historical: We simulate $x(t)$, $z(t)$ and $y(t)$ along two years (500 time steps). For each day, we compute the value of the CDS spread with equation (8) for different tenors, obtain their daily increments and the standard deviation of them. We repeat this exercise 10,000 times and return the average of the standard deviations.
2. One-day: We run 10,000 simulations of $x(t)$, $z(t)$ and $y(t)$ for one day. We compute the value of the CDS with equation (8) for different tenors, obtain their daily increments and get the standard deviation.
3. Theoretical: We obtain the standard deviations with equation (10).

We repeat the exercise for three references: BBVA, Santander and iTraxx, using the parameters obtained calibrating the period 2007-2009. As shown in Table 6, differences among the three approaches are minimal.

BBVA	Historical	One-day	Theoretical
3Y	0.0540%	0.0539%	0.0540%
5Y	0.0524%	0.0524%	0.0524%
7Y	0.0508%	0.0509%	0.0508%
10Y	0.0485%	0.0484%	0.0486%
Santander	Historical	One-day	Theoretical
3Y	0.0542%	0.0544%	0.0543%
5Y	0.0532%	0.0536%	0.0533%
7Y	0.0523%	0.0519%	0.0524%
10Y	0.0506%	0.0507%	0.0508%
iTraxx	Historical	One-day	Theoretical
3Y	0.04575%	0.04593%	0.04569%
5Y	0.04446%	0.04459%	0.04445%
7Y	0.04308%	0.04331%	0.04321%
10Y	0.04119%	0.04113%	0.04130%

Table 6: Standard deviations of CDS increments

We will observe historical prices of CDS spreads of different maturities, T_1 and T_2 , and different counterparties, $C_1, C_2 \in \{I, C\}$. Then, the instantaneous covariance of their increments will be given by:

$$\Sigma_{T_1, C_1}(t, E[\bar{X}_t]) \Omega \Sigma_{T_2, C_2}(t, E[\bar{X}_t])^\top dt \quad (10)$$

where

$$\Omega = \begin{bmatrix} 1 & \rho_{12} & \rho_{1I} & \rho_{1C} \\ \rho_{12} & 1 & \rho_{1I} & \rho_{1C} \\ \rho_{1I} & \rho_{1I} & 1 & \rho_{IC} \\ \rho_{1C} & \rho_{1C} & \rho_{IC} & 1 \end{bmatrix}$$

Therefore, we will be able to calibrate the model by taking the variance-covariance matrix of the daily increments of historical prices of CDS and fitting it to the parametric matrix. Since we are dealing with instantaneous covariances, our reasoning is measure independent, so we can get rid of the market price of risk and estimate the model in a consistent manner.

5.6 Estimation results

For each period (before and along the Spanish recession), we take two years of data of CDS prices from BBVA and Santander, the two leading Spanish banks. As a proxy for a general counterparty, we will use iTraxx Europe. iTraxx is the brand name for the family of several credit default swap index products. The most widely traded of the indices is the iTraxx Europe index, also known simply as "The Main", composed of the most liquid 125 CDS referencing European investment grade. Since we will calibrate the three references simultaneously, we will incorporate correlation among the two investors' intensities, i.e., BBVA and Santander. Therefore, we will obtain the corresponding β for each reference, plus some additional cross-correlations. Let us call ϱ the vector containing all these parameters.

In our minimization problem, we will match standard deviations for the same CDS contract and correlations for different ones. Since they have a different order of magnitude, we will need to weight their contributions to the minimization function differently. The problem can be stated as:

$$\hat{\varrho} = \underset{\varrho}{\operatorname{argmin}} \quad \omega_{Corr} \sum_{i=1}^n \sum_{j=i}^n [Corr^{Hist}(S_i, S_j) - Corr(S_i, S_j; \varrho)]^2 \\ + \omega_{Std} \sum_{i=1}^n [Std^{Hist}(S_i) - Std(S_i; \varrho)]^2$$

We first observe CDS prices on 7 February 2013 for the three references: BBVA, Santander and iTraxx, and tenors 3y, 5y, 7y and 10y. If we extract hazard rates for that date, we observe that the implied default probabilities of both Spanish banks are much higher than those of an average European counterparty, as seen in Figure 3.

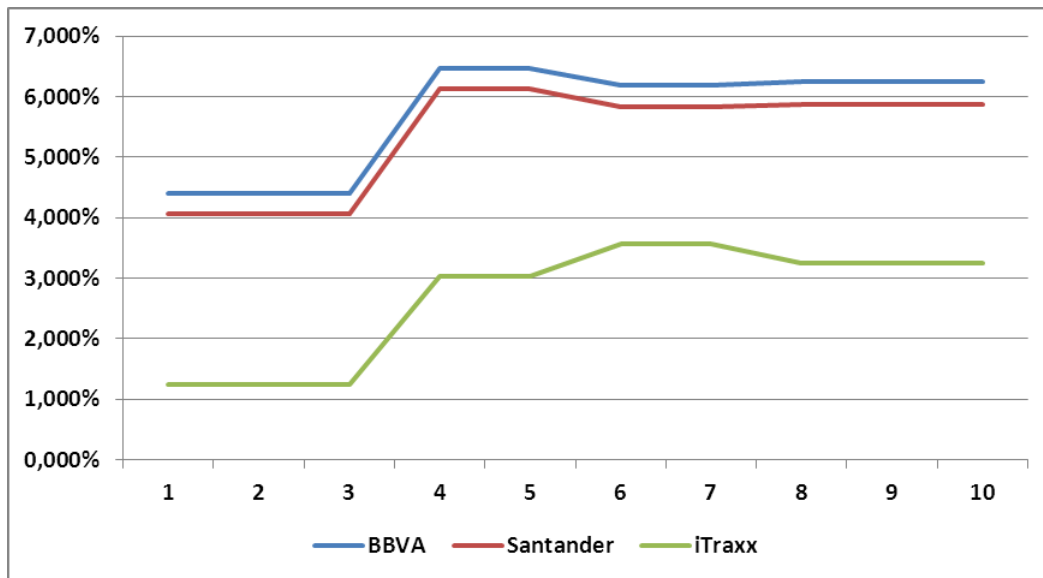


Figure 3: Hazard rates. 7-Feb-2013

However, if we repeat this exercise on 6 February 2009, we observe a completely different situation. As seen in Figure 4, at the beginning of the crisis the implied default probability of an average European counterparty was much higher than those of BBVA or Santander, especially for short maturities. This can be explained by the then ongoing mistrust environment, with entities questioning each other's balance sheets, but hoping that the situation would be clarified in the medium term.

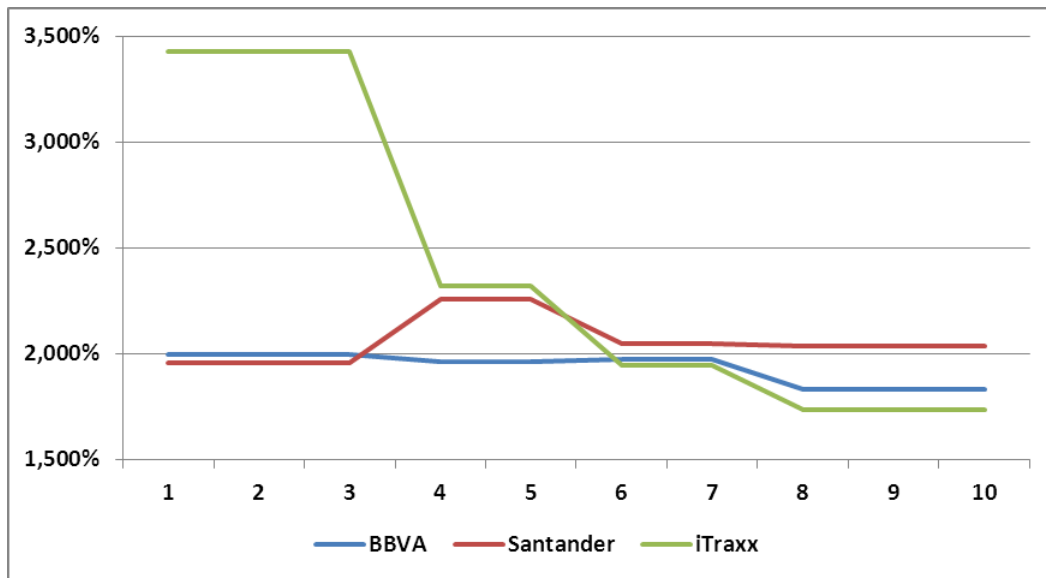


Figure 4: Hazard rates. 6-Feb-2009

We use daily data for two periods: from 7 February 2013 to 7 February 2011, and from 6 February 2009 to 6 February 2007. With the interest rate parametrization detailed above for the corresponding periods, we calibrate the historical correlation matrix. Calibrated parameters are shown on Table 7. For a comparison between historical and parametric matrices, see Appendix C.

6 A Spanish case study: interest rate swap

6.1 The payoff

The Bank for International Settlements reported in a recent review⁷ that over 60% of the entire notional amount outstanding in over-the-counter (OTC) derivatives corresponds to interest rate swaps (IRS), becoming by far the largest category by instrument (the next one, forward rate agreements, adds up to less than 9% of the aggregated notional). Thus, as a testing payoff, we will consider a 10-year par interest rate swap with both legs paying annually and compute the

⁷BIS Quarterly Review, June 2013. Downloadable at www.bis.org/statistics/dt1920a.pdf (last accessed: 29 September 2013)

Parameter	2007-2009	2011-2013
BBVA		
κ^1	2.7866%	2.5387%
μ^1	11.8385%	22.8575%
ν^1	5.2150%	9.0083%
Santander		
κ^2	1.2831%	10.5391%
μ^2	11.3247%	20.0987%
ν^2	5.2416%	11.5031%
iTraxx		
κ^3	2.8553%	2.7701%
μ^3	6.4543%	8.9746%
ν^3	5.9834%	4.6954%
ρ_{13} (Rates-BBVA)	-20.0656%	-22.2291%
ρ_{14} (Rates-Santander)	-25.2547%	-29.3652%
ρ_{15} (Rates-iTraxx)	-25.9896%	-18.4379%
ρ_{34} (BBVA-Santander)	91.1790%	90.9663%
ρ_{35} (BBVA-iTraxx)	81.3762%	82.4476%
ρ_{45} (Santander-iTraxx)	78.8252%	87.1892%

Table 7: Calibrated parameters

adjustments in several scenarios. We will also consider the case in which credit spreads are not simulated, and survival probabilities are extracted as seen the initial day. This approach is the one followed by most of the industry, since only one-third of banks model general WWR within CVA calculation⁸, which is achieved by jointly simulating credit spreads and underlying risk factors, linking credit worthiness and exposure. We will compute the adjustments in both the contingent and non-contingent framework.

When computing the adjustments, default times will be bucketed by assuming that default events can occur only on a time grid $\{T_i : 0 \leq i < n\}$, with $T_0 = t$ and $T_n = T$, anticipating each default event to the last T_i preceding it. A monthly time-grid spacing will be used to do the calculations. Thus, credit valuation adjustments in the contingent case will be calculated as follows:

⁸According to [DeloitSolum]

$$CVA(t, T) \simeq \text{LGD} \sum_{i=0}^{n-1} \mathbb{E}[1_{\{C \cup D\}} D(t, T_i) \mathbb{E}_{T_i}[NPV(T_i)^+]]$$

$$DVA(t, T) \simeq \text{LGD} \sum_{i=0}^{n-1} \mathbb{E}[1_{\{A \cup B\}} D(t, T_i) \mathbb{E}_{T_i}[(-NPV(T_i))^+]]$$

Similar expressions apply to the non-contingent case. Notice that $\mathbb{E}_{T_i}[\cdot]$ refers to forward expectations of the NPV of the underlying product. Since we are dealing with a simple IRS, these expectations have a closed-form solution in our Gaussian framework. Had we dealt with a more complicated payoff, we would have needed to use least-square regressions to estimate these forward expectations, as usually done to price Bermudan options with the Least Squared Monte Carlo Method.

There is no way to accurately estimate ρ_{Cop} , that is, the correlation among default times of the references. We will compute the adjustments for the contingent case in two scenarios, either assuming it to be zero or equal to the estimated spread correlation. We consider iTraxx as the investor and the Spanish banks as the counterparties. Finally, we also consider the case in which BBVA plays the role of the investor with Santander as its counterparty.

6.2 Main findings

Results are shown in Tables 8 to 10. They confirm that both before and after the crisis, credit valuation adjustments are relevant. If we recall that the risk-free price in all cases was zero since the fixed leg of the IRS paid the initial swap rate, the adjustments, especially in 2013, look even more impressive.

Both contingent and non-contingent adjustments reveal how difficult the situation became in 2013 for the Spanish financial sector relative to the European market. While in 2009 both CVA

and DVA were similar in absolute terms (especially when adding volatility), in 2013 the average European counterpart was charging a CVA of around -80bp to a Spanish bank, while her DVA was just around 15bp, and it was lowered to a few basis points if default correlation was added. This makes sense: if my counterparty is much worse than me, and indeed we are positively correlated (the spread correlation, which was used as default correlation too, was around 80%), then my counterparty will default before than me in almost all cases, so DVA will be non-existent in the contingent approach.

We also notice the importance of modeling the dynamics on rigorous CVA valuation. Despite being an interest rate payoff and a low estimated correlation between default intensities and interest rates, including credit volatilities can impact CVA up to 30% (from -76bp to -57bp for BBVA vs Santander in 2013, with contingency and no default correlation) or DVA up to 60% (from near 20bp to near 35bp for iTraxx vs Santander in 2009, non-contingent).

	2009		2013	
	No Vol	Vol	No Vol	Vol
iTraxx vs BBVA				
CVA	-0.36796%	-0.31464%	-0.91614%	-0.82017%
DVA	0.19196%	0.29932%	0.13868%	0.15685%
iTraxx vs Santander				
CVA	-0.39038%	-0.29407%	-0.87382%	-0.71121%
DVA	0.19166%	0.29589%	0.14076%	0.15308%
BBVA vs Santander				
CVA	-0.40694%	-0.29678%	-0.75580%	-0.57484%
DVA	0.14649%	0.24778%	0.29640%	0.36900%

Table 8: Contingent CVA for IR Swap. Copula correlation zero

7 Another case study: CDS

Furthermore, we explore the effects of incorporating credit dynamics when computing counterparty valuation adjustments in credit derivatives. Since we have already developed an expression for the valuation of a Credit Default Swap (CDS) in our modeling framework, we can use it to compute the exposures embedded in the CVA-DVA formula.

	2009		2013	
	No Vol	Vol	No Vol	Vol
iTraxx vs BBVA				
CVA	-0.22224%	-0.22241%	-0.90653%	-0.84264%
DVA	0.16376%	0.25718%	0.05115%	0.05691%
iTraxx vs Santander				
CVA	-0.24812%	-0.21071%	-0.87177%	-0.72775%
DVA	0.16380%	0.25050%	0.04322%	0.03821%
BBVA vs Santander				
CVA	-0.28124%	-0.20970%	-0.51686%	-0.42515%
DVA	0.09875%	0.16936%	0.22340%	0.28221%

Table 9: Contingent CVA for IR Swap. Copula correlation equal to spread correlation

	2009		2013	
	No Vol	Vol	No Vol	Vol
iTraxx vs BBVA				
CVA	-0.41894%	-0.37074%	-1.01071%	-0.93383%
DVA	0.20494%	0.34698%	0.17533%	0.23514%
iTraxx vs Santander				
CVA	-0.44534%	-0.34706%	-0.96482%	-0.81212%
DVA	0.20487%	0.34688%	0.17536%	0.23521%
BBVA vs Santander				
CVA	-0.44524%	-0.34697%	-0.96500%	-0.81231%
DVA	0.15843%	0.29997%	0.35567%	0.52179%

Table 10: Non Contingent CVA for IR Swap

In an in-depth study on CDS transactions, [Chenetal] found that both single-name and index contracts were most frequently traded in 5 year maturities, with 47% of all single-name transactions and 84% of indices being traded in the 5 year tenor. Therefore, as a testing payoff we will consider 5-year par CDS's and compute the adjustments in several scenarios. Contracts will be written on the third reference not involved in the transaction, i.e., in the case of iTraxx vs Santander, we are referring to iTraxx (the investor) selling protection to Santander (the counterparty) on BBVA⁹. As we did above, we will also consider the case in which credit spreads are not simulated at all. Here we introduce a further case in which spreads are simulated when computing exposures but *not* when calculating the survival probabilities for the CVA-DVA formula, which

⁹This situation is quite unlikely, since both Spanish banks are highly correlated and buying default protection on each other would be regarded as unsafe. However, we use this example to highlight the effects of high correlation levels on the adjustments.

are extracted as seen the initial day. Despite the clear inconsistency existing in this scenario, that we will call "Volatility only in Expected Exposure" or "Vol EE", we compute it for didactical purposes.

As shown on Tables 11 to 13, the effect of modeling the dynamics of credit spreads is much stronger when referred to credit derivatives. CVA and DVA, almost non existent in all cases under static credit spreads, jumps in the dynamic scenarios. As an example, in 2013 CVA for iTraxx vs BBVA in the contingent case drops from -1bp with no volatility to -6bp (to -8bp if default correlation is set to zero). Even more astounding is the effect on DVA in, for instance, iTraxx vs Santander, where DVA jumps from less than 2bp to around 24bp if default correlation is set to zero. The same case highlights the relevance of the distinction between contingent and non-contingent adjustments, since DVA drops back to 2bp if default correlation is considered.

	2009			2013		
	No Vol	Vol EE	Vol	No Vol	Vol EE	Vol
iTraxx vs BBVA						
CVA	-0.0052%	-0.0555%	-0.0127%	-0.0129%	-0.2669%	-0.0585%
DVA	0.0007%	0.0915%	0.1442%	0.0159%	0.1302%	0.2465%
iTraxx vs Santander						
CVA	-0.0104%	-0.0588%	-0.0141%	-0.0141%	-0.2588%	-0.0436%
DVA	0.0000%	0.0857%	0.1376%	0.0150%	0.1316%	0.2359%
BBVA vs Santander						
CVA	-0.0362%	-0.0668%	-0.0282%	-0.0016%	-0.0720%	-0.0139%
DVA	0.0000%	0.0360%	0.0815%	0.0550%	0.1394%	0.2508%

Table 11: Contingent CVA for CDS. Copula correlation zero

	2009			2013		
	No Vol	Vol EE	Vol	No Vol	Vol EE	Vol
iTraxx vs BBVA						
CVA	-0.0033%	-0.0353%	-0.0086%	-0.0119%	-0.2244%	-0.0783%
DVA	0.0006%	0.0788%	0.1294%	0.0063%	0.0475%	0.0843%
iTraxx vs Santander						
CVA	-0.0064%	-0.0353%	-0.0072%	-0.0133%	-0.2140%	-0.0326%
DVA	0.0000%	0.0707%	0.1124%	0.0017%	0.0144%	0.0150%
BBVA vs Santander						
CVA	-0.0243%	-0.0465%	-0.0175%	-0.0011%	-0.0335%	-0.0033%
DVA	0.0000%	0.0261%	0.0539%	0.0406%	0.0825%	0.1233%

Table 12: Contingent CVA for CDS. Copula correlation equal to spread correlation

	2009			2013		
iTraxx vs BBVA	No Vol	Vol EE	Vol	No Vol	Vol EE	Vol
CVA	-0.0056%	-0.0595%	-0.0134%	-0.0133%	-0.2751%	-0.0597%
DVA	0.0008%	0.0953%	0.1572%	0.0183%	0.1474%	0.3121%
iTraxx vs Santander	No Vol	Vol EE	Vol	No Vol	Vol EE	Vol
CVA	-0.0113%	-0.0633%	-0.0149%	-0.0146%	-0.2671%	-0.0446%
DVA	0.0000%	0.0892%	0.1499%	0.0171%	0.1478%	0.3026%
BBVA vs Santander	No Vol	Vol EE	Vol	No Vol	Vol EE	Vol
CVA	-0.0381%	-0.0700%	-0.0296%	-0.0018%	-0.0789%	-0.0148%
DVA	0.0000%	0.0374%	0.0892%	0.0621%	0.1546%	0.3118%

Table 13: Non Contingent CVA for CDS

8 Conclusions

We have analyzed the effects of the financial crisis in counterparty credit risk valuation adjustments. Following the arbitrage-free valuation framework presented in [Brigoetal09], we considered a model with stochastic Gaussian interest rates and CIR++ default intensities. Departing from previous literature, we have been able to calibrate default intensities profiting from Gaussian mapping techniques presented in [BrigoAlfonsi04], and reproduce the historically observed instantaneous covariances of CDS prices. To test the calibration procedure, we tracked the Spanish financial sector, who has behaved in a singular manner through the crisis, regarded among the safest in Europe at the beginning, and in need of a partial bailout few years later. We calculated adjustments involving the two major Spanish banks and a generic European counterpart before and along the Spanish recession in a plain vanilla interest rate swap and a Credit Default Swap (CDS).

Our results confirm credit valuation adjustments to be quite sensitive to dynamics parameters such as volatilities and correlations, in line with existing literature. The impact of the parameters is both relevant and financially logical, especially for credit derivatives.

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A Swaption prices under G2++

We recall that we had described the short-interest rate as the sum of two Gaussian processes:

$$r(t) = x(t) + z(t) + \varphi(t, \alpha)$$

where

$$\begin{cases} dx(t) = -ax(t)dt + \sigma dW_1(t), & x(0) = 0 \\ dz(t) = -bz(t)dt + \eta dW_2(t), & z(0) = 0 \end{cases}$$

and (W_1, W_2) is a two-dimensional Brownian motion under the risk-neutral measure with instantaneous correlation ρ_{12} .

Consider a European swaption with strike rate S_K and maturity T , which gives the holder the right to enter at time $t_0 = T$ an interest rate swap with payment times $\mathcal{T} = \{t_1, \dots, t_n\}$, $t_1 > T$, where he pays (receives) at the fixed rate S_K and receives (pays) LIBOR set in arrears. We denote by τ_i the year fraction from t_{i-1} to t_i , $i = 1, \dots, n$ and set $c_i = S_K \tau_i$ for $i = 1, \dots, n-1$ and $c_n = 1 + S_K \tau_n$. Then, as shown in [BrigoMercurio], the arbitrage-free price at time $t = 0$ of a European swaption under G2++ can be numerically computed as:

$$ES^{G2++}(0, T, \mathcal{T}, S_K, \omega; \alpha) = \omega P(0, T) \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2} \left(\frac{x - \mu_x}{\sigma_x} \right)^2}}{\sigma_x \sqrt{2\pi}} \left[\Phi(-\omega h_1(x)) - \sum_{i=1}^n \lambda_i(x) e^{-\kappa_i(x)} \Phi(-\omega h_2(x)) \right] dx$$

where $\omega = 1$ ($\omega = -1$) for a payer (receiver) swaption, $P(0, T)$ is the discount factor between 0 and T , and

$$\begin{aligned}
h_1(x) &= \frac{\bar{z} - \mu_z}{\sigma_z \sqrt{1 - \rho_{xz}^2}} - \frac{\rho_{xz}(x - \mu_x)}{\sigma_x \sqrt{1 - \rho_{xz}^2}} \\
h_2(x) &= h_1(x) + B(b, T, t_i) \sigma_z \sqrt{1 - \rho_{xz}^2} \\
\lambda_i(x) &= c_i A(T, t_i) e^{-B(a, T, t_i)x} \\
\kappa_i(x) &= -B(b, T, t_i) \left[\mu_z - \frac{1}{2}(1 - \rho_{xz}^2) \sigma_z^2 B(b, T, t_i) + \rho_{xz} \sigma_z \frac{x - \mu_z}{\sigma_x} \right] \\
A(t, T) &= \frac{P(0, T)}{P(0, t)} \exp \left(\frac{1}{2} [V(t, T) - V(0, T) + V(0, t)] \right) \\
B(s, t, T) &= \frac{1 - e^{-s(T-t)}}{s}
\end{aligned}$$

and

$$\begin{aligned}
V(t, T) &= \frac{\sigma^2}{a^2} \left(T - t + \frac{2}{a} e^{-a(T-t)} - \frac{1}{2a} e^{-2a(T-t)} - \frac{3}{2a} \right) \\
&+ \frac{\eta^2}{b^2} \left(T - t + \frac{2}{b} e^{-b(T-t)} - \frac{1}{2b} e^{-2b(T-t)} - \frac{3}{2b} \right) \\
&+ 2\rho_{12} \frac{\sigma\eta}{ab} \left(T - t + \frac{e^{-a(T-t)} - 1}{a} + \frac{e^{-b(T-t)} - 1}{b} - \frac{e^{-(a+b)(T-t)} - 1}{a+b} \right)
\end{aligned}$$

$\bar{z} = \bar{z}(x)$ is the unique solution of the following equation:

$$\sum_{i=1}^n c_i A(T, t_i) e^{-B(a, T, t_i)x - B(b, T, t_i)\bar{z}} = 1$$

and

$$\begin{aligned}
\mu_x &= -M_x^T(0, T) \\
\mu_z &= -M_z^T(0, T) \\
\sigma_x &= \sigma \sqrt{B(2a, 0, T)} \\
\sigma_z &= \eta \sqrt{B(2b, 0, T)} \\
\rho_{xz} &= \frac{\rho_{12} \sigma \eta}{\sigma_x \sigma_z} B(a + b, 0, T)
\end{aligned}$$

where

$$\begin{aligned}
M_x^T(s, t) &= \left(\frac{\sigma^2}{a^2} + \rho \frac{\sigma\eta}{ab} \right) \left[1 - e^{-a(t-s)} \right] - \frac{\sigma^2}{2a^2} \left[e^{-a(T-t)} - e^{-a(T+t-2s)} \right] \\
&\quad - \frac{\rho\sigma\eta}{b(a+b)} \left[e^{-b(T-t)} - e^{-bT-at+(a+b)s} \right] \\
M_z^T(s, t) &= \left(\frac{\eta^2}{b^2} + \rho \frac{\sigma\eta}{ab} \right) \left[1 - e^{-b(t-s)} \right] - \frac{\eta^2}{2b^2} \left[e^{-b(T-t)} - e^{-b(T+t-2s)} \right] \\
&\quad - \frac{\rho\sigma\eta}{a(a+b)} \left[e^{-a(T-t)} - e^{-aT-bt+(a+b)s} \right]
\end{aligned}$$

B CDS Pricing under CIR++ stochastic intensity and G2++ interest rates

B.1 Introduction

We recall our modeling framework. Interest rates under the risk-neutral measure are described by:

$$r(t) = x(t) + z(t) + \varphi(t, \alpha)$$

where processes x and z are \mathcal{F}_t adapted and satisfy

$$\begin{cases} dx(t) = -ax(t)dt + \sigma dW_1(t), & x(0) = 0 \\ dz(t) = -bz(t)dt + \eta dW_2(t), & z(0) = 0 \end{cases}$$

where (W_1, W_2) is a two-dimensional Brownian motion with instantaneous correlation ρ_{12} .

For the stochastic intensity model we set

$$\lambda_t = y_t + \psi(t, \beta)$$

y is assumed to be a Cox-Ingersoll-Ross process under the risk-neutral measure:

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dW_3(t)$$

where W_3 is a standard Brownian motion process such that:

$$dW_j dW_3 = \rho_{j3} dt, \quad j \in \{1, 2\}$$

CDS prices could be computed as:

$$CDS(t, T, T, S, \text{LGD}) = 1_{\{\tau > t\}} \mathbb{E} \left[D(t, \tau) (\tau - T_{\beta(t)-1}) 1_{\{\tau < T_n\}} S \right. \\ \left. + \sum_{i=\beta(t)}^n D(t, T_i) \alpha_i 1_{\{\tau > T_i\}} S - 1_{\{\tau < T\}} D(t, \tau) \text{LGD} \middle| \mathcal{G}_t \right]$$

where \mathbb{E} is the risk-neutral expectation, \mathcal{G}_t is the filtration modeling the flow of information of the whole market, including credit, and \mathcal{F}_t is a complete sub-filtration representing all the observable market quantities but the default events.

As shown in [BrigoMercurio], filtration switching techniques between \mathcal{G}_t and \mathcal{F}_t lead to a general formula for the price of a CDS:

$$CDS(t, T, T, S, \text{LGD}) = S \int_t^T \mathbb{E} \left[\exp \left(- \int_t^u (r_s + \lambda_s) ds \right) \lambda_u \right] (u - T_{\beta(u)-1}) du \\ + S \sum_{i=1}^n \alpha_i \mathbb{E} \left[\exp \left(- \int_t^{T_i} (r_s + \lambda_s) ds \right) \right] \\ - \text{LGD} \int_t^T \mathbb{E} \left[\exp \left(- \int_t^u (r_s + \lambda_s) ds \right) \lambda_u \right] du$$

Let us call

$$H_1(u) = \mathbb{E} \left[\exp \left(- \int_t^u (r_s + \lambda_s) ds \right) \lambda_u \right] \\ H_2(u) = \mathbb{E} \left[\exp \left(- \int_t^u (r_s + \lambda_s) ds \right) \right]$$

In general, $H_i(u)$, $i = 1, 2$ admit no closed form, and hence one is forced to solve

$$CDS(t, T, T, S, \text{LGD}) = 0$$

numerically to compute prices of CDS and use them to calibrate the model, which can be a rather dramatic task. Instead, we will use the Gaussian mapping technique shown in [BrigoAlfonsi04] to approximate those expectations with a closed form solution.

B.2 Gaussian mapping

We recall that $r(t) = x(t) + z(t) + \varphi(t, \alpha)$, and $\lambda_t = y_t + \psi(t, \beta)$. In this setup,

$$\begin{aligned} H_1(u) &= e^{-\int_t^u (\varphi(s, \alpha) + \psi(s, \beta)) ds} \left(\psi(u, \beta) \mathbb{E} \left[\exp \left(- \int_t^u (x(s) + z(s) + y(s)) ds \right) \right] \right. \\ &\quad \left. + \mathbb{E} \left[y(u) \exp \left(- \int_t^u (x(s) + z(s) + y(s)) ds \right) \right] \right) \\ H_2(u) &= e^{-\int_t^u (\varphi(s, \alpha) + \psi(s, \beta)) ds} \mathbb{E} \left[\exp \left(- \int_t^u (x(s) + z(s) + y(s)) ds \right) \right] \end{aligned}$$

Denoting by

$$\begin{aligned} \widehat{H}_1(u) &= \mathbb{E} \left[y(u) \exp \left(- \int_t^u (x(s) + z(s) + y(s)) ds \right) \right] \\ \widehat{H}_2(u) &= \mathbb{E} \left[\exp \left(- \int_t^u (x(s) + z(s) + y(s)) ds \right) \right] \end{aligned}$$

we have that

$$\begin{aligned} H_1(u) &= e^{-\int_t^u (\varphi(s, \alpha) + \psi(s, \beta)) ds} \left(\psi(u, \beta) \widehat{H}_2(u) + \widehat{H}_1(u) \right) \\ H_2(u) &= e^{-\int_t^u (\varphi(s, \alpha) + \psi(s, \beta)) ds} \widehat{H}_2(u) \end{aligned}$$

These expectations have no closed formula when $\rho_{i3} \neq 0$ and CIR processes are involved. The idea in [BrigoAlfonsi04] is to "map" CIR dynamics in an analogous tractable Gaussian one that preserves as much as possible the original CIR structure, and then do the calculations in a fully Gaussian setup. In their case, both interest rates and intensities are described by a CIR++, and still they show that converting their dynamics into equivalent Gaussian ones does not affect the pricing of a CDS. In our case, only intensities are to be converted, so the validity of their result is even more justified.

Let us consider a Vasicek process given by

$$dy_t^V = \kappa(\mu - y_t^V)dt + \nu^V dW_3(t)$$

ν^V , the volatility of the Vasicek process, is computed by matching bond prices obtained with y (a CIR process) and y^V (Vasicek). That is, we solve the equation:

$$\mathbb{E}\left[\exp\left(-\int_t^T y_s ds\right)\right] = \mathbb{E}\left[\exp\left(-\int_t^T y_s^V ds\right)\right]$$

Expectations on both sides are analytically known. If we name $P^{CIR}(t, T; \beta)$ the price of a zero-coupon bond under CIR, then, using the formula for the right-hand side shown in, for example, [BrigoMercurio], we have that the former equation reads:

$$P^{CIR}(t, T; \beta) = \exp\left[\left(\mu - \frac{(\nu^V)^2}{2\kappa^2}\right)[g(\kappa, T-t) - T + t] - \frac{(\nu^V)^2}{4\kappa}g(\kappa, T-t)^2 - g(\kappa, T-t)y_t^V\right]$$

where $g(a, s) = (1 - e^{-as})/a$. Using that $g(a, s)^2 = (2/a)(g(a, s) - g(2a, s))$, we can solve for ν^V :

$$\nu^V = k\sqrt{2\frac{\log(P^{CIR}(t, T; \beta)) + \mu(T-t) - (\mu - y_t^V)g(\kappa, T-t)}{(T-t) - 2g(\kappa, T-t) + g(2\kappa, T-t)}}$$

Next, following [BrigoAlfonsi04], we take the following approximations:

$$\widehat{H}_2(u) \approx \mathbb{E} \left[\exp \left(- \int_t^u (x(s) + z(s) + y^V(s)) ds \right) \right]$$

$$\widehat{H}_1(u) \approx \mathbb{E} \left[y^V(u) \exp \left(- \int_t^u (x(s) + z(s) + y^V(s)) ds \right) \right] + \Delta$$

where

$$\begin{aligned} \Delta &= \mathbb{E} \left[\exp \left(- \int_t^u (x(s) + z(s)) ds \right) \right] \left(\mathbb{E} \left[y(u) \exp \left(- \int_t^u y(s) ds \right) \right] \right. \\ &\quad \left. - \mathbb{E} \left[y^V(u) \exp \left(- \int_t^u y^V(s) ds \right) \right] \right) \\ &= \mathbb{E} \left[\exp \left(- \int_t^u (x(s) + z(s)) ds \right) \right] \left[\left(- \frac{\partial P^{CIR}}{\partial u}(t, u; \beta) \right) \right. \\ &\quad \left. - \mathbb{E} \left[y^V(u) \exp \left(- \int_t^u y^V(s) ds \right) \right] \right] \end{aligned}$$

where in the last equality we have differentiated under the expectation sign as in [Mamon].

B.3 Computing expectations

We need to compute four expectations based on three correlated Vasicek processes:

$$\begin{aligned} E_1(u) &= \mathbb{E} \left[y^V(u) \exp \left(- \int_t^u (x(s) + z(s) + y^V(s)) ds \right) \right] \\ E_2(u) &= \mathbb{E} \left[\exp \left(- \int_t^u (x(s) + z(s) + y^V(s)) ds \right) \right] \\ E_3(u) &= \mathbb{E} \left[y^V(u) \exp \left(- \int_t^u y^V(s) ds \right) \right] \\ E_4(u) &= \mathbb{E} \left[\exp \left(- \int_t^u (x(s) + z(s)) ds \right) \right] \end{aligned}$$

Following the same procedure as in [BrigoAlfonsi04], we first state the following lemma:

Lemma B.1. *Let x_t , z_t and y_t^V be three Vasicek processes defined as follows:*

$$\begin{cases} dx_t = -ax_t dt + \sigma dW_1(t) \\ dz_t = -bz_t dt + \eta dW_2(t) \\ dy_t^V = \kappa(\mu - y_t^V) dt + \nu^V dW_3(t) \end{cases}$$

with $dW_1(t)dW_2(t) = \rho_{12}dt$, $dW_i(t)dW_3(t) = \rho_{i3}dt$, $i = 1, 2$. Then, $A = \int_t^T (x_s + z_s + y_s^V) ds$ and $B = y_T^V$ are Gaussian random variables with respective means:

$$\begin{aligned} m_A &= g(a, T-t)x_t + g(b, T-t)z_t + g(\kappa, T-t)y_t^V + \mu \left[(T-t) - g(\kappa, T-t) \right] \\ m_B &= \mu - (\mu - y_t^V)e^{-(\kappa(T-t))} \end{aligned}$$

respective variances:

$$\begin{aligned} \sigma_A^2 &= \left(\frac{\sigma}{a}\right)^2 \left[(T-t) - 2g(a, T-t) + g(2a, T-t) \right] \\ &+ \left(\frac{\eta}{b}\right)^2 \left[(T-t) - 2g(b, T-t) + g(2b, T-t) \right] \\ &+ \left(\frac{\nu^V}{\kappa}\right)^2 \left[(T-t) - 2g(\kappa, T-t) + g(2\kappa, T-t) \right] \\ &+ \frac{2\rho_{12}\sigma\eta}{ab} \left[(T-t) - g(a, T-t) - g(b, T-t) + g(a+b, T-t) \right] \\ &+ \frac{2\rho_{13}\sigma\nu^V}{a\kappa} \left[(T-t) - g(a, T-t) - g(\kappa, T-t) + g(a+\kappa, T-t) \right] \\ &+ \frac{2\rho_{23}\eta\nu^V}{b\kappa} \left[(T-t) - g(b, T-t) - g(\kappa, T-t) + g(b+\kappa, T-t) \right] \\ \sigma_B^2 &= (\nu^V)^2 g(2\kappa, T-t) \end{aligned}$$

and correlation

$$\bar{\rho} = \frac{1}{\sigma_A\sigma_B} \left[\frac{\nu^V\sigma\rho_{13}}{a} \left(g(\kappa, T-t) - g(\kappa+a, T-t) \right) + \frac{\nu^V\eta\rho_{23}}{b} \left(g(\kappa, T-t) - g(\kappa+b, T-t) \right) + \frac{(\nu^V)^2}{\kappa} \left(g(\kappa, T-t) - g(2\kappa, T-t) \right) \right]$$

Proof. Let us define $\bar{y}_t = y_t^V e^{\kappa t}$. Using Ito's lemma, we have that:

$$d\bar{y}_t = \kappa\mu e^{\kappa t} dt + \nu^V e^{\kappa t} dW_3(t)$$

Integrating from t to a given $s > t$ to reach \bar{y}_s and substituting back for y_s^V yields:

$$y_s^V = y_t^V e^{-\kappa(s-t)} + \mu(1 - e^{-\kappa(s-t)}) + \nu^V \int_t^s e^{-\kappa(s-u)} dW_3(u)$$

Similar calculations can be made for x_t and z_t , also Vasicek processes, and we can calculate m_A as:

$$\begin{aligned} m_A &= \mathbb{E} \left[\int_t^T (x_s + z_s + y_s^V) ds \right] = \\ & \int_t^T \left[x_t e^{-a(s-t)} + z_t e^{-b(s-t)} + y_t^V e^{-\kappa(s-t)} + \mu(1 - e^{-\kappa(s-t)}) \right] ds = \\ & g(a, T-t)x_t + g(b, T-t)z_t + g(\kappa, T-t)y_t^V + \mu \left[(T-t) - g(\kappa, T-t) \right] \end{aligned}$$

m_B and σ_B^2 are as defined in [BrigoAlfonsi04]. To compute σ_A^2 we first calculate the variance of $\int_t^T y_s^V ds$:

$$\begin{aligned} Var \left[\int_t^T y_u^V du \right] &= Var \left[\nu^V \int_t^T \int_t^u e^{-\kappa(u-s)} dW_3(s) du \right] = \\ & Var \left[\nu^V \int_{s=t}^T \left(\int_s^T e^{-\kappa(u-s)} du \right) dW_3(s) \right] = \\ & \left(\frac{\nu^V}{\kappa} \right)^2 Var \left[\int_t^T \left(1 - e^{-\kappa(T-s)} \right) dW_3(s) \right] = \\ & \left(\frac{\nu^V}{\kappa} \right)^2 \int_t^T \left(1 - e^{-\kappa(T-s)} \right)^2 ds = \\ & \left(\frac{\nu^V}{\kappa} \right)^2 \left[(T-t) - 2g(\kappa, T-t) + g(2\kappa, T-t) \right] \end{aligned}$$

Similar arrangements for covariances among processes yield σ_A^2 . We illustrate one of them by calculating the covariance between A and B.

$$\begin{aligned}
Cov(A, B) = & Cov \left[\nu^V \int_t^T e^{-\kappa(T-u)} dW_3(u), \int_t^T \sigma \int_t^u e^{-a(u-s)} dW_1(s) + \right. \\
& \left. \eta \int_t^u e^{-b(u-s)} dW_2(s) + \nu^V \int_t^u e^{-\kappa(u-s)} dW_3(s) du \right] = \\
& \frac{\nu^V \sigma \rho_{13}}{a} \int_t^T e^{-\kappa(T-u)} \left(1 - e^{-a(T-u)} \right) du + \frac{\nu^V \eta \rho_{23}}{b} \int_t^T e^{-\kappa(T-u)} \left(1 - e^{-b(T-u)} \right) du + \\
& \frac{(\nu^V)^2}{\kappa} \int_t^T e^{-\kappa(T-u)} \left(1 - e^{-\kappa(T-u)} \right) du = \\
& \frac{\nu^V \sigma \rho_{13}}{a} \left[g(\kappa, T-t) - g(\kappa+a, T-t) \right] + \frac{\nu^V \eta \rho_{23}}{b} \left[g(\kappa, T-t) - g(\kappa+b, T-t) \right] + \\
& \frac{(\nu^V)^2}{\kappa} \left[g(\kappa, T-t) - g(2\kappa, T-t) \right]
\end{aligned}$$

Dividing $Cov(A, B)$ between $\sigma_A \sigma_B$ yields $\bar{\rho}$.

□

We can compute expectations E_1, E_2, E_3 and E_4 by using our lemma jointly with Lemma 3.1 from [BrigoAlfonsi04]:

$$\begin{aligned}
E_1(T) &= m_B \exp \left[-m_A + \frac{1}{2} \sigma_A^2 \right] - \bar{\rho} \sigma_A \sigma_B \exp \left[-m_A + \frac{1-\bar{\rho}^2}{2} \sigma_A^2 \right] \\
E_2(T) &= \exp \left[-m_A + \frac{1}{2} \sigma_A^2 \right] \\
E_3(T) &= m_B \exp \left[-m_A^{deg,3} + \frac{1}{2} (\sigma_A^{deg,3})^2 \right] - \bar{\rho}^{deg,3} \sigma_A^{deg,3} \sigma_B \exp \left[-m_A^{deg,3} + \frac{1-(\bar{\rho}^{deg,3})^2}{2} (\sigma_A^{deg,3})^2 \right] \\
E_4(T) &= \exp \left[-m_A^{deg,4} + \frac{1}{2} (\sigma_A^{deg,4})^2 \right]
\end{aligned}$$

where we have taken a degenerate case of x_t and z_t to apply the former lemma to compute E_3 , that is,

$$\begin{aligned}
m_A^{deg,3} &= g(\kappa, T-t) y_t^V + \mu \left[(T-t) - g(\kappa, T-t) \right] \\
(\sigma_A^{deg,3})^2 &= \left(\frac{\nu^V}{\kappa} \right)^2 \left[(T-t) - 2g(\kappa, T-t) + g(2\kappa, T-t) \right] \\
\bar{\rho}^{deg,3} &= \frac{1}{\sigma_A^{deg} \sigma_B} \left[\frac{(\nu^V)^2}{\kappa} \left(g(\kappa, T-t) - g(2\kappa, T-t) \right) \right]
\end{aligned}$$

and another degenerate case of y_t^V to apply the former lemma to compute E_4 , that is,

$$\begin{aligned}
m_A^{deg,4} &= g(a, T-t)x_t + g(b, T-t)z_t \\
(\sigma_A^{deg,4})^2 &= \left(\frac{\sigma}{a}\right)^2 \left[(T-t) - 2g(a, T-t) + g(2a, T-t) \right] \\
&\quad + \left(\frac{\eta}{b}\right)^2 \left[(T-t) - 2g(b, T-t) + g(2b, T-t) \right] \\
&\quad + \frac{2\rho_{12}\sigma\eta}{ab} \left[(T-t) - g(a, T-t) - g(b, T-t) + g(a+b, T-t) \right]
\end{aligned}$$

C Credit calibration results

	BBVA				Santander				iTraxx			
	3Y	5Y	7Y	10Y	3Y	5Y	7Y	10Y	3Y	5Y	7Y	10Y
3Y	0.049%	93.455%	93.247%	92.810%	84.450%	92.378%	90.448%	88.689%	78.031%	80.918%	80.886%	81.200%
5Y		0.054%	97.874%	97.267%	87.403%	95.495%	93.598%	91.634%	78.850%	83.129%	82.967%	83.438%
7Y			0.053%	98.065%	86.074%	94.620%	93.817%	92.280%	78.193%	82.415%	82.730%	83.149%
10Y				0.054%	85.516%	93.814%	93.188%	91.899%	77.849%	81.742%	81.855%	82.276%
3Y					0.050%	89.280%	88.066%	86.076%	72.003%	73.867%	72.723%	73.290%
5Y						0.054%	97.897%	96.029%	80.132%	83.765%	83.379%	83.645%
7Y							0.054%	98.107%	79.742%	83.303%	83.076%	83.202%
10Y								0.054%	78.614%	82.046%	81.818%	82.084%
3Y									0.042%	98.101%	96.824%	96.071%
5Y										0.045%	99.629%	99.290%
7Y											0.045%	99.799%
10Y												0.043%

Table 14: Historical correlation matrix (standard deviations in diagonal). 2007-2009

	BBVA				Santander				iTraxx			
	3Y	5Y	7Y	10Y	3Y	5Y	7Y	10Y	3Y	5Y	7Y	10Y
3Y	0.054%	99.999%	99.999%	99.999%	91.172%	91.158%	91.155%	91.143%	81.359%	81.365%	81.370%	81.374%
5Y		0.052%	99.999%	99.999%	91.172%	91.158%	91.155%	91.143%	81.359%	81.365%	81.370%	81.375%
7Y			0.051%	99.999%	91.175%	91.161%	91.158%	91.146%	81.362%	81.369%	81.373%	81.378%
10Y				0.049%	91.173%	91.159%	91.156%	91.144%	81.360%	81.366%	81.371%	81.376%
3Y					0.054%	99.991%	99.988%	99.977%	78.786%	78.754%	78.727%	78.693%
5Y						0.053%	99.997%	99.987%	78.804%	78.772%	78.744%	78.711%
7Y							0.052%	99.990%	78.807%	78.775%	78.747%	78.714%
10Y								0.051%	78.820%	78.788%	78.760%	78.727%
3Y									0.046%	99.999%	99.998%	99.996%
5Y										0.044%	99.999%	99.998%
7Y											0.043%	99.999%
10Y												0.041%

Table 15: Parametric correlation matrix (standard deviations in diagonal). 2007-2009

	BBVA				Santander				iTraxx			
	3Y	5Y	7Y	10Y	3Y	5Y	7Y	10Y	3Y	5Y	7Y	10Y
3Y	0.121%	98.790%	98.291%	97.747%	96.073%	96.566%	96.213%	95.601%	84.470%	84.466%	84.114%	83.680%
5Y		0.121%	99.750%	99.292%	96.053%	97.642%	97.434%	96.875%	84.536%	84.767%	84.479%	84.118%
7Y			0.122%	99.824%	95.633%	97.315%	97.197%	96.704%	84.205%	84.464%	84.213%	83.870%
10Y				0.122%	95.075%	96.830%	96.770%	96.364%	83.564%	83.843%	83.645%	83.320%
3Y					0.117%	98.229%	97.506%	96.626%	84.709%	84.646%	84.188%	83.781%
5Y						0.117%	99.718%	99.102%	84.915%	85.085%	84.736%	84.364%
7Y							0.118%	99.689%	84.305%	84.503%	84.189%	83.821%
10Y								0.119%	83.615%	83.836%	83.541%	83.184%
3Y									0.039%	99.867%	99.517%	99.225%
5Y										0.039%	99.771%	99.547%
7Y											0.037%	99.926%
10Y												0.036%

Table 16: Historical correlation matrix (standard deviations in diagonal). 2011-2013

	BBVA				Santander				iTraxx			
	3Y	5Y	7Y	10Y	3Y	5Y	7Y	10Y	3Y	5Y	7Y	10Y
3Y	0.129%	99.999%	99.999%	99.999%	90.992%	91.009%	91.045%	91.124%	82.439%	82.486%	82.522%	82.542%
5Y		0.122%	99.999%	99.999%	91.016%	91.033%	91.070%	91.149%	82.440%	82.487%	82.524%	82.544%
7Y			0.117%	99.999%	91.026%	91.042%	91.079%	91.159%	82.440%	82.488%	82.525%	82.545%
10Y				0.109%	91.037%	91.055%	91.091%	91.171%	82.439%	82.488%	82.526%	82.547%
3Y					0.137%	99.999%	99.999%	99.999%	87.150%	87.240%	87.310%	87.350%
5Y						0.121%	99.999%	99.999%	87.105%	87.195%	87.266%	87.306%
7Y							0.109%	99.999%	87.056%	87.147%	87.218%	87.259%
10Y								0.095%	86.961%	87.052%	87.124%	87.165%
3Y									0.042%	99.999%	99.996%	99.992%
5Y										0.041%	99.999%	99.997%
7Y											0.039%	99.999%
10Y												0.037%

Table 17: Parametric correlation matrix (standard deviations in diagonal). 2011-2013

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