

Topological characterisation of weakly compact operators

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Abstract

Let X be a Banach space. Then there is a locally convex topology for X , the “Right topology,” such that a linear map T , from X into a Banach space Y , is weakly compact, precisely when T is a continuous map from X , equipped with the “Right” topology, into Y equipped with the norm topology. When T is only sequentially continuous with respect to the Right topology, it is said to be *pseudo weakly compact*. This notion is related to Pelczynski’s Property (V).

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1. Introduction

One of the fundamental theorems of classical measure theory is the Riesz Representation Theorem, first obtained for scalar and then for vector measures. Let T be a compact Hausdorff space; $C(T)$ the space of complex valued continuous functions on T ; $B(T)$ the space of bounded Baire measurable functions on T , Y a Banach space and $T : C(T) \rightarrow Y$ a bounded linear operator. The generalised Riesz Representation Property holds for T if there exists an operator $T^\infty : B(T) \rightarrow Y$ which extends T and is such that, when (f_n) ($n = 1, 2, \dots$) is a bounded sequence in $B(T)$ which converges pointwise to f , then $\|T^\infty(f_n) - T^\infty(f)\| \rightarrow 0$. When Y is finite dimensional then this is true for any T , but for a general Banach space, Y , the appropriate condition needed is that T be a weakly compact operator.

So, when generalising vector measure theory to non-commutative C^* -algebras, a natural setting was to consider a weakly compact operator T mapping a C^* -algebra A into a Banach space Y . From considerations of non-commutative measure theory we have been led to the observation that weakly compact operators from a fixed Banach space X to an arbitrary Banach space Y can be characterised by a continuity property. More precisely, there is a “Right topology” for X , such that, a linear map from X into Y is weakly compact precisely when it is a continuous map from X , equipped with the Right topology, into Y , equipped with the norm topology. There are several different ways of proving the results below. We have strived for maximal clarity rather than maximal generality or maximal conciseness; but the arguments are brief. We present proofs which use only well-known classical results in Functional Analysis; more sophisticated approaches are possible. In a sequel to this note we will give a number of applications, generalisations and further developments, particularly for the multi-linear situation.

When T is only sequentially continuous with respect to the Right topology, it is said to be *pseudo weakly compact*. When a Banach space X has the property that every pseudo weakly compact operator from X to another Banach space is weakly compact, then X is said to be *sequentially Right*. It is proved that every Banach space possessing Pelczynski’s Property (V) must be sequentially Right.

2. Continuity from the Right topology to the norm topology

In this note let X and Y be Banach spaces. Whenever Z is a Banach space, let its closed unit ball be denoted by Z_1 . Let us recall [4,6] that the Mackey topology for the dual pair (X^{**}, X^*) is the topology of uniform convergence on sets $K \subset X^*$, where K is absolutely convex and $\sigma(X^*, X^{**})$ compact. That is, where K is a weakly compact, absolutely convex subset of the Banach space X^* . We denote this topology by $\tau(X^{**}, X^*)$; it is the finest locally convex topology for the dual pair (X^{**}, X^*) . We identify X with its canonical embedding in X^{**} and call the relative topology induced on X by $\tau(X^{**}, X^*)$, the “Right topology” for X . See also [5], where this topology is discussed and called the “quasi-Mackey topology.” We shall call a linear map between Banach spaces an *operator* precisely when it is bounded.

Theorem 1. *Let $T : X \rightarrow Y$ be a weakly compact operator. Then T^{**} is a continuous map from X^{**} , equipped with the $\tau(X^{**}, X^*)$ topology, into Y , equipped with the norm topology.*

Proof. Let K be the weak closure of $T^*[Y_1^*]$ in X^* . By Gantmacher’s theorem [3], $T^* : Y^* \rightarrow X^*$ is weakly compact. So K is an absolutely convex, weakly compact set. That is, K is $\sigma(X^*, X^{**})$ compact.

Let (x_λ) be a net in X^{**} which converges to x in the Mackey topology, $\tau(X^{**}, X^*)$. Then $(x_\lambda - x) \rightarrow 0$, uniformly on K , and hence

$$\lim_{\lambda \rightarrow \infty} \sup \{ |\langle (x_\lambda - x), \phi \rangle| : \phi \in K \} = 0.$$

But

$$\begin{aligned} \sup \{ |\langle (x_\lambda - x), \phi \rangle| : \phi \in K \} &\geq \sup \{ |\langle (x_\lambda - x), T^* \psi \rangle| : \psi \in Y_1^* \} \\ &= \sup \{ |\langle T^{**}(x_\lambda - x), \psi \rangle| : \psi \in Y_1^* \} = \|T^{**}(x_\lambda - x)\|. \end{aligned}$$

Therefore $\|T^{**}(x_\lambda - x)\| \rightarrow 0$. \square

Lemma 2. Let (y_λ) be a Cauchy net in a Banach space Y . Then there exists $z \in Y$ such that $\|y_\lambda - z\| \rightarrow 0$.

Proof. See [3, p. 28]. \square

Lemma 3. Let $T : X \rightarrow Y$ be a linear map whose restriction to X_1 is a continuous map from X_1 , equipped with the topology induced by the Right topology on X , into Y , equipped with its norm topology. Then T is a bounded linear operator.

Proof. Let (x_n) be a sequence in X which converges in norm to 0. Let K be any absolutely convex, weakly compact subset of X^* . Then K is bounded in norm. So $(x_n) \rightarrow 0$ uniformly on K . Thus $(x_n) \rightarrow 0$ in the Right topology. Since $\|x_n\| \rightarrow 0$, $\|x_n\| \leq 1$ for all sufficiently large n . So $\|Tx_n\| \rightarrow 0$. Thus T is a bounded linear operator. \square

Theorem 4. Let $T : X \rightarrow Y$ be a linear map whose restriction to X_1 is a continuous map from X_1 , equipped with the topology induced by the Right topology on X , into Y , equipped with its norm topology. Then T is weakly compact.

Proof. By Lemma 3, T is a bounded linear operator from X into Y . As usual, we identify X with its canonical image in X^{**} . Then, by Goldstine's theorem, see [3, p. 424], X_1 is $\sigma(X^{**}, X^*)$ dense in X_1^{**} . Since $\tau(X^{**}, X^*)$ is a locally convex topology for the dual pair (X^{**}, X^*) , the closure of the convex set X_1 in the $\tau(X^{**}, X^*)$ -topology is also X_1^{**} , see [3, Corollary 14, p. 418]. So, given any $z \in X_1^{**}$, there exists a net (x_λ) in X_1 which converges to z in the $\tau(X^{**}, X^*)$ topology. So this net also converges to z in the $\sigma(X^{**}, X^*)$ topology.

Fix ψ in Y^* , then $T^* \psi \in X^*$. So $\langle x_\lambda, T^* \psi \rangle \rightarrow \langle z, T^* \psi \rangle$, and thus $\langle T^{**}x_\lambda, \psi \rangle \rightarrow \langle T^{**}z, \psi \rangle$. Hence $\langle Tx_\lambda, \psi \rangle \rightarrow \langle T^{**}z, \psi \rangle$.

Fix $\varepsilon > 0$. Then, because of the continuity hypothesis imposed on T , the set $T^{-1}\{y : \|y\| < \varepsilon\} \cap X_1$ is of the form $V \cap X_1$, where V is open in the $\tau(X^{**}, X^*)$ topology. So there exists U such that U is an open absolutely convex neighbourhood of 0 in $\tau(X^{**}, X^*)$ and $U \subset V$. Then $\frac{1}{2}U$ is an open absolutely convex neighbourhood of 0, in the $\tau(X^{**}, X^*)$ topology. So there exists λ_0 such that $\lambda \geq \lambda_0$ implies $x_\lambda - z \in \frac{1}{2}U$. Thus, for $\mu \geq \lambda_0$ and $\lambda \geq \lambda_0$, $x_\lambda - x_\mu(x_\lambda - z) + (z - x_\mu) \in U$. So $\frac{1}{2}(x_\lambda - x_\mu) \in U \cap X_1$. Hence $\|T(x_\lambda - x_\mu)\| \leq 2\varepsilon$. It follows from this that (Tx_λ) is a Cauchy net (with respect to the norm) in Y . So it converges in norm to an element of Y , say w . But it follows from the preceding paragraph that $w = T^{**}z$. So $T^{**}z \in Y$. Hence T^{**} maps X^{**} into Y , which implies that T is weakly compact. \square

Corollary 5. *Let $T : X \rightarrow Y$ be a linear map. Then the following conditions are equivalent:*

- (i) *T is continuous from X , equipped with the Right topology, into Y , equipped with the norm topology.*
- (ii) *T is continuous from X_1 , equipped with the relative topology induced by the Right topology, into Y , equipped with the norm topology.*
- (iii) *T is weakly compact.*
- (iv) *T is a bounded linear operator and $T^{**} : X^{**} \rightarrow Y^{**}$ is continuous from X^{**} , equipped with the $\tau(X^{**}, X^*)$ topology, into Y^{**} equipped with the norm topology.*

Proof. (i) implies (ii) trivially and (ii) implies (iii) by Theorem 4. Suppose that (iii) holds. Then T maps X_1 into a weakly compact subset of Y . But weakly compact subsets of a Banach space are norm bounded. So T is a bounded linear operator. Hence T^{**} is well defined. By well-known properties of weakly compact operators, T^{**} maps X^{**} into Y . Then (iv) follows by applying Theorem 1. Clearly (iv) implies (i). \square

Corollary 6. *A Banach space X is reflexive if, and only if, the Right topology for X coincides with the norm topology.*

Proof. The space is reflexive precisely when the identity map on X is weakly compact (see [3, Theorem V.4.7]). By Corollary 5 the identity map is weakly compact, if, and only if, the norm topology is weaker than the Right topology. Since the Right topology is always weaker than the norm topology, the result follows. \square

3. Pseudo weakly compact operators

By the Eberlein Šmulian theorem we know that weak compactness is, in some sense, a sequential property. We have already seen in the previous section that a (bounded) linear operator, T , between two Banach spaces, X and Y , is weakly compact if and only if it is continuous from X , equipped with the Right topology, into Y , equipped with the norm topology. Clearly such an operator T is sequentially continuous from X , equipped with the Right topology, into Y , equipped with the norm topology. It is natural to ask if the converse is true.

Definition 7. Let X and Y be Banach spaces. Let $T : X \rightarrow Y$ be a linear map such that, when $x_n \rightarrow 0$ in the Right topology then $\|Tx_n\| \rightarrow 0$. Then we call T pseudo weakly compact.

By (the proof of) Lemma 3, a pseudo weakly compact linear map between Banach spaces is always a bounded operator.

The following example shows that, in general, not every pseudo weakly compact operator is weakly compact.

Example 8. Let T be the identity map from ℓ^1 onto ℓ^1 . Since ℓ^1 is not reflexive, its unit ball is not weakly compact, see [3, Theorem V.4.7]. So T is not a weakly compact operator.

On the other hand, when $x_n \rightarrow 0$, in the Right topology then $x_n \rightarrow 0$, in the $\tau((\ell^1)^{**}, (\ell^1)^*)$ -topology. So $x_n \rightarrow 0$, in the $\sigma((\ell^1)^{**}, (\ell^1)^*)$ -topology. Hence $x_n \rightarrow 0$, in the weak topology of ℓ^1 . But, by [3, IV.8.14], this implies that $x_n \rightarrow 0$, in the norm topology, that is, $\|Tx_n\| \rightarrow 0$. So T is pseudo weakly compact.

When X is a C^* -algebra, then its second dual, X^{**} , can be identified with the von Neumann envelope of X , when X is represented on its universal representation (Hilbert) space [7]. Then it follows from a fundamental result of Akemann, see Theorem III.5.7 in [7], that when the σ -strong* topology of X^{**} is restricted to the unit ball of X , it coincides with the restriction of the Right topology to X_1 (see [7] for the basic definitions). In [8] the last two authors of the present note introduced the notion of *quasi completely continuous* linear operators from a C^* -algebra into a Banach space. It turns out that an operator from a C^* -algebra into a Banach space is quasi completely continuous if, and only if, it is pseudo weakly compact. This is because, as remarked above, the Right topology coincides, on the norm closed unit ball, with the restriction of the σ -strong* topology of X^{**} . Therefore, from [8, Proposition 2.2], for a linear operator T from a C^* -algebra into a Banach space the following are equivalent: T is weakly compact; T is quasi completely continuous; T is pseudo weakly compact.

The following definition now makes sense.

Definition 9. A Banach space X is said to be *sequentially Right* if every pseudo weakly compact operator on X is weakly compact; in other words, if each operator on X which is sequentially continuous with respect to the Right topology is also continuous with respect to the Right topology.

We have already seen in the above paragraph that every C^* -algebra is sequentially Right. Clearly every reflexive Banach space is also sequentially Right. We also know that every Banach space can be embedded isometrically in the commutative C^* -algebra of weak* continuous functions on its dual ball. So it follows from Example 8, that the property of being sequentially Right is not inherited by subspaces. But we do have the following result.

Proposition 10. *Every closed complemented subspace of a sequentially Right Banach space is, itself, sequentially Right.*

Proof. Let X be a sequentially Right Banach space and let Z be a closed subspace of X for which there exists a bounded projection P from X onto Z . Let Y be a Banach space and $T: Z \rightarrow Y$ a pseudo weakly compact operator. We show that $TP: X \rightarrow Y$ is pseudo weakly compact. Let (x_n) be a sequence in X converging to 0 with respect to the Right topology of X . Let A be an absolutely convex $\sigma(Z^*, Z^{**})$ -compact subset of Z^* . Writing $W = P^{-1}(\{0\})$, we can identify X (via a topological isomorphism) with the ℓ^1 -direct sum $W \oplus Z$, and then X^* is identified with the ℓ^∞ -direct sum $W^* \oplus Z^*$ and X^{**} with the l^1 -direct sum $W^{**} \oplus Z^{**}$. Then the sequence (Px_n) in Z converges to 0 with respect to the Right topology of Z , because any absolutely convex $\sigma(Z^*, Z^{**})$ -compact set $A \subset Z^*$ can be interpreted as a $\sigma(X^*, X^{**})$ -compact subset of $X^* = W^* \oplus Z^*$, and we have $\langle Px_n, \phi \rangle = \langle x_n, \phi \rangle$, uniformly on $\phi \in A$. Hence $\lim_{n \rightarrow \infty} \|TPx_n\| = 0$. Thus TP is pseudo weakly compact on X , and so it is weakly compact by assumption. But the restriction of TP to Z is T which is therefore weakly compact. \square

Corollary 11. *Every closed complemented subspace of a C^* -algebra is sequentially Right.*

Lemma 12. *Let T be a linear operator between two Banach spaces X and Y . Then T is Right-Right continuous if, and only if, it is bounded.*

Proof. Suppose first that $T : X \rightarrow Y$ is a bounded linear operator. Let (x_λ) be a Right-null net in X . We shall see that $T(x_\lambda)$ is Right-null in Y . Indeed, let K be an absolutely convex weakly compact subset of Y^* , that is, a $\sigma(Y^*, Y^{**})$ -compact subset in Y^* . Since $T^* : Y^* \rightarrow X^*$ is weakly continuous, it follows that $T^*(K)$ is an absolutely convex $\sigma(X^*, X^{**})$ -compact subset in X^* . Let $K_1 = T^*(K)$. Since $(x_\lambda) \rightarrow 0$ in the Right-topology of X , we have

$$0 = \lim_{\lambda \rightarrow \infty} \sup_{\psi \in K_1} |\psi(x_\lambda)| = \lim_{\lambda \rightarrow \infty} \sup_{\phi \in K} |T^*(\phi)(x_\lambda)| = \lim_{\lambda \rightarrow \infty} \sup_{\phi \in K} |\phi(T(x_\lambda))|,$$

which shows that $(T(x_\lambda)) \rightarrow 0$, uniformly on K , and hence $(T(x_\lambda)) \rightarrow 0$, in the Right-topology of Y . This proves that T is Right-Right continuous.

To prove the converse, let us first assume that every Right-null sequence in Y is norm-null. We shall prove that, in this case, every Right-Right continuous operator $T : X \rightarrow Y$ is bounded. Let (x_n) be a norm-null sequence in X . Clearly (x_n) is Right-null in X and, by the hypothesis on T , we have $T(x_n) \rightarrow 0$ in the Right topology of Y , therefore, by the assumptions on Y , $\|T(x_n)\| \rightarrow 0$.

We notice that in every reflexive Banach space every Right-null sequence is norm-null, this is because the closed unit ball of its dual space is absolutely convex and weakly compact. Suppose that $T : X \rightarrow Y$ is a linear operator which is Right-Right continuous. From the first part of the proof, for each ϕ in Y^* , we have ϕ Right-Right continuous. Therefore, $\phi \circ T : X \rightarrow \mathbb{K}$ is a Right-Right continuous operator and hence, from the comments preceding this paragraph, $\phi \circ T$ is bounded. This proves that T is weakly continuous and so, by the Uniform Boundedness Theorem, bounded. \square

Let X be a Banach space. A series $\sum_n x_n$ in X is called *weakly unconditionally Cauchy* (w.u.C.) if there exists $C > 0$ such that for any finite subset $F \subset \mathbb{N}$ and $\varepsilon_n = \pm 1$, we have $\|\sum_{n \in F} \varepsilon_n x_n\| \leq C$. From [2, Exercise 2, p. 113] we know that when $\sum_n x_n$ is a w.u.C. series in X , and (e_n) is the canonical basis of c_0 , then there exists a unique bounded linear operator $U : c_0 \rightarrow X$, mapping each e_n to x_n . Since (e_n) is bounded and σ -strong*-null in (the commutative C^* -algebra) c_0 , and the σ -strong* and the Right topologies coincide on the closed unit ball of c_0 , it follows that (e_n) tends to 0 in the Right topology of c_0 . Now Lemma 12 tells us that $(U(e_n)) = (x_n)$ is Right-null. We have proved:

Lemma 13. *Let X be a Banach space and $\sum_n x_n$ a w.u.C. series in X . Then (x_n) is a Right-null sequence in X .*

Let X and Y be Banach spaces and $T : X \rightarrow Y$ a linear mapping. We say that T is *unconditionally converging* if, for every w.u.C. series $\sum_n x_n$ in X , the series $\sum_n T(x_n)$ is unconditionally convergent.

Proposition 14. *Every pseudo weakly compact operator between two Banach spaces is unconditionally converging.*

Proof. Let $T : X \rightarrow Y$ be a pseudo weakly compact operator. Let us assume that T is not unconditionally converging, then, by [2, Exercise 8, p. 54], T fixes a copy of c_0 . Thus there exist bounded linear operators $U : c_0 \rightarrow X$ and $V : Y \rightarrow c_0$, such that VTU is an isomorphism. Let (e_n) denote the canonical basis of c_0 . From Lemma 13 and the comments preceding it, $U(e_n) = x_n$ is a (bounded) Right-null sequence in X and $T(x_n)$ does not tend to 0 in the norm

topology of Y . However this contradicts T being pseudo weakly compact, therefore, T must be unconditionally converging. \square

Let us recall that a Banach space X is said to have Pelczynski's *Property (V)* if, for every Banach space Y , every unconditionally converging operator is weakly compact. We clearly have:

Corollary 15. *Every Banach space satisfying property (V) is sequentially Right.*

Since every JB*-triple satisfies property (V) (see [1]), we obtain:

Corollary 16. *Every JB*-triple is sequentially Right.*

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