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ON THE BEZOUT THEOREM IN THE REAL CASE

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ON THE BEZOUT THEOREM IN THE REAL CASE

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§1. Introduction.

The central result of these notes is one of Hovanskii (§2), which is an analogue of Bezout's theorem for polynomials over the reals; the number of common roots of a system of n real polynomial equations is bounded not in terms of the degrees but rather in terms of the total number of monomials appearing in the equations.

The notion of additive complexity of polynomials is then introduced (§3) and the result of Hovanskii is used to prove that the number of real roots of $P \in \mathbb{R}[X]$ is bounded in terms of its additive complexity; this result is then generalized by replacing the real roots of $P \in \mathbb{R}[X]$ by the number of connected components of the zero-set of $P \in \mathbb{R}[X_1,...,X_n]$.

In §§5 and 6, Liouville functions are considered; these functions are analytic on $\mathbb{R}$ and satisfy some finiteness property; finally, a number of open problems about these functions is proposed.

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§2. Hovanskii's theorem ([H_1] and [H_2]).

The goal of this paragraph is to prove the following theorem, due to Hovanskii

(2.1) Theorem.- Let $F_1,\ldots,F_n \in \mathbb{R}[X_1,\ldots,X_n,y_1,\ldots,y_k]$ where $X_1,\ldots,X_n$ are indeterminates and $y_i = e^{\langle a^i,X\rangle}$ $i=1,\ldots,k$ with $\langle a^i,X\rangle = a_1^iX_1+\ldots+a_n^iX_n$, $a_j^i \in \mathbb{R}$, all i,j. Then the number of non degenerated solutions of the system

$$(1)\begin{cases} F_1(X,y(X)) = 0 \\ \vdots \\ F_n(X,y(X)) = 0 \end{cases}$$

is finite and less than or equal to $(\prod_{i=1}^{n} m_i)(1+\Sigma m_i)^k.2^{k(k-1)/2}$, where m_i is the degree of F_i.

(2.2) Remark.- A point $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ is a non-degenerated solution of (1) if the Jacobian of $F_1(X,y(X)),\ldots,F_n(X,y(X))$ is not zero at x.

Proof of 2.1. To begin with we shall assume that the number of non-degenerated solutions of any system $F(X,y(X)) = b$ fullfilling the conditions of (2.1) is finite and bounded by a number independent of b. Under this hypothesis we shall prove the bound stated in the theorem. Later, taking into account this bound, we shall see that, in fact, this number is always finite.

Let N be the number of non-degenerate solutions of (1). We work by induction on the number k of exponentials. If

$k = 0$ we have a system of n polynomials with n indeterminates and we have to prove that $N \leq \prod_{i=1}^{n} m_i$. This is nothing but Bezout's theorem. So, let us assume the result holds for any system with k exponentials and consider the system $F(\underline{X},\underline{y}) = 0$, defined by:

$$(2) \begin{cases} F_1(X_1,\ldots,X_n,y_1,\ldots,y_k,y_{k+1}) = 0 \\ \vdots \\ F_n(X_1,\ldots,X_n,y_1,\ldots,y_k,y_{k+1}) = 0 \end{cases}$$

where $y_i = e^{\langle a^i,X\rangle}$.

Let us still denote by N the number of non-degenerated solutions of (2). We shall go on by bounding N in terms of the number of solutions of auxiliar systems till find one with only k-exponentials (although n+1 indeterminates) to which we shall apply the induction hypothesis.

Consider the functions

$$F: \mathbb{R}^n \to \mathbb{R}^n: x \to F(x,y(x)),$$

$$G: \mathbb{R}^{n+1} \to \mathbb{R}^n: (x,t) \to F(x,ty(x)),\ ty(x) = (y_1(x),\ldots,y_k(x),ty_{k+1}(x)),$$

and

$$\det(\partial G/\partial X): \mathbb{R}^{n+1} \to \mathbb{R}: (x,t) \to \det\left(\frac{\partial(F_1,\ldots,F_n)}{\partial(X_1,\ldots,X_n)}\right)(x,ty(x)).$$

We have

(2.3) Lemma. There exists $(b,c) \in \mathbb{R}^n \times \mathbb{R}$ arbitrarily close to the origin such that:

(a) b is a regular value of F and G, and (b,c) is a regular value of $(G, \det(\partial G/\partial X)): \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$.

(b) N is less than or equal to the number of solutions of the system $F(X,y(X)) = b$ (which are all non-degenerated by (a)).

(c) The number of non-degenerated solutions of $F(X,y(X)) = b$ is less than or equal to N_3+q where N_3 is the number of solutions of the system

$$(3) \qquad \begin{cases} G(X,t) = b \\ \det(\partial G/\partial X) = c \end{cases}$$

(which are non-degenerated by (a)), and where q is the number of non-compact connected components of the curve Γ defined by $G(X,t) = b$ (Notice that Γ is a curve because of (a); recall that $G(X,t) = F(X,ty(X))$).

Proof. (a) Let us define the vector field on $\mathbb{R}^{n+1}$:

$$\xi(X,t) = (\xi_1(X,t),\ldots,\xi_n(X,t),\xi_t(X,t)),$$

$$\xi_i(X,t) = \det\left(\frac{\partial(G_1,\ldots,G_n)}{\partial(X_1,\ldots,\hat{X}_i,\ldots,X_n,t)}(X,t)\right) \qquad i=1,\ldots,n$$

$$\xi_t(X,t) = \det\left(\frac{\partial(G_1,\ldots,G_n)}{\partial(X_1,\ldots,X_n)}(X,t)\right)$$

For each regular value b of G, the equation $G(X,t) = b$ defines a smooth curve Γ, and the vector field $\xi(X,t)$ is tangent to Γ. Note also that the solutions of $F(X,y(X)) = b$ are the intersection points of Γ with the hyperplane $t=1$.

If b is a regular value of F in all these points we have:

$$\xi_t(X,1) = \frac{\partial(F_1,\ldots,F_n)}{\partial(X_1,\ldots,X_n)}(X) \neq 0$$

and therefore Γ is transversal to t=1.

Let $\varepsilon(b) = \inf |\xi_t(X,1)|$, the infimum being taken over all roots of F(X,y(X)) = b. If b is a regular value of F(X,y(X)), then $\varepsilon(b)$ is non-zero, since we have assumed that the number of solutions of F(X,y(X)) = b is finite; furthermore, in any open U of regular values of F(X,y(X)) there is a non-empty open subset $V \subset U$ such that the number of solutions of F(X,y(X)) = b is constant for $b \in V$ (since we have assumed that the number of solutions of F(X,y(X)) = b is bounded by a constant independent of b). Then, for $b \in V$, $\varepsilon(b)$ is a continuous function by the implicit function theorem.

It is now possible to choose $(b,c) \in \mathbb{R}^n \times \mathbb{R}$ such that:

(i) b is a regular value of F(X,y(X))

(ii) (b,c) is a regular value of $(G, \det(\partial G/\partial X))$.

(These conditions are fulfilled in an open dense set of $\mathbb{R}^n \times \mathbb{R}$).

(iii) b is arbitrarily close to the origin

(iv) the number of solutions of F(X,y(X)) = b is locally constant around b

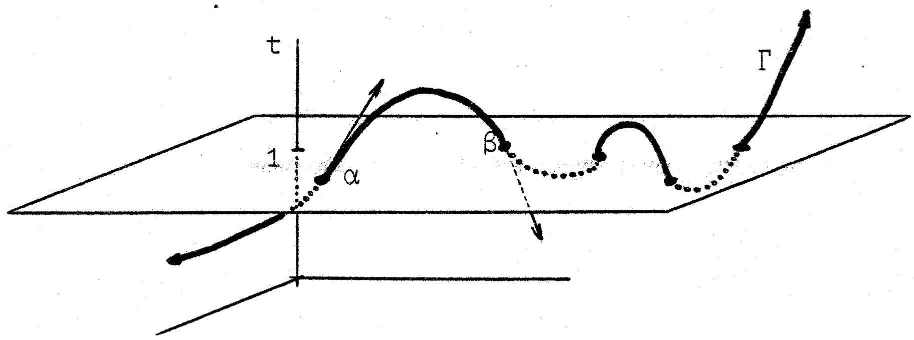
(v) $|c| < \varepsilon(b)$

(vi) b is a regular value of G(X,t) (this condition is also fullfilled in a dense subset of R^n by Sard's theorem).

(b) In a neighborhood of each non-degenerated solution of (2), the function F is a difeomorphism onto a neighborhood of the origin. Since we are assuming that N is finite, there is a neighborhood Ω of the origin such that for each $b \in \Omega$

$F(X,y(X)) = b$ has at least N solutions, which are non-degenerated. It is enough to take $b \in \Omega$, and this can be done by (iii) of (a).

(c) Consider the orientation of Γ determined by ξ, and let α,β be two consecutive points of intersection of Γ with $t=1$. Is is now easy to see that $\xi_t(\alpha,1).\xi_t(\beta,1) < 0$; it



follows by (v) of (a) that there exists ρ on Γ between α and β such that $\xi_t(\rho) = c$. Now ρ is a non-degenerated solution of (3) by (vi) of (a) and if we denote by q the number of non-compact connected components of Γ the result follows at once.

Before returning to the proof of theorem 2.1, we shall prove another lemma:

(2.4) <u>Lemma</u>.- Let $\Gamma \subset \mathbb{R}^n$ be a smooth curve. Assume that every hyperplane H transversal to Γ meets it in at most q points. Then Γ has at most q non-compact connected components.

<u>Proof</u>. Assume that there are q+1 non-compact components. Since each of them is closed in $\mathbb{R}^n$ and homeomorphic to a line it is clear that it has two half branches going to

infinite. Then, to each non-compact component we associate two points on the sphere S^{n-1}: the accumulation points of the image under the map $X \mapsto X/||X||$ of two sequences going to infinity along the respective half branches. Thus, counting these points with their multiplicities we obtain $2q+2$ points on S^{n-1}. Now let H be any hyperplane of $\mathbb{R}^n$ through the origin. It is clear that H leaves at least q+1 points on one of the half-spheres which determines. But by construction this means that moving H parallely far enough in the direction of this half-sphere, H meets Γ in at least q+1 points, contradiction.

Now back to the proof of 2.1, let us make the following change of variables:

$$U = ty_{k+1}(X) = t.e^{\langle a^{k+1},X\rangle}.$$

Then, for every $i,j=1,\ldots,n$ it is

$$\frac{\partial F_j}{\partial X_i}(X,ty(X)) = \frac{\partial F_j}{\partial X_i} + \sum_{r=1}^{k} \frac{\partial F_j}{\partial y_r}.\frac{\partial y_r}{\partial X_i} + \frac{\partial F_j}{\partial y_{k+1}}.\frac{\partial y_{k+1}}{\partial X_i}\, t =$$

$$= \frac{\partial F_j}{\partial X_i} + \sum_{r=1}^{k} \frac{\partial F_j}{\partial y_r} a_i^r\, y_r + \frac{\partial F_j}{\partial y_{k+1}} a_i^{k+1}\, U.$$

Consequently the system (3) becomes

$$(4)\quad \begin{cases} F(X,U,y_1,\ldots,y_k) = b \\ \det\left(\dfrac{\partial(F_1,\ldots,F_n)}{\partial(X_1,\ldots,X_n)}\right)(X,U,y_1,\ldots,y_k) = c \end{cases}$$

which has n+1 indeterminates but only k exponentials. It is obvious that $N_3 = N_4$. On the other hand, by lemma 2.4, it follows that the constant q introduced in lemma 2.3(b) is less than or equal to the number of solutions of the system

$$(5)\qquad \begin{cases} F(X,U,y_1,\dots,y_k) = b \\ \ell_1 x_1+\dots+\ell_n x_n+\ell_U U+\ell_o = 0 \end{cases}$$

with arbitrary $\ell_o,\dots,\ell_n,\ell_U$, which we may assume such that all the solutions are non-degenerated (we can choose (b,ℓ_o) like (b,c) in (ii) of (a) above).

Therefore, after 2.3 and 2.4, we have that N is less than or equal to the number N_6 of non-degenerated solutions of the system

$$(6)\qquad \begin{cases} F(X,U,y_1,\dots,y_k)-b = 0 \\ (\Sigma\ell_i X_i+\ell_U U+\ell_o)(\det(\frac{\partial F}{\partial X}(X,U,y))-c) = 0 \end{cases}$$

which has only k exponentials. Since the degree of $\det(\frac{\partial F}{\partial X}(X,U,y))$ is less than or equal to $\sum_{i=1}^{n} m_i$ we have by induction

$$N \leqq N_6 \leqq m_1 \dots m_n \,(1+\sum_{i=1}^{n} m_i)(1+m_1+\dots+m_n+1+\sum_{i=1}^{n} m_i)^k . 2^{k(k-1)/2} =$$

$$= \prod_{i=1}^{n} m_i (1+\Sigma m_i)^{k+1}\, 2^{k(k+1)/2} .$$

To complete the proof of 2.1 we must prove that, indeed, the number of roots of (1) is always finite, and bounded by a number independant of the second member. For that let us consider the system

$$(7)\qquad \begin{cases} F(X,y_1,\dots,y_k) = 0 \\ X_o^2 + \sum_{i=1}^{n} X_i^2 - R^2 = 0 \end{cases} ,$$

where X_o is a new indeterminate and $R \in \mathbb{R}^+$. It is obvious that the number N_7 of non-degenerated solutions of (7) is the double of the number of non-degenerated solutions of (1) which lie in the interior of the ball of radio R. But since the set of non-degenerated solutions of (7) is semianalytic, 0-dimensional and contained in the ball of radio R of $\mathbb{R}^{n+1}$ it must be finite. Thus applying the theorem in the case already proven we have a bound of N_7 which does not depend on R. Therefore it remains valid for any R and consequently for the whole space $\mathbb{R}^n$.

(2.5) Remark.- Hovanskii's theorem can be generalized to the case where the exponentials $y_1(x),\dots,y_k(x)$ are replaced by functions $f_1(x),\dots,f_k(x)$ with the property that for any $j=1,\dots,k$, and any $i=1,\dots,n$, there exists a polynomial $P_{ij}(X_1,\dots,X_n,T_1\dots,T_j)$ such that

$$\frac{\partial f_i}{\partial X_i} = P_{ij}(X_1,\dots,X_n,f_1(x),\dots,f_k(x))$$

(these systems are called Pfaffians in [H_1]). In this case the final bound depends on the degrees of the polynomials P_{ij}.

From theorem 2.1 we shall derive now a Bezout's theorem which bounds the number of non-degenerated roots of a system of polynomials in terms of the number of monomials they involve.

(2.6) Theorem.- The number of non-degenerate solutions of a polynomial system $P_1=\dots=P_n=0$ lying in the positive octant of $\mathbb{R}^n$ is less than or equal to $(n+1)^k.2^{k(k-1)/2}$, where k is the number of different monomials which occur in $P_1,\dots,P_n$.

Proof. We make the change $x_i = \exp(y_i)$. Then the original system transforms in a system of the type of theorem 2.1 with k exponentials and which gives all the solutions of $P_1 = \ldots = P_n = 0$ in the positive octant of $\mathbb{R}^n$: $x_1 > 0, \ldots, x_n > 0$. Also, in the new system all the equations have degree 1 in the exponentials. Then, by 2.1 we have

$$N \leq (1+n)^k \; 2^{k(k-1)/2}$$

what proves 2.6.

Theorem 2.6 introduces the "complexity" of the polynomials as an essential factor to bound from above the number of real roots of a system of polynomials, rather than the degree. A detailed study of the notion of "complexity" of polynomials will be done in §3 where some other results in the vein of 2.6 will be proved. To finish this paragraph let us point out that the bound given in 2.6 seems to be quite big. In this sense there exists a conjecture of Kusnirenko, according to which it can be lowered to $\prod_{i=1}^{n} (k_i - 1)$ where k_j is the number of monomials of P_j. For n=1 this follows from Descartes rule. For $n > 1$ the conjecture still remains unsolved.

§3. Additive complexity of polynomials.

(3.1) Definition.- Let K be a field and $p \in K[x]$. We say that the additive complexity over K of p is less than or equal to $k \in \mathbb{N} \cup \{0\}$ if there exist $c_0,\ldots,c_k$, $d_0,\ldots,d_{k-1} \in K$, $n_{ij}, n'_{ij} \in Z$ such that

$$(3.1.1)\quad \begin{cases} s_0 = x \\ s_1 = c_0 s_0^{n_{01}} + d_0 s_0^{n'_{01}} \\ s_2 = c_1 s_0^{n_{02}} s_1^{n_{12}} + d_1 s_0^{n'_{02}} s_1^{n'_{12}} \\ \vdots \\ p = s_{k+1} = c_k \prod_{i=0}^{k} s_2^{n_{i,k+1}} \end{cases}$$

, or, equivalently

$$(3.1.2)\quad \begin{cases} T_0 = x \\ T_1 = \mu_0 T_0^{n_{01}} + T_0^{n'_{01}} \\ T_2 = \mu_1 T_0^{n_{02}} T_1^{n_{12}} + T_0^{n'_{02}} T_1^{n'_{12}} \\ \vdots \\ p = T_{k+1} = \mu_k \prod_{i=0}^{k} T_i^{n_{i,k+1}} \end{cases}$$

The additive complexity $c_+^K(p)$ of p is defined as the minimum integer k such that the additive complexity of p is less than or equal to k.

We will call $K_+(k) = \{p \in K[x] : c_+^K(p) \leqq k\}$.

(3.2) Proposition.- Let K be a field and $\alpha_0,\ldots,\alpha_n \in K$ algebraically independent over $\mathbb{Q}$. If $p = \alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n$, then $c_+^{\mathbb{Q}}(p) = n$.

Proof. If $k = c_+^{\mathbb{Q}}(p)$ by equating coefficients in the last equation of 3.1.2 there are $P_o,\ldots,P_n \in \mathbb{Q}[x_o,\ldots,x_k]$ such that $\alpha_j = P_j(\mu_o,\ldots,\mu_k)$

So, $n+1 = \text{tr.d.}\mathbb{Q}(\alpha_o,\ldots,\alpha_n) \leqq \text{tr.d.}\mathbb{Q}(\mu_o,\ldots,\mu_k) \leqq k+1$

(3.3) Theorem ([B-K]).- Let n be a natural number and k the integral part of $\sqrt{n} - 2$. There exists $F \in Z[x_o,\ldots,x_n]$ such that, if $p = \alpha_o+\alpha_1 x+\ldots+\alpha_n x^n \in$ $\in K_+(k)$, $F(\alpha_o,\ldots,\alpha_n) = 0$.

Proof. Let us give an idea of the proof in the case where all the m_{ij}, m'_{ij} are in $\mathbb{N}$; for the general case, see [B-C]. Let us take $x = y+\theta$. We develop $x^{n_{ij}} = (y+\theta)^{n_{ij}}$ in powers of y. Substituting in 3.1.2. and equating coefficients in the last equation, we obtain a system with parameters $\mu_o,\ldots,\mu_k$, $\{n_{ij},n'_{ij}\}$ and indeterminates $\alpha_o,\ldots,\alpha_n$. Now we conclude using $(k+2)^2 < n$ and eliminating the parameters.

(3.4) Examples.- (1) ([W]). Let us consider Tchebyshev's polynomials $T_n(x) = \cos(n.\text{arc}\cos x)$ -i.e. $T_n(\cos t) =$ $= \cos nt$.

So, $T_1(x) = x$, $T_3(x) = 4x^3 - 3x,\ldots,T_{3^k}(x) = 4\,T^3_{3^{k-1}}(x)-3T_{3^{k-1}}(x)$.

Then: (a) $T_{3^k} \in \mathbb{R}_+(k)$, (b) $3^k = \deg T_{3^k}$ = number of distinct real roots of T_{3^k}.

The proof of (a) is trivial by induction on k, because $T_{3^k}(x)$ is computed with one $\pm$ sign from $T_{3^{k-1}}(x)$.

Part (b) is clear: for each natural number n, the roots of T_n are $x_j = \cos|(2j+1)n/2n|$, $0 \leqq j \leqq n-1$

(2) ([B-C]). Let $\{\varepsilon_n : n \geqq 1\}$ be a sequence of "small enough" positive real numbers. We construct:

$$P_o(x) = x$$
$$P_1(x) = (x^2-\varepsilon_1^2)x$$
$$\vdots$$
$$P_{k+1}(x) = (P_k^2(x)-\varepsilon_k^2)P_k(x)$$

The same argument above proves $P_k \in \mathbb{R}_+(k)$. Let us prove also that P_k has 3^k distinct real roots for a convenient choice of the ε_i. The result being obvious when k=0, let us assume it for P_k. Then, the fiber of 0 in $P_k\colon \mathbb{R} \to \mathbb{R}$ has 3^k distinct elements and so, the same happens with the fiber of ε_k and $-\varepsilon_k$ for ε_k small enough, because P_k is of degree 3^k and so all his 3^k roots are simple. Consequently P_{k+1} has $3.3^k = 3^{k+1}$ real roots.

(3.5) Definition.- Let $n_K(p)$ be the number of distinct roots in a field K of $p \in K[x]$ and, for each $k \in \mathbb{N}$

$$\rho_K(k) = \sup \{n_K(p)\colon p \in K_+(k)\}$$

The examples above show $3^k \leqq \rho_{\mathbb{R}}(k)$, and this is the best known lower bound for $\rho_{\mathbb{R}}(k)$.

(3.6) Example ([W]).- Let us take, for each $n \in \mathbb{N}$,

$$p_n(x) = [(1-ix)^{n+1} - (1+ix)^{n+1}]^2$$

Clearly, if

$$s_o = x$$
$$s_1 = 1-ix = s_o^o + (-i)s_o$$
$$s_2 = 1+ix = s_o^o s_1 + 2is_o s_i^o$$
$$s_3 = s_o^o\, s_1^{n+1}\, s_2^o + (-1)s_o^o\, s_1^o\, s_2^{n+1}$$
$$p_n = s_4 = s_o^o\, s_1^o\, s_2^o\, s_3^2$$

we conclude $p_n \in \mathbb{C}_+(3)$ for every $n \in \mathbb{N}$.

Since $p_n = -4.f^2$, $f = \sum_{\substack{j \text{ odd} \\ j \leq n+1}} \binom{n+1}{j} i^{j-1} x^j \in \mathbb{R}[x]$,

$p_n \in \mathbb{R}[x]$.

Moreover, via: $ix \to T$, the real roots of p_n correspond bijectively with the roots of

$$F_n(T) = (1-T)^{n+1} - (1+T)^{n+1}$$

in the imaginary axis. So, distinguishing the cases n is odd or even, we conclude the number of real roots of p_n is n. Consequently, the additive complexity of a polynomial depends heavily on the field of constants. In fact we saw $C_+^{\mathbb{C}}(p_n) \leq 3$, while, p_n having n distinct real roots and $3^k \leq \rho_{\mathbb{R}}(k)$ we conclude $C_+^{\mathbb{R}}(p_n)$ go to infinite with n.

Upper bounds of $\rho_{\mathbb{R}}(k)$ are obtained in the following:

(3.7) <u>Theorem</u> ([R]).- There exists $c \in \mathbb{R}^+$ such that $\rho_{\mathbb{R}}(k) \leq c^{k^2}$ for every $k \in \mathbb{N}$.

<u>Proof</u>. Let k be a natural number and $p \in \mathbb{R}[x]$ with $\rho_{\mathbb{R}}(k)$ real distinct roots, $p \in \mathbb{R}_+(k)$.

Hence, we have a system

$$(3.7.3) \quad \begin{cases} T_o = x \\ T_1 = \mu_o T_o^{m_{o1}} + T_o^{m'_{o1}} \\ \cdots\cdot \\ T_k = \mu_{k-1} \prod_{i=0}^{k-1} T_i^{m_{ik}} + \prod_{i=0}^{k-1} T_i^{m'_{ik}} \\ 0 = \mu_k \prod_{i=0}^{k} T_i^{m_{i,k+1}} \end{cases}$$

and the number of simple roots of P equals the number of non-degenerate solutions of 3.7.3. This will follow if we prove that $P'(X)$ is the jacobian determinant of 3.7.3 for any solution $(X, T_1, \ldots, T_k)$ of 3.7.3.

But if

$$\begin{aligned} W_1(X) &= T_1(X) \\ W_2(X) &= T_2(X, W_1(X)) \\ &\cdots \\ W_k(X) &= T_k(X, T_1(X), \ldots, T_{k-1}(X)) \\ P_*(X) &= \mu_k \prod_{i=o}^{k} T_i(X)^{m_{i,k+1}} \end{aligned}$$

we have $P'(X) = \dfrac{\partial P_*}{\partial X} + W_1'(X) \dfrac{\partial P_*}{\partial T_1} + \ldots + W_k'(X) \dfrac{\partial P_*}{\partial T_k}$, and

(arguing by induction) $W_j'(X)$ is the jacobian determinant of

$$(3.7.4) \quad \begin{cases} T_o = x \\ T_1 = \mu_o T_o^{m_{o1}} + T_1^{m'_{01}} \\ \cdots \\ T_j = \mu_{j-1} \prod_{i=0}^{j-1} T_i^{m_{ij}} + \prod_{i=0}^{j-1} T_i^{m'_{ij}} \end{cases}$$

Finally, notice that the jacobian determinant of 3.7.3 is

$$\begin{vmatrix} \dfrac{\partial T_1}{\partial X}, & -1, & 0, & \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial T_2}{\partial X}, & \dfrac{\partial T_2}{\partial T_1}, & -1, & 0, & \cdot & \cdot & \\ \vdots & & & & & & \\ \dfrac{\partial P_*}{\partial X}, & \dfrac{\partial P_*}{\partial T_1}, & \cdot & \cdot & \cdot & \cdot & \dfrac{\partial P_*}{\partial T_k} \end{vmatrix}$$

Now in order to apply 2.1 we must suppose that all the real roots of P are simple. This will be possible by the next

(3.7.5) Lemma. There exists ε (with $|\varepsilon| = 1$) such that for $t \in \mathbb{R}$ small enough:

(a) All the roots of $P(X) - \varepsilon|t|$ are simple.

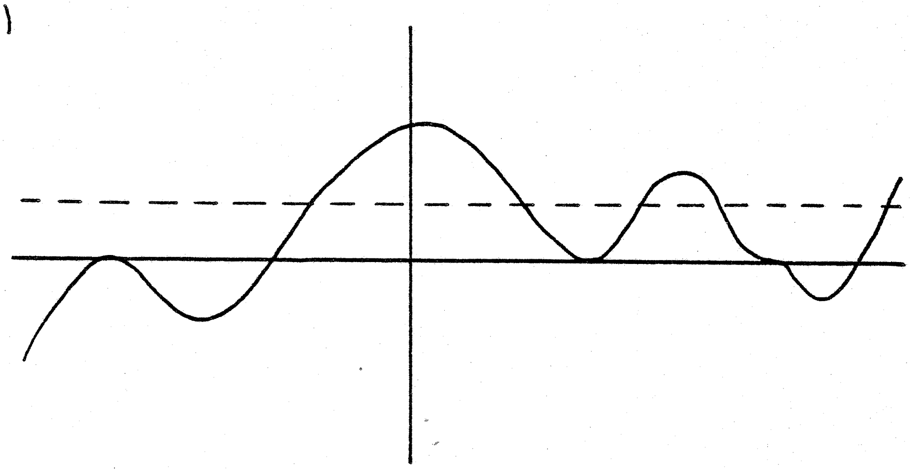
(b) the number of roots of $P(X) - \varepsilon|t|$ is greater than or equal to the number of roots of $P(X)$.

Proof. (a) An elementary version of Sard's theorem assures (a) is true if t is not equal to some $P(X_\alpha)$, where X_α are the roots of $P'(X)$.

For (b) one has to count the number of intersection points of $y = \varepsilon|t|$ with $y = P(X)$. Let us classify the n intersection points of $y = 0$ with $y = P(X)$ into n_1 transversal points; n_2 local minima of P; n_3 local maxima of P and n_4 inflection points of P. Then, taking $\varepsilon = 1$ if $n_2 \geqq n_3$ and $\varepsilon = -1$ if $n_2 \leqq n_3$, the number of distinct roots of $P(X) - \varepsilon|t|$ equals

$$n_1 + n_4 + 2 \max\{n_2, n_3\} \geqq n. \quad \text{(see figure below)}$$

$(n_2 \geqq n_3)$



So, to finish the proof, we apply 2.1 to a system like 3.1.3 changing the last line by $C = \mu_k \prod_{i=0}^{k} T_i^{m_{i,k+1}}$.

This system has 3k+1 non constant monomials and so if $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, the number of solutions in $(\mathbb{R}^*)^{k+1}$ is less than $2^{k+1}(k+2)^{3k+1}\, 2^{3k(3k+1)} = (k+2)^{3k+1}\, 2^{\frac{9k^2+5k+2}{2}}$.

Since constant monomials does not have influence in Hovanskiĭ theorem, we can make a little perturbation of the system and assume that all the solutions are in $(\mathbb{R}^*)^{k+1}$ -see the proof of 2.1-.

We have then $\rho(k) \leqq (k+2)^{3k+1}\, 2^{\frac{9k^2+5k+2}{2}} \leqq C^{k^2}$ for k big enough, and $C < 32$.

(3.8) <u>Remark</u>.- At least from a computational point of view, it seems interesting to know whether there is an upper bound of $\rho_{\mathbb{R}}(k)$ of the form d^k, $d \in \mathbb{R}$. In case Kusnirenko's conjecture were true, we would obtain by the same argument above

$$\rho_{\mathbb{R}}(k) \leqq 2^{k+1} . 5^{k+1} = 10 . 10^k \leqq 100^k$$

§4. Complexity of polynomials of $\mathbb{R}[x_1,\ldots,x_n]$

(4.1) Definition.- The additive complexity of $p \in \mathbb{R}[x_1,\ldots,x_n]$, is less than or equal to $k \in \mathbb{N} \setminus \{0\}$ if there exist a monomial S_o and a sum of two monomials S_1 in $\mathbb{R}[x_1,\ldots,x_n]$ such that

$$S_2 = c_1 S_o^{m_{02}} S_1^{m_{12}} + c_2 S_o^{m'_{02}} S_1^{m'_{12}}$$

$$\vdots$$

$$p = S_{k+1} = c_k \prod_{i=0}^{k} S_i^{m_{i,k+1}} ,$$

where m_{ij}, $m'_{ij} \in \mathbb{Z}$ and $c_i \in \mathbb{R}$.

We recall this definition agrees with 3.1 in case n=1.

We define $c_+^{\mathbb{R}}(P)$ = additive complexity of P, the minimum integer k such that the additive complexity of $P \leq k$, and

$$\mathbb{R}_+(k) = \{P \in \mathbb{R}[x_1,\ldots,x_n] : c_+^{\mathbb{R}}(P) \leq k\}$$

The result we want to prove in this paragraph is:

(4.2) Theorem ([R]).- There exists $\psi: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that given $(n,k) \in \mathbb{N} \times \mathbb{N}$ and $P \in \mathbb{R}[x_1,\ldots,x_n] \cap \mathbb{R}_+(k)$,

$$c(P) \leq \psi(k,n) \leq c^{k^4+n^4}$$

where c(P) denotes the number of connected components of $P=0$, and $0 < c < 32$.

Proof. By induction on n.
When n=1 the connected components of $\{P=0\}$ are the distinct roots of P. So, by theorem 3.7, $\psi(1,k) = c^{k^2}$ is a solution.

Let us assume the result is proved for n-1, and set $P \in \mathbb{R}[x_1,\dots,x_n] \cap \mathbb{R}_+(k)$.

We suppose $\{P=0\}$ is non singular - otherwise, by choosing a small enough regular value ε of P, the number of connected components of $\{Q = P-\varepsilon = 0\}$ is greater than or equal to $c(P)$ and $c_+(Q) \leqq k+1$; so taking into account that $c^{(k+1)^4} < d^{k^4}$ for a suitable d, we can change P by Q.

We denote $c_b(P)$ the number of bounded components of $\{P=0\}$ and $c_n(P)$ the number of unbounded components.

From lemma 2.4 there exists an affine hyperplane $H \subset \mathbb{R}^n$ that meets, at least, $\frac{c_n(P)}{2}$ unbounded components of $\{P=0\}$. If $H: a_o+a_1x_1+\dots+a_nx_n = 0$ and $a_i \neq 0$ for some $i=1,\dots,n$, the polynomial P_1 obtained from P substituting x_i by $-\frac{a_o + \sum_{j\neq i} a_jx_j}{a_i}$ verifies $c_+(P_1) \leqq c_+(P)+n-1 \leqq k+n-1$. Thus, by our induction hypothesis, $c_n(P) \leqq 2c(P_1) \leqq 2\psi(k+n-1,n-1)$.

Now we must bound $c_b(P)$. Let L be the number of distinct roots of

$$(4.2.1)\qquad \begin{cases} P(x_1,\dots,x_n) = 0 \\ \dfrac{\partial P}{\partial X_1}(x_1,\dots,x_n) = 0 \\ \vdots \\ \dfrac{\partial P}{\partial X_n}(x_1,\dots,x_n) = 0 \end{cases}$$

The solutions of 4.2.1 are, clearly, the critical points of the restriction of x_n to $P=0$. Since x_n has at least two critical points over each bounded component we deduce $2c_b(P) \leqq L$. But, after a linear change of coordinates $X_i = M(X_i')$ we may assume that all the solutions of 4.2.1 are non-degenerated, and we can apply theorem 2.1 to bound L. -this is a "Bertini's type theorem".

Now we show that $C_+^{\mathbb{R}}(\frac{\partial P}{\partial x_i}) \leqq \frac{3k(k+1)}{2}$, $i=1,\ldots,n$.

Indeed, considering P as a polynomial in one single variable X_i, which we denote by X, we have a system like 3.1.1.

Let $t_1(x) = s_1(x)$, $t_2(x) = s_2(x,t_1(x)),\ldots,$ $t_k(x) = s_k(x,\ldots,t_{k-1}(x))$.

Then, $t_k'(x) = \frac{\partial s_k}{\partial x} + t_1'(x)\frac{\partial s_k}{\partial s_1} + \ldots + t_{k-1}'(x)\frac{\partial s_k}{\partial s_{k-1}}$ and

$$P'(x) = \frac{\partial P_*}{\partial x} + t_1'(x)\frac{\partial P_*}{\partial s_1} + \ldots + t_k'(x)\frac{\partial P_*}{\partial s_k} .$$

By the very definition of t_k we have $C_+^{\mathbb{R}}(t_{k-1}) \leqq k-1$; so by the induction hypothesis on k (which is actually somewhat stronger) we get t_{k-1}, t_{k-1}' with no more than $3k(3k-1)/2$ operations plus (+) or minus (-). Then t_k needs one operation more, t_{k-1}' requires $(2k-1)$ more and finally we need k more in order to evaluate $P'(x)$.

$$\text{So, } C_+^{\mathbb{R}}(P) \leqq \frac{3k(k-1)}{2} + 3k = \frac{3k(k+1)}{2}$$

Now this result -and specially the last step- is used to compute a bound of L, by applying 2.1 to the system

$$\begin{cases} x_i = M(x_i') \\ P(x') = 0 \\ \frac{\partial P}{\partial x_i'} = 0 \\ s_i = 0 & 1 \leq i \leq k \\ \frac{\partial \bar{s}_i}{\partial s_j} = 0 & 1 \leq i \leq k, \quad 1 \leq j \leq i \\ \frac{\partial s_i}{\partial x_j'} = 0 & 1 \leq i \leq k, \quad 1 \leq j \leq n-1 \\ \frac{\partial t_i}{\partial x_j'} = 0 & 1 \leq i \leq k, \quad 1 \leq j \leq n-1 \end{cases}$$

where each of the letters introduced is considered as a new variable.

So the number of variables is $2n(k+1) + \frac{k(k-3)}{2} = \mu(k,n)$, and the total number of monomials is $n(n+1) + \frac{3nk(k+1)}{2} + (3n-1)(\frac{k(k-3)}{2}) = \phi(k,n)$. Since we consider each partial derivative as a new variable, the complexity of $\frac{\partial s_i}{\partial x_j}$, $\frac{\partial s_i}{\partial s_j}$, $\frac{\partial t_i}{\partial x_j}$ is computed as above. So

$$C(P) \leq 2\Psi(k+n-1,n-1) + \\ + 2^{\mu(k,n)}.(1+\mu(k,n))^{\phi(k,n)}.2^{\frac{\phi(k,n)(\phi(k,n)-1)}{2}} \leq C^{n^4+k^4}$$

by the induction hypothesis on ϕ and from the fact that μ and ϕ have degree less than or equal to 3.

5. Liouville Functions.

Let $I \subset \mathbb{R}$ be an open interval, maybe infinite. We denote by $\mathcal{A}$ the ring of all analytic functions defined in some subset $D_f \subset I$ whose complement $P_f = I \setminus D_f$ is finite.

Let $f \in \mathcal{A}$. Each point of P_f is called a pole of f and their number denoted by p(f) (note that a pole in this sense is not necessarily a pole in the classical sense). The domain D_f of f is a union of p(f)+1 open intervals. Over each of them, f either vanishes or has a discrete set of zeroes. We denote by Z_f this set of discrete zeroes and write $z(f) = \# Z_f$. Finally, put $n(f) = z(f) + p(f)$.

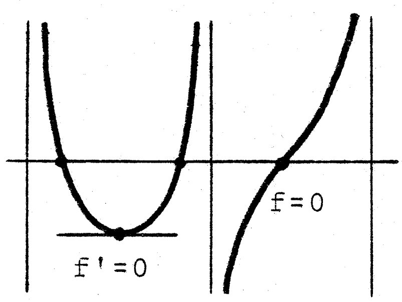
If a function $f \in \mathcal{A}$ has only discrete zeroes and $z(f) < +\infty$, then f^{-1} is a well defined function of $\mathcal{A}$. We have then:

$$D_{f^{-1}} \supset D_f \setminus Z_f \quad , \quad Z_{f^{-1}} \subset P_f$$

Let $f \in \mathcal{A}$. Then $f' \in \mathcal{A}$ and $D_{f'} \supset D_f$. Besides:

(5.1) Lemma.- $z(f) \leqq p(f) + z(f') + 1$

For, if $z(f) > p(f)$, by Rolle's theorem, two consecutive zeroes of f, without poles between, determine a zero of f', and the remaining zeroes and poles cancel pairwise.



In particular, if $z(f') < +\infty$ then $z(f) < +\infty$ too.

The first examples of functions in $\mathcal{A}$ are rational functions. But they all have the stronger property $z(f) < +\infty$.

Our aim here is to study this condition for arbitrary functions in $\mathcal{A}$. So, we shall say that a set $A \subset \mathcal{A}$ has the finiteness property if $z(f) < +\infty$ for all $f \in A$. With this terminology we have the following:

(5.2) Proposition ([G-H]).- Let A be a subring of $\mathcal{A}$ stable under derivation, which has the finiteness property. Let $g \in \mathcal{A}$ verify any of the conditions (i), (ii) or (iii) below:

(i) $g^{-1} \in A$; (ii) $g' \in \mathcal{A}$; (iii) $g'g^{-1} \in \mathcal{A}$.

Then $A[g] \subset \mathcal{A}$ is stable under derivation and has the finiteness property.

Proof.- First of all, to see that $A[g]$ is stable under derivation it suffices to check that $g' \in A[g]$. But this is obvious in cases (ii) and (iii), and follows from the formula $g' = -g^2(g^{-1})'$ in case (i). Let us show now that $z(f) < +\infty$ for $f = a_0 g^k + \ldots + a_k \in A[g]$.

Case (i). We have: $f = g^k(a_0+a_1g^{-1}+\ldots+a_kg^{-k}) = g^k h$, $h \in A$. Clearly: $z(f) \leqq z(g) + z(h) < +\infty$.

Case (ii). We argue by induction on the degree k. If k=0, the result is trivial, so we suppose it proved for degrees < k.

Let $J_1,\ldots,J_r$ be the subintervals of D_{a_0} over which a_0 vanishes. Then each $f|J_i$ has degree < k and by induction hypothesis, $z(f|J_i)$ is finite. But

$z(f) \leqq n(a_0) + \sum_{i=1}^{r} z(f|J_i) + \sum_{\ell=1}^{s} z(f|J'_\ell)$, where $J'_1,\ldots,J'_s$ are the connected components of $I \setminus \bigcup_{i=1}^{r} \overline{J_i}$. So we may assume $a_o^{-1} \in \mathcal{A}$ is well defined. Now

$$a_o^{-1} f = g^k + b_1 g^{k-1} + \ldots + b_k \in A[a_o^{-1}][g]$$

and by case (i) already solved, we are allowed to take $a_o = 1$. Then, the derivative of f is:

$$f' = (kg'+a_1')g^{k-1}+\ldots+(2a_{k-2}g'+a_{k-1}')g+(a_{k-1}g'+a_k) \in A[g]$$

because $g' \in A$. So, degree $(f') < k$ and, by induction again $z(f') < +\infty$. From 5.1 we conclude that $z(f)$ is finite.

<u>Case (iii)</u>. An induction on k applies here too. After a decomposition of I to have a_k^{-1} in $\mathcal{A}$, we write

$$a_k^{-1} f = b_o g^k + \ldots + b_{k-1} g + 1 \in A[a_k^{-1}][g]$$

which shows we may suppose $a_k=1$. Then:

$$g^{-1} f' = (a_o'+ka_1g'g^{-1})g^{k-1}+\ldots+(a_1'+a_1g'g^{-1}) \in A[g],$$

and, as degree $(g^{-1}f') < k$, we have $z(g^{-1}f') < +\infty$. From 5.1 we get:

$$z(f) \leqq p(f)+z(f')+1 \leqq p(f)+z(g)+z(g^{-1}f')+1$$

so we must prove $z(g) < +\infty$. But $g = c + \exp h$, where $c \in \mathbb{R}$ $h' = g'g^{-1} \in A$ and h may be chosen such that $p(h) = p(h')$. We conclude $z(g) \leqq p(h)+1 < +\infty$.

From 5.2 it follows inmediately:

(5.3) <u>Corollary</u>.- Let A be a subring of $\mathcal{A}$ stable under derivation which has the finiteness property. Let $g_1,\ldots,g_r$

be functions in $\mathcal{A}$, each verifying some condition amid (i), (ii) and (iii) in 5.2. Then the subring $B = A[g_1,\ldots,g_r] \subset \mathcal{A}$ is stable under derivation and has the finiteness property.

The previous results give some motivation to define:

(5.4) Definition.- Let A be a subring of $\mathcal{A}$. A subring B of $\mathcal{A}$ is a Liouville extension of A if it is a reunion of finite A-algebras $A[g_1,\ldots,g_r]$, where each $A_\ell = A_{\ell-1}[g_\ell]$ $(A_o = A)$ is like in 5.2.

Furthermore:

(5.5) Definition.- Let A be a subring of $\mathcal{A}$. An $f \in \mathcal{A}$ is a Liouville function over A if it lies in some Liouville extension of A.

A Liouville function is a Liouville function over $A = \mathbb{R}$.

(5.6) Remarks.- Let A be a subring of $\mathcal{A}$ stable under derivation, which has the finiteness property.

(1) Any Liouville extension of A is stable under derivation and has the finiteness property.

(2) A function $f \in \mathcal{A}$ is Liouville over A if and only if either f' is, or Ln f is (because $(\mathrm{Ln}\, f)' = f'f^{-1}$), or exp f is (now because $f' = (\exp f)^{-1}(\exp f')$), whenever these functions are in $\mathcal{A}$.

(3) The previous remarks show that the ring consisting of all Liouville functions over A is stable under most elementary operations, and has the finiteness property.

Finally, we look at the finiteness property "$z(f) < +\infty$" in a more algebraic way. We denote by $\mathcal{O}$ the ring of analytic functions on I. (Notice that $\mathcal{O} \subset A$ is the subring of all functions of A with no poles). Then:

(5.7) Proposition.- Let A be a subring of $\mathcal{O}$ which contains the polynomials and such that $A = \mathcal{O} \cap \mathrm{qf}A$. Then A is noetherian if and only if A has the finiteness property.

Proof.- Suppose first there is $f \in A$ with infinitely many zeroes a_n, $n \geq 1$. For each $m \geq 1$ we define the ideal

$$I(m) = \{g \in A : g(a_n) = 0 \ (n \geq m)\}.$$

Clearly, $I(m) \subset I(m+1)$ and the inclusion is strict. For, if a_m is a zero of order p_m of f, then $g = f.(t-a_m)^{-p_m} \in \mathcal{O} \cap \mathrm{qf}A = A$ and $g \in I(m+1) \setminus I(m)$. This shows A is not noetherian.

Conversely, assume $z(f) < +\infty$ for all $f \in A$. By Cohen's theorem it is enough to prove that all prime ideals $\mathfrak{p} \neq \{0\}$ of A are finitely generated. Actually, we shall see $\mathfrak{p} = \{t-t_o\}.A$ for some $t_o \in I$.

Indeed, choose $f \in \mathfrak{p}$, $f \neq 0$ and let $Z_f = \{t_1, \ldots, t_s\}$. Then for some i, $t_o = t_i$ is a common zero of all $g \in \mathfrak{p}$. For, if there were $g_1, \ldots, g_s \in \mathfrak{p}$ with $g_i(t_i) \neq 0$, then $h = f^2 + \sum_{j=1}^{s} g_j^2 \in \mathfrak{p}$ would not have any real root and $h^{-1} \in \mathcal{O} \cap \mathrm{qf}A = A$, a contradiction as $\mathfrak{p}$ is proper. Finally, if $g \in \mathfrak{p}$, we have $g.(t-t_o)^{-1} \in \mathcal{O} \cap \mathrm{qf}A = A$, which means $\mathfrak{p} \subset \{t-t_o\}.A$. As $\mathfrak{p}$ is prime, $\mathfrak{p} = \{t-t_o\}.A$.

(5.8) Remarks.- As a matter of fact, the precedent proof shows that if A has the finiteness property, then it is regular of dimension 1. The proof also checks that the factorization of an $f \in A$ is the same both in A and $\mathcal{O}$.

It follows from 5.7 that $\mathcal{O}$ is not noetherian, because there always are analytic functions on I with infinitely many zeroes. (Actually, any closed discrete set $P \subset I$ is the zero set of some $f \in \mathcal{O}$). The interesting point here is that if we restrict our attention to Liouville functions this failure disappears.

Let us denote by $\mathcal{L}$ the ring of all analytic Liouville functions of $\mathcal{A}$: $\mathcal{L} \subset \mathcal{O}$ is a Liouville extension of $\mathbb{R}$ (but not the largest one), and so has the finiteness property. Thus, we deduce from 5.7:

(5.9) Corollary.- $\mathcal{L}$ is a noetherian ring.

Finally, we would like to say a word on an interesting question. Liouville functions have been introduced in reference with their remarkable finiteness properties. This is a standard feature of Nash functions, which, as is well known, were introduced in real geometry to better the relationship between algebraic sets and their rings of functions. But Nash functions fail to have a good cohomological behaviour ([Hu] or [L-T]). Thus the point is, would things go better with Liouville functions? This topic falls off our lectures, but it is worth quoting it here.

§6. Thom's Lemma and Liouville extensions.

In this section we describe a kind of separating algorithm inspired on Thom's classical lemma. We keep all notations of §5.

(6.1) Definition.- We say that $f_1,\ldots,f_p \in \mathcal{A}$ form a separating family if for any choice of symbols $?_i$ among >, =, <, the set

$$\bigcap_{i=1}^{p} \{f_i \; ?_i \; 0\}$$

is connected and its closure is given by relaxing inequalities, provided it is not empty.

Then, Thom's lemma just gives an effective method to extend any collection of polynomials to a separating family, namely, by adding all of their derivatives. The constructive aspect is of course a most interesting part of this lemma.

Following this idea, we shall say a ring $A \subset \mathcal{A}$ has a separating algorithm if there is an algorithm which extends any given functions $f_1,\ldots,f_q \in A$ to a separating family $\{f_1,\ldots,f_q,f_{q+1},\ldots,f_p\}$.

With this terminology:

(6.2) Proposition.- Let A be a subring of $\mathcal{O}$ stable under derivation. Let $B \subset \mathcal{O}$ be a Liouville extension of A. Then, if A has a separating algorithm, so does B.

Proof.- Let $f_1,\ldots,f_q \in B$. By definition of Liouville extension, there are $g_1,\ldots,g_r \in B$ such that

$f_1,\dots,f_q \in A[g_1,\dots,g_r] \subset B$, and for each $\ell=1,\dots,r$, either g_ℓ^{-1}, or g_ℓ', or $g_\ell' g_\ell^{-1}$ is in $A_\ell = A[g_1,\dots,g_{\ell-1}]$ ($A_o=A$). As our proof is constructive, we may merely assume a g is given with $B = A[g]$ and either g^{-1} or g' or $g'g^{-1}$ is in A. We distinguish the three possibilities:

<u>Case $g^{-1} \in A$.</u> Then $f_i = g^{k_i} h_i$, for some $k_i \geq 0$, $h_i \in A$, $i=1,\dots,q$. The algorithm of A provides us with a separating family

$$\{h_1,\dots,h_q,h_{q+1},\dots,h_p\},$$

and we claim that $\{f_1,\dots,f_q,h_{q+1},\dots,h_p\}$ is separating.

For, consider any non-empty intersection

$$J = \bigcap_{i=1}^{q} \{f_i \ ?_i \ 0\} \cap \bigcap_{i=q+1}^{p} \{h_i \ ?_i \ 0\} \neq \emptyset$$

As $g^{-1} \in A \subset \mathcal{O}$, g has no zeroes in I. That means g has constant sign on I. We conclude

$$J = \bigcap_{i=1}^{q} \{h_i ?_i' \ 0\} \cap \bigcap_{i=q+1}^{p} \{h_i \ ?_i \ 0\}$$

where the symbols $?_i'$ are choosen according to the sign of g on I. As $\{h_i=0\} = \{f_i=0\}$, our claim follows.

<u>Case $g' \in A$.</u> Then $f_i = a_{io} g^{k_i} + \dots + a_{ik_i} \in A[g]$, $a_{io} \neq 0$. We use induction on $\max\{k_1,\dots,k_q\}$ and a similar trick to the one in 5.2. Consider:

$$h_i = a_{io} f_i' - a_{io}' f_i = (k_i a_{io} g' + a_{io} a_{i1}' - a_{io}' a_{i1}) g^{k_i-1} + \dots$$

$$\dots + (a_{ik_i-1} g' + a_{ik_i}') = b_{io} g^{k_i-1} + \dots + b_{ik_i-1} \in A[g]$$

and notice that

(6.2.1) $$h_i = a_{io}^2(a_{io}^{-1}f)' \qquad (\text{in } \{a_{io} \neq 0\})$$

Now, by induction, $\{a_{1o},\dots,a_{qo},h_1,\dots,h_q\}$ may be extended to a separating family by adding, let us say, $h_{q+1},\dots,h_p \in A[g]$. We claim then that $\{f_1,\dots,f_q,a_{1o},\dots,a_{qo},h_1,\dots,h_p\}$ is separating too.

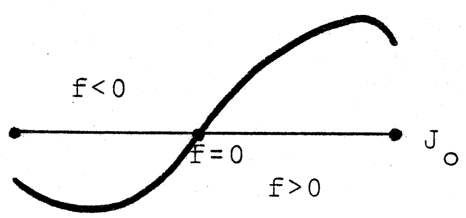
Indeed let

(6.2.2) $$J = \bigcap_{i=1}^{q}\{f_i\ ?_i\ 0\} \cap \bigcap_{i=1}^{q}\{a_{io}\ ?_i^*\ 0\} \cap \bigcap_{i=1}^{p}\{h_i\ ?_i^!\ 0\} \neq \emptyset$$

We know $J_o = \bigcap_{i=1}^{q}\{a_{io}\ ?_i^*\ 0\} \cap \bigcap_{i=1}^{p}\{h_i\ ?_i^!\ 0\} \neq \emptyset$ is a single point or an open interval whose closure is given by relaxing inequalities. In the former case, the conclusion is clear. Otherwise, no $?_i^*$, $?_i^!$ is = and the a_{io}'s and h_i's have constant sign in I_o. In particular, each a_{io}^{-1} is an analytic function on J_o and, by 6.2.1:

$$(a_{io}^{-1}f)' = a_{io}^{-2}h_i$$

has constant sign on J_o. Whence, $a_{io}^{-1}f_i$ has at most one zero in $\overline{J_o}$ and this zero lies in J_o. Of course, the same is true for f_i, from which we deduce that each $\{f_i\ ?_i\ 0\} \cap J_o$ is a single point t_i or an open subinterval of J_o, whose closure is obtained by relaxing inequalities.



Notice now that since $J \neq \emptyset$, all t_i coincide, and then, if there is some t_i, relaxing inequalities in 6.2.2 does not add any point to J. Finally, if all $\{f_i\ ?_i\ 0\} \cap J_o$ are open intervals the conclusion also follows, because the closure of their intersection is the intersection of their closures.

Case $g'g^{-1} \in A$. First notice $g = c+\exp h$, $c \in \mathbb{R}$, $h \in \mathcal{O}$, $h' = g'g^{-1} \in A$. As $A[g] = A[g-c]$ we assume $c=0$. Then write $f_i = a_{io}g^{k_i}+\ldots+a_{ik_i} \in A[g]$ and argue by induction on $\max\{k_1,\ldots,k_q\}$. Take

$$h_i = \sum_{j=o}^{k_i-1} (a'_{ij}a_{ik_i}-a_{ij}a'_{ik_i})g^{k_i-j-1} + \sum_{j=0}^{k_i-1} (k_i-j)a_{ij}a_{ik}g'g^{-1} \in A[g]$$

The reason behind this choice is that $\mathrm{degree}(h_i) < k_i$, and

(6.2.3) $$gh_i = a_{ik_i}^2(a_{ik_i}^{-1}f)' \qquad (\text{in } \{a_{ik_i}\neq 0\})$$

Then, by induction, there are $h_{q+1},\ldots,h_p \in A[g]$ such that the family $\{a_{1k_1},\ldots,a_{qk_q},h_1,\ldots,h_p\}$ is separating, and we claim so is $\{f_1,\ldots,f_q,a_{1k_1},\ldots,a_{qk_q},h_1,\ldots,h_p\}$. For, the argument in the previous case applies here too, because, $g = \exp h$ being positive on I, the factor g in 6.2.3 has no meaning.

Finally, with respect to the finiteness property, we have:

(6.3) Proposition.- Let A be a subring of $\mathcal{O}$ which has a separating algorithm. Then A has the finiteness property.

Proof.- For, let $f \in A$. By hypothesis there are $f_1,\ldots,f_p \in A$ such that the family $\{f,f_1,\ldots,f_r\}$ is separating. Then

$$\{f = 0\} \cap \{f_1 ?_1 0,\ldots,f_p ?_p 0\}$$

must be empty or a point for any choice of $?_1,\dots,?_p$. But there are finitely many choices and so if $z(f) = +\infty$, there would be a zero t_o of f such that

$$t_o \notin \{f_1 ?_1 0, \dots, f_p ?_p 0\}$$

for all possible $?_1,\dots,?_p$. This is absurd, and we conclude $z(f) < +\infty$.

It follows from 6.3 that $\mathcal{O}$ has no separating algorithm. On the contrary, the ring $\mathcal{L} \subset \mathcal{O}$ of Liouville functions does have one, because of 6.2. Again, as in §5 with respect to neotherianity, $\mathcal{L}$ has a better behaviour than $\mathcal{O}$.

(6.4) Final remarks and some problems.- The contents in §§5 and 6 raise interesting questions in connection with the results of the previous sections. We shall quote some here.

The proof of 6.2 gives a quite constructive way of extending any Liouville function $f \in \mathcal{L}$ to obtain a separating family. Clearly, this construction is closely related to the complexity of f, that means, the number of elementary operations (≡ arithmetic operations, derivation, integration, exponentiation ...) needed to get f from the constants. Of course, a first problem here is to define properly this concept.

Then, it turns out that the minimal number s(f) of functions we have to add to f to obtain a separating family might be bounded in terms of that complexity. Besides, from the argument in 6.3 it follows an upper bound for the number of zeroes, namely, $z(f) \leq 3^{s(f)}$. This suggests that, like for polynomials in §§3 and 4, complexity could give bounds of z(f).

Thus we come to the point of extending the main result, Hovanskii's theorem to Liouville functions, what is most natural after remark 2.5.

Firstly, we need a precise definition of what Liouville function of several variables means. A possible choice would be the following. Let $U \subset \mathbb{R}^n$ be an open connected set. Then:

(i) A Liouville function of order 0 (in U) is a rational function.

We denote by $L_o(U)$ the set of all of them, and define recursively $L_k(U)$, $k \geq 1$:

(ii) <u>A Liouville monomial of order k</u> (in U) is a composition $\psi \circ f$, where $f \in L_{k-1}(U)$ and ψ is a Liouville function in some interval $I \supset f(U)$.

<u>A Liouville function of order k</u> (in U) is a rational function of Liouville monomials of order k.

So, $L_k(U)$ is the set of all Liouville functions of order k. Finally, we have:

(6.4.1) <u>Definition</u> ([H_2]).- The ring of functions in U generated by the $L_k(U)$, $k \geq 0$, is called ring of <u>Liouville functions of U</u> and denoted by $L(U)$. Its members are the <u>Liouville functions</u> of U.

Once again, the <u>complexity</u> appears naturally in working with this definition, involved with two main questions:

(A) As was said before a Hovanskii type theorem should be available for Liouville functions. Then the problem is to give explicit bounds by means of complexity.

(B) Secondly, it would be useful to know a separating algorithm for Liouville functions of several variables with a constructive description related to their complexity.

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