Aggregation Operators for Fuzzy Rationality Measures

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Abstract. Fuzzy rationality measures represent a particular class of aggregation operators. Following the axiomatic approach developed in [1,3,4,5] *rationality* of fuzzy preferences may be seen as a fuzzy property of fuzzy preferences. Moreover, several rationality measures can be aggregated into a global rationality measure. We will see when and how this can be done. We will also comment upon the feasibility of their use in real life applications. Indeed, some of the rationality measures proposed, though intuitively (and axiomatically) sound, appear to be quite complex from a computational point of view.

Keywords: aggregation rules, fuzzy preferences, decision making.

1 Introduction

Suppose we are given a finite set of alternatives X. We can consider all complete fuzzy binary relations [21] defined on X, that is, mappings

$$\mu : X \times X \to [0,1]$$

such that each value $\mu(x,y)$ represents the intensity value to which alternative y is not worse than alternative x (see also [11,16]). The completeness assumption guarantees that

$$\mu(x,y) + \mu(y,x) \geq 1$$

for all $x, y \in X$. Without such an assumption we would have to deal with possible incomparability problems.

An axiomatic characterization for the notion of *rationality* measure has been given in [4], where *consistency* was viewed as a fuzzy property of fuzzy binary preference relations. Moreover, in [5] it was studied the problem of logical combinations of such rationality measures along with some definitions of functional equivalence. Since each fuzzy preference relation can be measured according to different rationality criteria, we should be willing to check if and when we can aggregate different rationality criteria into one.

On the other hand, fuzzy rationality measures can be seen as a special class of aggregation operators. Therefore, we should be willing to investigate on the feasibility of their use in real life applications.

2 Fuzzy rationality measures

Max-min transitivity is one of the most common rationality measures in the context of fuzzy binary preference relations. However, the property of being max-min transitive is crisp, that is to say each relation either is or is not max-min transitive, despite the fact that we intuitively realize that some fuzzy preferences are much closer to max-min transitivity than others. We will see that there exist other measures which allow a fuzzy classification of fuzzy preferences. These measures verify a set of axioms which have been formally introduced in [4]. In details, rationality measures are characterized as mappings

$$\rho : \mathcal{P} \to [0,1]$$

where

$$\mathcal{P} = \bigcup_{X \text{ finite}} \mathcal{P}(X)$$

represents the family of all complete fuzzy binary relations. The following conditions were assumed in order for such a mapping to define a *rationality measure*.

(R1) $\rho(\mu) = 1$ for any μ defining a crisp strict chain on X.

(R2) Given $\mu \in \mathcal{P}$ and a permutation $\pi : X \to X$ then

$$\rho(\mu^\pi) = \rho(\mu)$$

where $\mu^\pi(x,y) = \mu(\pi(x),\pi(y))$ for all $x, y \in X$.

(R3) For all $\mu \in \mathcal{P}$, $\rho(\mu^{-1}) = \rho(\mu)$, where $\mu^{-1}(x,y) = \mu(y,x)$ for all $x, y \in X$.

(R4) Let Y be a non-empty finite set of alternatives and let x be an extra alternative not belonging to Y. Let us consider a fuzzy preference $\mu : Y \times Y \to [0,1]$ such that $\mu(y,z) = 1, \mu(z,y) = 0, \forall y \in Y_1, \forall z \in Y_2$ for some Y_1, Y_2 partition of Y, and an extension μ' such that

$$\mu'(y,z) = \mu(y,z), \forall y,z \in Y$$
$$\mu'(y,x) = 1, \mu'(x,y) = 0, \forall y \in Y_1$$
$$\mu'(x,z) = 1, \mu'(z,x) = 0, \forall z \in Y_2$$
$$\mu'(x,x) = 1$$

Then it must be

$$\rho(\mu') \geq \rho(\mu).$$

(R5) Let $\mu \in \mathcal{P}(X)$ be fixed. Given an arbitrary ordered pair of alternatives $(\bar{x},\bar{y})$, an arbitrary point $(\bar{a},\bar{b}) \in [0,1] \times [0,1]$, and real numbers γ and λ such that $0 \leq \bar{a} + \lambda \cos\gamma \leq 1$, $0 \leq \bar{b} + \lambda \sin\gamma \leq 1$ and $\bar{a} + \bar{b} + \lambda(\sin\gamma + \cos\gamma) \geq 1$, we denote by $\Gamma_\mu((\bar{x},\bar{y}),(\bar{a},\bar{b}),\gamma,\lambda)$ the fuzzy preference relation defined as

$$\Gamma_\mu((\bar{x},\bar{y}),(\bar{a},\bar{b}),\gamma,\lambda)(x,y) = \begin{cases} \bar{a} + \lambda\cos\gamma & \text{if } (x,y) = (\bar{x},\bar{y}) \\ \bar{b} + \lambda\sin\gamma & \text{if } (x,y) = (\bar{y},\bar{x}) \\ \mu(x,y) & \text{otherwise} \end{cases}.$$

Let $(\bar{x}, \bar{y})$, $(\bar{a}, \bar{b})$, γ be fixed and let us consider the fuzzy preference relation $\Gamma^*(\lambda)$ defined as

$$\Gamma^*(\lambda)(x, y) = \Gamma_\mu((\bar{x}, \bar{y}), (\bar{a}, \bar{b}), \gamma, \lambda)(x, y)$$

Then, one of the following two properties must be verified by ρ.
(R5.1) there is no value λ such that

$$\rho(\Gamma^*(\lambda_1)) > \rho(\Gamma^*(\lambda))$$
$$\rho(\Gamma^*(\lambda)) < \rho(\Gamma^*(\lambda_2))$$

for some λ_1, λ_2 such that $\lambda_1 < \lambda < \lambda_2$.
(R5.2) there is no value λ such that

$$\rho(\Gamma^*(\lambda_1)) < \rho(\Gamma^*(\lambda))$$
$$\rho(\Gamma^*(\lambda)) > \rho(\Gamma^*(\lambda_2))$$

for some λ_1, λ_2 such that $\lambda_1 < \lambda < \lambda_2$.

We then have the following definition.

Definition 1. Any mapping $\rho : \mathcal{P} \to [0, 1]$ is a

1. pessimistic fuzzy rationality measure if it verifies conditions (R1)-(R5.1);
2. optimistic fuzzy rationality measure if it verifies conditions (R1)-(R5.2).

A fuzzy rationality measure being both pessimistic and optimistic will be said *normal*. □

For example, max min transitivity rationality condition is a *pessimistic* fuzzy rationality.

(Ex1)

$$\rho_{maxmin}(\mu) = \begin{cases} 1 \text{ if } \mu(x, z) \geq \min(\mu(x, y), \mu(y, z)), \forall x, y, z \in X, \\ 0 \text{ otherwise} \end{cases}$$

Some more examples of fuzzy rationality measures can be found in [4].

3 Equivalent rationality measures

As pointed out in [3], two fuzzy rationality measures can be considered equivalent if they always agree when comparing individual rationalities. In this case, it can be supposed that the discrepance between both rationality measures is just due to different underlying scales.

Definition 2. Given two fuzzy rationality measures ρ_1 and ρ_2 we will say that ρ_1 and ρ_2 are equivalent if and only if for any pair of individuals μ_1 and μ_2, $\rho_1(\mu_1) \geq \rho_1(\mu_2)$ if and only if $\rho_2(\mu_1) \geq \rho_2(\mu_2)$. □

In this particular context, it was proven that any class of equivalent fuzzy rationality measures is closed with respect to the most important logical compositions.

Theorem 3. *Let us consider $\rho_1, \rho_2, \cdots, \rho_k$, k equivalent fuzzy rationality measures. Let*

$$H : [0,1]^k \to [0,1]$$

be a strictly nondecreasing mapping, in such a way that

$$H(a_1, \cdots, a_k) \geq H(b_1, \cdots, b_k) \text{ if } a_j \geq b_j \text{ for all } j = 1, 2, \cdots, k$$

and

$$H(a_1, \cdots, a_k) > H(b_1, \cdots, b_k) \text{ whenever } a_j > b_j \text{ for all } j = 1, 2, \cdots, k.$$

Let also assume

$$H(1, 1, \cdots, 1) = 1.$$

Then the mapping

$$H(\rho_1, \rho_2, \cdots, \rho_k) : \mathcal{P} \to [0,1]$$

defined as

$$H(\rho_1, \rho_2, \cdots, \rho_k)(\mu) = H(\rho_1(\mu), \rho_2(\mu), \cdots, \rho_k(\mu)), \forall \mu$$

is a fuzzy rationality measure equivalent to $\rho_1, \rho_2, \cdots, \rho_k$.

Since OWA operators [18,19,20] do verify the conditions of the above theorem, they can be used to obtain new measures from finite collections of equivalent fuzzy rationality measures. T-norms and T-conorms can be used whenever they verify the above strict monotonicity condition.

4 General results on min and max operators

Max and *Min* operators have been justified in the fuzzy literature in several ways (see, e.g., [17,19]). The following results show the good performance of these two operators when restricted to pessimistic and optimistics rationality measures.

Theorem 4. *Let ρ_1 and ρ_2 be two pessimistic [resp. optimistic] fuzzy rationality measures. Then the mapping $\min(\rho_1, \rho_2)$ [resp. $\max(\rho_1, \rho_2)$] defines a pessimistic [resp. optimistic] fuzzy rationality measure.*

Proof: We will just prove the min case, since the other case is perfectly similar. Properties (R1)-(R4) are immediate. In order to check regularity, let us assume that

$$\min(\rho_1, \rho_2)(\Gamma^*(\lambda_1)) > \min(\rho_1, \rho_2)(\Gamma^*(\lambda)) < \min(\rho_1, \rho_2)(\Gamma^*(\lambda_2))$$

for some $\lambda, \lambda_1, \lambda_2$ such that $\lambda_1 < \lambda < \lambda_2$. Let us assume also that

$$\min(\rho_1, \rho_2)(\Gamma^*(\lambda)) = \rho_1(\Gamma^*(\lambda)).$$

Since ρ_1 is pessimistic but

$$\min(\rho_1, \rho_2)(\Gamma^*(\lambda_1)) \leq \rho_1(\Gamma^*(\lambda_1))$$

and

$$\min(\rho_1, \rho_2)(\Gamma^*(\lambda_2)) \leq \rho_1(\Gamma^*(\lambda_2))$$

always hold, then we have reached to a contradiction. □

Analogous closure result is obtained for OWA operators if restricted to more particular families of pessimistic or optimitic rationality measures (see [5]).

Theorem 5. *Let $\rho_1, \rho_2, \cdots, \rho_k$ be k fuzzy rationality measures, all of them pessimistic (optimistic), in such a way that -according to notation given in (R5)- every mapping $\rho_i(\Gamma^*(\lambda))$ is concave (convex) in λ. Then $H(\rho_1, \rho_2, \cdots, \rho_k)$ is also a pessimistic (optimistic) fuzzy rationality measure, if H is an OWA operator with increasing (decreasing) associated weights.*

5 The computational problem

Let us consider the fuzzy rationality measure ρ_{maxmin}. Though semantically such a rationality measure is quite unsatisfactory, since it requires a crisp property, from a computational point of view is instead a good example. Indeed, as it can easily be seen (see, e.g., [9]), for any μ the value of $\rho_{maxmin}(\mu)$ can be computed in time $O(|X|^3)$. Therefore, we have a polynomial time computable fuzzy rationality measure which makes it easy to use in real life applications. This is not always the case.

The example that follows is based upon Orlovsky's choice set of unfuzzy nondominated alternatives (see, e.g., [16]). Given a fuzzy preference

$$\mu : X \times X \rightarrow [0, 1]$$

for any arbitrary non-empty subset Y of alternatives we define

$$Y^{\mu}_{UND} = \{x \in Y | \mu(x, y) > \mu(y, x), \forall y \in Y\}.$$

Then the following map is an optimistic fuzzy rationality measure (see also [4]):

$$\rho_N(\mu) = \begin{cases} 0 \text{ if } \exists Y^{\mu}_{UND} = \emptyset, \text{ for some } Y \neq \emptyset \\ \min\{\frac{1}{|Y^{\mu}_{UND}|} / Y^{\mu}_{UND} \neq \emptyset\} \text{ otherwise} \end{cases}.$$

The problem of computing such a rationality measures can be rewritten in a graph-theoretical manner (see, e.g., [10]. Given μ and X we can build a graph G_μ whose vertices are the elements of X and such that there exists an edge from x to y if and only if $\mu(x, y) > \mu(y, x)$. We then have

Proposition 6. *$\rho_N(\mu) = 0$ if and only if there exists a cycle in G_μ.* □

As a consequence, it is relatively easy to check whether μ is absolutely irrational (acording to this particular rationality measure). However, once we know that there are no cycles in G_μ, computing the value of $\rho_N(\mu)$ becomes a computationally hard problem. Indeed, the following holds.

Proposition 7. *Given a positive integer K, $\rho_N(\mu) \leq \frac{1}{K}$ if and only if G_μ has an independent set of size K or more.* □

We recall that an independent set is a subset of vertices such that no two vertices in it are connected by an edge. Moreover, it is well known that the problem of deciding whether a graph has an independent set of size K or more belongs to the class of $\mathcal{NP}$-complete problems.

If we introduce the following definition

Definition 8. A fuzzy rationality measure is computationally hard if given any $0 \leq r \leq 1$ and fuzzy preference relation μ the problem of deciding whether $\rho(\mu) \geq r$ is NP-hard. □

then we have that ρ_N is computationally hard.

5.1 Polynomial Constructability

We end this section by remarking a property of the fuzzy rationality measure ρ_N which will be used later on. Given any positive integer K it is possible in polynomial time to produce a fuzzy preference relation μ such that $\rho_N(\mu) = \frac{1}{K}$. By using the graph G_μ, we build a graph on X with a maximum independent set of size K as follows:

- choose randomly a set Y of K vertices;
  add an edge from each vertex of Y to each vertex not in Y;
- if $K \geq \frac{|X|}{2}$ then stop, else recursively, build a graph on $X \setminus Y$ with an independent set of size K.

Definition 9. A fuzzy rationality measure ρ is [polynomially] constructible if there exists a [polynomial] algorithm which given any number $0 \leq r < 1$ decides whether there exists μ such that $\rho(\mu) = r$ and it outputs an example. □

Obviously, ρ_N is polynomially constructible.

In an attempt to bypass the hardness problem such as the one above seen, one may think to find a rationality measure, equivalent to a given computationally hard one but that instead can be computed easily. In fact, this approach is in general doomed to failure. Indeed the following holds.

Theorem 10. *Let ρ be a computationally hard fuzzy rationality measure. Suppose that ρ is polynomially constructible. Then, if ρ' is equivalent to ρ, is computationally hard as well.*

6 Final Remarks

Consistency of fuzzy preferences use to be defined according to the main application the decision maker is thinking on. When our objective is just descriptive, we should realize that depending on the potential application our idea of consistency may be changing. In this case we should keep the information of an adequate number of rationality measures allowing a right description for all those potential purposes. Then, we may be interested in some global index, which will be some amalgamated value of those partial indices.

Computational aspects should be always addressed in the aggregation procedure of fuzzy rationality measures, but also to each rationality measure itself. Computational difficulties should be taken into account, for example, when considering Montero's rationality measure [2,12,13,14], where completeness of fuzzy preferences was also assumed. We must point out the importance of such a completeness assumption. Rationality under incomparability as modeled in [6,7,8] (see also [15]) appears as a need in future research.

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