



Completeness: From Husserl to Carnap

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Abstract. In his *Doppelvortrag* (1901), Edmund Husserl introduced two concepts of “definiteness” which have been interpreted as a vindication of his role in the history of completeness. Some commentators defended that the meaning of these notions should be understood as categoricity, while other scholars believed that it is closer to syntactic completeness. A detailed study of the early twentieth-century axiomatics and Husserl’s *Doppelvortrag* shows, however, that many concepts of completeness were conflated as equivalent. Although “absolute definiteness” was principally an attempt to characterize non-extendible manifolds and axiom systems (different from Hilbert’s axiom of completeness), an absolutely definite theory has a unique model and, thus, it is non-forkable and semantically complete (decidable). Non-forkability and decidability were formally delimited by Fraenkel and Carnap almost three decades later and, in fact, they mentioned Husserl as precursor of the latter. Therefore, this paper contributes to a reassessment of Husserl’s place in the history of logic.

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1. Introduction

The history of completeness during the first decades of the twentieth century has been extensively researched. The work of the Hilbert school in Göttingen, specially Hilbert’s lecture notes to the winter semester 1917–1918 (cf. [31]) and Bernays’s *Habilitationschrift* (cf. [6]), is one field of investigation.¹ A second important topic was the discussion on the writings of Huntington (cf.

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¹Cf., for instance, Moore [47], Zach [70], Ewald [15] and Ewald and Sieg [16].

[33, 34]) and Veblen (cf. [67, 68]), known as “American postulate theorists”.² Finally, in the last few years, much attention has been devoted to Carnap’s *Untersuchungen* (cf. [8]), which remained unpublished until 2000.³

Many scholars have connected these threads in order to explain the development of the different notions of completeness inherited by contemporary logic. Awodey and Reck (cf. [3, 4]), Mancosu (cf. [44]) and Manzano and Alonso (cf. [46]), among others, carried out this task. In fact, [3] goes back to the study of axiomatic systems in the last part of the nineteenth century, when the fundamental notions of categoricity and completeness were conceptualized for the first time. Centrone argues that [3] is “a technically detailed and historically well documented study of the various notions of completeness that occurred in connection with the development of the axiomatic method” ([9], p. 168, fn. 78). However, she regrets, “it is a pity that Husserl’s notions of “definiteness⁴” are not mentioned at all” (Centrone [9], p. 168, fn. 78).

Although Awodey and Reck ([3], p. 12, fn. 38) mentioned Husserl while discussing Hilbert’s axiom of completeness, Majer also thinks that the Double Lecture in which these notions were introduced (known as *Doppelvortrag*⁵) is “a particular piece of Husserl’s work which has been dreadfully neglected” ([43], p. 37). As a result, a consensus has recently emerged that Husserl played an important role in the history of completeness. Contributions by Da Silva (cf. [11, 14]), Hartimo (cf. [22–24]) and Centrone (cf. [9]) have provided us a more detailed picture of Husserl’s *Doppelvortrag*. Nevertheless, there is a heated controversy on the exact meaning of “absolute definiteness” and “relative definiteness”. In this paper, I will restrict myself to the meaning of the former. “Absolute definiteness” means, according to Da Silva, syntactic (or deductive) completeness,⁶ but, according to both Hartimo and Centrone, it means categoricity.⁷

Definition 1 (*Syntactic completeness*). A theory Γ in a language \mathcal{L} is syntactically complete if, for every sentence φ of \mathcal{L} , either φ or $\neg\varphi$ is provable from Γ , but not both.

Definition 2 (*Categoricity*). A theory Γ in a language \mathcal{L} is categorical if, for every pair of models \mathcal{M} and \mathcal{N} of Γ , there is an isomorphism from \mathcal{M} to \mathcal{N} .

In my view, some historical objections can be raised against all of these interpretations. I will argue that between the intuitive notion of completeness

²Cf. Corcoran [10] and Scanlan [50, 51].

³Cf. Awodey and Carus [2], Schiemer [52] and Schiemer, Zach and Reck [53].

⁴“Finally, I further distinguish relatively and absolutely definite axiom systems” (Husserl [41], p. 426).

⁵Cf. Husserl [40], pp. 430–451 (german version) and [41], pp. 409–473 (english version).

⁶“The notion of absolute definiteness is identical with Hilbert’s notion of deductive or syntactic completeness” (Da Silva [11], p. 417).

⁷“Centrone [9] defends an interpretation of relative definiteness as syntactic completeness and absolute definiteness as categoricity. The present approach is in agreement with her account of absolute definiteness, but holds that also the former, relative definiteness, should be understood as categoricity” (Hartimo [24], p. 1522, fn. 22).

introduced by Husserl and the formal concepts of syntactic completeness and categoricity there is still a big gap. Absolute definiteness is, in my opinion, a property of those theories whose domain of interpretation is non-extendible. I will also argue that Husserl assumed that the uniqueness of the model could be inferred from its non-extendibility. As a result, he thought that an absolutely definite theory had two more properties, which Carnap (cf. [8]) and Fraenkel (cf. [18]) identified as non-forkability and decidability⁸ (what Tarski ([62], p. 489) called “semantic completeness”) almost thirty years later.

Definition 3 (*Non-forkability*). A theory Γ in a language \mathcal{L} is non-forkable if there is no sentence φ of \mathcal{L} such that both $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are satisfiable.

Definition 4 (*Semantic completeness*). A theory Γ in a language \mathcal{L} is semantically complete if, for every sentence φ of \mathcal{L} , either φ or $\neg\varphi$ is a logical consequence of Γ , but not both.

Tarski ([62], p. 490) stated that every theory that is non-forkable is also semantically complete, although the converse does not hold *in general*.⁹ The notion of semantic completeness is obtained by replacing the formal concept of provability with its semantical counterpart, i.e. that of logical consequence. The notion of non-forkability expresses the fact that a theory is not satisfiable jointly with a sentence as well as with its negation. If a theory Γ is satisfiable jointly with φ as well as with $\neg\varphi$, then Γ “forks” (exactly as a road forks) at φ (cf. Awodey and Reck [3], pp. 3–4). In first-order logic, syntactic completeness, non-forkability and semantic completeness are equivalent, for the reason that *the logic* is complete. As an example of a first-order theory which is forkable (and, therefore, incomplete), let Γ be the axioms for the class of lineal orders. Assume that \mathcal{K} is the class of lineal orders, and let φ be the following sentence:

$$\varphi := \forall xy(R(x, y) \wedge x \neq y \rightarrow \exists z(R(x, z) \wedge z \neq x \wedge R(z, y) \wedge z \neq y).$$

Obviously, Γ “forks” at φ , because some structures of \mathcal{K} satisfy $\Gamma \cup \{\varphi\}$ (those which are dense lineal orders), but other structures of \mathcal{K} make $\Gamma \cup \{\neg\varphi\}$ true (those which are not dense).¹⁰ By contrast, in the case of a non-forkable theory, further axioms will be redundant. Husserl’s early attempt to define a notion of completeness foreshadow later developments on non-forkability and semantic completeness (cf. *infra*).

⁸A decidable theory, in this context, is an early characterization of semantic completeness, rather than decidability in the contemporary sense of computability theory. This notion of semantic completeness (of a theory) must not be confused with the semantic completeness of a logic. A logic is semantically complete if the set of logical theorems includes the class of truths of that logic.

⁹Tarski [62] speaks of “relative completeness”, but he remarked, in a letter to Quine, that relative completeness and non-forkability (cf. [64], pp. 390–392) are equivalent. This remark is indeed correct (cf. Mancosu [44], p. 476). For an example of a theory that is semantically complete and forkable, cf. Mancosu [44], pp. 478–479.

¹⁰As another example of a forkable theory, let Γ be the axioms for the class of groups. Γ is forkable at the (first-order) sentence that axiomatizes the property of Abelian groups. On the other hand, second-order Peano arithmetic is non-forkable (and, therefore, semantically complete).

Thus, the first part of this paper points out that “absolute definiteness” cannot mean neither categoricity nor syntactic completeness. The second one focuses on non-extendibility and the consequences Husserl draws from it. For this purpose, I will stress the fact that a notion of isomorphism is not present in the *Doppelvortrag*, and also that the metatheoretical concept of syntactic completeness could have only emerged from 1917 onwards. Then, I will argue that Husserl is conflating his own notion of absolute definiteness with certain consequences that are supposed to follow from non-extendibility. Veblen (cf. [67], pp. 346–347) is also conflating non-extendibility, categoricity and semantic completeness. For both historical and logical reasons, it will be useful to start with some terminological clarifications concerning Husserl’s *Doppelvortrag*.

2. Preliminary Notions

It is well known that Husserl addressed the problem of imaginary numbers¹¹ in the *Doppelvortrag*. Starting from an axiom system for the natural numbers (which for Husserl were the *genuine* numbers), all possible theories that result when new numbers are successively added should not lead to contradictions. Thus, the operations defined on, for instance, the integers, must preserve the same value in every system of numbers containing the integers. There is some agreement in the literature as to Husserl’s solution: if a theory Γ is consistent and relatively definite, and if the extension Γ^* of Γ is relatively or absolutely definite, then Γ^* must also be consistent.¹²

The concept of “theory” in the *Doppelvortrag* is different from the one developed in modern logic. A formal theory is not a set of sentences of a given formal language (closed under provability), but *the form* of a theory. Husserl explained that, if the content is abstracted from a particular theory, then the “pure form of the theory” is obtained (cf. [41], p. 410). The constituents of a theory are, therefore, “formal propositions”, the most basic of which are the formal axioms:

¹¹The problem of imaginary numbers is associated with the Principle of Permanence. And, almost thirty years after the *Doppelvortrag*, Husserl ([39], p. 97) explained that his concept of “definite theory” was originally introduced with the aim of clarifying Hankel’s Principle of Permanence of the Formal Laws. The issue of how to guarantee that certain arithmetical laws are preserved while extending the concept of number was raised by Peacock (cf. Kleiner [42], p. 13). In the words of Peacock: “Whatever form is algebraically equivalent to another when expressed in general symbols, *must continue to be* equivalent, whatever those symbols denote” (Peacock [49], p. 198). In the words of Hankel: “Wenn zwei in allgemeinen Zeichen der arithmetica universalis ausgedruckte Formen einander gleich sind, so sollen sie einander auch gleich bleiben, wenn die Zeichen aufhören, einfache Grossen zu bezeichnen, und daher auch die Operationen einen irgend welchen anderen Inhalt bekommen” (Hankel [21], p.11).

¹²According to Centrone, “the thesis that Husserl proposes in the *Doppelvortrag* is a conditional claim: “if Γ is consistent and syntactically complete (definite) then every consistent extension of Γ is conservative, so that the transition through the imaginary is justified” (Centrone [9], p. 178).

“A systematically elaborated theory in this sense is defined by a totality of formal axioms, i.e., by a limited number of purely formal basic propositions, mutually consistent and independent of one another.” (Husserl [41], p. 410)

However, not every formal theory was considered to be of mathematical interest. “We restrict ourselves to axiom systems that have a domain. (Why not directly to totalities of objects which satisfy the axiom system?)” (Husserl [41], pp. 437–438). This idea of Husserl is at the root of the modern concept of *intended* (or standard) *model*, which is the particular structure that a theory is supposed to describe. Husserl used the terms “domain” (“*s. Gebiet*”) and “object domain” (“*s. Objektgebiet*”) to refer to the intended model \mathcal{A} of a theory Γ . Consider, for instance, the following quote from the *Doppelvortrag*:

“The object domain is defined through the axioms in the sense that it is delimited as a certain sphere of objects in general, irrespective of whether real or Ideal, for which basic propositions of such and such forms hold true.” (Husserl [41], p. 410)

Da Silva also believes that the object domain of an axiom system should be understood as its intended model. “An interpreted theory is, for Husserl, one that describes a given specific domain, or, as we would say today, a theory with an *intended* model” (Da Silva [13], p. 141). The distinction between the intended model of a theory and the theory itself goes back to the last decades of the nineteenth century, when algebraists and geometers classified different “systems of objects” (“*e. Systeme von Dingen*”) attending to the formal laws satisfied by these systems.¹³ In the *Doppelvortrag*, a system of objects is called “sphere of objects” (“*e. Sphäre von Objekten*”), and even “manifold¹⁴” (“*e. Mannigfaltigkeit*”).

Although the first occurrence of the expression “ φ is true in \mathcal{A} ” was due to Skolem (cf. Skolem [55], p. 352), the idea of a proposition being true in a system of objects was common currency at the turn of the century (Mancosu [44], p. 10). The Peano school, and the American postulate theorists, talked about a set of postulates being “verified”¹⁵ by a system. In fact, as presented in the above quote, Husserl alludes to the axioms that “hold true” in a certain sphere of objects. Hence, he was familiar with the concept that preceded the formal notion of *truth in a structure*.

Definition 5 (*Theory of a structure*). Let \mathcal{L} be a language and \mathcal{A} an structure. The \mathcal{L} -theory of \mathcal{A} is the set of all sentences φ of \mathcal{L} such that $\models_{\mathcal{A}} \varphi$.

¹³Cf. Hodges [32], pp. 135–136.

¹⁴The word “structure” was not used, during the first part of the twentieth century, as an equivalent of “system of objects” (cf. Mancosu [44], p. 100). Instead, Husserl preferred the word “manifold”, which could be taken from Schröder, as well as the notion of “domain”, since he wrote a review (cf. [35]) of Schröder’s [54]. However, other german mathematicians, like Fries, Riemann and Cantor, used the term “*Mannigfaltigkeit*”, probably borrowed from Kant’s philosophy (cf. Torretti [66], pp. 7–8, fn. 1).

¹⁵Cf., for instance, Padoa [48].

Finally, it should be stressed that “definiteness” is not only a property of the theories, but also of the manifolds. This will lead to a parallel distinction between relatively and absolutely definite “spheres of objects”:

“Relatively definite is the sphere of the whole and the fractional numbers, of the rational numbers [...] I call a manifold absolutely definite if there is no other manifold which has the same axioms (all together) as it has. Continuous number sequence, continuous sequence of ordered pairs of numbers.” (Husserl [41], p. 426)

Intuitively, an absolutely definite sphere of objects has no “gaps”, for it is continuous. By contrast, a relatively definite manifold has “discontinuities”, just like the rational number line has (uncountable) many gaps corresponding to each irrational number. Thus, an absolutely definite manifold will have the property of the reals called “Dedekind completeness¹⁶”, which is equivalent to the well known least-upper-bound property. Hilbert’s axiomatization of the reals imposed this condition to its intended model by means of his axioms of continuity and, more specifically, by the axiom of completeness. The meaning of this axiom will be discussed at length in the next sections (cf. p. 11).

Now, I will explain why, strictly speaking, Husserl’s notion of “absolute definiteness” cannot mean neither categoricity nor syntactic completeness. I will show, then, how to trace this concept back to the intuitions from which categoricity, non-forkability and decidability directly derives.

3. Categoricity and Syntactic Completeness

3.1. “Absolute Definiteness” as Categoricity

The logical notion of “categoricity” involves a precise concept of isomorphism between the models of a theory (cf. Definition 2). For this reason, if Husserl meant categoricity by his notion of “absolute definiteness”, then the concept of isomorphism would be buried somewhere in the *Doppelvortrag*. Since this concept is simply not there, “absolute definiteness” cannot mean categoricity.

The lack of an articulated concept of isomorphism in the *Doppelvortrag* is already known in the literature. For instance, Hartimo ([22], pp. 252–253) admitted that, without a sharpened notion of isomorphism, the exact nature of “absolute definiteness” remains regrettably unclear. She believes that this objection can be removed by pointing out that even David Hilbert lacked the resources that would allow him to express this idea more accurately. However, Giovannini ([20], p. 154) has showed that, around 1897, Hilbert had already developed a relatively precise concept of isomorphism. Let me reproduce the relevant passage (quoted by Giovannini) from *Zahlbegriff und Quadratur des Kreises*:

“The axioms define a system of objects univocally [*eindeutig*], i.e. if there is another system of objects which satisfies all these axioms,

¹⁶“Dedekind continuity: For all cuts (A, B) in R , there is an element c in R such that $a \leq b \leq c$ for all $a \in A$ and all $b \in B$ ” (Awodey and Reck [3], p. 13).

then the objects of the first system can be mapped one-to-one into the objects of the second system.” (cf. Giovannini [20], p. 154, fn. 22; my translation)

It is thus clear that, in 1897, Hilbert attempted to clarify what is meant by the univocal definition of a “system of objects”. This attempt was clearly formal, and appealing to a one-to-one mapping seems a major step forward in the delimitation of the concept of isomorphism.¹⁷ The way of arriving at our modern definition introduces the following condition: the one-to-one mapping must *respect* addition, order, etc.—e.g. Let \mathcal{A} be a group with the operation \star , and \mathcal{B} a group with the operation \bullet . An isomorphism h from \mathbf{A} to \mathbf{B} is a bijection that satisfies, for every $a, b \in \mathbf{A}$, $h(a \star b) = h(a) \bullet h(b)$. Surprisingly, this new requirement was formulated in an exact way at the beginning of the twentieth century. Consider for instance the following quote by Huntington:

“We notice that the two systems mentioned in the proof of theorem I are *equivalent* (simply isomorphic); that is, the first can be put into one-to-one correspondence with the second in such a way that when a corresponds with a' and b with b' then $a \circ b$ will correspond with $a' \circ' b'$.” (Huntington [33], p. 268)

This quote comes from a paper published in 1902 and, two years later, Veblen described a system of twelve axioms which he called “categorical”:

“It is part of our purpose however to show that there is *essentially only one* class of which the twelve axioms are valid. In more exact language, any two classes K and K' of objects that satisfy the twelve axioms are capable of a one-to-one correspondence such that if any three elements A, B, C of K are in the order ABC , the corresponding elements of K' are also in the order ABC .” (Veblen [67], p. 346)

Huntington ([34], p. 26) also used the term “categorical¹⁸” to refer to a set of postulates whose models are “isomorphic with respect to addition and multiplication”. Therefore, it is not true that a precise notion of isomorphism could not be formally posed during the first decade of the twentieth century. Hilbert, Huntington and Veblen *had* the mathematical resources to develop a formal analogue of the intuitive concept of “univocal definition” (of a system of objects), whereas Husserl did not. This explains the absence of a rigorous concept of isomorphism in the 1901 *Doppelvortrag*, and shows why “absolute definiteness” cannot mean categoricity.

Consequently, there is little evidence in favor of Hartimo’s reading. The most convincing evidence for her claim appeared in the following passage: “I call a manifold absolutely definite if there is no other manifold which has the

¹⁷Actually, the first appearance of the term “isomorphism” occurred in the early part of the 1900s. Mancosu ([44], pp. 13–14) attributed the notion of an *isomorphism* between arbitrary systems of objects to Bôcher [7], and a fully generalized concept to Weyl [69].

¹⁸Tarski ([59], p. 174 and [57], p. 282) credits Veblen [67] with the concept of categoricity, who attributed the idea to Huntington and the word to the philosopher J. Dewey. It seems that Dewey suggested this word to him in private communication (cf. Veblen [67], p. 346).

same axioms (all together) as it has” (Husserl [41], p. 426). However, it does not follow from this quote that “absolute definiteness” means categoricity. It follows, on the contrary, that Husserl assumed that there is a unique manifold satisfying the axioms of an absolutely definite theory. Thus, Husserl’s concept of absolute definiteness is much more informal than the modern categoricity.

Hartimo ([24], pp. 1518–1520) has supported her interpretation with the attribution of an *Euclidean ideal* to Husserl. She takes her start from a paper by Tennant [65], where he introduced the term “monomathematics” to refer to an old aspiration of the logical community. “Success in monomathematics requires both *expressive power* (the power to describe structures exactly) and *deductive power* (the power to prove whatever follows from one’s description)” (Tennant [65], p. 257). According to Hartimo, Husserl was a perfect example of a monomathematician. Hence, she concludes, an absolutely definite theory must describe its models univocally, and also *prove* whatever follows from its axioms. Although I agree with her that Husserl really believed so, the reason behind this belief was not an ancient aspiration. My preferred explanation is that the distinction between categoricity and syntactic completeness had not been even formulated¹⁹ in 1901.

Furthermore, the evidence speaks against the attribution of this “monomathematics” point of view to Husserl. In 1929, he did mention an Euclidean ideal that has continually guided mathematics from the Greeks. However, the ideal is not the combination of both expressive and deductive power, but the *axiomatization* (cf. Manzano [45], p. 128) of a mathematical structure:

“If the *Euclidean ideal* were actualized, then the whole infinite system of space-geometry could be derived from the irreducible finite system of axioms by purely syllogistic deduction [...] and thus the *a priori essence of space could become fully disclosed in a theory.*” (Husserl [39], p. 95)

Let \mathcal{A} be “the whole infinite system of space-geometry”. Husserl is saying that there exists an irreducible and finite set of axioms Δ , from which “the *a priori essence of space*” can be fully disclosed by syllogistic deduction. In fact, Da Silva [12] has showed that, in *Ideas I* (cf. [36]), the notion of “definiteness” was presented again, but with an important difference. “This time a mathematical *structure* is under consideration, and it is called a *definite manifold* [...] if its complete theory (i.e. the set of all sentences which are true in this structure) is *finitely* axiomatizable” (Da Silva [12], p. 133). Therefore, if \mathcal{A} is definite, then $Cn(\Delta)$ is the set of all sentences φ of \mathcal{L} such that $\models_{\mathcal{A}} \varphi$.

According to Husserl [37], a theory from which the “*a priori essence*” of a manifold could be fully disclosed determines “completely and unambiguously” all possible operations in the domain. For this reason, “nothing further remains open within it” (cf. [12] pp. 133). In the *Doppelvortrag*, Husserl stated that an axiom system is definite if it leaves “no relation undetermined for the formal objects of its domain” (Husserl [41], p. 451). Thus, as I defend below, Husserl meant non-extendibility by “(absolute) definiteness”; uniqueness is a

¹⁹The first clear awareness of this distinction is, as far as I know, in Fraenkel [17].

consequence. He thought that, if the *Euclidean ideal* is to be actualized, then these non-extendible manifolds are completely and unambiguously described by absolutely definite theories.

Let me conclude this section by pointing to Centrone's position, as she also maintains that "absolute definiteness" was intended to mean categoricity. She believes that the aim of giving mathematical definitions of Husserl's two notions of definiteness "seems to be worth pursuing", so Centrone claims that an absolutely definite axiom system "is categorical, i.e. it individuates, up to isomorphism, only one model" (Centrone [9], p. xvii). The lack of a rigorous concept of isomorphism in the *Doppelvortrag* refutes this reading –just in the same way it refutes Hartimo's.

3.2. "Absolute Definiteness" as Syntactic Completeness

The logical notion of "syntactic completeness" involves a very precise concept of the provability of a sentence from a set of sentences (cf. Definition 1). The textual evidence for the interpretation of "absolute definiteness" as syntactic completeness may appear in the following passage:

"Absolutely definite: (1) [...] An axiom system is absolutely definite if every proposition meaningful according to it is decided in general." (Husserl [41], p. 427)

"An axiom system that delimits a domain is said to be "definite" if [...] Only two things are possible, either the proposition follows from the axioms or contradicts them." (Husserl [41], p. 438)

It is doubtful whether this description of "definiteness" can be labeled as "syntactical". However, as Centrone (cf. [9], p. 169) has already showed, other characterizations of definiteness are rather semantical than syntactical²⁰:

"If I suppose some meaningful sentence constructed, then I can ask whether it is valid if I take it to be a sentence about the objects of the domain, in the previously defined sense. The domain is definite if the truth or falsity of any such sentence is decided for the domain on the basis of the axioms." (Husserl [41], p. 439)

For this reason, she rightly argues that one cannot seriously defend that Husserl had at his disposal the "standard and clear cut" between provability and logical consequence ([9], pp. 167–168). A full awareness of this distinction had to wait until the works of Tarski [61] and the theorems of completeness (of a logic).²¹ In my view, Centrone's argument (which taken literally rejects the identification of absolute definiteness with syntactic completeness) can in fact be strengthened. Husserl was not only unable to separate the syntactical and the semantical sides of the concept of *consequence*, but also to incorporate its formal counterpart to his investigations.

²⁰"The exact interpretation of what Husserl means by definiteness has created much discussion. One reason behind the confusion is that the rigorous concepts of logic, and that of consequence and deduction, were not in place at the time." (Hartimo [24], p. 1516)

²¹Manzano and Alonso [46] provide an historical overview of the theorems of completeness, from the early works of Bernays and Post to the new proofs discovered by Henkin.

In 1913, Husserl argued that the “*a priori* essence” of a definite manifold can be derived from a system of axioms by means of syllogistic deduction (cf. supra). I find plausible that Husserl did not read Whitehead’s and Russell’s *Principia*—the first volume had been published in 1910—while he was writing *Ideas I*, because it is natural to think that he would have taken from it both a formal language and a notion of provability. But there are reasons to doubt that he had incorporated these resources. Husserl was aware of Frege’s work, which is of course prior to the 1901 *Doppelvortrag*, and he refused the so-called “logistic project”²²:

“What Frege has aimed at is absolutely not a psychological analysis of the concept of number [...] A founding of arithmetic on a sequence of formal definitions, out of which all the theorems of that science could be deduced purely syllogistically, is Frege’s ideal.

Surely no extensive discussion is necessary to show why I cannot share this view, especially since all of the investigations which I have carried out to this point present nothing but arguments in refutation of it.” (Husserl [41], p. 124)

It is easy to see that Husserl was not sympathetic to Frege’s approach, who thought that the laws of arithmetic were a simple development of logic. Every arithmetical proposition is either a law of logic or a derived proposition from the laws of logic (cf. Frege [19], p. 99). Therefore, it is very unlikely that Husserl had adopted Frege’s axiomatic logic, and hence his understanding of logical consequence remained regrettably obsolete—Husserl did not allude to formal derivations, but to “syllogistic deduction”.

Now, I present textual evidence to support the conclusion that, between Husserl’s intuitive notion of consequence and the formal notion of provability, there is a big gap. Husserl said that the proposition $7 + 5 = 12$ holds true in an (arithmetical) axiom system if it can be proved as “necessarily true, from the concepts 7, 5, 12 and the concept of addition” (Husserl [41], p. 194). The deduction of $7 + 5 = 12$ from an axiom system does not make use of a sound calculus (i.e. a set of rules of inference), but on the contrary it is justified by appealing to the definitions of 7, 5, 12 and addition. Consequently, “absolute definiteness” cannot mean syntactic completeness.

Da Silva’s [14] defense of his interpretation of “absolute definiteness” as syntactic completeness is that he had preferred to concentrate on conceptual analyses, rather than on historical matters.²³ Hence, his reading would be the most satisfactory solution to understand Husserl’s answers to the problem of imaginary numbers. “We take the risk of interpreting Husserl’s (and Hilbert’s) contributions to axiomatics from the perspective of modern logic” (Da Silva [14], p. 1928). However, Da Silva (cf. [11], p. 417) had argued that the concept

²²In particular, Husserl cited *Die Grundlagen der Arithmetik* (cf. [41], pp. 123–130).

²³“Of course, text and context are important and will be taken into consideration, but sometimes they can mislead rather than lead. A good deal of attention must be given first to the problems Husserl was facing and what they involved conceptually.” (Da Silva [14], p. 1926).

of syntactic completeness—which he assumed to be identical with “absolute definiteness”—was also Hilbert’s concept of completeness. Since this paper is more historically oriented than Da Silva’s, let me make a few remarks on the attribution of “syntactic completeness” to Hilbert.

In Hilbert’s “Über den Zahlbegriff” [27], he presented an axiomatization of the real numbers that included the metalogical²⁴ “axiom of completeness”, which will be also added (in a French edition) to his axiom system for geometry (cf. [25], p. 123). However, the axiom does not guarantee the completeness of a theory in the sense that it proves or disproves every sentence of its language (i.e. syntactic completeness), but the non-extendibility of its intended model. An intended model is non-extendible if it is impossible to extend its universe without falsifying any of the axioms:

“IV 2. (Axiom of Completeness). It is not possible to add to the system of numbers another system of things so that the axioms I, II, III and IV 1 are also all satisfied in the combined system; in short, the numbers form a system of things which is incapable of being extended while continuing to satisfy all the axioms.” (Hilbert [27], p. 1094)

Thus, Hilbert’s concept of completeness was not referring to “syntactic completeness” at the beginning of the twentieth century. Contrary to claims by Da Silva, completeness was primarily a property of the models (not of the theories) when Husserl was investigating the problem of imaginary numbers. Zach [70] has already pointed out the similarities between Post completeness, which states the impossibility of adding a non-provable sentence to a theory without rendering it inconsistent, and the requirement imposed by Hilbert’s axiom. Post completeness was first formulated²⁵ in the winter semester 1917–1918, so Hilbert’s shift from the completeness of an intended model to the completeness of a theory had to wait until two decades later. Concerning syntactic completeness, Zach argues that:

“By 1921 at least, Hilbert is well aware of the difference between the requirement expressed by the completeness axiom and completeness of axiom system in the syntactic sense [...] The latter requirement is obviously closely related to Hilbert’s “no ignorabimus”, the conviction that every well-posed mathematical question can be answered positively or negatively.” (Zach [70], p. 354)

For this reason, the notion of “absolute definiteness”, assuming that it was taken from Hilbert’s concept of completeness, cannot mean (at least from an historical point of view) syntactic completeness. The question of syntactic

²⁴Present-day commentators agree that Hilbert’s axiom of completeness holds a peculiar metalinguistic form and a clearly “metalogical character”. This axiom does not speak about the laws governing the intended model of the theory, but about the model itself. (cf. Moore [47] and Giovannini [20])

²⁵“Wir wollen das vorgelegte Axiomen-System vollständig nennen, falls durch die Hinzufügung einer bisher nicht ableitbaren Formel zu dem System der Grundformeln stets ein widerspruchsvolles Axiomen-System entsteht” (Hilbert [31], p. 157).

completeness could have only emerged from 1917 onwards, when Hilbert was strongly influenced by Whitehead's and Russell's *Principia*. The work of the Hilbert school, in Göttingen, was essential to formulate the rules of inference accurately, to introduce recursive definitions and, in short, to find metalogical results. "Both ingredient concepts—categoricity of a theory, and completeness of a system of proof—were well in place by the winter of 1917–1918" (Tennant [65], p. 258).

4. Non-extendibility and Its Consequences

4.1. Absolute Definiteness and the Axiom of Completeness

I will consider here my account of "absolute definiteness". I claim that Husserl meant non-extendibility of the model,²⁶ rather than categoricity or syntactic completeness. Nevertheless, Husserl thought that the uniqueness of the model could be inferred from its non-extendibility. It seems that he also thought, as Veblen (cf. [67], p. 346), that a theory whose intended model is both non-extendible and unique must be non-forkable and decidable (cf. Definitions 3 and 4). However, Husserl's characterizations of "non-forkability" and "decidability" are pretty intuitive and still largely informal (cf. 4.3 and 4.4).

First of all, let me introduce the evidence in favor of my interpretation of "absolute definiteness" as non-extendibility. In the *Doppelvortrag*, Husserl explicitly said that "absolutely definite = complete, in Hilbert's sense" ([41], p. 427). Since Hilbert's concept of completeness referred—at least in 1901—to non-extendible models, if absolute definiteness is equivalent to completeness "in Hilbert's sense", then it is obvious that "absolute definiteness" must also refer to non-extendibility.

Almost thirty years after his *Doppelvortrag*, Husserl explained the origin of the notion of "definiteness", which he had introduced to characterize some systems of axioms. He also made reference to his unpublished investigations, which were probably his lectures on the problem of imaginary numbers:

"Throughout the present exposition I have used the expression "complete system of axioms", which was not mine originally but derives from Hilbert. Without being guided by the philosophico-logical considerations that determined my studies, Hilbert arrived at his concept of completeness (naturally, quite independently of my still unpublished investigations); he attempts, in particular, to complete a system of axioms by adding a separate "axiom of completeness"." (Husserl [39], pp. 96-97)

This quote confirms that, when Husserl talked about Hilbert's sense of completeness, he was thinking on the axiom of completeness. For this reason, it is not surprising that the "sphere of objects" that Husserl considered to be

²⁶In short, I argue that, for Husserl, an absolutely definite theory is equivalent to a theory containing Hilbert's axiom of completeness. And, "by imposing non-extendibility, Hilbert's axiom of completeness grants uniqueness of domain" (Da Silva [14], p. 1935).

absolutely definite was the continuum, i.e. the real numbers. The—intended—model of an absolutely definite theory (or, equivalently, a theory containing the axiom of completeness) is an absolutely definite manifold, that is, a non-extendible one. “The peculiar character of the Hilbertian closed axiom system is this, that it [...] admits of no expansion of the operational domain by new objects brought under the same prevailing operations” (Husserl [41], p. 451).

It is thus clear that “absolute definiteness” was formulated as a property of the models, but also as a property of the theories. Hilbert’s shift from the completeness of an intended model to the completeness of a theory occurred in the winter semester 1917-18. However, for Husserl a theory can be “definite”, just like an intended model is non-extendible:

“Definite? I cannot add to the “axioms”, i.e., to the forms of basic principles hypothetically taken for a basis, any new “axiom”, any new statement of substance, without evoking a contradiction.” (Husserl [41], p. 426)

“An axiom system is definite if the addition of a new axiom restricted to those objects signifies a contradiction with the axioms defining the objects.” (Husserl [41], p. 451)

Zach ([70], p. 354) suggested that one possible way to explain the shift from the axiom of completeness to Post completeness is that Hilbert became aware that the latter is saying, about formulas, “exactly the same thing” that the former says about the reals. In other words, the impossibility of extending a model without contradiction led to the impossibility of extending a system of formal axioms. But, in 1901, Husserl had already proposed the idea of non-extendible theories by means of his concept of definiteness. And, in particular, “if a manifold is absolutely definite, then there is, in general, no further axiom which could be added to the axioms” (Husserl [41], p. 426).

Therefore, Husserl anticipated Hilbert’s shift from talking about adding *numbers* (or geometrical objects) to talking about adding *axioms*, and he did it almost two decades before Hilbert’s *Prinzipien* [31]. There is more evidence that points in this way. Zach ([70], p. 354) gave another possible explanation to the shift from the axiom of completeness to the completeness of a theory. Perhaps, Hilbert and Bernays started to think that non-extendibility should be obtained as a theorem *about* the axiom systems, instead of being imposed as an extra (metalogical) condition. Once again, Husserl anticipated, in 1901, this criticism of the axiom of completeness:

“On my view, completeness is never an axiom—but rather a theorem, for definite axiom systems and manifolds.” (Husserl [41], p. 426)

Husserl’s skepticism as to the use of the axiom of completeness is really well known in the literature. Hartimo, for instance, remarked that the notions of relative and absolute definiteness were introduced when he was discussing Hilbert’s axiomatic method. “Husserl, critical of Hilbert’s axiom of completeness, writes that completeness should not be an axiom, but a theorem” ([24],

p. 1521). It seems obvious that, if these doubts concerning the relevance²⁷ of the axiom of completeness were necessary for the emergence of the concept of completeness—of a theory—, then this concept could have perfectly arisen in the *Doppelvortrag*. In fact, “definiteness” is a property of both axiom systems and manifolds.

To sum up: Absolutely definite theories are those whose intended model is non-extendible. This is the reason why Husserl said that these theories are complete in Hilbert’s sense. Non-extendible models (the ordered field of real numbers) are called “absolutely definite manifolds”, but its non-extendibility should not be simply stipulated. On the contrary, it should be obtained as a (meta)theorem of every theory that does not admit additional axioms without contradiction.

4.2. The Uniqueness of the Model as a Corollary

In my view, Husserl assumed that the uniqueness of an intended model was a consequence of its non-extendibility. The idea of a unique manifold satisfying all the axioms of a theory was common to both Veblen (cf. [67]) and Carnap (cf. [8]), who did study the formal notion of “categoricity” (Carnap preferred to use the term “monomorphicity”). I will show that Veblen believed, in 1904, that every axiom system containing Hilbert’s axiom of completeness must be categorical, what makes plausible that Husserl had reached basically the same conclusion a few years before.

But let me now present the similarities between one of the (many) senses of “absolute definiteness” and the informal explanation of categoricity. In the *Doppelvortrag*, Husserl said that a manifold is absolutely definite if “there is no other manifold which has the same axioms (all together) as it has” ([41], p. 426). This amounts to say that there is exactly one manifold satisfying all the axioms of an (absolutely definite) theory. Consider Veblen’s introduction to what he called a “categorical” system of axioms, which I quoted above (cf. Sect. 3.1):

“It is part of our purpose however to show that there is *essentially only one* class of which the twelve axioms are valid. In more exact language, any two classes K and K' of objects that satisfy the twelve axioms are capable of a one-to-one correspondence such that if any three elements A, B, C of K are in the order ABC , the corresponding elements of K' are also in the order ABC .” (Veblen [67], p. 346)

It is evident that the same idea was used to define the mentioned sense of “absolute definiteness”. However, and contrary to Husserl, Veblen was trying to express it “in more exact language” and thus he appealed to the notion of isomorphism. Therefore, Veblen’s concept of categoricity is technically more sophisticated than Husserl’s notion of definiteness, but both mathematicians aimed at the intuitive idea of “essentially only one” model. Similarly, Carnap

²⁷As pointed out in Giovannini [20], there were also explicit criticisms to the impossibility of proving the independence of the axiom of completeness.

argued, almost thirty years later, that the model of a monomorphic theory is an isomorphic class.²⁸ His definition of isomorphism is given in full generality (cf. Carnap [8], p. 71).

Without the idea of a (preserving) one-to-one correspondence in hand, I think that Husserl could have only inferred the uniqueness of a model from its non-extendibility. There is a passage where Veblen says things that are in agreement with this reasoning. He attributes to Hilbert the idea of categorical system of axioms, and he makes reference to the axiom of completeness:

“The categorical property of a system of propositions is referred to by Hilbert in his “Axiom der Vollständigkeit”, which is translated by Townsend²⁹ into “Axiom of Completeness”.” (Veblen [67], p. 346)

Mancosu has already pointed out that this quote reveals “an inaccurate understanding of Hilbert’s completeness axiom and of its consequences” ([44], p. 10). In 1928, the German mathematician R. Baldus³⁰ showed that Hilbert’s axiomatization of geometry, which contains the axiom of completeness, is not categorical. Thus, non-extendibility does not imply categoricity. Nevertheless, Hilbert’s system of axioms for the reals is indeed categorical when the axiom of completeness is added (cf. Giovannini [20], pp. 154–156).

The belief that non-extendibility implies categoricity *in general* rested upon the fact that the ordered field of the reals is essentially the only model of Hilbert’s axioms.³¹ Since the counterexample had to wait until the last years of the 1920s, it is very plausible that many mathematicians felt that ensuring non-extendibility was the way to guarantee the uniqueness of the model. The full description of a “system of objects” (or “manifold”) had always been an ideal to strive for.

In particular, this would explain why Husserl did not distinguish clearly between absolute definiteness as *non-extendibility* and absolute definiteness as *uniqueness* of the model. Husserl, like Hilbert³² and Veblen did, assumed that a theory whose model is a manifold that is incapable of being extended (an absolutely definite one) should single out essentially one model (the intended one). “I call a manifold absolutely definite if there is no other manifold which

²⁸“Satz 3.2.3. Dass ein Axiomensystem formal und monomorph ist, ist äquivalent damit, dass sein Umfang aus genau einer Isomorphieklasse besteht” (Carnap [8], p. 129).

²⁹He was the English translator of Hilbert’s *Grundlagen der Geometrie*.

³⁰“Bezeichnet man ein Axiomensystem als monomorph, wenn es nur noch isomorphe Deutungen zulässt (Satz von Nr. 8), dann ist damit gezeigt: *Das Axiomensystem der absoluten Geometrie ist vollständig im Sinne des Vollständigkeitsaxioms, aber nicht monomorph. Erst durch Hinzufügung eines (Euklidischen oder hyperbolischen) Parallelenaxioms wird das Axiomensystem monomorph*” (Baldus [5], pp 328–329).

³¹“Hilbert is aware, again, that adding these two axioms [axioms of continuity] rules out all unintended models. That is to say, he notes that any system of numbers satisfying all of his axioms is essentially the same as the familiar system of real numbers.” (Awodey and Reck [3], p. 14).

³²“Wie man erkennt, gibt es unendlich viele Geometrien, die den Axiomen I–IV, V,1 genügen, dagegen nur eine, nämlich die Cartesische Geometrie, in der auch zugleich das Vollständigkeitsaxiom V,2 gültig ist.” (Hilbert [28], p. 20)

has the same axioms (all together) as it has” (Husserl [41], p. 426). Thus, absolutely definite theories ensure the uniqueness of the intended model, but, just as well as in Hilbert’s axiomatization of the reals, uniqueness is a consequence of its non-extendibility.

The confusion found in Hilbert [28], Husserl [41] and Veblen [67] between non-extendibility and uniqueness is completely removed in Carnap [8]. In his *Untersuchungen*, he mentioned, apart from the notions of “monomorphicity”, “non-forkability” and “decidability”, a fourth concept of completeness which is associated with the “extremal axioms” (cf. Schiemer et al. [53], pp. 40–41). These axioms are used to impose minimality or maximality conditions to the models of a theory (for instance, Hilbert’s axiom of completeness and Peano’s induction axiom). Carnap ([8], p. 127) remarks the importance of separating the completeness of a *theory* from the completeness of its *model* (the system of objects the theory is talking about). Monomorphicity is not considered to be a corollary to the completeness (non-extendibility) of a model.

Although the intuitive idea of a theory having essentially one model is at the basis of absolute definiteness, categoricity and monomorphicity, there are significant differences between these notions. Husserl’s “absolute definiteness” is the only one that does not involve a mathematical concept of isomorphism, but he and Veblen believed that uniqueness follows from non-extendibility. Contrary to this widespread belief, Carnap did not mention non-extendibility while he was discussing monomorphicity in his 1927–1929 *Untersuchungen*, what is compatible with the fact that non-extendibility does not imply categoricity in general.

4.3. “Absolute Definiteness” as Non-forkability

Tarski ([64], p. 390, th. 9) proved that every categorical theory is non-forkable (cf. Definitions 2 and 3). I will argue that Husserl is conflating the uniqueness of an intended model with two consequences of it, namely the non-forkability of the theory and its decidability (understood as semantic completeness). In this section, I will show that one of the senses of “absolute definiteness” seems to be the intuitive counterpart of non-forkability. Veblen [67] also attributed to categorical theories the property of being non-forkable, but without making a distinction between categoricity and non-forkability.

In the third edition of Fraenkel’s *Einleitung in die Mengenlehre*, which was published in 1928, he added a third notion of completeness besides those of syntactic completeness and categoricity. This is the notion of a non-forkable theory, which does not involve any concept of provability—it is, so to speak, “purely semantical”:

“The system is to be called *complete* if, no matter whether we in fact succeed to deduce the truth or falsity of φ from the system or are able to secure its deducibility theoretically, only *either* the truth *or* the falsity of φ —but not both possibilities— is *compatible* with the system.” (Fraenkel [18], pp. 347–348; quoted from Awodey and Reck [3], p. 23)

According to Definition 3, a theory Γ is non-forkable if there is no sentence φ of its language such that both $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are satisfiable. This definition is equivalent to saying that Γ is compatible with φ or $\neg\varphi$, but not with both possibilities. If a theory is compatible with the truth of both φ and $\neg\varphi$, then there will be two models \mathcal{A}, \mathcal{B} of Γ such that \mathcal{A} satisfies φ and \mathcal{B} satisfies $\neg\varphi$. It follows that, in this case, φ will be independent from Γ (as $\Gamma \not\models \varphi$ and $\Gamma \not\models \neg\varphi$). Therefore, from a semantical point of view, there is no sentence of the language of a non-forkable theory which could be independent from that theory.³³ Husserl himself had no doubts about the impossibility of adding independent axioms to a “definite” axiom system:

“An axiom system is definite if it delimits an object domain as existing, and indeed in such a way that for that domain no new axiom (deductively independent of the axiom system) is possible.”
(Husserl [41], p. 436)

Veblen ([67], p. 346) called some axiom systems “disjunctive” as opposed to categorical theories. While it is always possible to add independent axioms to a disjunctive axiom system, every further axiom will be “redundant” if the theory is categorical. Stated in modern terminology: if Γ is categorical, then the model of $\Gamma \cup \{\varphi\}$ will be the (only) model of Γ or, on the contrary, $\Gamma \cup \{\varphi\}$ will not have a model at all. A disjunctive axiom system *forks* at some φ of its language, so the first-order theory of groups could be called “disjunctive” (it “leaves open” the possibility of having Abelian and non-Abelian models). “One [axiom system] to which it is possible to add independent axioms (and which therefore leaves more than one possibility open) is called *disjunctive*” (Veblen [67], p. 346).

In the same way, Husserl also talked about axiom systems that does not leaves open (or “undecided”) any question related to its intended model:

“An axiom system with a domain is “definite” if it leaves open or undecided no question related to the domain and meaningful in terms of this system of axioms.

Equivalent to this, once again, is the following crucial statement: An axiom system is definite if it delimits an object domain as existing, and indeed in such a way that for that domain no new axiom (deductively independent of the axiom system) is possible.”
(Husserl [41], p. 436)

One can adduce that here Husserl is not explaining a concrete meaning of “absolute definiteness”, but introducing “definiteness” in general. However, there is enough evidence to conclude that every axiom system which does not leave open any question is absolutely definite. As I have argued in Sect. 4.1, an absolutely definite theory is, for Husserl, equivalent to a theory containing Hilbert’s axiom of completeness. Consider now Husserl’s description of those axiom systems including this axiom: “The peculiar character of the Hilbertian

³³Although the second-order Peano arithmetic is non-forkable, Gödel’s formula is obviously *deductively* independent from it, for the reason that, by his (first) incompleteness theorem, $\mathbf{PA}^2 \not\vdash g$ and $\mathbf{PA}^2 \not\vdash \neg g$.

closed axiom system is this, that it leaves open no question whatsoever that the operation system offers [...] and consequently it admits of no expansion” (Husserl [41], p. 451). Hence, if “absolutely definite = complete, in Hilbert’s sense”, then an absolutely definite theory does not leave anything open (like Veblen’s categorical theories).

Furthermore, Husserl also thought that some axiom systems could leave something open. “An axiom system can delimit a sphere of existence and leave open a vague, broader sphere” (Husserl [41], p. 437). These theories could be extended by adding independent axioms³⁴ which, together with the previous ones, would describe a different model. “An axiom system is *relatively definite* if [...] it does admit that for a broader domain the same, and then of course also new, axioms are valid” (Husserl [41], p. 426). Therefore, relatively definite theories are disjunctive (or, in modern terminology, forkable). This is not the place to give an account of “relative definiteness”, but it is clear, I think, that for an absolutely definite theory no further axiom could be considered (again like Veblen’s categorical theories, for which additional axioms are redundant).

In 1928, Fraenkel called an axiom system “incomplete” if it is compatible with two contradictory sentences. More formally, a theory Γ is incomplete if there is a sentence φ of its language such that $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both consistent. This notion of incompleteness was identified by him and Carnap³⁵ as “forkability”. Fraenkel argued that these incomplete (forkable) theories do not only leave open the question of the provability of φ from Γ , but also every possible choice between φ and $\neg\varphi$ (this choice is open in an *absolute* sense). In the words of Veblen, that theory “leaves more than one possibility open” ([67], p. 346).

To summarize, Husserl and Veblen pointed out that some axiom systems do not admit independent axioms, as opposed to those which are “relatively definite” or “disjunctive”. An absolutely definite theory does not leave “open” any question related to its intended model, just like a categorical one. On the contrary, a disjunctive theory is compatible with two contradictory sentences φ and $\neg\varphi$, and thus it was called “forkable” and “incomplete” by Fraenkel. In 1928, he separated non-forkability from other concepts of completeness which were not clearly distinguished in the works of Husserl and Veblen.

4.4. “Absolute Definiteness” as Decidability

Tarski ([62], p. 491, th. 2) proved that every categorical theory is semantically complete (cf. Definitions 2 and 4). I will show that Husserl and Veblen treated non-forkability and semantic completeness as equivalent, but without a precise delimitation of these notions. Semantic completeness was hence attributed to absolutely definite and categorical theories, respectively. However, in [41,

³⁴“In the broadened domain it cannot be that merely the old axioms are valid. Otherwise the domain would not be broadened. Therefore, in the broader domain, in addition to the old axioms, yet further propositions must be valid—and, indeed, propositions that are not mere consequences of the old axioms.” (Husserl [41], p. 427).

³⁵Carnap ([8], p. 131) also argued that a theory f (a conjunction of propositional functions) is forkable at a propositional function g if it is compatible with both g and $\neg g$.

[67], there is only an implicit concept of logical consequence, although Veblen ([68], p. 28) anticipated that logical consequence may differ from its syntactic counterpart—i.e. a syllogistic process of deduction (cf. Awodey and Reck [3], p. 19). Finally, I will give textual evidence for concluding that Fraenkel’s (and Carnap’s) notion of “decidability” goes back to this last meaning of absolute definiteness.

Veblen obtained semantic completeness as a consequence of categoricity, but without proof. A theory that has essentially only one model decides “the validity of any possible statement”:

“It is part of our purpose however to show that there is *essentially only one* class of which the twelve axioms are valid [...] Consequently, any proposition which can be made in terms of points and order either is in contradiction with our axioms or is equally true of all classes that verify our axioms. The validity of any possible statement in these terms is therefore completely determined by the axioms.” (Veblen [67], p. 346)

A number of things are relevant here. If a theory Γ is categorical, then, for every sentence φ of its language, either $\Gamma \cup \{\varphi\}$ or $\Gamma \cup \{\neg\varphi\}$ does not have a model (as categoricity implies non-forkability). The reason is that the (only) model of Γ cannot satisfy φ and $\neg\varphi$ simultaneously. Thus, the (only) model of Γ will make true every sentence of the language of Γ (“any proposition which can be made in terms of points and order”) or its negation. This explains, of course, why Veblen argues that the validity of every statement is “completely determined” by the axioms of Γ .

Now, if for every φ of the language of Γ it is true that either $\Gamma \cup \{\varphi\}$ or $\Gamma \cup \{\neg\varphi\}$ is not satisfiable, then it follows that φ or $\neg\varphi$ is a logical consequence of Γ . In particular, if $\Gamma \cup \{\varphi\}$ is not satisfiable, then $\Gamma \models \neg\varphi$; and, if $\Gamma \cup \{\neg\varphi\}$ is not satisfiable, then $\Gamma \models \varphi$. In 1906, Veblen arrived at the same conclusion (i.e. he inferred semantic completeness from categoricity), but it is formulated informally:

“If we have before us a categorical system of axioms, every proposition which can be stated in terms of our fundamental (undefined) symbols either is or is not true of the systems of objects satisfying the axioms. In this sense it either is a consequence of the axioms or is in contradiction with them.” (Veblen [68], p. 28)

I assume that the expression “ φ is in contradiction with Γ ” is equivalent to “ $\neg\varphi$ is a consequence of Γ ”. The point is that Veblen himself is aware that—when Γ is categorical—every system of objects satisfying Γ is either a model of φ or a model of $\neg\varphi$, so either φ or $\neg\varphi$ is a consequence of Γ . Surprisingly, the same argument was found in Husserl’s philosophy of arithmetic. He explained, in clear terms, that a definite axiom system is what Tarski ([62], p. 489) called “semantically complete”:

“An axiom system that delimits a domain is said to be “definite” if every proposition intelligible on the basis of the axiom system, understood as a proposition of the domain, is either true on the

basis of the axioms or false on the basis of them. Or, put otherwise:
 If only two things are possible, either the proposition follows from
 the axioms or contradicts them.” (Husserl [41], p. 438)

In the *Doppelvortrag*, Husserl explicitly defended that “an axiom system is absolutely definite if every proposition meaningful according to it is decided in general” ([41], p. 427). It is important to emphasize the close similarities between Husserl’s definition of “absolute definiteness” and Veblen’s approach to categoricity. The validity of every sentence is “completely determined” by the axioms of a categorical theory. Clearly, this requirement is connected with Hilbert’s conviction that every mathematical question can be answered either positively or negatively.³⁶ An absolutely definite theory is also “complete” in this last sense.

I have recently proposed that this sense of “absolute definiteness” should be understood as semantic completeness, rather than syntactic completeness, to make more plausible Husserl’s solution to the problem of imaginary numbers (cf. [1]). Although (in 1901) the “standard cut” between provability and logical consequence had not yet been defined, I think that there are reasons to accept that Husserl’s notion of consequence (explicitly given in the above quote) was basically semantical, at least in the context of absolute definiteness.

Firstly, it must be pointed out that there is a unique manifold satisfying the axioms of an absolutely definite theory. For this reason, Husserl could have noticed that, for every sentence φ of its language (or “intelligible on the basis of the axiom system”), this manifold makes true either φ or $\neg\varphi$ (or “is either true on the basis of the axioms or false on the basis of them”). And, if every manifold that satisfies the axioms of the theory also satisfies φ (or $\neg\varphi$), then φ (or $\neg\varphi$) is a consequence of the axioms. This is a semantical conception of “consequence” and, consequently, absolute definiteness foreshadows semantic completeness. Husserl concluded that “only two things are possible, either the proposition follows from the axioms or contradicts them” ([41], p. 438).

Secondly, Fraenkel [18] quoted Husserl’s *Ideas I* to explain the meaning of a “definite” manifold, because this notion is very close to completeness as “decidability” (“*Entscheidungsdefinitheit*”). His concept of consequence was, according to what Fraenkel said, defined in semantical terms (it involves the notion of *truth*):

“Here it is formulated [decidability] in such a way that, in a definite “manifold”, every proposition expressed in the relevant terms must be “either a purely formal consequence of the axioms or a counter-consequence, i.e. in contradiction with the axioms”. Hence, “true” and “formal consequence of the axioms” are equivalent terms.³⁷”
 (Fraenkel [18], p. 352, fn. 1)

³⁶“After all, one of the things that attract us most when we apply ourselves to a mathematical problem is [...] here is the problem, search for the solution; you can find it by pure thought, for in mathematics there is no *ignorabimus*.” (Hilbert [30], p. 384).

³⁷“Es wird hier so formuliert, dass in einer derart definierten „Mannigfaltigkeit“ jeder in den einschlägigen Begriffen ausgedrückte Satz „entweder eine pure formallogische Folge der Axiome oder eine ebensolche Widerfolge, d. h. den Axiomen formal widersprechend“ sein

From this passage it is also clear that “decidability” has one of its roots in Husserl’s concept of “definiteness”. In fact, Carnap also mentioned Husserl as precursor to one of the three senses of completeness:

“*Terminologie*: (1) „monomorph“: Veblen „kategorisch“, Huntington „hinreichend“, Fraenkel und Weyl „vollständig“; (2) „nichtgabelbar“; (3) „entscheidungsdefinit“: so Husserl’s und Becker; Hilbert „vollständig.“ (Carnap [8], p. 128, fn. 1)

This paper is, as far as I know, the first historical study on completeness where the connection between Husserl, Fraenkel and Carnap is documented. Notice that for Carnap ([8], pp. 143–144, Def. 3.6.1) a satisfiable axiom system Γ is decidable if, for every propositional function φ , either φ or $\neg\varphi$ will be a consequence of Γ (it is k -decidable if it is possible to specify a procedure that allows to prove either φ or $\neg\varphi$). Husserl said that, if a proposition has sense according to the axioms of a “definite” theory, then its truth or falsity will be decided (“*entschieden*”) by them.³⁸ Hence, (absolutely) definite theories are “*entscheidungsdefinit*”.

However, the many senses in which an axiom system is “complete” (or, better, “definite”) are not clearly distinguished in Husserl’s texts. Apart from the confusion between uniqueness and non-extendibility, Husserl also thought that decidability was equivalent to non-forkability:

“An axiom system that delimits a domain is said to be “definite” if every proposition intelligible on the basis of the axiom system [...] either follows from the axioms or contradicts them.

Equivalent to this is the following statement: An axiom system with a domain is definite if it leaves open or undecided no question related to the domain [...]

Equivalent to this, once again, is the following crucial statement: an axiom system is definite if it delimits an object domain as existing, and indeed in such a way that for that domain no new axiom [...] is possible.” (Husserl [41], p. 438)

Consequently, it is not surprising that Carnap ([8], p. 59) believed that the main difficulty of undertaking metalogical investigations lied in the vagueness of the concepts used at that time. In particular, he noticed that the term “complete” was (simultaneously) applied to categorical, non-forkable and decidable theories. Although Carnap failed to prove the equivalence of the three properties (cf. Awodey and Reck [3], p. 26), it is clear that Husserl represents a conflation of the concepts of completeness that was typical at the beginning of the twentieth century.

muss. Hiemach werden „wahr“ und „formallogische Folge der Axiome“ zu äquivalenten Begriffen.” (Fraenkel [18], p. 352, fn. 1).

³⁸“Absolut definit ist ein Axiomensystem, wenn jeder nach ihm sinnvolle Satz überhaupt entschieden ist” (Husserl [40], p. 440).

“Ein solches Operationssystem ist definit auch in dem Sinne, dass jedem Satz nun angesehen werden kann, ob er in das Gebiet füllt oder nicht [...] Hat ein Satz Sinn gemäss den Axiomen, dann ist er durch die Axiome in Wahrheit und Falschheit entschieden” (Husserl [40], p. 455).

Conclusions

In light of the discussion above, the conclusion is, I think, a mixed assessment of Husserl's *Doppelvortrag*. This *Doppelvortrag* has been largely neglected by the analytic tradition, so the commentators could not give a full explanation of the development of the notion(s) of completeness. Nevertheless, "absolute definiteness" cannot mean neither syntactic completeness nor categoricity, as the discussion made evident. In order to assess Husserl's place in the history of logic, a more balanced account of "absolute definiteness" is needed.

On the positive side, Husserl addressed different questions regarding the completeness of the axiom systems in the context of the problem of imaginary numbers. This discussion—which was central to Husserl's contemporaries—is rarely seen as a historical antecedent for the modern notion of completeness. However, Husserl introduced absolute definiteness not only as a property for non-extendible "manifolds" (like the real numbers), but also for those theories that contain the axiom of completeness. These theories have a non-extendible model, although he believed that such a requirement should not be stipulated by means of an axiom. The rejection of Hilbert's axiom could explain his shift from the completeness (definiteness) of the models to the completeness of the axiom systems, what occurred years later in Hilbert's work. In 1901, Husserl attributed to absolutely definite theories different properties that were finally identified (by Fraenkel and Carnap) as the various concepts of completeness. I have argued that categoricity, non-forkability and decidability are the formal counterparts of the intuitions that are conflating in Husserl's idea of absolute definiteness. This is confirmed by the fact that Fraenkel and Carnap mention Husserl as precursor to decidability.

On the negative side, Husserl's approach to completeness is still largely informal. An articulated concept of isomorphism (which is not present in the *Doppelvortrag* and was not so uncommon during that period) could have been useful to formulate absolute definiteness mathematically. Husserl also lacked the tools that would allow him to incorporate a formal language and a sharp notion of provability. Since these resources were developed by Frege, Husserl had them at his disposal. Although he did not separate the uniqueness of a model from its non-extendibility, the belief that, in general, non-extendibility implies categoricity seems to have been quite widespread among mathematicians until 1928. In the same way, a comparison between the works of Husserl and Veblen shows that the uniqueness of a model was not clearly distinguished from its consequences. Categorical and absolutely definite theories were what Fraenkel and Carnap called "non-forkable" and "decidable" (i.e. semantically complete). However, the proofs had to wait until Tarski's contributions.

As a result, I firmly believe that Husserl's notion of definiteness deserves a place in the history of completeness alongside the metalogical investigations carried out (by Hilbert, Huntington, Veblen, etc.) in the late nineteenth and early twentieth century. This notion is far from being formally delimited, but it was a clear step forward in the understanding of the role of non-extendible models and axiom systems in mathematics.

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