OPTIMAL TIME-CONSISTENT FISCAL POLICY IN AN ENDOGENOUS GROWTH ECONOMY WITH PUBLIC CONSUMPTION AND CAPITAL

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ABSTRACT

In an endogenous growth model with public consumption and public investment, we explore the time-consistent optimal choice for two policy instruments: an income tax rate and the split of government spending between consumption and investment. We show that under the time-consistent, Markov policy, the economy lacks any transitional dynamics and also that there is local and global determinacy of equilibrium. We compare the Markovian optimal policy with the Ramsey policy as well as with the solution to the planner's problem under lump-sum taxation. For empirically plausible parameter values we find that the Markov-perfect policy implies a higher tax rate and a larger proportion of government spending allocated to consumption than those chosen under a commitment constraint. As a result, economic growth is slightly lower under the Markov-perfect policy than under the Ramsey policy, with growth under lump-sum taxes being highest.

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1. Introduction

The relevance of time consistent policies stems from the fact that the government has no incentive to change its policy once private agents have made their decisions conditional on that announced policy. Unfortunately, the difficulty in solving the time consistent optimal policy problem that defines the Markov policy has generally led academic research into the characterization of the more limited Ramsey optimal policies, that assume commitment and are hence subject to potential deviations by the government from the previously announced policy rule. The same difficulty also explains that most research on time-consistent optimal policies has been done in exogenous growth environments.

Ortigueira (2006) and Klein, Krusell and Rios-Rull (2008) consider the same stylized exogenous growth model, with leisure and public consumption in the utility function, to characterize the optimal time-consistent tax policy under two different game designs. Klein, Krusell and Rios-Rull (2008) consider a game in which the government is a dominant player that takes the optimal reaction of private agents as given when deciding the optimal policy. Ortigueira (2006) compares the results obtained under the structure in Klein, Krusell and Rios-Rull with those from an alternative design of the game in which the government and private agents make their respective decisions simultaneously, characterizing the behavior of the economy along the transition to the optimal steady-state. These authors consider alternative fiscal structures, always with a single instrument: either a single tax levied on total income, a single tax on capital income or a single tax on labor income. Martin (2010) follows the same game structure as Klein, Krusell and Rios-Rull (2008), extending the analysis to the simultaneous consideration of different tax rates for capital and labor income, solving for the optimal time consistent choice for both fiscal instruments. A further exogenous growth analysis is done by Azzimonti et al. (2009), who characterize the Markovian tax rate raised on total income when used to finance public investment.

For the literature on optimal taxation, it is of central importance to overcome the two limitations mentioned above, by describing how to characterize the optimal time consistent fiscal policy under endogenous growth. Endogenous growth models not only allow for a more plausible representation of actual economies, but also for explicitly taking into account the effect of fiscal policy on the rate of growth.

That is the type of analysis in Malley et al. (2002), who characterize the Markov tax policy in an endogenous growth economy where the government raises tax revenues on total income, using the proceeds to finance public consumption and government production services. However, their setup is still too restrictive in two directions: *i*) the split of government spending between consumption and investment is exogenously given, and *ii*) private agents are supposed to have a logarithmic utility function and physical capital is supposed to fully depreciate each period. Under these parametric restrictions, the Ramsey policy is not subject to a time consistency problem and it coincides with the Markov perfect solution, a result that we show later on.[1]

In our paper we overcome these two additional limitations: First, we consider an economic environment with a CRRA utility function defined on private and public consumption, with incomplete depreciation of capital. Second, we incorporate an endogenously time-varying split of government spending between public consumption and public capital. For standard parameterizations of our model we show that a reasonable time consistent optimal policy exists and it is described by the optimal choice of both, the income tax and the split of public spending between consumption and production activities. Under the Markov solution the state of the economy, represented by the public to private capital ratio, is always on the Balanced Growth Path. Additionally, we numerically show that there is not indeterminacy of equilibrium and hence, the Markov solution lacks any transitional dynamics.

Under this more general economic framework, when comparing the optimal Markov-perfect and Ramsey policies, we find that: *i*) the income tax rate is higher under the time consistent policy, since the Markov government cannot internalize the distortionary effects of the current tax on the level of investment undertaken in previous periods (as in Ortigueira (2006), in a neoclassical growth framework), *ii*) the proportion of public resources devoted to consumption is higher under the Markov government than under the Ramsey government, since the former only commits to current policies, thereby giving priority to current consumption, with an immediate effect on utility, rather than to investment, whose effects on production and welfare will mainly take place in future periods, and *iii*) as a result, economic growth is slightly lower under the Markov-perfect policy than under the Ramsey policy, with the growth rate under lump-sum taxes being the highest.

The implication is that a government that is aware that society knows its inability to pledge future policy decisions should impose a slightly higher tax rate and devote a higher share of public resources to consumption, with a relatively lower implied rate of growth.

[1] Azzimonti et al. (2009) also show this result for an exogenous growth economy.

In section 2 we describe the model economy and analyze the competitive equilibrium conditions. The alternative optimal policy problems, with and without commitment on the part of the government, are presented in section 3. In section 4 we characterize the time-consistent optimal policy in an economy with logarithmic preferences and full depreciation of private capital, which allow for an analytical solution to exist. In section 5 we analyze a more general economy with CRRA preferences and incomplete depreciation of private capital. For this more general economic environment, we numerically characterize the optimal fiscal policies obtained with and without commitment of the government in section 6. The welfare loss under the time-consistent optimal policy, relative to the planner's allocation under lump-sum taxes is discussed in section 7. Finally, the paper closes with some conclusions.

2. The model economy

We consider an economy in which firms maximize profits subject to a technology that produces the single consumption commodity. The stocks of private and public capital, K_t and $K_{p,t}$, are used together with labor, L_t, as production inputs in an aggregate technology: $Y_t = BK_t^{\alpha}(L_t K_{p,t})^{1-\alpha}$. Firms pay rents $r_t K_t + w_t$ to households for the use of private capital and labor, solving each period the static profit optimization problem:

$$\underset{\{K_t, L_t\}}{Max}\, \Pi_t = BK_t^{\alpha}(L_t K_{p,t})^{1-\alpha} - r_t K_t - w_t L_t .$$

We assume in what follows that population does not grow and that it is equal to the labor supply, which we normalize to 1: $L_t = 1, \forall t$.

Markets for production inputs are competitive. At each point in time, input prices are equal to their marginal product:

$$r_t = r\left(K_t, K_{p,t}\right) = \alpha B\left(K_t / K_{p,t}\right)^{\alpha-1}, \tag{1}$$

$$w_t = w\left(K_t, K_{p,t}\right) = (1-\alpha)B\left(K_t / K_{p,t}\right)^{\alpha} K_{p,t}, \tag{2}$$

so that real interest rates and wages are functions of the two state variables, private and public capital.

The government uses the proceeds from income taxes to finance public consumption and to accumulate public capital. We denote by η_t the proportion of revenues used at time *t* to purchase public consumption, the remaining public resources being used to pay for public investment. The government budget constraint is $\tau_t\left(r_t K_t + w_t\right) = g_t + K_{p,t}$, where

$$g_t = \eta_t \tau_t \left(r_t K_t + w_t\right), \tag{3}$$

$$K_{p,t} = (1-\eta_t)\tau_t\left(r_t K_t + w_t\right). \tag{4}$$

In line with Barro (1990), and Cazzavillan (1996), it is public investment that is productive, since the same $K_{p,t}$ variable enters as an argument in the production function and as an expenditure component in the government budget constraint. Alternatively, we could think of public capital as fully depreciating each period.

Households maximize their life-time discounted aggregate utility, $\sum_{t=0}^{\infty}\rho^t U(c_t, g_t)$, defined over private and public consumption, c_t, g_t, subject to a flat tax τ_t on total income. They know the current values of τ_t and η_t, and expect future governments to follow policies $\tau_{t+1} = \mathrm{T}(K_{t+1}, K_{p,t+1})$ and $\eta_{t+1} = \mathrm{H}(K_{t+1}, K_{p,t+1})$. The typical household solves the problem:

$$\upsilon\left(k_t, K_t, K_{p,t}; \tau_t; \eta_t; \mathrm{T}; \mathrm{H}\right) = \underset{\{c_t, k_{t+1}\}}{Max}\left[U\left(c_t, g_t\right) + \rho\tilde{\upsilon}(k_{t+1}, K_{t+1}, K_{p,t+1}; \mathrm{T}; \mathrm{H})\right] \tag{5}$$

given k_0, and subject to the budget constraint,

$$c_t + K_{t+1} - (1-\delta)K_t = (1-\tau_t)\left[w_t\left(K_t, K_{p,t}\right) + r_t\left(K_t, K_{p,t}\right)K_t\right], \tag{6}$$

leading to the following Euler equation,[2] in which we have already taken (1) into account:

$$U_c\left(c_t, g_t\right) = \rho U_c\left(c_{t+1}, g_{t+1}\right)\left[1-\delta+\left(1-\tau_{t+1}\right)\alpha B\left(K_{t+1}/K_{p,t+1}\right)^{\alpha-1}\right]. \tag{7}$$

With homogeneous households, together with the normalization $L_t = 1$, we have in equilibrium: $K_t = k_t, K_{pt} = k_{pt}$, and all the variables in the model can be regarded either as per capita or in aggregate terms. In what follows, they will all be denoted by lower case letters.

From the government budget expenditure rules and the optimality conditions for the competitive firms we get,

$$g_t = \mathrm{G}(k_t, k_{p,t}; \tau_t, \eta_t) = \eta_t \tau_t B k_t^{\alpha} k_{p,t}^{1-\alpha}, \tag{8}$$

$$k_{p,t} = (1-\eta_t)\tau_t B k_t^{\alpha} k_{p,t}^{1-\alpha}, \tag{9}$$

as well as the global constraint of resources:

[2] Along the paper we denote partial derivatives by $F_v \equiv \dfrac{\partial F}{\partial v}$.

$$k_{t+1} = (1-\delta)k_t + (1-\tau_t)Bk_t^{\alpha} k_{p,t}^{1-\alpha} - \mathsf{C}\left(k_t, k_{p,t}; \tau_t, \eta_t\right), \tag{10}$$

where $\mathsf{C}\left(k_t, k_{p,t}; \tau_t, \eta_t\right)$ is the consumption function that solves the Euler equation (7).

Substituting (9) in the technology function, aggregate output is given by $y_t = B^{1/\alpha}\left[(1-\eta_t)\tau_t\right]^{\frac{1-\alpha}{\alpha}} k_t$. As a consequence, in the competitive equilibrium allocation, *i*) the ratio of public capital to output is equal to $(1-\eta_t)\tau_t$, an extension of the result in Barro (1990), and *ii*) the ratio of private capital to output is a function of $(1-\eta_t)\tau_t$ and structural parameters α and B. As is typical in the Barro family of AK models, the constant returns to scale in the cumulative factors are the sources of endogenous growth in our model economy.

Finally, in competitive equilibrium we will have

$$\upsilon\left(k_t, k_{p,t}; \mathsf{T}\left(k_t, k_{p,t}\right); \mathsf{H}\left(k_t, k_{p,t}\right); \mathsf{T}; \mathsf{H}\right) = \tilde{\upsilon}\left(k_t, k_{p,t}; \mathsf{T}; \mathsf{H}\right).$$

3. Optimal policy

3.1 The time-consistent optimal policy

We use the same equilibrium concept as Klein, Krusell and Ríos-Rull (2008) and the "government-moves-first" case in Ortigueira (2006).[3] We consider a dynamic game played by a sequence of governments, each one of them choosing current period policies on the basis of the state of the economy in the current period, as summarized by the aggregate stock of private and public capital. Hence, each government chooses the current tax rate τ_t and the proportion of revenues used to purchase public consumption, η_t, before the household decides on consumption and savings. When making optimal policy choices, the government is able to correctly anticipate the reaction of the household to policy decisions.

The government knows that the consumption policy function of the household is the solution to (7). Acting as a leader, it chooses the current tax rate and the split of public resources taking as given the policies followed by future governments and taking into account that reaction of the household to the policy choices, as follows:

$$V\left(k_t, k_{p,t}\right) = \underset{\{\tau_t, \eta_t\}}{Max}\left[U\left(\mathsf{C}\left(k_t, k_{p,t}; \tau_t, \eta_t\right), \mathsf{G}\left(k_t, k_{p,t}; \tau_t, \eta_t\right)\right) + \rho V(k_{t+1}, k_{p,t+1})\right] \tag{P1}$$

where $\mathsf{G}\left(k_t, k_{p,t}; \tau_t, \eta_t\right)$, $k_{p,t}$ and k_{t+1} are given by (8), (9) and (10), respectively.

[3] Which is also used in Krusell and Ríos-Rull (1999), and Krusell, Quadrini and Ríos-Rull (1996).

Proposition 1. *The time consistent policy corresponding to the Markov equilibrium is the solution to the set of Generalized Euler Equations (GEE):*

$$\frac{U_{c_t} C_{\tau_t} + U_{g_t} G_{\tau_t}}{C_{\tau_t} + \Lambda(\tau_t)\,\Omega(\tau_t,\eta_t)\,k_t} = \frac{U_{c_t} C_{\eta_t} + U_{g_t} G_{\eta_t}}{C_{\eta_t} + \frac{1-\alpha}{\alpha}\frac{1}{(1-\eta_t)}\Omega(\tau_t,\eta_t)\,k_t}, \tag{11}$$

and

$$\frac{U_{c_t} C_{\tau_t} + U_{g_t} G_{\tau_t}}{\Omega(\tau_t,\eta_t)\Lambda(\tau_t)k_t + C_{\tau_t}} = \rho\left[\begin{array}{c} U_{c_{t+1}} C_{k_{t+1}} + U_{g_{t+1}} G_{k_{t+1}} + \dfrac{U_{c_{t+1}} C_{\tau_{t+1}} + U_{g_{t+1}} G_{\tau_{t+1}}}{\Omega(\tau_{t+1},\eta_{t+1})\Lambda(\tau_{t+1})k_{t+1} + C_{\tau_{t+1}}} \times \\ \left[1-\delta+\Omega(\tau_{t+1},\eta_{t+1}) - C_{k_{t+1}}\right] \end{array}\right], \tag{12}$$

where $\Lambda(\tau) \equiv \frac{\tau-(1-\alpha)}{\alpha\tau(1-\tau)}$, $\quad \Omega(\tau,\eta) \equiv (1-\tau)\left[(1-\eta)\tau\right]^{(1-\alpha)/\alpha} B^{1/\alpha}$.

Proof.- *See Appendix* 1.

Equation (11) is a condition relating the optimal choice of the two policy instruments at a given point in time, while equation (12) characterizes the optimal intertemporal choice of income tax rates.

From the aggregate constraint of resources we get the size of the reduction in time t investment from an increase in taxes is: $\partial(k_{t+1}-k_t)/\partial\tau_t =_{\text{t}} -\left[C_{\tau_t} + \Lambda(\tau_t)\,\Omega(\tau_t,\eta_t)\,k_t\right]$. Hence, the left hand side at (11) gives the change in utility produced by a tax increase, per unit of crowded-out investment. This is what Ortigueira (2006) calls today's marginal value of taxation. By a similar argument, the right hand side at (11) is the change in utility from an increase in the share of resources devoted to public consumption, per unit of crowded-out investment. The optimal choices of the two policy instruments must satisfy the equality between these two marginal effects on utility.

The left hand side at (12) is again the marginal change in utility per unit of crowded out investment implied by a decrease in the tax rate. Lower taxes at t+1 stimulate investment, and an additional unit of capital at t+1 has a direct effect on utility of $U_{c_{t+1}} C_{k_{t+1}} + U_{g_{t+1}} G_{k_{t+1}}$ through its effect on private and public consumption and an indirect effect through its impact on time t+2 capital stock, $\frac{\partial k_{t+2}}{\partial k_{t+1}} = 1-\delta+\Omega(\tau_{t+1},\eta_{t+1}) - C_{k_{t+1}}$, which needs to be appropriately discounted. The total effect is given by the square bracket at the right hand side of (12). It shows that the change in utility per unit of crowded-out investment at time t

implied by a marginal change in the optimal tax rate must be equal to the discounted change in utility resulting at time t+1.

Definition.- A Markov-Perfect equilibrium is a set of functions $\mathrm{C}(k_t, k_{p,t}; \tau_t, \eta_t)$, $\mathrm{T}(k_t, k_{p,t})$, $\mathrm{H}(k_t, k_{p,t})$ and $V(k_t, k_{p,t})$ such that:

i) Given government rules (8) and (9), $\mathrm{C}(k_t, k_{p,t}; \tau_t, \eta_t)$ solves the Euler equation (7) subject to the global constraint of resources (10),

ii) $\mathrm{T}(k_t, k_{p,t})$, $\mathrm{H}(k_t, k_{p,t})$ satisfy conditions (8), (9), the global constraint of resources (10), as well as the Generalized Euler Equations (11) and (12); and

iii) $V(k_t, k_{p,t})$ is the value function of government obtained as a solution to [P1]:

$$V(k_t, k_{p,t}) = U\left(\mathrm{C}\left(k_t, k_{p,t}; \mathrm{T}(k_t, k_{p,t}),\ \mathrm{H}(k_t, k_{p,t})\right), \mathrm{G}\left(k_t, k_{p,t}; \mathrm{T}(k_t, k_{p,t}),\ \mathrm{H}(k_t, k_{p,t})\right)\right) + \rho V(k_{t+1}, k_{p,t+1}).$$

3.2 The Ramsey policy

As usual, we define the benchmark “Ramsey equilibrium” as the solution to an optimal-policy problem where the government can commit to future policies. The Ramsey optimal policy is then the solution to the problem of maximizing the time aggregate utility of the household, subject to the equilibrium conditions (7), (8) and (10) as constraints:

$$\begin{aligned}
&\underset{\{c_t, g_t, \tau_t, \eta_t, k_{t+1}\}}{Max} \sum_{t=0}^{\infty} \rho^t U\left(c_t, g_t\right) \\
&\text{subject to}: \\
&k_{t+1} = (1-\delta)k_t + \Omega(\tau_t, \eta_t)k_t - c_t \qquad\qquad [\mathrm{P2}] \\
&U_{c_t} = \rho U_{c_{t+1}}\left[1-\delta+\alpha\, \Omega\left(\tau_{t+1}, \eta_{t+1}\right)\right] \\
&g_t = \eta_t \tau_t \left[(1-\eta_t)\tau_t\right]^{\frac{1-\alpha}{\alpha}} B^{\frac{1}{\alpha}} k_t \ .
\end{aligned}$$

where (9) has been substituted in (7), (8), (10), and $\Omega\left(\tau_t, \eta_t\right) \equiv (1-\tau_t)\left[(1-\eta_t)\tau_t\right]^{(1-\alpha)/\alpha} B^{1/\alpha}$.

The Ramsey policy takes into account the optimal reactions of private agents. However, it is time inconsistent, since once private agents adjust their decisions to the announced economic policy it will be optimal for the government to change policy.

Given the complexity involved in characterizing optimal policy under lack of commitment, attention has often been restricted to Ramsey policies, in spite of their well-known limitation of assuming commitment on the part of the current government on future periods. It is therefore important to evaluate to what extent the Markov-perfect fiscal policy differs from the Ramsey policy in our setup. We will perform such analysis in Section 6. In Appendix 2 we characterize the first order conditions and the balanced growth path for the Ramsey problem.

4. An analytical solution: logarithmic utility and full depreciation of private capital

We consider in this section the special case of logarithmic preferences that are separable in private and public consumption, $U\left(c_t, g_t\right) = \ln c_t + \theta \ln g_t$, together with full depreciation of private capital. The two assumptions together allow us to obtain an analytical characterization of the time consistent optimal fiscal policy that we can compare with the Ramsey solution as well as with the allocation that would be obtained under lump-sum taxes.

Under this utility function, the competitive equilibrium allocation is characterized by the system:

$$\left.\begin{aligned} k_{t+1} &= \Omega(\tau_t, \eta_t) k_t - c_t, \\ \frac{c_{t+1}}{c_t} &= \rho\left[\alpha\, \Omega(\tau_{t+1}, \eta_{t+1})\right] \end{aligned}\right\} \tag{13}$$

Proposition 2. *Under full depreciation of private capital and a logarithmic utility function, the competitive equilibrium allocations are given by:*

$$k_{t+1} = \rho\alpha\, \Omega(\tau_t, \eta_t)\, k_t, \tag{14}$$

$$c_t = \left(1 - \rho\alpha\right) \Omega(\tau_t, \eta_t)\, k_t. \tag{15}$$

Proof. *Plugging in the previous system* (13) *a guess for the functional form for the competitive equilibrium allocation as:* $k_{t+1} = A\Omega_t k_t$ *, it is easy to show that A=ρα.*□

Expressions (8) and (15) for g_t, c_t allow us to compute the partial derivatives that enter into the Generalized Euler equations (11)-(12), to find an analytical solution to the time consistent optimal policy problem.

The next set of results shows that the Markov solution in this economy has η_t and τ_t being constant from the initial period, implying that under the time consistent optimal policy, the state of the economy, defined as the ratio of public to private capital, lacks any transitional dynamics. We also show that there is no indeterminacy of equilibria. Hence, the economy itself lacks transitional dynamics, being on the balanced growth path from the initial period onwards.

Proposition 3. *Under full depreciation of private capital and a logarithmic utility function, separable in private and public consumption, the optimal time-consistent fiscal policy satisfies:*

$$\tau_t^M = \frac{1-\alpha}{1-\eta_t^M} \ \forall t \tag{16}$$

Proof.- *See Appendix* 3.

Corollary 1.- *Under the optimal Markov policy the state of the economy, defined as the ratio of public to private capital, is constant for all t.*

Proof.- *Using (16) in (9), we get:* $\frac{k_{p,t}}{k_t} = (1-\alpha)^{1/\alpha} B^{1/\alpha}, \forall t$.

Proposition 4.- *Under full depreciation of private capital and a logarithmic utility function, separable in private and public consumption,*

i) There is no indeterminacy of equilibria,

ii) The economy lacks transitional dynamics,

iii) The optimal Markov policy is:

$$\tau_t^M = \tau^M = 1 - \frac{\alpha(1+\rho\theta)}{1+\theta}, \quad \forall t, \tag{17}$$

$$\eta_t^M = \eta^M = \frac{\alpha\theta(1-\rho)}{1-\alpha+\theta(1-\alpha\rho)}, \quad \forall t. \tag{18}$$

Proof-. *See Appendix 3.*

Notice that the optimal split of resources between public consumption and investment is well defined, taking values between 0 and α, while the optimal income tax rate is always between 1-α and one.

We now characterize the optimal allocation of resources in terms of the ratios of private and public consumption to the stock of private capital: $\chi_t \equiv \frac{c_t}{k_t}, \phi_t \equiv \frac{g_t}{k_t}$. These ratios must remain constant along the balanced growth path.

Proposition 5. *The optimal allocation of resources under the Markov-perfect optimal policy is given by*:

$$\gamma_t^M \equiv \left(\frac{k_{t+1}}{k_t}\right)^M = \gamma^M = \rho\alpha\,\Omega(\tau^M,\eta^M) = \frac{\rho\alpha^2(1+\rho\theta)}{1+\theta}(1-\alpha)^{\frac{1-\alpha}{\alpha}} B^{1/\alpha} \quad \forall t,$$

$$\chi_t^M \equiv \left(\frac{c_t}{k_t}\right)^M = \chi^M = (1-\rho\alpha)\Omega(\tau^M,\eta^M) = \frac{(1-\rho\alpha)\alpha(1+\rho\theta)}{1+\theta}(1-\alpha)^{\frac{1-\alpha}{\alpha}} B^{1/\alpha} \quad \forall t,$$

$$\phi_t^M \equiv \left(\frac{g_t}{k_t}\right)^M = \phi^M = \frac{\eta^M\tau^M}{1-\tau^M}\Omega(\tau^M,\eta^M) = (1-\alpha)^{\frac{1-\alpha}{\alpha}} B^{1/\alpha} \quad \forall t.$$

Proof. *Their expressions can readily be obtained from* (8), (14) *and* (15).

The three following corollaries can be readily shown from (17) and (18):

***Corollary* 2.** *When public consumption does not enter as an argument into the utility function (θ=0), the Markov-perfect optimal tax rate coincides with that in Barro* (1990)*:* $\tau = 1-\alpha$. *In that situation, public resources are fully devoted to investment*.

***Corollary* 3.** *The Markov-perfect optimal tax rate converges to the Barro tax as the discount rate approaches 1, with public resources again being fully devoted to public investment.*

***Corollary* 4.** *i*) *The proportion of public resources devoted to public consumption increases with θ and α, and it decreases with ρ; ii) the optimal time consistent income tax increases with θ, and it decreases with α and ρ.*

As expected, the proportion of public resources devoted to consumption increases with the relative importance of public consumption in the utility function. It also increases with the output elasticity of private capital. A more productive private capital, relative to public capital, allows for a higher share of public resources being consumed, rather than invested. Turning the argument around, the more productive is public capital relative to private capital, the more interesting is to allocate resources to productive activities rather than to consumption. The share of public resources dedicated to consumption decreases for a larger *ρ*. We then tend to value future consumption almost as much as current consumption, and it becomes interesting to increase investment and defer consumption for the future.

As public consumption is more appreciated by consumers for higher values of *θ* and lower values of *ρ*, it is appropriate to raise higher tax revenues to finance that component of public spending. On the contrary, a high elasticity of private capital, *α*, leads the private sector to allocate more resources to investment, and taxes can be lower.

4.1 Comparing Ramsey and Markov policies under logarithmic utility and full depreciation of private capital

The following proposition shows that, for this special case, the Ramsey and Markov policies coincide.

Proposition 6. *Under a logarithmic utility function and full depreciation, the optimal Ramsey policy, becomes:*

$$\tau^R = 1 - \frac{\alpha(1+\rho\theta)}{1+\theta},$$
$$\eta^R = \frac{\alpha\theta(1-\rho)}{1-\alpha+\theta(1-\alpha\rho)}.$$

Proof: *See Appendix 3.*

The income tax and the proportion of public resources devoted to public consumption under the Ramsey policy coincide with the values obtained under the time-consistent policy, so the properties analyzed in Proposition 4 and Corollaries 2 to 4 for the Markov-perfect optimal policy apply to the Ramsey policy as well. The equality of solutions arises because under a logarithmic utility and complete depreciation of physical capital the Ramsey policy is time consistent, a result shown by Azzimonti et al. (2009) in a neoclassical growth model.

5. Optimal time-consistent fiscal policy under CRRA preferences and incomplete depreciation of private capital

The Generalized Euler conditions (11) and (12) should incorporate the consumption decision rule of private agents, which is characterized as the solution to the Euler equation (7) of the competitive equilibrium. Unfortunately, it is not possible to find the analytical solution to (7) in general, and that precludes us from obtaining an analytical characterization of the transition towards the balanced growth path.

Assuming a CRRA utility: $U(c_t, g_t) = \dfrac{c_t^{1-\sigma} g_t^{\theta(1-\sigma)} - 1}{1-\sigma}, \sigma > 0$ the Euler condition of the competitive equilibrium becomes,

$$c_t^{-\sigma} g_t^{\theta(1-\sigma)} = \rho c_{t+1}^{-\sigma} g_{t+1}^{\theta(1-\sigma)} \left[1 - \delta + \alpha\left(1-\tau_{t+1}\right)\left[\left(1-\eta_{t+1}\right)\tau_{t+1}\right]^{\frac{1-\alpha}{\alpha}} B^{1/\alpha} \right],$$

or, in ratios,

$$\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)} = \rho\chi_{t+1}^{-\sigma}\phi_{t+1}^{\theta(1-\sigma)}\left[1-\delta+\alpha\left(1-\tau_{t+1}\right)\left[\left(1-\eta_{t+1}\right)\tau_{t+1}\right]^{\frac{1-\alpha}{\alpha}} B^{1/\alpha}\right], \qquad (19)$$

where the growth rate is obtained from the resources constraint,

$$\gamma_t = (1-\tau_t)\Omega(\tau_t,\eta_t) + 1 - \delta - \chi_t \qquad (20)$$

and from (8):

$$\phi_t = \eta_t \tau_t^{1/\alpha}(1-\eta_t)^{(1-\alpha)/\alpha} B^{1/\alpha}. \qquad (21)$$

The right-hand side at (19) involves values at time t+1 of policy variables, η_{t+1}, τ_{t+1}, and ratios of decision variables to the stock of private capital, χ_{t+1}, ϕ_{t+1}. Each one of these must be a function of the state of the economy, $k_{p,t}/k_t$, so that we can think of the right-hand side at (19) as a function $F(k_{p,t}/k_t)$.

$$\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)} = F(k_{p,t}/k_t) \qquad (22)$$

We can now characterize the optimal Markov policy in the more general set up considered in this Section. We start by showing that the relationship between the two policy instruments is the same we found under logarithmic preferences and full depreciation of private capital. As shown in Appendix 4, the function $F(k_{p,t}/k_t)$ cancels out in the Generalized Euler equation (11) that relates both policy choices. As a consequence, $F(k_{p,t}/k_t)$ does not play any role in the characterization of the relationship between τ_t and η_t.

Proposition 7.- *The time-consistent optimal choice of the two policy instruments satisfies:*

$$\tau_t = \frac{1-\alpha}{1-\eta_t}, \forall t, \qquad (23)$$

***Proof**.- See Appendix 4.*

Therefore, the two policy variables satisfy at all periods the same relationship we already obtained for the simple economy with logarithmic preferences and full depreciation of capital. Again, optimal income tax rates will be above $1-\alpha$, whereas the optimal proportion of public resources devoted to consumption will always be below α.

Even more importantly, the relationship (23) implies again that the ratio of the two state variables, private and public capital, is constant over time.

***Corollary 5**.- In the absence of equilibrium indeterminacy, the Markov perfect solution lacks any transitional dynamics.*

***Proof:** Since the ratio of each control variable to private capital can only depend on the ratio of the two state variables,* $k_{p,t}/k_t$*, the fact that this ratio is constant implies that the ratios of private and public consumption to capital will also be constant. Hence, in the absence of indeterminacy of equilibria, the economy will always be on its balanced growth path, displaying no transition.*□

Equilibrium indeterminacy in this model can only be analyzed numerically. Local indeterminacy depends on the dynamic properties of the solution to the second Generalized Euler condition (12). Substituting (22) in (12) and using the fact that the ratio $k_{p,t}/k_t$ is constant over time, equation (12) becomes,

$$\frac{\chi_{\tau_t} + \chi_t \theta \dfrac{1}{\alpha \tau_t}}{\chi_{\tau_t} + \dfrac{\tau_t - (1-\alpha)}{\alpha \tau_t (1-\tau_t)} \Omega(\tau_t, \eta_t)} = \rho \gamma_{t+1}^{\theta(1-\sigma)-\sigma} \left[(1+\theta)\chi_{t+1} + \frac{\chi_{\tau_{t+1}} + \chi_{t+1}\theta \dfrac{1}{\alpha \tau_{t+1}}}{\chi_{\tau_{t+1}} + \dfrac{\tau_{t+1} - (1-\alpha)}{\alpha \tau_{t+1}(1-\tau_{t+1})} \Omega(\tau_{t+1}, \eta_{t+1})} \gamma_{t+1} \right]. \qquad (24)$$

The partial derivative χ_{τ_t} is shown in Appendix 4 (expression A4.2) to be a function of γ_t and χ_t. Thus, using (A4.2), as well as (20)-(23) in equation (24), we obtain a single, first order, nonlinear difference equation in η_t.

The dynamic properties of (24) can be analyzed through the eigenvalue of the linearized version of this equation. An eigenvalue below 1 would suggest an indetermination of equilibrium since in that case, we would need an initial condition for η (that is, η_0) in order to compute the time series for η_t. Any arbitrary choice for η_0 would yield a valid Markov equilibrium, and we would then have indeterminacy of equilibria. That would produce transitional dynamics, as the trajectory followed by the economy would depend on the choice for η_0. On the other hand, an eigenvalue greater than one would imply that the only stable solution is obtained with η_t being constant over time: $\eta_t = \eta, \forall t$, without indeterminacy of equilibria.

We have numerically computed such eigenvalue for wildly different parameterizations, obtaining always a value above one, even for empirically implausible parameter values. Lacking an analytical proof, our numerical analysis suggests that there is not indeterminacy of equilibria, with $\eta_t = \eta, \forall t$, and hence, $\tau_t = \tau, \gamma_t = \gamma, \phi_t = \phi, \chi_t = \chi \;\; \forall t$, the economy being at each point in time on its balanced growth path.

5.1 Solving for the Markov equilibrium

The consumption policy rule $C(k_t, k_{p,t}; \tau_t, \eta_t)$ is the solution to the competitive equilibrium Euler equation under the global constraint of resources. Since the ratio of private to public capital and the optimal policy instruments τ_t, η_t are constant over time, we have: $\frac{C(k_t, k_{p,t}; \tau_t, \eta_t)}{k_t} = \chi(\tau, \eta)$.

Evaluating the Euler equation in ratios (19) along the balanced growth path and using (23), we get: $\gamma = \left[\rho\left(1-\delta+\alpha\ \Omega(\tau,\eta)\right)\right]^{1/(\sigma-\theta(1-\sigma))}$. Taking this expression to (20), we get:

$$\chi(\tau,\eta) = 1-\delta+\Omega(\tau,\eta)-\left[\rho\left(1-\delta+\alpha\ \Omega(\tau,\eta)\right)\right]^{1/(\sigma-\theta(1-\sigma))}. \quad (25)$$

with partial derivatives χ_τ and χ_η :

$$\chi_\tau \equiv \frac{\partial\chi}{\partial\tau} = \Lambda(\tau)\Omega(\tau,\eta)\left(\frac{\rho\alpha\gamma^{1-\sigma+\theta(1-\sigma)}}{\sigma-\theta(1-\sigma)}-1\right), \quad (26)$$

$$\chi_\eta \equiv \frac{\partial\chi}{\partial\eta} = \frac{1-\alpha}{\alpha(1-\eta)}\Omega(\tau,\eta)\left(\frac{\rho\alpha\gamma^{1-\sigma+\theta(1-\sigma)}}{\sigma-\theta(1-\sigma)}-1\right). \quad (27)$$

where $\Lambda(\tau) \equiv \frac{\tau-(1-\alpha)}{\alpha\tau(1-\tau)}$.

Finally, the Markov equilibrium $\left\{\gamma^M, \chi^M, \phi^M, \tau^M, \eta^M\right\}$ is obtained as the solution to the system (20), (21), (23), (24) and (25) evaluated along the balanced growth path. The system can only be solved numerically and the following section is devoted to analyze its properties under several parameterizations. Under all parameterizations considered, the system has been shown to have a single solution,[4] suggesting that the equilibrium is globally determined.

6. Comparing the Ramsey and Markov solutions in the general case

Let us now compare the Markov and Ramsey solutions between themselves, as well as with the allocation of resources that would be achieved by a benevolent planner using

[4] When solving the nonlinear system of equations, we have tried very different sets of initial conditions, always reaching the same solution shown in the Tables.

lump-sum taxes, which is characterized in Appendix 5. We will use $\tau_t^P = \frac{g_t + k_{p,t}}{y_t}$ as a measure of the size of the public sector in the planner solution and we will use $\eta_t^P = \frac{g_t}{g_t + k_{p,t}}$ for the composition of public expenditures. Both of them will be used in the graphs and tables we present below.

Let us now examine the values taken by the main variables in the economy along the balanced growth path under the three alternative fiscal policies: *i*) the planner's policy under lump-sum taxes, *ii*) the Ramsey policy and *iii*) the time-consistent policy, all of them under the more general setup, with a CRRA utility function and incomplete depreciation of private capital. Unfortunately, our results are not readily comparable with those in the literature because numerical results are usually derived using a logarithmic, separable utility function, whereas our results correspond to general CRRA utility functions, and also because of our consideration of endogenous growth.

The Markov equilibrium is obtained as explained in section 5.1. As shown in Appendix 2, the solution to the Ramsey problem [P2] is characterized by a system of 8 dynamic equations in $\{\gamma, \chi, \phi, \eta, \tau, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3\}$ that allows us to compute the balanced growth path for the Ramsey policy (τ^R, η^R) as well as the implied allocation of resources, characterized by $(\gamma^R, \chi^R, \phi^R)$ and three multipliers, $\{\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3\}$. That system is made up only by control variables, with no participation of any state variable. Hence, in the absence of local indeterminacy of equilibrium, the only possible solution is that control variables stay on the balanced growth path (BGP) from the initial period, with no transition.

Under incomplete depreciation of private capital, the choice of parameter values: $\theta = 0.4$, $1-\alpha = 0.20$, $\rho = 0.99$, $\delta = 0.10$, $B = 0.4555$, when generating annual data lead to sensible properties of the Markov solution. Parameter values are standard in the literature for annual data except for θ, which is chosen so that the ratio of public consumption to private consumption for the Markov solution is in line with data for the postwar US economy (g/c=0.25). For instance, for $\sigma = 2$, we get a ratio of public to private consumption around 0.25, an annual growth rate γ=1.5%, and a gross real interest rate: $1/\left(\rho\gamma^{(1-\sigma)(1+\theta)}\right) \approx 1.03$. The value chosen for α is consistent with a broad concept of capital that includes both physical and human components, as it is commonly established in endogenous growth models with public and private capital (see Cazzavillian, 1996). As to the elasticity of output with respect to public capital, 1-α, we set a value which is in line with previous literature: Azzimonti et al. (2009) takes a benchmark elasticity of 0.25, but the range of values varies significantly across authors between the 0.03 estimated by Eberts (1986), and the 0.39 estimated by Aschauer (1989).

Figure 1 shows values for the main variables in the economy under the three equilibrium concepts as a function of the risk aversion parameter, σ. Over the whole range of values considered, the optimal income tax increases with risk aversion. It always falls between 20% and 30%, being higher under the Markov-perfect policy than under the time-inconsistent Ramsey policy. The proportion of public resources devoted to consumption, relative to investment, is also increasing in σ, staying between 6% and 32%. It is also higher under the Markov-perfect solution than under the Ramsey policy.

Steady state growth is slightly higher under the Ramsey policy. Growth rates are large for low values of the risk aversion parameter, but they become quite realistic for values of σ above 1.5. As a proportion of output, private consumption is higher under the Ramsey policy, while public consumption is higher under the Markov policy. In terms of specific values, private consumption never exceeds 35% of output under either policy, while public consumption remains below 10% of output, both observations below the levels observed in actual economies. However, the public to private consumption ratio is around 25%, as in observed data. For the Markov and Ramsey solutions we could obtain ratios of public and private consumption to output similar to those in actual data, at the expense of getting income tax rates implausibly high.

A planner with access to lump-sum taxes under commitment would devote an even higher proportion of public resources to consumption than the Markov and Ramsey solutions, and the growth rate would be considerably higher than under the alternative solutions.

That the income tax is higher under the Markov-perfect policy than under the Ramsey solution is consistent with the result obtained by Ortigueira (2006) in an exogenous growth economy under inelastic labor supply.[5] This result arises because the Markovian government cannot internalize the distortionary effects of current taxation on past investment, while in the Ramsey solution, the government takes fully into account the negative effect of the income tax on future investment. A similar argument explains that the

[5] Even though the two results are not strictly comparable, since one of them refers to an exogenous growth economy and the other to an endogenous growth economy.

Markov government devotes a higher proportion of public resources to consumption, which has a direct impact on current utility, to the expense of public investment, which would have a positive effect on future utility. A lower income tax rate and a higher share of investment in public expenditures make the growth rate to be higher under the Ramsey than under the Markov solution.

Figure 2 presents results for σ = 2, and values of the relative weight of public consumption in the utility function, θ, between 0.2 and 1.5, the remaining parameters being as in Figure 1. As expected, public consumption as a share of total public spending increases with θ. Qualitative results stay the same, with the Markov-perfect policy imposing a higher income tax than the Ramsey policy and devoting a higher proportion of public resources to consumption. The growth rate is again higher under the Ramsey than under the Markov policy.

Table 1 summarizes the results by displaying a single point from Figure 1 and Figure 2. Table 2 analyzes the effects of a change in α. The value of B has been chosen to guarantee positive growth rates under the Markov and Ramsey solutions.

Since the resource allocations obtained under the three solution concepts satisfy the conditions for competitive equilibrium, the fact that the ratio of public capital to output is the same for the three solutions means that the product $(1-\eta)\tau$ is also the same for the three solution concepts. This property implies that the ratio of private capital to output is also the same for the three solutions under any parameterization. The common value of $(1-\eta)\tau$ turns out to be equal to the elasticity of output with respect to public capital, again an extension of the result obtained by Barro (1990) in a model with just public capital.

The solution under lump-sum taxes leads to the largest public sector and devotes a lowest share of public resources to investment. Since taxes are not distortionary under the planner's solution, a larger proportion of resources extracted by the public sector can be made compatible with a higher rate of growth.

The comparison between the two panels in Table 1 shows what happens as public consumption becomes more important in the utility function: while the ratios of both types of capital to output remain unchanged, the optimal tax rate increases, as it does the proportion of public resources devoted to consumption. These two changes lead to a lower rate of growth.

Table 1. Values for the main variables under the three solution concepts. Effects of a change in θ						
	B = 0.4555, σ = 2.00, θ = 0.40, α = 0.80, δ = 0.10, ρ = 0.99			B = 0.4555, σ = 2.00, θ = 1.00, α = 0.80, δ = 0.10, ρ = 0.99		
	Planner	*Markov*	*Ramsey*	*Planner*	*Markov*	*Ramsey*
η (%)	26.7	24.9	20.4	41.6	38.7	30.9
τ (%)	27.3	26.6	25.1	34.2	32.6	28.9
γ (%)	3.6	1.5	1.6	2.9	0.8	1.1
c/y(%)	18.3	27.4	28.4	14.3	24.2	26.9
g/y(%)	7.3	6.6	5.1	14.3	12.6	8.9
kp/y(%)	20.0	20.0	20.0	20.0	20.0	20.0
k/y	4.0	4.0	4.0	4.0	4.0	4.0

Table 2. Values for the main variables under the three solution concepts. Effects of a change in α						
	B = 0.658, σ = 2.00, θ = 0.40, α = 0.80, δ = 0.10, ρ = 0.99			B =0.658, σ = 2.00, θ = 0.40, α = 0.70, δ = 0.10, ρ = 0.99		
	Planner	*Markov*	*Ramsey*	*Planner*	*Markov*	*Ramsey*
η (%)	32.9	32.1	28.9	19.2	16.9	13.1
τ (%)	29.8	29.4	28.1	37.1	36.1	34.5
γ (%)	8.1	4.5	4.7	4.8	1.5	1.6
c/y(%)	24.6	33.9	34.8	17.9	28.9	30.0
g/y(%)	9.8	9.5	8.1	7.1	6.1	4.5
kp/y(%)	20.0	20.0	20.0	30.0	30.0	30.0
k/y	2.5	2.5	2.5	3.0	3.0	3.0

Note to the tables: for the planner solution $\tau_t^P = \frac{g_t + k_{p,t}}{y_t}$ and $\eta_t^P = \frac{g_t}{g_t + k_{p,t}}$.

Table 2 shows that an increase in the productivity of public capital (lower α) leads to higher tax rates. The government detracts more aggregate resources from the economy and devotes a larger proportion of them to investment. Because of the increase in the tax rate generated by a lower α parameter, the productivity of private capital and hence, the rate of growth, both decrease.

7. Welfare

In this section we compute the level of welfare that would arise along the balanced growth path under the time consistent Markov policy and compare it with the level of welfare that would be obtained under lump-sum taxes.[6] As in Lucas (1987), what we compute is the consumption compensation (as a percentage of output) that would be needed

[6] We do not consider the level of welfare under the Ramsey solution because of its time-inconsistent nature.

under the Markov rule to achieve the same level of welfare than under the resource allocation of the planner with non-distortionary taxation.

Under a CRRA utility, welfare can be written,

$$W_i = \sum_{t=0}^{\infty} \rho^t \frac{c_{t,i}^{1-\sigma} g_{t,i}^{\theta(1-\sigma)} - 1}{1-\sigma} = \frac{1}{1-\sigma}\left[\frac{\chi_i^{1-\sigma}\phi_i^{1-\sigma}}{1-\rho\,\gamma_i^{(1-\sigma)(1+\theta)}} - \frac{1}{1-\rho}\right], i = Planner, Markov.$$

Let $\{c_{t,i}, g_{t,i}\}$, i=P,M, be the optimal path for private and public consumption for the planner's solution and the Markov solution, respectively, that is:

$$\begin{aligned} c_{t,i} &= \chi_i k_{t,i} = \chi_i k_0 \gamma_i^t \underbrace{=}_{k_0=1} \chi_i \gamma_i^t, \\ g_{t,i} &= \phi_i k_{t,i} = \phi_i k_0 \gamma_i^t \underbrace{=}_{k_0=1} \phi_i \gamma_i^t, \quad i = P, M \end{aligned}$$

where we have indicated the normalization k_0 =1.

The consumption compensation λ needed for the Markov solution to achieve the same level of welfare as under the planner's allocation can be obtained by solving the following equation:

$$W_P = \sum_{t=0}^{\infty} \rho^t \frac{(1+\lambda)^{1-\sigma} c_{t,M}^{1-\sigma} g_{t,M}^{\theta(1-\sigma)} - 1}{1-\sigma},$$

that is,

$$\frac{1}{1-\sigma}\left[\frac{\chi_P^{1-\sigma}\phi_P^{1-\sigma}}{1-\rho\,\gamma_P^{(1-\sigma)(1+\theta)}} - \frac{1}{1-\rho}\right] = \frac{1}{1-\sigma}\left[\frac{(1+\lambda)^{1-\sigma}\chi_M^{1-\sigma}\phi_M^{1-\sigma}}{1-\rho\,\gamma_M^{(1-\sigma)(1+\theta)}} - \frac{1}{1-\rho}\right],$$

and finally,

$$1+\lambda = \left[\frac{1-\rho\,\gamma_M^{(1-\sigma)(1+\theta)}}{1-\rho\,\gamma_P^{(1-\sigma)(1+\theta)}}\right]^{\frac{1}{1-\sigma}} \frac{\chi_P}{\chi_M}\left(\frac{\phi_P}{\phi_M}\right)^{\theta}. \tag{28}$$

To translate this compensation into output units, we have to compute $100\lambda \frac{c_{t,M}}{y_{t,M}}$, which is the compensation shown in Figure 3.

As the risk aversion parameter changes between 1 and 5, the Markov consumption compensation falls from 45% to 3% of output. In particular, for σ = 2, the compensation that would be necessary to achieve the planner's welfare is around 8% of output. By and large, the decrease in consumption compensation is due to the decline in the value of the first factor in (28).[7]

The consumption compensation increases with θ. For σ=2, the Markov consumption compensation increases from 6% to 23% of output. Again, this increase in the consumption compensation is mainly due to the first factor in (28).[8] So, the difference in growth rates is the main determinant of the welfare loss of the Markov solution relative to the planner's solution, over and above the effects of differences in the ratios of private or public consumption to output.

8. Conclusions

We have characterized the optimal Markov-perfect fiscal policy in an endogenous growth economy with public consumption and capital in which the fiscal authority cannot commit to policy choices beyond the current period. We have considered two policy variables: a single tax on total income and the split of public resources between investment and consumption.

Under logarithmic preferences and full depreciation of capital, we can analytically characterize the optimal values of the two policy variables. With that particular specification, we show that the Markov-perfect policy coincides with the optimal Ramsey policy that would arise by imposing commitment. The optimal policy reduces to that of Barro if we assume away public consumption.

For the more general case of a CRRA utility function and less than perfect depreciation of private capital, we show the economy to be on its balanced growth path from the initial period onwards. In this case there is no closed form solution, but we compute numerical values for the Markov-perfect and the Ramsey optimal policies under parameter values calibrated to the US economy. We also explore the sensitivity of the numerical solutions to the values of three parameters: the intertemporal elasticity of substitution of consumption, the relative weight of public consumption in agents' utility function and the elasticity of output with respect to private capital. For empirically plausible parameter

[7] The first factor, which depends on growth rates, falls from 17.13 for σ=1.1, to 1.23 for σ=5. The second factor increases from 0.29 to 0.86, while the third factor initially increases from its starting value of 1.018 to 1.054, and it decreases after that to essentially its same initial level.

[8] The first factor increases from 1.72 to 3.02 as θ changes from 0.2 to 1.5. The second factor gradually decreases from 0.70 to 0.54, and the third factor shows a moderate increase, from 1.13 to 1.23

values, the income tax is higher under the Markov policy than under the Ramsey solution, and a higher proportion of public resources are devoted to consumption. Consequently, the growth rate is lower under the Markov policy than under the Ramsey solution.

The welfare loss of the Markov solution relative to the planner's allocation is mainly determined by the differences in growth rates, more than by differences in the ratios of private or public consumption to output.

The implication of our results is that if the private sector is aware of the government's inability to pledge future policy decisions, then the government should impose a slightly higher tax rate and devote a higher share of public resources to consumption, with a relatively low cost in terms of growth.

Considering a more complex tax structure, as well as non-trivial transitional dynamics in an endogenous growth model with public debt, are left as future extensions of this work.

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Appendix 1: *Proof of Proposition* 1

First order optimality conditions for the government's problem are:

- with respect to τ:

$$U_{c_t} C_{\tau_t} + U_{g_t} G_{\tau_t} + \rho V_{k_{t+1}} \left(\frac{\partial \Omega(\tau_t, \eta_t)}{\partial \tau_t} k_t - C_{\tau_t} \right) = 0 \,,$$

where:

$$\frac{\partial \Omega(\tau_t, \eta_t)}{\partial \tau_t} = (1-\tau_t)\left[(1-\eta_t)\tau_t\right]^{\frac{1-\alpha}{\alpha}} B^{1/\alpha} \frac{\tau_t - (1-\alpha)}{\alpha \tau_t (1-\tau_t)} = -\Omega(\tau_t, \eta_t)\Lambda(\tau_t) \,,$$

so that:

$$U_{c_t} C_{\tau_t} + U_{g_t} G_{\tau_t} = \rho V_{k_{t+1}} \left(\Omega(\tau_t, \eta_t)\, \Lambda(\tau_t) k_t + C_{\tau_t} \right)$$

- with respect to η:

$$U_{c_t} C_{\eta_t} + U_{g_t} G_{\eta_t} + \rho V_{k_{t+1}} \left(\frac{\partial \Omega(\tau_t, \eta_t)}{\partial \eta_t} k_t - C_{\eta_t} \right) = 0 \,,$$

where:

$$\frac{\partial \Omega(\tau_t, \eta_t)}{\partial \eta_t} = -(1-\tau_t)\left[(1-\eta_t)\tau_t\right]^{\frac{1-\alpha}{\alpha}} \frac{1-\alpha}{\alpha} \frac{1}{1-\eta_t} B^{1/\alpha} = -\Omega(\tau_t, \eta_t) \frac{1-\alpha}{\alpha} \frac{1}{1-\eta_t} \,,$$

so that:

$$U_{c_t} C_{\eta_t} + U_{g_t} G_{\eta_t} = \rho V_{k_{t+1}} \left(\Omega(\tau_t, \eta_t) \frac{1-\alpha}{\alpha} \frac{1}{1-\eta_t} k_t + C_{\eta_t} \right).$$

The envelope condition is:

$$V_{k_t} = U_{c_t} C_{k_t} + U_{c_t} \left(C_{\tau_t} \frac{\partial \tau_t}{\partial k_t} + C_{\eta_t} \frac{\partial \eta_t}{\partial k_t} \right) + U_{g_t} G_{k_t} + U_{g_t} \left(G_{\tau_t} \frac{\partial \tau_t}{\partial k_t} + G_{\eta_t} \frac{\partial \eta_t}{\partial k_t} \right) +$$
$$\rho V_{k_{t+1}} \left[1 - \delta + \Omega(\tau_t, \eta_t) - C_{k_t} + \left(\frac{\partial \Omega(\tau_t, \eta_t)}{\partial \tau_t} k_t - C_{\tau_t} \right) \frac{\partial \tau_t}{\partial k_t} + \left(\frac{\partial \Omega(\tau_t, \eta_t)}{\partial \eta_t} k_t - C_{\eta_t} \right) \frac{\partial \eta_t}{\partial k_t} \right],$$

which, after using the first order conditions derived above, it can be written as

$$V_{k_t} = U_{c_t} C_{k_t} + U_{g_t} G_{k_t} + \rho V_{k_{t+1}} \left[1 - \delta + \Omega(\tau_t, \eta_t) - C_{k_t} \right].$$

From the optimality conditions above we get,

$$\rho V_{k_{t+1}} = \frac{U_{c_t} C_{\tau_t} + U_{g_t} G_{\tau_t}}{\Omega\left(\tau_t,\eta_t\right)\Lambda\left(\tau_t\right)k_t + C_{\tau_t}},$$

$$\rho V_{k_{t+1}} = \frac{U_{c_t} C_{\eta_t} + U_{g_t} G_{\eta_t}}{\Omega\left(\tau_t,\eta_t\right)\frac{1-\alpha}{\alpha}\frac{1}{1-\eta_t}k_t + C_{\eta_t}},$$

which leads to condition (11).

Plugging the first equation into the envelope condition we get,

$$V_{k_t} = U_{c_t} C_{k_t} + U_{g_t} G_{k_t} + \frac{U_{c_t} C_{\tau_t} + U_{g_t} G_{\tau_t}}{\Omega\left(\tau_t,\eta_t\right)\Lambda\left(\tau_t\right)k_t + C_{\tau_t}}\left[1-\delta+\Omega(\tau_t,\eta_t) - C_{k_t}\right],$$

and, finally, we get equation (12):

$$\frac{U_{c_t} C_{\tau_t} + U_{g_t} G_{\tau_t}}{\Omega\left(\tau_t,\eta_t\right)\Lambda\left(\tau_t\right)k_t + C_{\tau_t}} = \rho\left[\begin{array}{l} U_{c_{t+1}} C_{k_{t+1}} + U_{g_{t+1}} G_{k_{t+1}} + \dfrac{U_{c_{t+1}} C_{\tau_{t+1}} + U_{g_{t+1}} G_{\tau_{t+1}}}{\Omega\left(\tau_{t+1},\eta_{t+1}\right)\Lambda\left(\tau_{t+1}\right)k_{t+1} + C_{\tau_{t+1}}}\times \\ \qquad\qquad\qquad\qquad \left[1-\delta+\Omega(\tau_{t+1},\eta_{t+1}) - C_{k_{t+1}}\right] \end{array}\right].$$

Appendix 2: *Optimal Ramsey policy under a CRRA utility function and incomplete depreciation of private capital*

The Ramsey optimal policy is the solution to the utility maximization problem, subject to the equilibrium conditions as constraints. Under the CRRA utility function, the Lagrangian for the Ramsey problem becomes:

$$L = \sum_{t=0}^{\infty}\rho^t \frac{c_t^{1-\sigma} g_t^{\theta(1-\sigma)} - 1}{1-\sigma} + \rho^t \mu_{1t}\left[\left(1-\delta+\Omega(\tau_t,\eta_t)\right)k_t - c_t - k_{t+1}\right] +$$
$$\rho^t \mu_{2t}\left[\eta_t\left(1-\eta_t\right)^{\frac{1-\alpha}{\alpha}}\tau_t^{1/\alpha}B^{1/\alpha}k_t - g_t\right] +$$
$$\rho^t \mu_{3t}\left[\rho c_{t+1}^{-\sigma} g_{t+1}^{\theta(1-\sigma)}\left(1-\delta+\alpha\,\Omega(\tau_{t+1},\eta_{t+1})\right) - c_t^{-\sigma} g_t^{\theta(1-\sigma)}\right].$$

Taking the derivatives with respect to $c_t, g_t, k_{t+1}, \tau_t, \eta_t$ to be equal to zero, we obtain the optimality conditions for the Ramsey problem:

$$c_t^{-\sigma} g_t^{\theta(1-\sigma)} = \mu_{1t} - \mu_{3t}\sigma c_t^{-\sigma-1} g_t^{\theta(1-\sigma)} + \sigma\mu_{3,t-1} c_t^{-\sigma-1} g_t^{\theta(1-\sigma)}\left(1-\delta+\alpha\,\Omega(\tau_t,\eta_t)\right),$$

$$\theta c_t^{-\sigma} g_t^{\theta(1-\sigma)} = \mu_{2t} - (1-\sigma)\theta c_t^{-\sigma-1} g_t^{\theta(1-\sigma)}\left[\mu_{3t} - \mu_{3,t-1}\left(1-\delta+\alpha\,\Omega(\tau_t,\eta_t)\right)\right],$$

$$\mu_{1t} = \rho\left[\mu_{1,t+1}\left(1-\delta+\Omega(\tau_{t+1},\eta_{t+1})\right) + \mu_{2,t+1}B^{1/\alpha}\tau_{t+1}^{1/\alpha}\eta_{t+1}(1-\eta_{t+1})^{\frac{1-\alpha}{\alpha}}\right],$$

$$\mu_{1t}k_t\left(\frac{1-\alpha}{\alpha}\frac{1-\tau_t}{\tau_t} - 1\right) + \mu_{2t}\eta_t k_t \frac{1}{\alpha} + \mu_{3t-1}c_t^{-\sigma} g_t^{\theta(1-\sigma)}\alpha\left(\frac{1-\alpha}{\alpha}\frac{1-\tau_t}{\tau_t} - 1\right) = 0,$$

$$-\mu_{1t}\frac{1-\alpha}{\alpha}k_t + \mu_{2t}\frac{\tau_t}{1-\tau_t}k_t\left(1-\frac{\eta_t}{\alpha}\right) - \mu_{3t-1}c_t^{-\sigma}g_t^{\theta(1-\sigma)}\left(1-\alpha\right) = 0\,.$$

Transforming the multipliers by: $\tilde{\mu}_{1t} \equiv \mu_{1t}k^{\sigma-\theta(1-\sigma)}$, $\tilde{\mu}_{2t} \equiv \mu_{2t}k^{\sigma-\theta(1-\sigma)}$, $\tilde{\mu}_{3t} \equiv \frac{\mu_{3t}}{k_t}$, and defining the rate of growth $\gamma_{t+1} = \frac{k_{t+1}}{k_t}$, the consumption to capital ratio $\chi_t = \frac{c_t}{k_t}$, and the ratio between public and private capital: $\phi_t = \frac{g_t}{k_t}$, we can get a system of equations in stationary ratios. First, from the global constraint of resources, we get an expression for the growth rate:

$$\gamma_{t+1} = 1-\delta+\Omega(\tau_t,\eta_t)-\chi_t\,.$$

Whereas from the government budget constraint, we can write the ratio of public to private capital:

$$\phi_t = B^{1/\alpha}\tau_t^{1/\alpha}\eta_t(1-\eta_t)^{\frac{1-\alpha}{\alpha}}\,.$$

From the Euler equation for the competitive equilibrium:

$$x_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_{t+1}^{\sigma-\theta(1-\sigma)} = \rho x_{t+1}^{-\sigma}\phi_{t+1}^{\theta(1-\sigma)}\left[\left(1-\delta+\alpha\,\Omega(\tau_{t+1},\eta_{t+1})\right)\right],$$

and from the set of optimality conditions above, we finally get the system of equations characterizing the optimal Ramsey policy represented in stationary ratios:

$$x_t^{-\sigma}\phi_t^{\theta(1-\sigma)} = \tilde{\mu}_{1t} - \sigma x_t^{-\sigma-1}\phi_t^{\theta(1-\sigma)}\left\{\tilde{\mu}_{3t} - \tilde{\mu}_{3t-1}\frac{1}{\gamma_t}\left[\left(1-\delta+\alpha\,\Omega(\tau_t,\eta_t)\right)\right]\right\},$$

$$\theta x_t^{1-\sigma}\phi_t^{\theta(1-\sigma)-1} = \tilde{\mu}_{2t} - (1-\sigma)\theta x_t^{-\sigma}\phi_t^{\theta(1-\sigma)-1}\left\{\tilde{\mu}_{3t} - \tilde{\mu}_{3t-1}\frac{1}{\gamma_t}\left[\left(1-\delta+\alpha\,\Omega(\tau_t,\eta_t)\right)\right]\right\},$$

$$\tilde{\mu}_{1t} = \rho\gamma_{t+1}^{\theta(1-\sigma)-\sigma}\left[\tilde{\mu}_{1t+1}\left(1-\delta+\Omega(\tau_{t+1},\eta_{t+1})\right) + \tilde{\mu}_{2t+1}B^{1/\alpha}\tau_{t+1}^{1/\alpha}(1-\eta_{t+1})^{\frac{1-\alpha}{\alpha}}\right],$$

$$\tilde{\mu}_{1t}\left(\frac{1-\alpha}{\alpha}\frac{1-\tau_t}{\tau_t}-1\right) + \tilde{\mu}_{2t}\eta_t\frac{1}{\alpha} + \tilde{\mu}_{3t-1}\frac{1}{\gamma_t}x_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\alpha\left(\frac{1-\alpha}{\alpha}\frac{1-\tau_t}{\tau_t}-1\right) = 0,$$

$$-\tilde{\mu}_{1t}\frac{1-\alpha}{\alpha} + \tilde{\mu}_{2t}\frac{\tau_t}{1-\tau_t}\left(1-\frac{\eta_t}{\alpha}\right) - \tilde{\mu}_{3t-1}\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\left(1-\alpha\right)\frac{1}{\gamma_t} = 0.$$

Along the balanced growth path, the system of equations for the Ramsey equilibrium becomes:

$$\gamma^{\sigma-\theta(1-\sigma)} = \rho\left(1-\delta+\alpha\,\Omega(\tau,\eta)\right),$$

$$\chi = 1-\delta+\Omega(\tau,\eta)-\gamma,$$

$$\phi = B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}},$$

$$1 = \tilde{\mu}_1 x^{\sigma}\phi^{-\theta(1-\sigma)} + \tilde{\mu}_3\sigma\frac{1}{\chi}\left[\frac{1}{\gamma}\big(1-\delta+\alpha\,\Omega(\tau,\eta)\big)-1\right],$$

$$1 = \tilde{\mu}_2\frac{1}{\theta}x_t^{\sigma-1}\phi^{1-\theta(1-\sigma)} - \tilde{\mu}_3\frac{1-\sigma}{\chi}\left[\frac{1}{\gamma}\big(1-\delta+\alpha\,\Omega(\tau,\eta)\big)-1\right],$$

$$\tilde{\mu}_1\left[1-\rho\gamma^{\theta(1-\sigma)-\sigma}\big(1-\delta+\Omega(\tau,\eta)\big)\right] = \rho\gamma^{\theta(1-\sigma)-\sigma}\tilde{\mu}_2 B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}},$$

$$\left(\frac{1-\alpha}{\alpha}\frac{1-\tau}{\tau}-1\right)+\tilde{\mu}_2\eta\frac{1}{\alpha}+\tilde{\mu}_3\frac{1}{\gamma}x^{-\sigma}\phi^{\theta(1-\sigma)}\alpha\left(\frac{1-\alpha}{\alpha}\frac{1-\tau}{\tau}-1\right)=0,$$

$$-\tilde{\mu}_1\frac{1-\alpha}{\alpha}+\tilde{\mu}_2\frac{\tau}{1-\tau}\left(1-\frac{\eta}{\alpha}\right)-\tilde{\mu}_3\chi^{-\sigma}\phi^{\theta(1-\sigma)}\big(1-\alpha\big)\frac{1}{\gamma}=0\,.$$

Denoting by:

$$\Psi = \frac{1}{\gamma}\big[1-\delta+\alpha\,\Omega(\tau,\eta)\big]-1,\; F = 1-\delta+\Omega(\tau,\eta),\text{ and }\Gamma = \frac{1-\rho\gamma^{\theta(1-\sigma)-\sigma}F}{\rho\gamma^{\theta(1-\sigma)-\sigma}\phi},$$

we characterize the balanced growth path of the Ramsey equilibrium by particularizing the system of equations above to:

$$\gamma = \left\{\rho\left[\big(1-\delta+\alpha\,\Omega(\tau,\eta)\big)\right]\right\}^{\frac{1}{\sigma-\theta(1-\sigma)}},$$

$$\chi = 1-\delta+\Omega(\tau,\eta)-\gamma,$$

$$\phi = B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}},$$

$$\tilde{\mu}_1 = \frac{1/\sigma}{\chi^{\sigma}\phi^{-\theta(1-\sigma)}\left[\frac{1}{\theta}\frac{\phi\Gamma}{\chi}+\frac{1-\sigma}{\sigma}\right]},$$

$$\tilde{\mu}_2 = \Gamma\tilde{\mu}_1,$$

$$\tilde{\mu}_3 = \frac{1-\chi^{\sigma}\phi^{-\theta(1-\sigma)}\tilde{\mu}_1}{\Psi\sigma/\chi},$$

$$\left[\tilde{\mu}_1+\tilde{\mu}_3\frac{1}{\gamma}\chi^{-\sigma}\phi^{\theta(1-\sigma)}\alpha\right]\left(\frac{1-\alpha}{\alpha}\frac{1-\tau}{\tau}-1\right)+\tilde{\mu}_2\eta\frac{1}{\alpha}=0,$$

$$-\tilde{\mu}_1\frac{1-\alpha}{\alpha}+\tilde{\mu}_2\frac{\tau}{1-\tau}\left(1-\frac{\eta}{\alpha}\right)-\tilde{\mu}_3\chi^{-\sigma}\phi^{\theta(1-\sigma)}(1-\alpha)\frac{1}{\gamma}=0,$$

a system of 8 equations in $\{\gamma,\chi,\phi,\eta,\tau,\tilde{\mu}_1,\tilde{\mu}_2,\tilde{\mu}_3\}$ that allows us to compute the balanced growth path for the Ramsey policy (τ^R,η^R) as well as the implied allocation of resources, characterized by (γ^R,χ^R,ϕ^R).

Appendix 3.- *Proofs of propositions 3, 4 and 6*

Proof of Proposition 3:

The problem solved by the government is:

$$V(k_t)=\underset{\{\tau_t,\eta_t\}}{\text{Max}}\left[\ln \mathrm{C}(k_t,\tau_t,\eta_t)+\theta\ln \mathrm{G}(k_t,\tau_t,\eta_t)+\rho\, V(k_{t+1})\right]$$

$$\text{where}\quad k_{t+1}=\Omega(\tau_t,\eta_t)k_t-\mathrm{C}(k_t,\tau_t,\eta_t),$$
$$\mathrm{C}(k_t,\tau_t,\eta_t)=(1-\rho\alpha)\,\Omega(\tau_t,\eta_t)k_t,$$
$$\mathrm{G}(k_t,\tau_t,\eta_t)=\frac{\eta_t\tau_t}{1-\tau_t}\Omega(\tau_t,\eta_t)\,k_t.$$

The first order conditions for this problem are:

$$\tau_t:\ -(1+\theta)\,\Lambda(\tau_t)+\frac{\theta}{\tau_t(1-\tau_t)}-\rho\,V_{k_{t+1}}\left[\rho\alpha\,\Omega(\tau_t,\eta_t)\,\Lambda(\tau_t)k_t\right]=0\,, \qquad \text{(A3.1)}$$

$$\eta_t:\ -(1+\theta)\frac{1-\alpha}{\alpha}\frac{1}{1-\eta_t}+\frac{\theta}{\eta_t}-\rho\,V_{k_{t+1}}\left[\rho\alpha\,\Omega(\tau_t,\eta_t)\frac{1-\alpha}{\alpha}\frac{1}{1-\eta_t}k_t\right]=0\,, \qquad \text{(A3.2)}$$

From (A3.1) and (A3.2) we obtain a relationship between the optimal values of the tax rate and the government spending split in the Markov-perfect equilibrium:

$$\tau_t^M=\frac{1-\alpha}{1-\eta_t^M}\ \forall t\,. \qquad \text{(A3.3)}$$

Proof of Proposition 4:

i) To examine the dynamic properties of the Markov solution, we consider the envelope condition :

$$V_{k_t}=(1+\theta)\frac{1}{k_t}+(1+\theta)\frac{1}{\Omega(\tau_t,\eta_t)}\frac{\partial\Omega(\tau_t,\eta_t)}{\partial\tau_t}\left(\frac{\partial\tau_t}{\partial k_t}+\frac{\partial\eta_t}{\partial k_t}\right)+$$
$$\theta\left(\frac{1}{\tau_t(1-\tau_t)}\frac{\partial\tau_t}{\partial k_t}+\frac{1}{\eta_t}\frac{\partial\eta_t}{\partial k_t}\right)+\rho V_{k_{t+1}}\left[\rho\alpha\left(\Omega(\tau_t,\eta_t)+\frac{\partial\Omega(\tau_t,\eta_t)}{\partial\tau_t}\left(\frac{\partial\tau_t}{\partial k_t}+\frac{\partial\eta_t}{\partial k_t}\right)\right)\right],$$

which, using conditions (A3.1) and (A3.2), the envelope can be written as,

$$V_{k_t} = (1+\theta)\frac{1}{k_t} + \rho V_{k_{t+1}}\left[\rho\alpha\ \Omega(\tau_t,\eta_t)\right]. \qquad \text{(A3.4)}$$

Using (A3.2) and (A3.3) in (A3.4), we obtain the dynamic equation:

$$\tilde{\eta}_t - \frac{1}{\rho}\tilde{\eta}_{t-1} + \frac{1+\theta}{\rho} = 0, \qquad \text{(A3.5)}$$

where $\tilde{\eta}_t \equiv \frac{\theta\alpha}{(1-\alpha)}\frac{1-\eta_t}{\eta_t}$. The solution to the difference equation (A3.5) is unstable, since $1/\rho > 1$.

Hence the only stable solution is that $\tilde{\eta}_t$ stays constant over time, and the same applies to η_t, that is, $\eta_t = \eta, \forall t$.

ii) The ratio to private capital of each control variable can only depend on the ratio of the two state variables, $k_{p,t}/k_t$. Since this ratio is constant under the optimal Markov policy (Corollary 1), χ_t and ϕ_t will also be constant. Together with the absence of indeterminacy, this result implies that the economy lacks transitional dynamics, being on the balanced growth path from the initial period on.

iii) From (A3.5) we obtain the value of η:

$$\eta^M = \frac{\alpha\theta(1-\rho)}{1-\alpha+\theta(1-\alpha\rho)},$$

and using (A3.3), we obtain the Markov perfect optimal tax rate:[9]

$$\tau^M = 1 - \frac{\alpha(1+\rho\theta)}{1+\theta}. \square$$

Proof of proposition 6:

Particularizing the system of equations for the balanced growth path under the optimal Ramsey policy obtained in Appendix 2 to the case of a logarithmic utility function (σ=1) and full depreciation (δ=1), we obtain:

[9] Malley et al. (2002) obtain a similar expression for the Markov perfect tax rate.

$$\gamma = \rho B^{1/\alpha}(1-\tau)\alpha\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}} \Rightarrow \gamma = \rho\alpha F,$$

$$\chi = B^{1/\alpha}(1-\tau)\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}} - \gamma,$$

$$\phi = B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}},$$

$$\Omega = \frac{1}{\gamma}\left(B^{1/\alpha}(1-\tau)\alpha\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}}\right) - 1 = \frac{1}{\rho} - 1,$$

$$F = B^{1/\alpha}(1-\tau)\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}},$$

$$\Gamma = \frac{1-\rho\gamma^{-1}F}{\rho\gamma^{-1}\phi} = \frac{1-1/\alpha}{1-\rho\gamma^{-1}F},$$

$$\tilde{\mu}_1 = \frac{1}{\chi\frac{1}{\theta}\frac{\phi\Gamma}{\chi}} = \frac{\theta\rho\gamma^{-1}}{1-\rho\gamma^{-1}F} = -\frac{\theta}{B^{1/\alpha}(1-\tau)(1-\alpha)\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}}},$$

$$\tilde{\mu}_2 = \Gamma\tilde{\mu}_1 = \frac{\theta}{\phi} = -\frac{\theta}{B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}}},$$

$$\tilde{\mu}_3 = \frac{1-\chi\tilde{\mu}_1}{\left(\frac{1}{\rho}-1\right)\frac{1}{\chi}},$$

together with:

$$\left(\tilde{\mu}_1 + \tilde{\mu}_3\frac{1}{\gamma\chi}\alpha\right)\frac{1-\alpha-\tau}{\alpha\tau} + \tilde{\mu}_2\eta\frac{1}{\alpha} = 0,$$

$$-\tilde{\mu}_1\frac{1-\alpha}{\alpha} + \tilde{\mu}_2\frac{\tau}{1-\tau}\left(1-\frac{\eta}{\alpha}\right) - \tilde{\mu}_3\chi(1-\alpha)\frac{1}{\gamma} = 0.$$

Substituting the expressions for the Lagrange multipliers into the last two equations gives us:

$$\left(-\frac{\theta}{B^{1/\alpha}(1-\tau)(1-\alpha)\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}}} + \frac{1-\chi\tilde{\mu}_1}{\frac{1}{\rho}-1}\frac{1}{\gamma}\alpha\right)\frac{1-\alpha-\tau}{\alpha\tau} + \frac{\theta}{B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}}}\eta\frac{1}{\alpha} = 0,$$

$$\frac{\theta}{B^{1/\alpha}(1-\tau)(1-\alpha)\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}}}\left(\frac{1-\alpha}{\alpha} - \frac{1-\frac{1}{\alpha}}{\rho\chi^{-1}\phi}\frac{\tau}{1-\tau}\left(1-\frac{\eta}{\alpha}\right)\right) - \frac{1-\chi\tilde{\mu}_1}{\left(\frac{1}{\rho}-1\right)\gamma}(1-\alpha) = 0.$$

Finally leading to the system:

$$-\frac{\theta}{1-\alpha}+\frac{1+\frac{1-\rho\alpha}{1-\alpha}\theta}{1-\rho}(1-\alpha-\tau^R)+\theta(1-\tau^R)=0,$$

$$\eta^R=\frac{\theta\alpha(1-\tau^R)}{\frac{(1+\rho\theta)(1-\alpha)}{1-\rho}+(1-\tau^R)(1-\alpha)}.$$

The first equation yields the Ramsey-optimal tax rate as a function of the structural parameters *α, θ, ρ,* while the second equation gives us the associated optimal split of public resources. It is easy to see that the solution to this system is given by,

$$\tau^R=1-\frac{\alpha(1+\rho\theta)}{1+\theta},$$

$$\eta^R=\frac{\alpha\theta(1-\rho)}{1-\alpha+\theta(1-\alpha\rho)}.$$

Appendix 4.- *Optimal Markov policy*

Proof of Proposition 7:

Taking into account that $\gamma_t=(1-\tau_t)\Omega(\tau_t,\eta_t)+1-\delta-\chi_t$, and $\phi_t=\eta_t\tau_t^{1/\alpha}(1-\eta_t)^{(1-\alpha)/\alpha}B^{1/\alpha}$, we get partial derivatives:

$$\phi_{\eta_t}=\phi_t\frac{\alpha-\eta_t}{\alpha\eta_t(1-\eta_t)}; \qquad \phi_{\tau_t}=\frac{1}{\alpha\tau_t}\phi_t;$$

$$\gamma_{\tau_t}=\frac{1-\alpha-\tau_t}{\alpha\tau_t(1-\tau_t)}\Omega(\eta_t,\tau_t); \quad \gamma_{\eta_t}=-\frac{1-\alpha}{\alpha}\frac{\Omega(\eta_t,\tau_t)}{1-\eta_t}; \quad \gamma_{\chi_t}=-1,$$

Under CRRA preferences the first Generalized Euler condition (11) becomes:

$$\frac{c_t^{-\sigma}g_t^{\theta(1-\sigma)}C_{\tau_t}+\theta c_t^{1-\sigma}g_t^{\theta(1-\sigma)-1}G_{\tau_t}}{C_{\tau_t}+\Lambda(\tau_t)\Omega(\tau_t,\eta_t)k_t}=\frac{c_t^{-\sigma}g_t^{\theta(1-\sigma)}C_{\eta_t}+\theta c_t^{1-\sigma}g_t^{\theta(1-\sigma)-1}G_{\eta_t}}{C_{\eta_t}+\frac{1-\alpha}{\alpha}\frac{1}{1-\eta_t}\Omega(\tau_t,\eta_t)k_t}$$

which can be written in terms of ratios using the equalities: $C_{\tau_t}=\chi_{\tau_t}k_t;C_{\eta_t}=\chi_{\eta_t}k_t;G_{\tau_t}=\phi_{\tau_t}k_t;G_{\eta_t}=\phi_{\eta_t}k_t$, as:

$$\frac{\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}k_t^{-\sigma+\theta(1-\sigma)}\chi_{\tau_t}+\theta\chi_t^{1-\sigma}\phi_t^{\theta(1-\sigma)-1}k_t^{-\sigma+\theta(1-\sigma)}\phi_{\tau_t}}{\chi_{\tau_t}+\Lambda(\tau_t)\Omega(\tau_t,\eta_t)}=\frac{\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}k_t^{-\sigma+\theta(1-\sigma)}\chi_{\eta_t}+\theta\chi_t^{1-\sigma}\phi_t^{\theta(1-\sigma)-1}\phi_{\eta_t}\frac{1}{\phi_t}}{\chi_{\eta_t}+\frac{1-\alpha}{\alpha}\frac{1}{1-\eta_t}\Omega(\tau_t,\eta_t)}$$

and, after cancelling out the product $\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)}$ at both sides of the equality, we are left with:

$$\frac{\chi_{\tau_t}+\theta\chi_t\dfrac{1}{\alpha\tau_t}}{\chi_{\tau_t}+\Lambda(\tau_t)\Omega(\eta_t,\tau_t)}=\frac{\chi_{\eta_t}+\theta\chi_t\dfrac{\alpha-\eta_t}{\alpha\eta_t(1-\eta_t)}}{\chi_{\eta_t}+\dfrac{1-\alpha}{\alpha}\dfrac{\Omega(\eta_t,\tau_t)}{1-\eta_t}} \tag{A4.1}$$

To further elaborate on this optimality condition, we need to compute the partial derivatives of χ_t with respect to the two policy variables τ_t,η_t. To that end, we differentiate in (22) to obtain:

$$\left[-\sigma\chi_t^{-\sigma-1}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)}-\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\left(\sigma-\theta(1-\sigma)\right)\gamma_t^{\sigma-\theta(1-\sigma)-1}\right]d\chi_t+$$
$$\left[\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)-1}\gamma_t^{\sigma-\theta(1-\sigma)}\theta(1-\sigma)\frac{1}{\alpha}\frac{\phi_t}{\tau_t}+\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\left(\sigma-\theta(1-\sigma)\right)\gamma_t^{\sigma-\theta(1-\sigma)-1}\frac{1-\alpha-\tau_t}{\alpha\tau_t(1-\tau_t)}\Omega(\tau_t,\eta_t)\right]d\tau_t=0$$

so that,

$$\chi_{\tau_t}=\frac{d\chi_t}{d\tau_t}=\frac{\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)}\left[\theta(1-\sigma)\dfrac{1}{\alpha\tau_t}+\dfrac{\sigma-\theta(1-\sigma)}{\gamma_t}\dfrac{1-\alpha-\tau_t}{\alpha\tau_t(1-\tau_t)}\Omega(\tau_t,\eta_t)\right]}{\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)}\left[\sigma\dfrac{1}{\chi_t}+\left(\sigma-\theta(1-\sigma)\right)\dfrac{1}{\gamma_t}\right]}, \tag{A4.2}$$

and, similarly, we would obtain:

$$\chi_{\eta_t}=\frac{d\chi_t}{d\eta_t}=\frac{\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)}\left[\theta(1-\sigma)\dfrac{\alpha-\eta_t}{\alpha\eta_t(1-\eta_t)}-\dfrac{\sigma-\theta(1-\sigma)}{\gamma_t}\dfrac{1-\alpha}{\alpha}\dfrac{\Omega(\tau_t,\eta_t)}{1-\eta_t}\right]}{\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)}\left[\sigma\dfrac{1}{\chi_t}+\left(\sigma-\theta(1-\sigma)\right)\dfrac{1}{\gamma_t}\right]}. \tag{A4.3}$$

As we can see, the product $\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)}$ cancels out again in both partial derivatives. As a consequence, the characterization of that product that we made at (22) as a function $F(k_{p,t}/k_t)$ of the state of the economy does not play any role in the first Generalized Euler equation that relates the optimal choice of the two policy variables τ_t,η_t .

Using now the partial derivatives $\chi_{\tau_t},\chi_{\eta_t}$ in the first Generalized Euler equation (A4.1), we finally get:

$$\frac{1+\dfrac{\sigma-\theta(1-\sigma)}{\theta\gamma_t}\left[\dfrac{1-\alpha-\tau_t}{1-\tau_t}\Omega(\eta_t,\tau_t)+\chi_t\theta\right]}{1-\sigma+\dfrac{\sigma}{\theta\chi_t}\dfrac{\tau_t-(1-\alpha)}{1-\tau_t}\Omega(\eta_t,\tau_t)}=\frac{1+\dfrac{\sigma-\theta(1-\sigma)}{\theta\gamma_t}\left[\chi_t\theta+\dfrac{(1-\alpha)\eta_t}{\alpha-\eta_t}\Omega(\eta_t,\tau_t)\right]}{1-\sigma+\dfrac{\sigma}{\theta\chi_t}\dfrac{(1-\alpha)\eta_t}{\alpha-\eta_t}\Omega(\eta_t,\tau_t)}$$

that can only hold if:

$$\tau_t=\frac{1-\alpha}{1-\eta_t},\forall t. \square$$

Appendix 5. *The planner's problem under lump-sum taxes*

A planner with access to lump-sum taxes would allocate resources so as to maximize time aggregate utility with the global constraint of resources as its sole restriction, thereby solving the problem,

$$\underset{\{c_t,k_{t+1},k_{pt},g_t\}}{Max} \sum_{t=0}^{\infty} \rho^t \frac{c_t^{1-\sigma} g_t^{\theta(1-\sigma)} - 1}{1-\sigma}$$

subject to:

$$k_{t+1} - (1-\delta)k_t + c_t + g_t + k_{p,t} = Bk_t^{\alpha} k_{p,t}^{1-\alpha},$$

leading to optimality conditions:

$$\frac{c_{t+1}}{c_t} = \left\{ \rho \left[\alpha (1-\alpha)^{\frac{1-\alpha}{\alpha}} B^{1/\alpha} + (1-\delta) \right] \right\}^{\frac{1}{\sigma - \theta(1-\sigma)}},$$

that defines the rate of growth γ_P, and

$$\frac{k_{t+1}}{k_t} - (1-\delta) + \chi_t + \theta\chi_t + \left[(1-\alpha)B\right]^{1/\alpha} = B^{1/\alpha}(1-\alpha)^{\frac{1-\alpha}{\alpha}}.$$

These relationships lead to expressions for the ratios of private and public consumption to private capital:

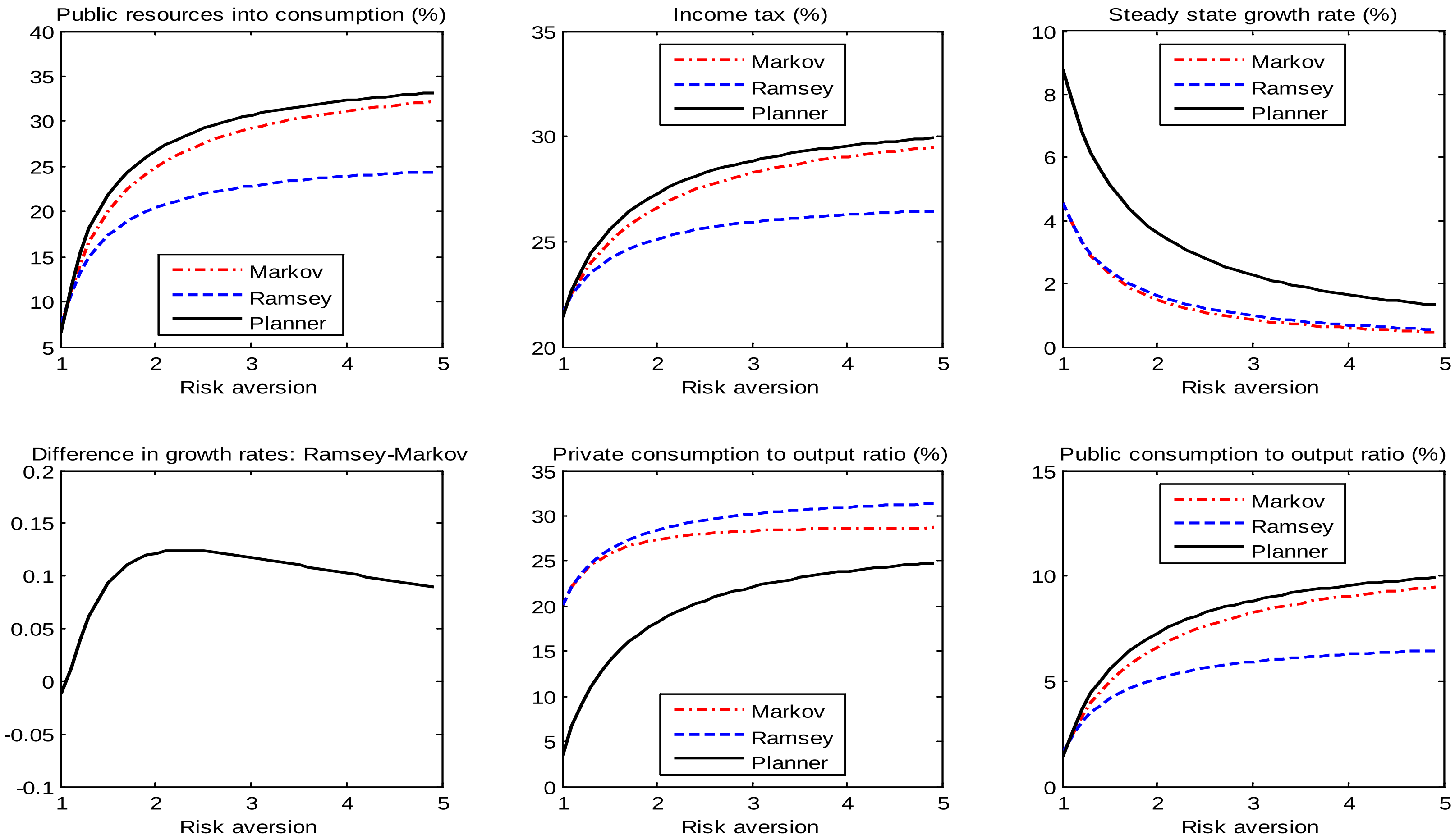
$$\chi_P = \frac{1}{1+\theta}\left[\alpha B^{1/\alpha}(1-\alpha)^{\frac{1-\alpha}{\alpha}} + (1-\delta) - \gamma_P \right],$$

$$\phi_P = \frac{g_t}{k_t} = \theta\chi_P.$$

For the purpose of comparison with the Markov and Ramsey equilibria, we can introduce a measure of the size of the public sector, as $\tau_t^P = \dfrac{g_t + k_{p,t}}{y_t}$ and the composition of public expenditures, $\eta_t^P = \dfrac{g_t}{g_t + k_{p,t}}$.

Figure 1

Values for the main variables in the economy under the three equilibrium concepts, for different values of the risk aversion parameter

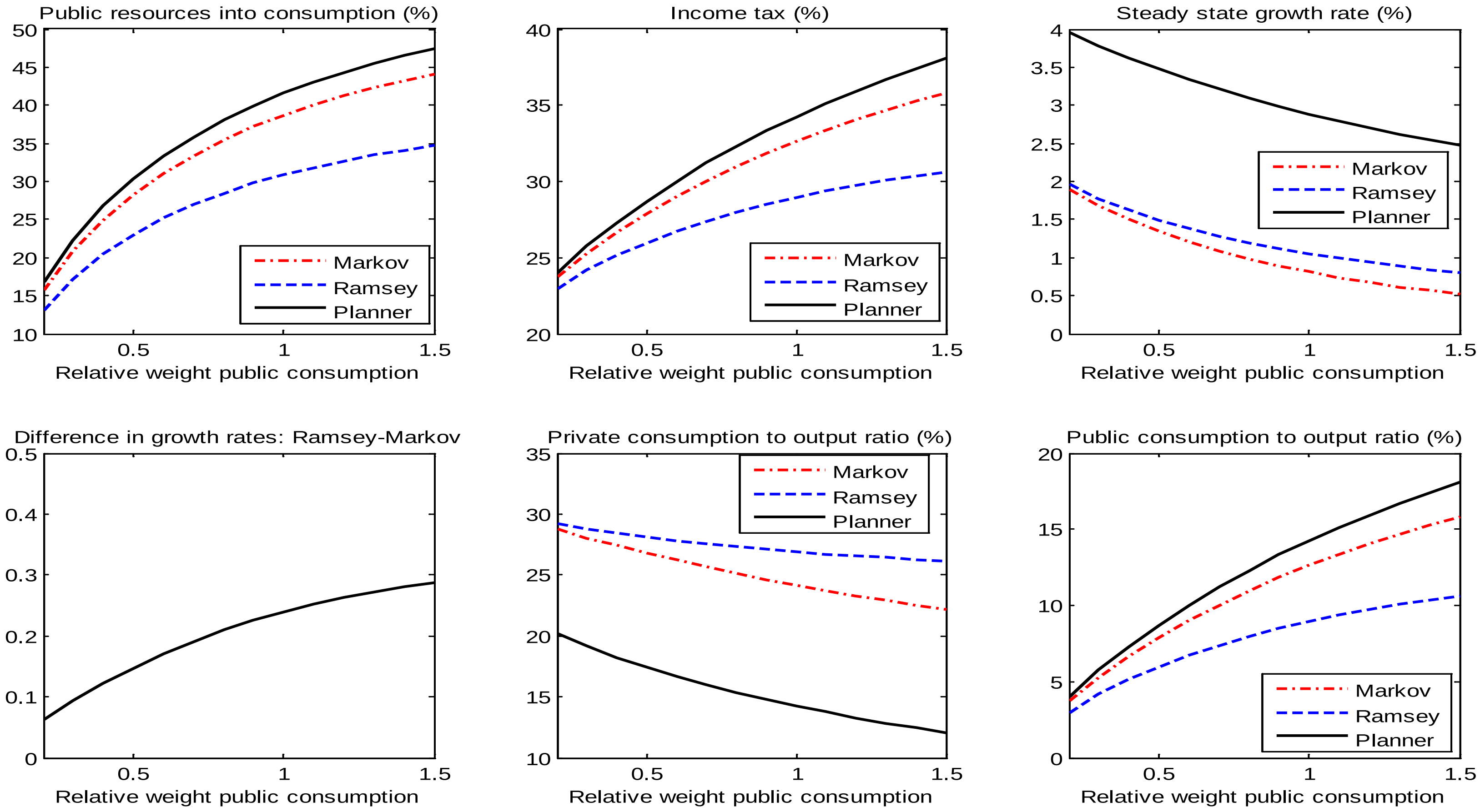


From left to right and from above to below, the graphs display: the share of public resources devoted to public consumption, the optimal income tax rate, the growth rate along the balanced path, the difference between the growth rates under the Ramsey and the Markov policies, and the ratios of private and public consumption to output.

$\theta = 0.40$	Relative weight of public consumption in utility function
$\alpha = 0.80$	Elasticity of private capital in production function
$\rho = 0.99$	Discount rate
$\delta = 0.10$	Depreciation rate
$B = 0.4555$	Productivity level

Figure 2

Values for the main variables in the economy under the three equilibrium concepts,
for different values of the relative weight of public consumption in the utility function

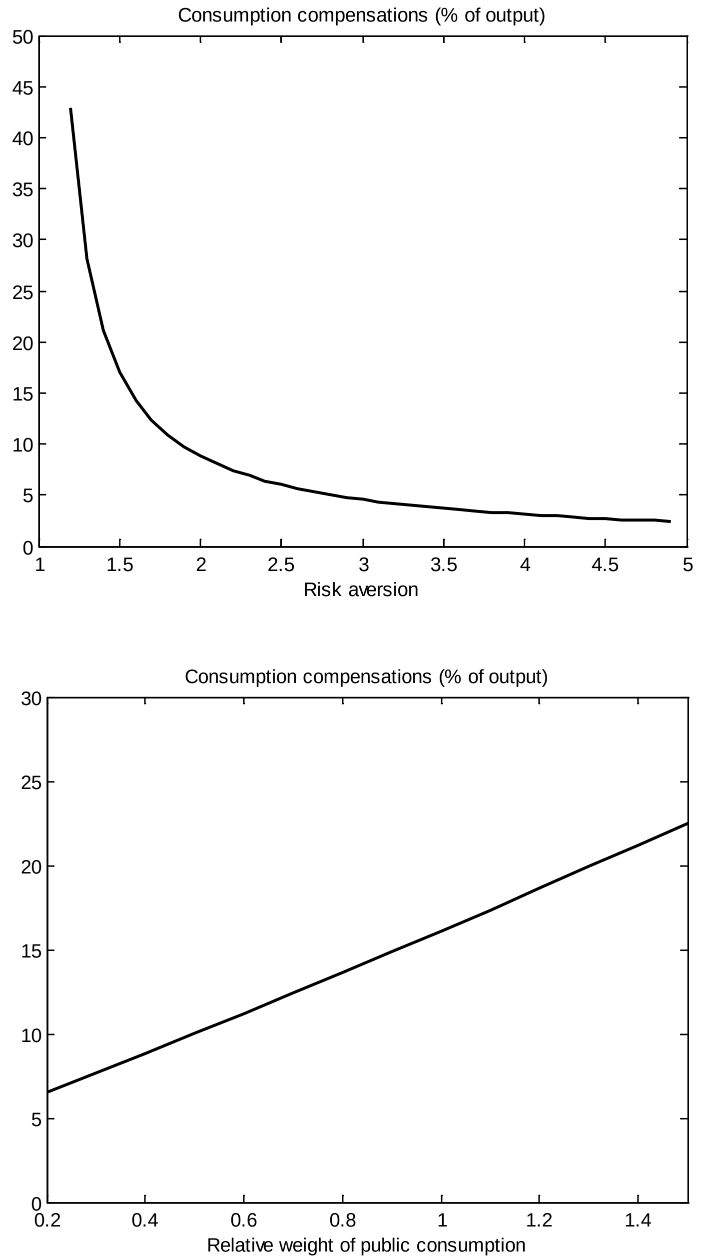


From left to right and from above to below, the graphs display: the share of public resources devoted to public consumption, the optimal income tax rate, the growth rate along the balanced path, the difference between the growth rates under the Ramsey and the Markov policies, and the ratios of private and public consumption to output.

$\sigma = 2.0$	Relative risk aversion
$\alpha = 0.80$	Elasticity of private capital in production function
$\rho = 0.99$	Discount rate
$\delta = 0.10$	Depreciation rate
$B = 0.4555$	Productivity level

Figure 3

Consumption compensation needed for the Markov policy to achieve the same level of welfare as the planner's allocation of resources



$\sigma = 2.00$	Relative risk aversion (in the second graph)
$\theta = 0.40$	Relative weight of public consumption in preferences (in the first graph)
$\alpha = 0.80$	Elasticity of private capital in production function
$\rho = 0.99$	Discount rate
$\delta = 0.10$	Depreciation rate
$B = 0.4555$	Productivity level