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On totally umbilical and minimal surfaces of the Lorentzian Heisenberg groups

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Abstract

This paper has manifold purposes. We first introduce a description of the Gauss map for submanifolds (both spacelike and timelike) of a Lorentzian ambient space and relate the conformality of the Gauss map of a surface to total umbilicity and minimality. We then focus on surfaces of the three-dimensional Heisenberg group, equipped with any of its left-invariant Lorentzian metrics. We prove that with the obvious exception of the flat case, no totally umbilical surfaces occur. On the other hand, we determine and explicitly describe several examples of minimal and constant mean curvature (CMC) surfaces.

KEYWORDS

CMC surfaces, Gauss map, Heisenberg group, minimal surfaces, totally umbilic surfaces

1 | INTRODUCTION

A submanifold M of a pseudo-Riemannian manifold is *totally umbilical* if its second fundamental form h is a multiple of the metric on M (equivalently, if its shape operator is a multiple of the identity). Consequently, totally geodesic submanifolds are totally umbilical. Totally umbilical surfaces have been extensively studied in Riemannian ambient spaces.

Besides its intrinsic interest and its relationship with total geodesicity, total umbilicity has been proved to be related to several other relevant geometric properties. Riemannian surfaces having a conformal Gauss map are either totally umbilical or minimal [11, 16]. Minimal surfaces and their generalization, namely, constant mean curvature (CMC) surfaces, are a very active research topic.

In [4], an interesting link was found between semiparallel surfaces and totally umbilical ones, where semiparallelism is a necessary condition for a surface to have a parallel second fundamental form. In fact, *a surface in a three-dimensional ambient space is semiparallel if and only if either it is flat or totally umbilical*.

Totally umbilical surfaces were classified in homogeneous Riemannian three-manifolds [12, 18]. In several cases, no totally umbilical surfaces occur. In the recent paper [6], parallel and totally geodesic hypersurfaces were classified in the four-dimensional Thurston geometry Sol_0^4.

On the other hand, to our knowledge, totally umbilical surfaces in three-dimensional homogeneous Lorentzian ambient spaces have not been investigated yet. The purpose of this paper is to undertake this study. We shall first obtain a suitable general description for the Gauss map of a nondegenerate (i.e., spacelike or timelike) submanifold in a Lorentzian ambient space. In particular, we shall prove that the Gauss map of a nondegenerate surface in a three-dimensional Lorentzian manifold is conformal if and only if the surface is either totally umbilical or minimal. We explicitly remark that these general results parallel their Riemannian analogs [11, 16], but they are obtained by independent arguments and formalized within the framework of principal bundles. We then investigate totally umbilical and minimal surfaces in the Lorentzian Heisenberg groups.

The study of totally umbilical surfaces of the Riemannian Heisenberg group was the first of such investigations in a three-dimensional homogeneous nonflat Riemannian manifold. Eberlein [7] and, more explicitly, Sanini [16] proved the nonexistence of any totally umbilical surfaces in the Riemannian Heisenberg group. We prove the Lorentzian analog of this result, investigating totally umbilical surfaces for the Heisenberg group H_3 equipped with its left-invariant Lorentzian metrics. H_3 admits three nonisometric classes of left-invariant Lorentzian metrics, known in literature as g_1, g_2, g_3 ([14, 15]). We shall restrict to the nonflat left-invariant metrics g_1 and g_2 on H_3, as g_3 is flat and so, it falls within the more general classifications given in [1] and [8].

Along such investigation of totally umbilical surfaces of the Lorentzian Heisenberg groups, we shall determine, study, and explicitly describe several classes of minimal and CMC surfaces. We point out that a first investigation of minimal surfaces for the left-invariant Lorentzian metrics on the Heisenberg group was done in [19], with the main results (Theorems 2.1, 2.4, 2.6) giving an implicit description for one-parameter subgroups corresponding to minimal surfaces.

The paper is organized in the following way. In Section 2, we shall report the needed information concerning the submanifolds in a Lorentzian ambient space and the geometry of the Lorentzian Heisemberg groups. The description in full generality of the Gauss map of a nondegenerate submanifold in a Lorentzian ambient space is achieved in Section 3, where we also characterize the conformality of the Gauss map of a nondegenerate surface in terms of total umbilicity and minimality. Sections 4 and 5 contain the nonexistence results for totally umbilical surfaces of (H_3, g_1) and (H_3, g_2), respectively, together with the geometric investigation and explicit description of several examples of minimal and CMC surfaces of these homogeneous Lorentzian spaces.

2 | PRELIMINARIES

2.1 | On pseudo-Riemannian submanifolds

We report some general essential information about submanifolds in pseudo-Riemannian ambient spaces, referring to [13] for more details. Throughout the paper, the arguments we shall apply will be purely local. We denote by $F : M \to N$ an isometric immersion of a pseudo-Riemannian submanifold M into a pseudo-Riemannian manifold N. g and g^M will, respectively, denote the metric tensor on N and its pullback to M by F. We naturally identify M with its image $F(M) \subset N$. Then, at any point $p \in M$, the tangent space T_pN decomposes orthogonally as $T_pN = T_pM \oplus (T_pM)^\perp$. The corresponding orthogonal decomposition extends to vector fields defined on M.

Denote by ∇^M and ∇ the Levi–Civita connections of M and N, respectively. They are related by the well-known *formula of Gauss*: For any vector fields X, Y on M,

$$\nabla_X Y = \nabla^M_X Y + \mathrm{II}(X, Y), \tag{2.1}$$

where $\nabla^M_X Y$ coincides with the component of $\nabla_X Y$ tangent to M, while its component normal to M defines the *second fundamental form* II of the immersion, which is a symmetric tensor defined on TM and taking values into $TM^\perp$.

In addition, the curvature tensor R of N satisfies the *equations of Gauss and Codazzi*, given, respectively, by

$$g(R(X,Y)Z,W) = g(R^M(X,Y)Z,W) + g(\mathrm{II}(X,Z),\mathrm{II}(Y,W)) - g(\mathrm{II}(X,W),\mathrm{II}(Y,Z)) \tag{2.2}$$

and

$$(R(X,Y)Z) \perp = -(\nabla^M_X \mathrm{II})(Y,Z) + (\nabla^M_Y \mathrm{II})(X,Z), \tag{2.3}$$

where X, Y, Z, W are vector fields tangent to M.

Next, the *mean curvature vector field* $\vec{\mathrm{H}}$ of M is defined by

$$\vec{\mathrm{H}} = \frac{1}{m}\sum_{i=1}^{m} \varepsilon_i \mathrm{II}(e_i, e_i),$$

where $m = \dim M$, $\{e_1, .., e_m\}$ is a local orthonormal frame on M, and $\varepsilon_i = (\mathrm{g}^M)^{-1}(e_i, e_i)$.

A point $p \in M$ is said to be *totally umbilical* if there is a normal vector $\zeta \in (T_pM)^\perp$ such that

$$\mathrm{II}_p(X_p, Y_p) = \mathrm{g}^M_p(X_p, Y_p)\zeta,$$

for all $X_p, Y_p \in T_pM$. We recall the following.

Definition 2.1. A submanifold M is said to be

1. *totally geodesic* if $\mathrm{II} = 0$;
2. *minimal* if $\vec{\mathrm{H}} = 0$;
3. *totally umbilical* if each point of M is totally umbilical;
4. *pseudo-umbilical* if $\mathrm{II} \cdot \vec{\mathrm{H}} = ||\vec{\mathrm{H}}||^2 g^M$.

Clearly, both totally umbilical and minimal submanifolds generalize the totally geodesic ones.

2.2 | Totally umbilical, minimal, and CMC surfaces

We shall now specialize and simplify the information concerning a pseudo-Riemannian submanifold to the case of a pseudo-Riemannian surface $F : M \to N$ of a three-dimensional Lorentzian space N. Let ξ denote a unit normal vector field on the surface, with $\mathrm{g}(\xi, \xi) = \varepsilon \in \{-1, 1\}$; when $\varepsilon = 1$ (respectively, -1), ξ is spacelike (respectively, timelike), and we call M a timelike (respectively, spacelike) surface, meaning that the immersion is Lorentzian (respectively, Riemannian).

Observe that ξ is uniquely determined, up to sign. Therefore, the formula of Gauss (2.1) now simplifies as

$$\nabla_X Y = \nabla^M_X Y + h(X, Y)\xi, \tag{2.4}$$

that is, $\mathrm{II}(X, Y) = h(X, Y)\xi$, where h is a symmetric bilinear form on TM.

Consequently, M is totally geodesic if and only if $h = 0$. Moreover, a point $p \in M$ is *totally umbilical* if there is a real number $\lambda(p)$ such that $h(X_p, Y_p) = \lambda(p)\mathrm{g}^M_p(X_p, Y_p)$. Then, M is totally umbilical if for every point of M, there exist a neighborhood $U \subseteq M$ and a function $\lambda : U \to \mathbb{R}$, such that

$$h(X, Y) = \lambda\, \mathrm{g}^M(X, Y), \tag{2.5}$$

for every $X, Y \in TM$. Clearly, Equation (2.5) is equivalent to requiring that the shape operator S satisfies $S = \lambda\, \mathrm{Id}$, which means that M "looks the same" in every direction.

We also recall that a surface M is said to be *parallel* (or having a parallel second fundamental form) if

$$\nabla^M h = 0.$$

For a parallel surface, all the extrinsic invariants derived from the second fundamental form are covariantly constant. In recent years, parallel surfaces of three-dimensional manifolds have been investigated and characterized in both Riemannian and Lorentzian ambient spaces (see, e.g., [2–5, 9, 10] and references therein).

Denoting by R^M and R the Riemann–Christoffel curvature tensors of M and N, respectively, the surface is said to be *semiparallel* if $R^M \cdot h = 0$, where

$$(R^M \cdot h)(X, Y, Z, W) = -h(R^M(X, Y)Z, W) - h(Z, R^M(X, Y)W),$$

for every $X, Y, Z, W \in TM$. The following result emphasizes the link between this class of surfaces and the totally umbilical ones.

Lemma2.2 ([4]). *Any parallel surface in a Lorentzian three-manifold* $(\bar{M}, g)$ *is semiparallel. Moreover, a surface* M *in a Lorentzian three-manifold is semiparallel if and only if it is either flat or totally umbilical.*

The classification of parallel surfaces of the Heisenberg group, equipped with any left-invariant Lorentzian metric, is included in the results of [4]. Consequently, the results of this paper also complete the classification of semiparallel surfaces of the Lorentzian Heisenberg groups.

Using (2.5) in (2.3), we obtain the following equation for totally umbilical surfaces:

$$g(R(X,Y)Z,\xi) = g(X,Z)Y(\lambda) - g(Y,Z)X(\lambda) \tag{2.6}$$

for every vector fields X, Y, Z tangent to M.

The mean curvature of a surface M in a three-dimensional ambient space N is determined by $\vec{\mathrm{H}} = \mathrm{H}\xi$, where

$$\mathrm{H} = \frac{1}{2}\mathrm{tr}_{g^M} h = \frac{1}{2}\sum (g^M)^{ij} h_{ij}$$

and $(g^M)^{ij}$ are the components of $(g^M)^{-1}$ with respect to a given basis of vector fields tangent to M. Then, the surface M is minimal if $\mathrm{H} = 0$, while it is *of CMC* if H is a constant. Observe that a surface in a three-dimensional ambient space is pseudo-umbilical if and only if it is either totally umbilical or minimal.

2.3 | Lorentzian metrics on the Heisenberg group

The *Heisenberg group* is the subgroup H_3 of $GL(3, \mathbb{R})$ whose matrices have the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

It is a three-dimensional, connected, simply connected, and 2-step nilpotent Lie group.

Lorentzian left-invariant metrics on H_3 were investigated in [14] and [15]. As proved in [15], any left-invariant Lorentzian metric on the Heisenberg group H_3 is isometric to one of the following metrics:

$$g_1 = -\frac{1}{\kappa^2}dx^2 + dy^2 + (xdy + dz)^2, \quad \kappa > 0, \tag{2.7}$$

$$g_2 = \frac{1}{\kappa^2}dx^2 + dy^2 - (xdy + dz)^2, \quad \kappa > 0, \tag{2.8}$$

$$g_3 = dx^2 + (xdy + dz)^2 - ((1 - x)dy - dz^2). \tag{2.9}$$

By contrast, H_3 admits just one kind of left-invariant Riemannian metric. Sanini [16] proved that differently from the three-dimensional space forms, the Riemannian Heisenberg group does not admit any totally umbilical surface.

We now describe the needed information concerning the Levi–Civita connection and Riemann curvature of the left-invariant Lorentzian metrics on H_3. We shall restrict ourselves to metrics g_1 and g_2. In fact, as proved in [15], the metric g_3 is flat. We may refer to [8] for the complete classification of totally umbilical Lorenztian surfaces of the Minkowski space (of arbitrary dimension).

With regard to the metric g_1, as described in (2.7), vector fields

$$e_1 = \partial_z, \quad e_2 = \partial_y - x\partial_z, \quad e_3 = \kappa\partial_x \tag{2.10}$$

form an orthonormal basis with e_3 timelike. Applying the *Koszul formula*, we find that with respect to this basis, the Levi–Civita connection is determined as follows:

$$\begin{aligned}
&\nabla_{e_1}e_1 = 0, && \nabla_{e_2}e_1 = \frac{\kappa}{2}e_3, && \nabla_{e_3}e_1 = \frac{\kappa}{2}e_2,\\
&\nabla_{e_1}e_2 = \frac{\kappa}{2}e_3, && \nabla_{e_2}e_2 = 0, && \nabla_{e_3}e_2 = -\frac{\kappa}{2}e_1,\\
&\nabla_{e_1}e_3 = \frac{\kappa}{2}e_2, && \nabla_{e_2}e_3 = \frac{\kappa}{2}, e_1 && \nabla_{e_3}e_3 = 0.
\end{aligned} \tag{2.11}$$

Consequently, the curvature tensor (which throughout the paper is taken with the sign convention $R(X,Y) = \nabla_{[X,Y]-[\nabla_X,\nabla_Y]}$) is determined by

$$R(e_1,e_2)e_1 = \frac{\kappa^2}{4}e_2, \qquad R(e_1,e_3)e_1 = \frac{\kappa^2}{4}e_3, \qquad R(e_2,e_3)e_2 = -\frac{3\kappa^2}{4}e_3. \tag{2.12}$$

With regard to the Lorentzian metric g_2, as described by (2.8), correspondingly we find an orthonormal basis $\{e_i\}$ with e_3 timelike, the Levi–Civita connection, and the curvature tensor, determined as in the following equations:

$$e_1 = \kappa\partial_x, \quad e_2 = -\partial_y + x\partial_z, \quad e_3 = \partial_z, \tag{2.13}$$

$$\begin{aligned}
&\nabla_{e_1}e_1 = 0, && \nabla_{e_2}e_1 = -\frac{\kappa}{2}e_3, && \nabla_{e_3}e_1 = \frac{\kappa}{2}e_2,\\
&\nabla_{e_1}e_2 = \frac{\kappa}{2}e_3, && \nabla_{e_2}e_2 = 0, && \nabla_{e_3}e_2 = -\frac{\kappa}{2}e_1,\\
&\nabla_{e_1}e_3 = \frac{\kappa}{2}e_2, && \nabla_{e_2}e_3 = -\frac{\kappa}{2}e_1, && \nabla_{e_3}e_3 = 0,
\end{aligned} \tag{2.14}$$

$$R(e_1,e_2)e_1 = -\frac{3\kappa^2}{4}e_2, \qquad R(e_1,e_3)e_1 = \frac{\kappa^2}{4}e_3, \qquad R(e_2,e_3)e_2 = \frac{\kappa^2}{4}e_3. \tag{2.15}$$

Remark 2.3. With respect to an orthonormal basis $\{e_i\}$, the *Ricci tensor* is given by $\varrho(X,Y) = \sum_{i=1}^{3} \varepsilon_i g(R(X,e_i)Y,e_i)$, where $\varepsilon_i = g(e_i,e_i)$. By the above equations (2.10), (2.12) for g_1 and (2.13), (2.15) for g_2, a straightforward calculation yields that the components of the Ricci tensor, with respect to the given orthonormal basis, are determined as follows:

$$\begin{cases}
\varrho_{11} = -\varrho_{22} = \varrho_{33} = \dfrac{\kappa^2}{2}, \quad \varrho_{ij} = 0 \ \text{ if } \ i \neq j & \text{for } g_1,\\[2ex]
\varrho_{11} = \varrho_{22} = \varrho_{33} = -\dfrac{\kappa^2}{2}, \quad \varrho_{ij} = 0 \ \text{ if } \ i \neq j & \text{for } g_2.
\end{cases}$$

Consequently, $\{e_i\}$ is a basis of eigenvectors for the Ricci operator and the Ricci eigenvalues are given by

$$\begin{cases}
\lambda_1 = \dfrac{\kappa^2}{2}, \quad \lambda_2 = \lambda_3 = -\dfrac{\kappa^2}{2} & \text{for } g_1,\\[2ex]
\lambda_1 = \lambda_2 = -\dfrac{\kappa^2}{2}, \quad \lambda_3 = \dfrac{\kappa^2}{2} & \text{for } g_2.
\end{cases}$$

The above descriptions of the Ricci curvature extend the ones given in [15] in the special case $\kappa = 1$.

3 | THE GAUSS MAPS OF SUBMANIFOLDS IN LORENTZIAN AMBIENT SPACES

3.1 | Indefinite Grassmann bundles

To our knowledge, the results in literature focus on the Riemannian case [17]. We will show how the key ideas can be extended to Lorentzian settings.

Let $\pi : ON \to N$ denote the $O(n,1)$-principal bundle of orthonormal frames on a Lorentzian manifold (N,g), of dimension $n+1$. The Levi–Civita connection, viewed as a principal connection on ON, gives an $O(n,1)$-invariant decomposition

$$T_uON = H_u \oplus V_uON, \qquad u \in ON,$$

where $V_uON = \ker d\pi_u$. Equivalently, we have a $\mathfrak{o}(n,1)$-valued 1-form ω, such that $R_\gamma^*\omega = Ad_{\gamma^{-1}} \circ \omega$, $\forall \gamma \in O(n,1)$, and $\omega(B_u^*) = B$, where

$$B_u^* = \frac{d}{dt}\bigg|_{t=0} R_{\exp(tB)}(u), \quad B \in \mathfrak{o}(n,1).$$

The relationship between these equivalent definitions is given by the fact that $H_u = \ker \omega_u$, $u \in ON$. Recall that

$$\mathfrak{o}(n,1) = \{B \in \mathfrak{M}(n+1,\mathbb{R}) : B^tJ + JB = 0\},$$

where

$$J = \mathrm{diag}(1, \dots, 1, -1).$$

We equip $\mathfrak{o}(n,1)$ with the adjoint invariant product

$$\begin{array}{rcl} \tilde{h} : \mathfrak{o}(n,1) \times \mathfrak{o}(n,1) & \to & \mathbb{R} \\ (A,B) & \mapsto & -\frac{1}{2}\mathrm{tr}(AB), \end{array}$$

which has signature $\left(\frac{n(n-1)}{2}, n\right)$. We thus define a metric $\tilde{g}$ in ON as

$$\begin{cases} \tilde{g}(X,Y) = g(d\pi_u(X), d\pi_u(Y)), & X, Y \in H_uON, \\ \tilde{g}(X,U) = 0, & X \in H_uON, \quad U \in V_uON, \\ \tilde{g}(U,V) = \tilde{h}(\omega_u(U), \omega_u(V)), & U, V \in V_uON, \end{cases}$$

of signature $\left(\frac{n(n+1)}{2}, n+1\right)$, which makes $\pi : ON \to N$ a pseudo-Riemannian submersion, and the action R_g, $g \in O(n,1)$ is by isometries of $\tilde{g}$.

We now consider, for $1 \le m < n$ and $1 < m \le n$, respectively, the subgroups

$$O(m) \times O(n-m,1) \subset O(n,1), \qquad O(n+1-m) \times O(m-1,1) \subset O(n,1).$$

In addition, we consider the $\tilde{h}$-orthogonal decompositions

$$\mathfrak{o}(n,1) = V \oplus V^\perp, \qquad V = \mathfrak{o}(m) \times \mathfrak{o}(n-m,1)$$

and

$$\mathfrak{o}(n,1) = \tilde{V} \oplus \tilde{V}^\perp, \qquad \tilde{V} = \mathfrak{o}(n+1-m) \times \mathfrak{o}(m-1,1),$$

respectively. Recall that if $A \in \mathfrak{o}(r,1)$, we have $A = (a_{ij})$, $a_{ij} = -a_{ji}$ and $a_{i,r+1} = a_{r+1,i}$ for $i, j \neq r+1$, $a_{r+1,r+1} = 0$. Then,

$$V = \left\{ A \in \mathfrak{o}(n,1) : A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, A_1 \in \mathfrak{o}(m), A_2 \in \mathfrak{o}(n-m,1) \right\},$$

$$V^\perp = \left\{ A \in \mathfrak{o}(n,1) : A = \begin{pmatrix} 0 & \bar{B}_1^t \\ B_1 & 0 \end{pmatrix}, B_1 \in \mathfrak{M}(m \times (n+1-m)) \right\},$$

$$\tilde{V} = \left\{ A \in \mathfrak{o}(n,1) : A = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}, C_1 \in \mathfrak{o}(n+1-m), C_2 \in \mathfrak{o}(m-1,1) \right\},$$

$$\tilde{V}^\perp = \left\{ A \in \mathfrak{o}(n,1) : A = \begin{pmatrix} 0 & \bar{D}_1^t \\ D_1 & 0 \end{pmatrix}, D_1 \in \mathfrak{M}((n+1-m) \times m) \right\},$$

where $\bar{B}_1^t$ (respectively, $\bar{D}_1^t$) is equal to $-B_1^t$ (respectively, to $-D_1^t$) but for the last column, which equals the last row of B_1 (respectively, D_1). These decompositions are invariant under the adjoint action since $\tilde{h}$ is adjoint invariant.

For $t \in \mathbb{R}^+$, we define the symmetric bilinear form

$$\begin{cases} Q_t(X,Y) = g(d\pi_u(X), d\pi_u(Y)), & X, Y \in H_uON, \\ Q_t(X,U) = 0, & X \in H_uON,\ U \in V_uON, \\ Q_t(U,W) = t^2\tilde{h}((pr_\perp \circ \omega)_u(U), (pr_\perp \circ \omega)_u(W)), & U, W \in V_uON, \end{cases}$$

where in both cases $pr_\perp$ denotes the canonical projection ($pr_\perp : \mathfrak{o}(n,1) \to V^\perp$ for the first decomposition and $pr_\perp : \mathfrak{o}(n,1) \to \tilde{V}^\perp$ for the second one). Of course, Q_t is not a metric since it is degenerate. In fact, its radical is given by

$$\mathrm{rad}(Q_t)_u = \left\{ B_u^* : B \in \begin{cases} V = \mathfrak{o}(m) \times \mathfrak{o}(n-m,1) \text{ in the first case,} \\ \tilde{V} = \mathfrak{o}(n+1-m) \times \mathfrak{o}(m-1,1) \text{ in the second case.} \end{cases} \right\} \tag{3.16}$$

We shall make use of the following general result.

Proposition 3.1. *Let $\mu : P \to \Sigma$ be a G-principal bundle and Q a G-invariant symmetric bilinear form such that* $\mathrm{rad}(Q) = VP$ *(i.e., the vertical tangent bundle of P). Then, there exists a metric $\bar{g}$ in Σ such that $\mu^*\bar{g} = Q$.*

Proof. A covariant tensor Q in P is the pullback of a tensor in Σ if and only if

$$i_{B^*}Q = 0 \quad \text{and} \quad L_{B^*}Q = 0,$$

for any vector $B \in \mathfrak{g}$, where the contraction i_{B^*} is with respect any entry of Q. In our case, the condition $i_{B^*}Q = 0$ follows at once from $\mathrm{rad}(Q) = VP$. On the other hand, the Lie derivative L_{B^*} is the derivative with respect to the flow of the action of G. Since Q is G-invariant, this Lie derivative vanishes, and we have that there is a symmetric tensor $\bar{g}$, such that $\mu^*\bar{g} = Q$. It is easy to see that the radical of the tensor $\bar{g}$ is zero, that is, $\bar{g}$ is a metric. □

We apply the above result to the following situation. We consider the principal bundles

$$\mu^+ : ON \to G_m^+(N) = \frac{ON}{O(m) \times O(n-m,1)},$$

$$\mu^- : ON \to G_m^-(N) = \frac{ON}{O(n+1-m) \times O(m-1,1)},$$

that is, respectively, $\Sigma = G_m^+(N)$ and $G = O(m) \times O(n-m,1)$ and $\Sigma = G_m^-(N)$ with $G = O(n+1-m) \times O(m-1,1)$. From (3.16), the radical of Q_t coincides with the vertical bundle of $OM \to \Sigma$. Furthermore, since $\tilde{h}$ is adjoint invariant, Q_t is $O(m) \times O(n-m,1)$-invariant (respectively, $O(n+1-m) \times O(m-1,1)$-invariant). Then, there exist a metric g_t^+ on $G_m^+(N)$ and a metric g_t^- on $G_m^-(N)$, such that

$$(\mu^+)^* g_t^+ = Q_t \qquad \text{and} \qquad (\mu^-)^* g_t^- = Q_t. \tag{3.17}$$

3.2 | The Gauss map(s)

Given an isometric immersion $F : (M, g^M) \to (N, g)$ of an m-dimensional Riemannian manifold, we define the *Gauss map* γ^+ of the Riemannian submanifold (M, g^M) by

$$\gamma^+ : M \to G_m^+(N),$$

$$p \mapsto F_*(T_pM),$$

which is a section of the bundle $f^*G_m^+(N) \to M$.

Proposition 3.2. *The Gauss map* γ^+ *of the Riemannian submanifold* (M, g^M) *satisfies*

$$(\gamma^+)^* g_t^+ = g + t^2\Big(\mathrm{Ric}^N + m\mathrm{II}_{\mathrm{H}} - \mathrm{Ric}^M + \mathrm{Ric}(F)\Big),$$

where $\mathrm{II}_{\mathrm{H}} = g(\mathrm{II}, \mathrm{H})$ *and* $\mathrm{Ric}(F) = C_{14} f^* R^N$ *is a contraction of the first and fourth components of* $F^* R^N$.

Proof. Given a point $p_0 \in M$, we consider an adapted orthonormal frame $(u_1, \dots, u_{n+1})$, $g(u_{n+1}, u_{n+1}) = -1$, on a neighborhood U around $F(p_0)$, such that $F_*(T_p M) = \mathrm{Span}\,(u_1, \dots, u_m)$, for any $p \in F^{-1}(U)$. This moving frame defines a section $s : U \to ON$ such that

$$\mu^+ \circ s = \gamma^+|_U.$$

Then, from (3.17), we have $(\gamma^+)^* g_t^+ = s^* \circ (\mu^+)^* g_t^+ = s^* Q_t^+$. For any u_i, u_j, $1 \le i, j \le m$, we have

$$\begin{aligned}(\gamma^+)^* g_t^+(u_i, u_j) &= Q_t(s_*(u_i), s_*(u_j))\\ &= g(\pi_*(s_*(u_i)), \pi_*(s_*(u_j)))\\ &\quad + t^2 \tilde{h}((pr_\perp \circ \omega)(s_*(u_i)), (pr_\perp \circ \omega)(s_*(u_j)))\\ &= g(u_i, u_j) + t^2 \tilde{h}((pr_\perp \circ \omega)(s_*(u_i))^v, (pr_\perp \circ \omega)(s_*(u_j))^v).\end{aligned}$$

But

$$\omega(s_*(u_i))^v = \omega(\nabla u_i) = (\omega^A_{iB}),$$

where $(\omega^A_{CB}) \in \mathfrak{o}(n, 1)$ is the matrix of the connection forms, defined by

$$\nabla_{u_B} u_C = \sum_{s=1}^{n+1} \omega^A_{CBj} u_A.$$

Then, for $i = 1, \dots, m$, we have

$$(pr_\perp \circ \omega)(s_*(u_i))^v = \begin{pmatrix} 0 & \bar{\omega}^\sigma_{ik} \\ \omega^\sigma_{ik} & 0 \end{pmatrix}, \qquad k = 1, \dots, m; \quad \sigma = m+1, \dots, n+1,$$

so that

$$\tilde{h}((pr_\perp \circ \omega)(s_*(u_i))^v, (pr_\perp \circ \omega)(s_*(u_j))^v) = \sum_{k=1}^{m} \sum_{\sigma=m+1}^{n+1} \omega^\sigma_{ik} \omega^\sigma_{jk}.$$

On the other hand,

$$\omega^A_{CB} = \varepsilon_A g(\nabla_{u_B} u_C, u_A)$$

whence,

$$\omega^\sigma_{ik} = \varepsilon_\sigma g(\nabla_{u_k} u_i, u_\sigma) = \varepsilon_\sigma \mathrm{II}^\sigma(u_i, u_k), \qquad i, k = 1, \dots, m, \;\; \sigma = m+1, \dots, n+1,$$

where we put

$$\mathrm{II}^\sigma(X, Y) = \varepsilon_\sigma g(\mathrm{II}(X, Y), u_\sigma).$$

(Observe that $\mathrm{II}(X, Y) = \sum_{\sigma=m+1}^{n+1} \mathrm{II}^\sigma(X, Y) u_\sigma$.) Thus,

$$\begin{aligned}(\gamma^+)^* g_t^+(u_i, u_j) &= g(u_i, u_j) + t^2 \sum_{k=1}^{m} \sum_{\sigma=m+1}^{n+1} \mathrm{II}^\sigma(u_i, u_k) \mathrm{II}^\sigma(u_j, u_k)\\ &= g(u_i, u_j) + t^2 \sum_{k=1}^{m} g(\mathrm{II}(u_i, u_k), \mathrm{II}(u_j, u_k)).\end{aligned}$$

Next, from the Gauss formula (2.2), contracting in $Y, Z \in TM$, we get

$$\begin{aligned}\mathrm{Ric}^N(X,W) - \sum_{\sigma=m+1}^{n+1} g(R^N(X,u_\sigma)u_\sigma, W)\\ = \mathrm{Ric}^M(X,W) + \sum_{k=1}^{m} g(\mathrm{II}(X,u_k),\mathrm{II}(u_k,W)) - mg(\vec{\mathrm{H}},\mathrm{II}(X,W)).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{m} g(\mathrm{II}(u_i,u_k),\mathrm{II}(u_j,u_k)) = \mathrm{Ric}^N(u_i,u_j) + mg(\vec{\mathrm{H}},\mathrm{II}(u_i,u_j))\\ -\mathrm{Ric}^M(u_i,u_j) - \mathrm{Ric}(F)(u_i,u_j)\end{aligned}$$

whence the conclusion follows. □

Correspondingly, if $F : (M, g^M) \to (N, g)$ is an isometric immersion of an m-dimensional Lorentzian manifold, we consider *the Gauss map* γ^- of (M, g^M), defined by

$$\begin{aligned}\gamma^- : M &\to G_m^-(N)\\ p &\mapsto F_*(T_pM),\end{aligned}$$

which defines a section of the bundle $F^*G_m^-(N) \to M$.

Given $p_0 \in M$, we now take an adapted orthonormal frame $(u_1, \dots, u_{n+1})$, with $g(u_{n+1}, u_{n+1}) = -1$, on a neighborhood U around $F(p_0)$, such that $F_*(T_pM) = \mathrm{Span}\,(u_1, \dots, u_{m-1}, u_{n+1})$, for any $p \in F^{-1}(U)$. The proof of the following result is then completely analog to the one of the above Proposition 3.2, simply taking into account that we now have to consider different ranges for the indices: $i, j, k \in \{1, .., m-1, n+1\}$ and $\sigma \in \{m, .., n\}$.

Proposition 3.3. *The Gauss map* γ^- *of the Lorentzian manifold* (M, g^M) *satisfies*

$$(\gamma^-)^* g_t^- = g + t^2\Big(\mathrm{Ric}^N + m\mathrm{II}_{\mathrm{H}} - \mathrm{Ric}^M + Ric(F)\Big),$$

where $\mathrm{II}_{\mathrm{H}} = g(\mathrm{II}, \mathrm{H})$*, and* $\mathrm{Ric}(F) = C_{14} f^* R^N$ *is a contraction of the first and fourth components of* $F^* R^N$.

Theorem 3.4. *Let* $F : (M, g^M) \to (N, g)$ *denote an isometric immersion of an* m*-dimensional Riemannian (respectively, Lorentzian) submanifold in a Lorentzian ambient space* (N, g)*. Then, any three of the following properties imply the remaining one:*

1. *The Gauss map* γ^+ *(respectively,* γ^-*) is conformal;*
2. $\mathrm{Ric}(F)$ *is a scalar multiple of* g^M*;*
3. M *is pseudo-umbilical;*
4. Ric^M *is a multiple of* g^M.

The following result now follows directly from the above Propositions 3.2 and 3.3.

As conditions (2) and (4) of Theorem 3.4 are automatically satisfied for a surface in a three-dimensional ambient space, we have at once the following.

Corollary 3.5. *Let* (M, g^M) *be a Riemannian (respectively, Lorentzian) surface in a three-dimensional Lorentzian ambient space* (N, g)*. Then, whatever the curvature of* (N, g)*, the Gauss map* γ^+ *(respectively,* γ^-*) is conformal if and only if* (M, g^M) *is pseudo-umbilical, that is, either minimal or totally umbilical.*

Remark 3.6. The existence of totally umbilical surfaces is a relatively rare phenomenon in ambient spaces of nonconstant sectional curvature. In any three-dimensional ambient space with no totally umbilical surfaces, Corollary 3.5 (together with its Riemannian analogue [11, 16]) yield that the Gauss map of a surface M is conformal if and only if M is minimal.

4 | TOTAL UMBILICITY OF SURFACES IN (H_3, g_1)

Let $F : M \to (H_3, g_1)$ denote the immersion of a surface M into H_3 and ξ the local unit normal vector field to the surface. We look for some necessary conditions on the components of ξ with respect to the orthonormal frame $\{e_1, e_2, e_3\}$ on H_3 described by (2.10), in order for M to satisfy Equation (2.6) for totally umbilical surfaces. In doing so, we shall use the following well-known general result.

Lemma 4.1. *Let $\ell : M \to \mathbb{R}$ denote an arbitrary continuous function. Then,*

$$\begin{aligned}\Omega &= \Omega_1 \cup \Omega_2 \\ &= \{p \in M : \ell \neq 0 \text{ in a neighborhood of } p\} \cup \{p \in M : \ell = 0 \text{ in a neighborhood of } p\}\end{aligned}$$

is a dense open subset of M.

We prove the following.

Proposition 4.2. *Let $F : M \to (H_3, g_1)$ be a surface with a second fundamental form satisfying (2.6) and ξ the local unit normal vector field, with $g_1(\xi, \xi) = \varepsilon \in \{-1, 1\}$. Consider the orthonormal frame $\{e_i\}$ on (H_3, g_1) introduced in (2.10). Then, there exists a dense open subset Ω of M, such that every point of Ω admits a neighborhood on which one of the following conditions holds:*

- *(I)* $\xi = be_2 + ce_3$ *for some functions* $b, c : U \to \mathbb{R}$ *and* λ *is constant on* M;
- *(II)* $\xi = ae_1 + ce_3$ *for some functions* $a, c : U \to \mathbb{R}$ *and* λ *satisfies the conditions* $e_2(\lambda) = 0$, $(ce_1 + ae_3)(\lambda) = \kappa^2 ac$ *along* M;
- *(III)* $\xi = ae_1 + be_2$ *for some functions* $a, b : U \to \mathbb{R}$ *and* λ *satisfies the conditions* $e_3(\lambda) = 0$, $(be_1 - ae_2)(\lambda) = \kappa^2 ab$ *along* M;
- *(IV)* $\xi = ae_1 + be_2 + ce_3$ *for some functions* $a, b, c : U \to \mathbb{R}$ *and* λ *satisfies the conditions* $(be_1 - ae_2)(\lambda) = \kappa^2 ab$, $(ce_1 + ae_3)(\lambda) = \kappa^2 ac$ *along* M.

Proof. Consider the normal vector field $\xi = ae_1 + be_2 + ce_3$, for some functions $a, b, c : U \to \mathbb{R}$ such that $g_1(\xi, \xi) = a^2 + b^2 - c^2 = \varepsilon = \pm 1$. Then, the following vector fields are tangent to the surface:

$$X_1 = be_1 - ae_2, \quad X_2 = ce_1 + ae_3, \quad X_3 = ce_2 + be_3.$$

By Equation (2.6), we have

$$g_1(R(X_i, X_j)X_k, \xi) = g_1(X_i, X_k)X_j(\lambda) - g_1(X_j, X_k)X_i(\lambda), \qquad i, j, k \in \{1, 2, 3\}.$$

Using (2.15) we then get

$$\kappa^2 a^3 c + bcX_1(\lambda) - (a^2 + b^2)X_2(\lambda) = 0, \tag{4.1}$$
$$\kappa^2 a^3 b + (c^2 - a^2)X_1(\lambda) - bcX_2(\lambda) = 0, \tag{4.2}$$
$$\kappa^2 a^2(b^2 - c^2) - abX_1(\lambda) + acX_2(\lambda) = 0, \tag{4.3}$$
$$\kappa^2 a^2 bc - acX_1(\lambda) - (a^2 + b^2)X_3(\lambda) = 0, \tag{4.4}$$
$$\kappa^2 a^2 b^2 - abX_1(\lambda) - bcX_3(\lambda) = 0, \tag{4.5}$$
$$\kappa^2 ab(b^2 - c^2) - (b^2 - c^2)X_1(\lambda) + acX_3(\lambda) = 0, \tag{4.6}$$
$$\kappa^2 a^2 c^2 - acX_2(\lambda) - bcX_3(\lambda) = 0, \tag{4.7}$$
$$\kappa^2 a^2 bc - abX_2(\lambda) - (c^2 - a^2)X_3(\lambda) = 0, \tag{4.8}$$
$$\kappa^2 ac(b^2 - c^2) - (b^2 - c^2)X_2(\lambda) + abX_3(\lambda) = 0. \tag{4.9}$$

Because of Equation (4.3), we shall consider separately two cases, depending on whether $a = 0$. Applying Lemma 4.1 to $\ell = a$, we have that there exists a dense open subset Ω of M, such that each point $p \in \Omega$ either admits a neighborhood where either $a = 0$ or a neighborhood where $a \neq 0$ at any point.

Case 1: $\boldsymbol{a = 0}$. In this case, (4.8) reduces to $c^2 X_3(\lambda) = 0$, which leads to two subcases: either $c = 0$ or $c \neq 0$ (applying Lemma 4.1 to $\ell = c$, we can assume these conditions in a neighborhood of each point of a dense open subset).

Subcase 1.1.: $\boldsymbol{a = c = 0}$. Then, $\xi = be_2$ and, without loss of generality, we can take $b = 1$; moreover, $\{X_1, X_3\}$ is a basis of vector fields tangent to M and from (4.4) and (4.6), we conclude that $X_1(\lambda) = X_3(\lambda) = 0$. Thus, λ is constant on M and we found a special case of (I) in the statement of the theorem.

Subcase 1.2.: $\boldsymbol{a = 0 \neq c}$. In this case, (4.8) implies that $X_3(\lambda) = 0$. Moreover, as $X_1 = be_1$, from (4.6) we have to consider two cases, depending on whether $b = 0$.

If $a = b = 0$, then by an argument similar to the one of Subcase 1.1, we get a special case of (I) of the statement. Finally, if $a =0\neq b, c$, from (4.6) we deduce $X_1(\lambda) = 0$, that is, $e_1(\lambda) = 0$ along M and since $X_3(\lambda) = 0$, we recover the case (I) of the statement.

Case 2: $\boldsymbol{a \neq 0}$. In this case, by (4.5) we have to consider separately two cases: either $b = 0$ or $b \neq 0$.

Subcase 2.1: $\boldsymbol{a \neq 0 = b}$. Now, from (4.2) we deduce $X_1(\lambda) = 0$, that is $e_2(\lambda) = 0$ along M. Moreover, from (4.1) we find $X_2(\lambda) = \kappa^2 ac$ and so, we obtain case (II) of the statement.

Subcase 2.2: $\boldsymbol{a, b \neq 0}$. In this case, because of (4.7) we shall consider separately two cases, depending on whether $c = 0$.

Case 2.2.1: $\boldsymbol{a, b \neq 0 = c}$. From (4.1) we now have $X_2(\lambda) = 0$, that is, $e_3(\lambda) = 0$ along M and from (4.2) we get $X_1(\lambda) = \kappa^2 ac$, that is, the case (III) of the statement of the Theorem.

Case 2.2.2: $\boldsymbol{a, b, c \neq 0}$. In this case, we use (4.1) and (4.2) to express $X_1(\lambda)$ and $X_2(\lambda)$ and we get the case (IV) of the statement. Observe that, as a byproduct, in this case we have $X_3(\lambda) = 0$. □

We shall now proceed case by case for the different possibilities listed in Proposition 4.2.

Case (I). In this case, as $\xi = be_2 + ce_3$, we shall consider separately two cases, depending on the sign of $b^2 - c^2 = g_1(\xi, \xi) = \varepsilon = \pm 1$. In fact, if $\varepsilon = 1$ (respectively, $\varepsilon = -1$), then M is timelike (respectively, spacelike) and there exists a smooth function $\theta : U \to \mathbb{R}$ such that $b = \cosh\theta$ (respectively, $\sinh\theta$) and $c = \sinh\theta$ (respectively, $\cosh\theta$).

In both cases, the vector fields

$$Y_1 = e_1, \qquad Y_2 = ce_2 + be_3 \tag{4.10}$$

span the tangent space to M at every point and with respect to this basis the metric on M is determined by

$$g^M = \begin{pmatrix} 1 & 0 \\ 0 & -\varepsilon \end{pmatrix}. \tag{4.11}$$

Moreover, it is worthwhile to observe that $Y_i(b) = cY_i(\theta)$ and $Y_i(c) = bY_i(\theta)$ for every $i = 1, 2$; then, a direct calculation, using (4.10) and (2.11), gives

$$\begin{aligned} \nabla_{Y_1} Y_1 &= 0, & \nabla_{Y_2} Y_1 &= \frac{\kappa}{2}\xi, \\ \nabla_{Y_1} Y_2 &= \left(Y_1(\theta) + \frac{\kappa}{2}\right)\xi, & \nabla_{Y_2} Y_2 &= Y_2(\theta)\xi. \end{aligned}$$

Because of the symmetry of the second fundamental form h, we deduce that $Y_1(\theta) = 0$ and h is determined by

$$h = \varepsilon \begin{pmatrix} 0 & \frac{\kappa}{2} \\ \frac{\kappa}{2} & Y_2(\theta) \end{pmatrix}. \tag{4.12}$$

Therefore, $h \neq \lambda g^M$ for every $\lambda \in \mathbb{R}$ and so, M cannot be a totally umbilical surface.

As $\nabla^M_{Y_i} Y_j = 0$ for all indices i, j, we deduce that M is flat. Moreover, vector fields

$$Y_1 = \partial_u, \quad Y_2 = \partial_v \tag{4.13}$$

may be taken as coordinate vector fields on M, as they commute.

From (4.11) and (4.12), we deduce that M is CMC if and only if $\partial_v \theta = \varepsilon m$ is a constant, that is,

$$\theta = \varepsilon m v + q,$$

for some real constant q. In particular, M is minimal if and only if $m = 0$. In this case, $\theta = q$ is a real constant.

When $m = 0$, denote by $F : M \to H_3,\ (u, v) \mapsto (F_1(u, v), F_2(u, v), F_3(u, v))$ the immersion of the minimal surface M in the local coordinates introduced above. By (2.10), (4.10), and (4.13), we obtain

$$\begin{aligned} (\partial_u F_1, \partial_u F_2, \partial_u F_3) = &\ (0, 0, 1), \\ (\partial_v F_1, \partial_v F_2, \partial_v F_3) = &\ (b\kappa, c, -cF_1). \end{aligned} \tag{4.14}$$

Then, by integration of (4.14), we find

$$F_1 = \kappa b v + c_1, \quad F_2 = cv + c_2, \quad F_3 = u - \kappa bc\frac{v^2}{2} - c_1 cv + c_3,$$

for some real constants c_1, c_2 and c_3. Thus, we obtained the immersion

$$F(u, v) = \begin{cases} \left(\kappa \cosh q\, v + c_1, \sinh q\, v + c_2, u - \frac{\kappa}{2} \sinh q \cosh q\, v^2 - c_1 \sinh qv + c_3\right) & \text{if } \varepsilon = 1 \\ \left(\kappa \sinh q\, v + c_1, \cosh q\, v + c_2, u - \frac{\kappa}{2} \sinh q \cosh q\, v^2 - c_1 \cosh qv + c_3\right) & \text{if } \varepsilon = -1, \end{cases}$$

for some real constants c_1, c_2, and c_3.

Setting $c_4 = \cosh q$ (respectively, $c_4 = \sinh q$) when $\varepsilon = 1$ (respectively, -1), we obtain the immersion

$$F(u, v) = \left(\kappa c_4\, v + c_1, \sqrt{c_4^2 - \varepsilon}\ v + c_2, u - \frac{\kappa}{2} c_4 \sqrt{c_4^2 - \varepsilon}\ v^2 - c_1 \sqrt{c_4^2 - \varepsilon}\ v + c_3\right),$$

for some real constants c_i, $i = 1, \ldots, 4$.

When $m \neq 0$, denote by $F : M \to H_3,\ (u, v) \mapsto (F_1(u, v), F_2(u, v), F_3(u, v))$ the immersion of the proper CMC surface M in the local coordinates introduced above. Using (2.10), (4.10), and (4.13), we obtain

$$\begin{aligned} (\partial_u F_1, \partial_u F_2, \partial_u F_3) = &\ (0, 0, 1), \\ (\partial_v F_1, \partial_v F_2, \partial_v F_3) = &\ (b(v)\kappa, c(v), -c(v)F_1), \end{aligned}$$

which, by integration, yield

$$F_1 = \varepsilon\kappa\frac{c(v)}{m} + c_1, \quad F_2 = \varepsilon\frac{b(v)}{m} + c_2, \quad F_3 = u - c_1\left(\varepsilon\frac{b(v)}{m} + c_2\right) - \frac{\kappa}{2m}\left(\frac{b(v)c(v)}{m} - v\right),$$

for some real constants c_1 and c_2. Therefore, we obtained the immersion

$$F(u, v) = \begin{cases} \left(\kappa\frac{\sinh(\theta(v))}{m} + c_1, \frac{\cosh(\theta(v))}{m} + c_2, u - c_1\left(\frac{\cosh(\theta(v))}{m} + c_2\right) - \frac{\kappa}{2m}\left(\frac{\sinh(2\theta(v))}{2m} - v\right)\right) & \text{if } \varepsilon = 1, \\ \left(-\kappa\frac{\cosh(\theta(v))}{m} + c_1, -\frac{\sinh(\theta(v))}{m} + c_2, u + c_1\left(\frac{\sinh(\theta(v))}{m} - c_2\right) - \frac{\kappa}{2m}\left(\frac{(\sinh 2\theta(v))}{2m} - v\right)\right) & \text{if } \varepsilon = -1, \end{cases}$$

for some real constants c_1, c_2 and for $\theta(v) = \varepsilon m v + q$.

Case (II). As $\xi = ae_1 + ce_3$, we consider separately two cases, depending on the sign of $a^2 - c^2 = g_1(\xi, \xi) = \varepsilon = \pm 1$. If $\varepsilon = 1$ (respectively, $\varepsilon = -1$), then M is timelike (respectively, spacelike) and there exists a smooth function $\theta : U \to \mathbb{R}$ such that $a = \cosh\theta$ (respectively, $\sinh\theta$) and $c = \sinh\theta$ (respectively, $\cosh\theta$).

In both cases, vector fields

$$Y_1 = ce_1 + ae_3, \qquad Y_2 = e_2 \tag{4.15}$$

span the tangent space to M at every point and the metric on M is determined by

$$g^M = \begin{pmatrix} -\varepsilon & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.16}$$

Moreover, we have $Y_i(a) = cY_i(\theta)$ and $Y_i(c) = aY_i(\theta)$ for every $i = 1, 2$. Then, a direct calculation, using (4.15) and (2.11), yields

$$\nabla_{Y_1} Y_1 = \kappa ac Y_2 + Y_1(\theta)\xi, \qquad \nabla_{Y_2} Y_1 = \left(\frac{\kappa}{2} + Y_2(\theta)\right)\xi,$$
$$\nabla_{Y_1} Y_2 = \varepsilon\kappa ac Y_1 - \varepsilon\frac{\kappa}{2}(a^2 + c^2)\xi, \qquad \nabla_{Y_2} Y_2 = 0.$$

By the symmetry of the second fundamental form h, we deduce that

$$Y_2(\theta) = -\varepsilon\kappa a^2 \tag{4.17}$$

and so, h is determined by

$$h = \varepsilon \begin{pmatrix} Y_1(\theta) & \frac{\kappa}{2} + Y_2(\theta) \\ \frac{\kappa}{2} + Y_2(\theta) & 0 \end{pmatrix}. \tag{4.18}$$

Then, since $h \neq \lambda g^M$ for every $\lambda \in \mathbb{R}$, M is not a totally umbilical surface.

Moreover, applying the Gauss formula (2.1), we find that the Levi–Civita connection on M is completely determined by the following possibly nonvanishing components:

$$\nabla^M_{Y_1} Y_1 = \kappa ac Y_2, \qquad \nabla^M_{Y_1} Y_2 = \varepsilon\kappa ac Y_1. \tag{4.19}$$

Consequently, M is not flat. More precisely, $R^M(Y_1, Y_2)Y_2 = -\kappa^2 a^4 Y_1$ and so, the Gaussian curvature of M is positive and given by

$$K^M = \kappa^2 a^4 > 0.$$

We now need to determine some local coordinates on M. Using Equations (4.19) for ∇^M, it is easy to check that vector fields

$$\partial_u = Y_2, \quad \partial_v = \frac{1}{a(u,v)} Y_1 \tag{4.20}$$

may be taken as coordinate vector fields on M.

From (4.16) and (4.18), we deduce that M is CMC if and only if

$$\partial_v \theta(u,v) = \varepsilon \frac{m}{a(u,v)}, \tag{4.21}$$

for some real constant m. In particular, M is minimal if and only if $m = 0$ (in this case, $\theta = \theta(u)$).

We now denote by $F : M \to H_3$, $(u,v) \mapsto (F_1(u,v), F_2(u,v), F_3(u,v))$ the immersion of the surface M in the local coordinates introduced above. By, (2.10), (4.15), and (4.20), we obtain

$$\begin{aligned} (\partial_u F_1, \partial_u F_2, \partial_u F_3) &= (0, 1, -F_1), \\ (\partial_v F_1, \partial_v F_2, \partial_v F_3) &= \left(\kappa, 0, \frac{c}{a}\right). \end{aligned} \tag{4.22}$$

Substituting a and c by their expressions in function of θ, by (4.17), we get $\partial_u \left(\frac{c}{a}\right) = -\kappa$, that is,

$$\frac{c}{a} = -\kappa u + f(v),$$

where $f = f(v)$ is a smooth function.

If $m = 0$, as $\theta = \theta(u)$, we get $a = a(u)$ and $c = c(u)$, that is, $f(v) = c_4$, for some real constant c_4. Therefore, by integration of (4.22), we obtain

$$F_1 = \kappa v + c_1, \quad F_2 = u + c_2, \quad F_3 = -\kappa uv - c_1 u + c_4 v + c_3,$$

that is, the immersion

$$F(u, v) = (\kappa v + c_1, u + c_2, -\kappa uv - c_1 u + c_4 v + c_3)$$

where $c_i, i = 1, .., 4$ are real constants.

On the other hand, when $m \neq 0$, from (4.21), we get

$$f'(v) = \partial_v\left(\frac{c}{a}\right) = \varepsilon\frac{\partial_v(\theta(u, v))}{a^2(u, v)} = \frac{m}{a^3(u, v)}.$$

Deriving the first and the last sides of the above equation with respect to u and using (4.17), we obtain

$$0 = \partial_u\left(f'(v)\right) = -\frac{3m}{a^4(u, v)}\partial_u a = \frac{3\varepsilon km}{a^2(u, v)}c,$$

which cannot occur, as $c \neq 0$. Therefore, this surface M cannot be proper CMC.

Case (III). This case can be treated similarly to the previous Case (II). We now have that $\xi = ae_1 + be_2$ is spacelike. So, M is timelike and there exists a smooth function $\theta : U \to \mathbb{R}$, such that $a = \cos\theta$ and $c = \sin\theta$. Moreover, vector fields

$$Y_1 = \sin\theta e_1 - \cos\theta e_2, \qquad Y_2 = e_3 \tag{4.23}$$

span the tangent space to M at every point and with respect to this basis, the metric on M is determined by

$$g^M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.24}$$

A direct calculation, using (4.23) and (2.11), gives

$$\begin{aligned} \nabla_{Y_1} Y_1 &= -\kappa \sin\theta\cos\theta Y_2 + Y_1(\theta)\xi, & \nabla_{Y_2} Y_1 &= \left(\tfrac{\kappa}{2} + Y_2(\theta)\right)N, \\ \nabla_{Y_1} Y_2 &= -\kappa \sin\theta\cos\theta Y_1 + \tfrac{\kappa}{2}(\sin^2\theta - \cos^2\theta)\xi, & \nabla_{Y_2} Y_2 &= 0. \end{aligned}$$

Then, by the symmetry of the second fundamental form h, we deduce that

$$Y_2(\theta) = -\kappa\cos^2\theta \tag{4.25}$$

and h is determined by

$$h = \begin{pmatrix} Y_1(\theta) & -\frac{\kappa}{2}\cos 2\theta \\ -\frac{\kappa}{2}\cos 2\theta & 0 \end{pmatrix}. \tag{4.26}$$

Then, since $h \neq \lambda g^M$ for every $\lambda \in \mathbb{R}$, M cannot be totally umbilical.

Applying the Gauss formula (2.1), we find that the Levi–Civita connection on M is completely determined by the following possibly nonvanishing components:

$$\nabla^M_{Y_1} Y_1 = -\kappa\sin\theta\cos\theta Y_2, \qquad \nabla^M_{Y_1} Y_2 = -\kappa\sin\theta\cos\theta Y_1. \tag{4.27}$$

Moreover, M is not flat. More precisely, $R^M(Y_1, Y_2)Y_2 = \kappa^2\cos^4\theta Y_1$. Therefore, the Gaussian curvature of M is positive and given by

$$K^M = \kappa^2\cos^4\theta > 0.$$

Starting from (4.27), we check that vector fields

$$\partial_u = Y_2, \quad \partial_v = \frac{1}{\cos\theta(u,v)} Y_1 \tag{4.28}$$

commute and so, they may be taken as coordinate vector fields on M.

From (4.24) and (4.26), we deduce that M is CMC if and only if

$$\partial_v \theta(u,v) = \frac{m}{\cos\theta(u,v)}, \tag{4.29}$$

for some real constant m. In particular, M is minimal if and only if $m = 0$, in which case, $\theta = \theta(u)$.

Denote by $F : M \to H_3,\ (u,v) \mapsto (F_1(u,v), F_2(u,v), F_3(u,v))$ the immersion of the surface M in the local coordinates introduced above. Using (2.10), (4.23), and (4.28), we obtain

$$\begin{aligned} (\partial_u F_1, \partial_u F_2, \partial_u F_3) = &\quad (\kappa, 0, 0), \\ (\partial_v F_1, \partial_v F_2, \partial_v F_3) = &\quad (0, 1, (\tan\theta + F_1)). \end{aligned} \tag{4.30}$$

Then, from (4.25), we have $\partial_u (\tan\theta) = -\kappa$, that is,

$$\tan\theta(u,v) = -\kappa u + f(v),$$

where $f = f(v)$ is a smooth function.

When $m = 0$, since $\theta = \theta(u)$, we get $f(v) = c_4$, for some real constant c_4. Therefore, by integration of (4.30), we obtain

$$F_1 = \kappa u + c_1, \quad F_2 = -v + c_2, \quad F_3 = (c_1 + c_4)v + c_3,$$

for some real constants $c_i, i = 1, \dots, 4$. So, we obtained the immersion

$$F(u,v) = (\kappa u + c_1, -v + c_2, (c_1 + c_4)v + c_3).$$

On the other hand, if $m \neq 0$, then from (4.29) we get

$$f'(v) = \frac{m}{\cos^3\theta(u,v)}.$$

Deriving with respect to u both sides of the above equation and using (4.25), we find

$$0 = \partial_u \big(f'(v)\big) = \frac{3m}{\cos^3\theta(u,v)} \sin\theta \partial_u \theta,$$

that cannot occur, as in this case, $0 = \partial_u (\tan\theta) = -\kappa$. So, such a surface M cannot be proper CMC.

Case (IV). In this case, $\xi = ae_1 + be_2 + ce_3$, with $a^2 + b^2 - c^2 = \varepsilon$ and we observe that the vector fields

$$Y_1 = be_1 - ae_2, \qquad Y_2 = ce_1 + ae_3 \tag{4.31}$$

span the tangent space to M at every point. The conditions we found in Case (IV) of Theorem 4.2 can be then rewritten as follows:

$$Y_1(\lambda) = \kappa^2 ab, \qquad Y_2(\lambda) = \kappa^2 ac. \tag{4.32}$$

Moreover, with respect to the basis $\{Y_1, Y_2\}$, the metric on M is determined by:

$$g^M = \begin{pmatrix} a^2 + b^2 & bc \\ bc & c^2 - a^2 \end{pmatrix}.$$

Using (4.31), (2.11), and applying the Gauss formula (2.1), we find that the Levi–Civita connection on M is completely determined by the following components:

$$\nabla^M_{Y_1} Y_1 = \varepsilon\left(Y_1(b)b - Y_1(a)\frac{c^2 - a^2}{a} + b^2 c\kappa \right) Y_1 - \varepsilon\left(Y_1(b)c - Y_1(a)\frac{bc}{a} + b(a^2 + b^2)\kappa \right) Y_2,$$

$$\begin{aligned}\nabla^M_{Y_1} Y_2 =&\varepsilon\left(\left(Y_1(c) - a^2\frac{\varkappa}{2}\right)b - \left(Y_1(a) - ac\frac{\varkappa}{2}\right)\frac{bc}{a} + b(c^2 - a^2)\frac{\varkappa}{2}\right)Y_1\\ &- \varepsilon\left(\left(Y_1(c) - a^2\frac{\varkappa}{2}\right)c - \left(Y_1(a) - ac\frac{\varkappa}{2}\right)\frac{a^2 + b^2}{a} + b^2c\frac{\varkappa}{2}\right)Y_2,\end{aligned}$$

$$\begin{aligned}\nabla^M_{Y_2} Y_1 =&\varepsilon\left(\left(Y_2(b) + a^2\frac{\varkappa}{2}\right)b - \left(Y_2(a) - ab\frac{\varkappa}{2}\right)\frac{c^2 - a^2}{a} + bc^2\frac{\varkappa}{2}\right)Y_1\\ &- \varepsilon\left(\left(Y_2(b) + a^2\frac{\varkappa}{2}\right)c - \left(Y_2(a) - ab\frac{\varkappa}{2}\right)\frac{bc}{a} + c(a^2 + b^2)\frac{\varkappa}{2}\right)Y_2,\end{aligned}$$

$$\nabla^M_{Y_2} Y_2 =\varepsilon\left(Y_2(c)b - Y_2(a)\frac{bc}{a} + c(c^2 - a^2)\varkappa\right)Y_1 - \varepsilon\left(Y_2(c)c - Y_2(a)\frac{a^2 + b^2}{a} + bc^2\varkappa\right)Y_2,$$

and the second fundamental form h is determined by

$$\begin{aligned}h(Y_1, Y_1) &= aY_1(b) - bY_1(a) + \varkappa abc,\\ h(Y_1, Y_2) &= aY_1(c) - cY_1(a) + \tfrac{\varkappa}{2}a(-a^2 + b^2 + c^2),\\ h(Y_2, Y_1) &= aY_2(b) - bY_2(a) + \tfrac{\varkappa}{2}a(a^2 + b^2 + c^2),\\ h(Y_2, Y_2) &= aY_2(c) - cY_2(a) + \varkappa abc.\end{aligned}$$

We now require that $h = \lambda g^M$, for some $\lambda \in \mathbb{R}$ and we get the following system of equations:

$$\begin{cases} aY_1(b) - bY_1(a) + \varkappa abc = \lambda(a^2 + b^2),\\ aY_1(c) - cY_1(a) + \tfrac{\varkappa}{2}a(-a^2 + b^2 + c^2) = \lambda bc,\\ aY_2(b) - bY_2(a) + \tfrac{\varkappa}{2}a(a^2 + b^2 + c^2) = \lambda bc,\\ aY_2(c) - cY_2(a) + \varkappa abc = \lambda(c^2 - a^2).\end{cases} \tag{4.33}$$

Taking into account that $Y_i(a)a - Y_i(b)b - Y_i(c)c = 0$, the solution of system (4.33) is given by

$$\begin{matrix} Y_1(a) = -\lambda b + \tfrac{\varkappa}{2}ac, & Y_1(b) = \lambda a - \tfrac{\varkappa}{2}bc, & Y_1(c) = \tfrac{\varkappa}{2}(a^2 - b^2),\\ Y_2(a) = -\lambda c + \tfrac{\varkappa}{2}ab, & Y_2(b) = -\tfrac{\varkappa}{2}(a^2 + c^2), & Y_2(c) = -\lambda a - \tfrac{\varkappa}{2}bc.\end{matrix} \tag{4.34}$$

From (4.32) and (4.34), we get

$$[Y_1, Y_2]^M(\lambda) = Y_1(Y_2(\lambda)) - Y_2(Y_1(\lambda)) = \varkappa^3 a(a^2 - b^2 + c^2). \tag{4.35}$$

On the other hand, using the above components of ∇^M and (4.34), we obtain

$$[Y_1, Y_2]^M = \left(\lambda\frac{c}{a} - \frac{\varkappa}{2}b\right)Y_1 - \left(\lambda\frac{b}{a} - \frac{\varkappa}{2}c\right)Y_2,$$

which, together with (4.32), yields

$$[Y_1, Y_2]^M(\lambda) = -\frac{\varkappa^3}{2}a(b^2 - c^2). \tag{4.36}$$

Comparing (4.35) and (4.36), we obtain

$$a^2 - b^2 + c^2 = -\frac{1}{2}(b^2 - c^2)$$

which gives

$$\frac{2}{3}\varepsilon = b^2 - c^2 = \varepsilon - a^2,$$

that is, $a \neq 0$ is a real constant and so, by (4.34), we obtain

$$0 = -\lambda b + \frac{\kappa}{2}ac, \quad 0 = -\lambda c + \frac{\kappa}{2}ab,$$

which implies $b^2 - c^2 = 0$, that is a contradiction.

Then we conclude that, also in this last case, the surface M can never be totally umbilical (in particular, totally geodesic).

The above calculations are summarized in the following main result of this section.

Theorem 4.3. *The Lorentzian Heisenberg group* (H_3, g_1) *does not admit any totally umbilical (in particular, any totally geodesic) surface. On the other hand, referring to the orthonormal frame field* $\{e_1, e_2, e_3\}$ *described in (2.10),*

- (H_3, g_1) *admits the following minimal surfaces:*

(I) *Any open subset of some (either spacelike or timelike) flat surfaces* M *normal to* $\xi = be_2 + ce_3$*, where* $b^2 - c^2 = \varepsilon = \pm 1$*. The immersion* $F : M \to H_3$ *is explicitly described by*

$$F(u, v) = \left(\kappa c_4\, v + c_1, \sqrt{c_4^2 - \varepsilon}\; v + c_2, u - \frac{\kappa}{2}c_4\sqrt{c_4^2 - \varepsilon}\; v^2 - c_1\sqrt{c_4^2 - \varepsilon}\; v + c_3\right).$$

(II) *Any open subset of some (either spacelike or timelike) surfaces* M *with positive Gaussian curvature and normal to* $\xi = ae_1 + ce_3$*, where* $a^2 - c^2 = \varepsilon = \pm 1$*. The immersion* $F : M \to H_3$ *is explicitly described by*

$$F(u, v) = (\kappa v + c_1, u + c_2, v(c_4 - \kappa u) - c_1 u + c_3).$$

(III) *Any open subset of some timelike surfaces* M *with positive Gaussian curvature and normal to* $\xi = ae_1 + be_2$*. The immersion* $F : M \to H_3$ *is explicitly described by*

$$F(u, v) = (\kappa u + c_1, -v + c_2, (c_1 + c_4)v + c_3).$$

In all the immersions described above, $c_i, i = 1, \dots, 4$ *are some real constants.*

- (H_3, g_1) *admits as proper* CMC *surfaces any open subset of some (either spacelike or timelike) flat surfaces* M *normal to* $\xi = be_2 + ce_3$*, where* $b^2 - c^2 = \varepsilon = \pm 1$*. The immersion* $F : M \to H_3$ *is explicitly described by*

$$F(u, v) = \begin{cases} \left(\kappa \frac{\sinh(\theta(v))}{m} + c_1, \frac{\cosh(\theta(v))}{m} + c_2, u - c_1\left(\frac{\cosh(\theta(v))}{m} + c_2\right) - \frac{\kappa}{2m}\left(\frac{\sinh(2\theta(v))}{2m} - v\right)\right) & \text{if } \varepsilon = 1 \\ \left(c_1 - \kappa \frac{\cosh(\theta(v))}{m}, c_2 - \frac{\sinh(\theta(v))}{m}, u + c_1\left(\frac{\sinh(\theta(v))}{m} - c_2\right) - \frac{\kappa}{2m}\left(\frac{\sinh(2\theta(v))}{2m} - v\right)\right) & \text{if } \varepsilon = -1, \end{cases}$$

for some real constants c_1, c_2 *and for* $\theta(v) = \varepsilon m v + q$*.*

Remark 4.4. Starting from their analytical representation, it is possible to give a geometric interpretation of the surfaces M described in the above Theorem 4.3. For example, in the global coordinates (x, y, z) of H_3, a flat, minimal surface M, as described in case (I), is characterized by the Cartesian equation

$$\sqrt{c_4^2 - \varepsilon}\,(x - c_1) - \kappa c_4 (y - c_2) = 0.$$

So, M may be interpreted as (a part of) a “plane” in H_3. (Observe that for $c_4 = 1$ and $c_4 = 0$, we, respectively, get, as special cases, the Lorentzian “plane” $y = c_2$ and the Riemannian “plane” $x = c_1$). By similar arguments, we have the following.

Corollary 4.5. *A minimal surface* M *of* H_3*, as described in Theorem 4.3*

(1) *in case* (I) *is (a part of) a “plane,” of equation* $\sqrt{c_4^2 - \varepsilon}\,(x - c_1) - \kappa c_4(y - c_2) = 0$ *which is Lorentzian if* $\varepsilon = 1$ *and Riemannian if* $\varepsilon = -1$*;*

(2) *in case* (II) *is (a part of) a "quadric" (more precisely, a "hyperbolic paraboloid") of equation* $z = \frac{c_4}{\kappa}(x - c_1) - c_1(y - c_2) - (x - c_1)(y - c_2) + c_3$;

(3) *in case* (III) *is (a part of) a Lorentzian "plane" of equation* $z = (c_1 + c_4)(c_2 - y) + c_3$.

Moreover, any CMC *surface* M *of* H_3 *described Theorem (4.3), is (a part of) a "cylinder" of equation*

$$(x - c_1)^2 - \kappa^2(y - c_2)^2 = -\varepsilon\frac{\kappa^2}{m^2},$$

which is Lorentzian if $\varepsilon = 1$ *and Riemannian if* $\varepsilon = -1$.

Clearly, these surfaces have different properties than the corresponding ones in a flat ambient space. For example, planes corresponding to Case (I) are flat and minimal but not totally geodesic; planes described in Case (III) are minimal and have positive Gaussian curvature.

5 | TOTAL UMBILICITY OF SURFACES IN (H_3, g_2)

Let $F : M \to (H_3, g_2)$ denote the immersion of a surface M into H_3 and ξ the unit normal vector field to the surface. Proceeding as in the previous section, we look for some necessary conditions on the components of ξ with respect to the orthonormal frame $\{e_1, e_2, e_3\}$ on H_3 described by (2.13), in order for M to be totally umbilical. The proof of the following result is completely analogous to the one of Proposition 4.2.

Proposition 5.1. *Let* $F : M \to (H_3, g_2)$ *be a surface with a second fundamental form satisfying (2.6) and* ξ *the local unit normal vector field, with* $g_2(\xi, \xi) = \varepsilon \in \{-1, 1\}$. *Consider the orthonormal frame* $\{e_i\}$ *on* (H_3, g_2) *introduced in (2.10). Then, every point of* M *admits a neighborhood* $U \subseteq M$, *on which one of the following conditions holds:*

(I) $\xi = ae_1 + be_2$ *for some functions* $a, b : U \to \mathbb{R}$ *and* λ *is constant on* M;

(II) $\xi = ae_1 + ce_3$ *for some functions* $a, c : U \to \mathbb{R}$ *and* λ *satisfies the conditions* $e_2(\lambda) = 0$, $(ce_1 + ae_3)(\lambda) = -\kappa^2 ac$ *along* M;

(III) $\xi = be_2 + ce_3$ *for some functions* $b, c : U \to \mathbb{R}$ *and* λ *satisfies the conditions* $e_3(\lambda) = 0$, $(ce_2 + be_3)(\lambda) = -\kappa^2 ab$ *along* M;

(IV) $\xi = ae_1 + be_2 + ce_3$ *for some functions* $a, b, c : U \to \mathbb{R}$ *and* λ *satisfies the conditions* $(ce_2 + be_3)(\lambda) = -\kappa^2 ab$, $(ce_1 + ae_3)(\lambda) = -\kappa^2 ac$ *along* M.

We then proceed again case by case for the different possibilities listed in Proposition 5.1.

Case (I). This case can be treated in a similar way to the Case (I) of the previous section, interchanging e_3 with e_1 and setting $a = \cos\theta$, $b = \sin\theta$. As $\xi = \cos\theta e_1 + \sin\theta e_2$, M is a timelike surface and the commuting vector fields

$$\partial_u = Y_1 = \sin\theta e_1 - \cos\theta e_2, \quad \partial_v = Y_2 = e_3 \tag{5.1}$$

span the tangent space to M and can be used as coordinate vector fields on M.

A direct calculation using (5.1) and (2.14) yields that M is flat and the symmetry of the second fundamental form reads $\partial_v(\theta) = 0$. Again, M cannot be totally umbilical. The surface M is proper CMC if and only if $\theta = mu + q$, for some real constants $m \neq 0$ and q. In this case, M is explicitly described by

$$F(u, v) = \left(-\frac{\kappa}{m}\cos(\theta(u)) + c_1, \frac{1}{m}\sin(\theta(u)) + c_2, u + \frac{\kappa}{m}\left(\frac{2\theta(u) + \sin(2\theta(u))}{4m} - \frac{c_1}{m}\sin(\theta(u))\right) + c_3\right),$$

for some real constants c_1, c_2, and c_3. Moreover, when $m = 0$, M is minimal, $\theta = q$ is a real constant, and the surface is explicitly described by

$$F(u, v) = \left(\kappa\sin(q)u + c_1, \cos(q)u + c_2, v - \frac{\kappa}{4}\sin(2q)u^2 - c_1\cos(q)u + c_3\right),$$

for some real constants c_1, c_2 and c_3.

Setting $c_4 = \sin q$ we obtain the immersion

$$F(u,v) = \left(\kappa c_4 u + c_1, \sqrt{1-c_4^2}\, u + c_2, v - \frac{\kappa}{2} c_4 \sqrt{1-c_4^2}\, u^2 - c_1\sqrt{1-c_4^2}\, u + c_3\right),$$

for some real constants c_i, $i = 1, \dots, 4$.

Case (II). It is completely analogous to the Case (II) of the previous section. Also in this case, M could be either timelike or spacelike but it has always positive Gaussian curvature $K^M = \kappa^2 c^4$. In particular, by the same argument of Case (II) in the previous section, we conclude that M cannot be totally umbilical. Moreover, M is minimal if and only if $\theta = \theta(u)$ and cannot be CMC. In this case, M is explicitly described by:

$$F(u,v) = (\kappa v + c_1, -u + c_2, \kappa uv + c_1 u + c_4 v + c_3),$$

for some real constants c_1, c_2, c_3 and c_4.

Case (III). This case can be treated as the Case (III) of the previous section, interchanging a with c and e_1 with e_3. As $\xi = be_2 + ce_3$, M can be either timelike or spacelike and we shall use opportunely the hyperbolic functions $\cosh\theta$ and $\sinh\theta$ to express function b and c. The vector fields

$$Y_1 = e_1, \qquad Y_2 = ce_2 + be_3 \tag{5.2}$$

span the tangent space to M at every point. After a computation using (5.2) and (2.14), we deduce that $\partial_u = Y_1$ and $\partial_v = c(u,v)^{-1} Y_2$ can be taken as coordinate vector fields on M. Moreover, a direct calculation gives that M has positive Gaussian curvature $K^M = \kappa^2 c^4$ and cannot be totally umbilical. Again, M is minimal if and only if $\theta = \theta(u)$ and cannot be CMC. In this case, M is explicitly described by:

$$F(u,v) = (\kappa u + c_1, -v + c_2, (c_1 + c_4)v + c_3),$$

for some real constants c_i, $i = 1, .., 4$.

Case (IV). The argument we followed is completely analogous to the one used for the Case (IV) of the previous section and leads to the same conclusion, that is, M cannot be totally umbilical in this case.

We summarize the conclusions obtained in the above cases in the following main result of this section.

Theorem 5.2. *The Lorentzian Heisenberg group (H_3, g_2) does not admit any totally umbilical (in particular, any totally geodesic) surface. On the other hand, referring to the orthonormal frame field $\{e_1, e_2, e_3\}$ described in (2.10),*

- *(H_3, g_2) admits the following minimal surfaces:*

(I) Any open subset of some (timelike) flat surfaces M normal to $\xi = \cos\theta e_1 + \sin\theta e_2$. The immersion $F : M \to H_3$ is explicitly described by

$$F(u,v) = \left(\kappa c_4 u + c_1, \sqrt{1-c_4^2}\, u + c_2, v - \frac{\kappa}{2} c_4 \sqrt{1-c_4^2}\, u^2 - c_1\sqrt{1-c_4^2}\, u + c_3\right).$$

(II) Any open subset of some (either spacelike or timelike) surfaces M with positive Gaussian curvature and normal to $\xi = ae_1 + ce_3$, where $a^2 - c^2 = \varepsilon = \pm 1$. The immersion $F : M \to H_3$ is explicitly described by

$$F(u,v) = (\kappa v + c_1, u + c_2, v(c_4 + \kappa u) + c_1 u + c_3).$$

(III) Any open subset of some (either spacelike or timelike) surfaces M with positive Gaussian curvature and normal to $\xi = be_2 + ce_3$, where $b^2 - c^2 = \varepsilon = \pm 1$. The immersion $F : M \to H_3$ is explicitly described by

$$F(u,v) = (\kappa u + c_1, -v + c_2, (c_1 + c_4)v + c_3).$$

In all the immersions given above c_i, $i = 1, \dots, 4$ are some real constants.

- *(H_3, g_2) admits as proper* CMC *surfaces any open subset of some (timelike) flat surfaces M normal to $\xi = \cos\theta e_1 + \sin\theta e_2$. The immersion $F : M \to H_3$ is explicitly described by*

$$F(u,v) = \left(-\frac{\kappa}{m}\cos(\theta(u)) + c_1, \frac{1}{m}\sin(\theta(u)) + c_2, u + \frac{\kappa}{m}\left(\frac{2\theta(u) + \sin(2\theta(u))}{4m} - \frac{c_1}{m}\sin(\theta(u))\right) + c_3\right),$$

for some real constants c_1, c_2, and c_3, $\theta(u) = mu + q$.

Also in this case we can give a geometric interpretation of the surfaces M described in the above Theorem 5.2, obtaining the following.

Corollary 5.3. *A minimal surface M of H_3, as described in Theorem 5.2*

(1) in case (I) *is (a part of) a Lorentzian "plane," of equation $\sqrt{1 - c_4^2}\,(x - c_1) - \kappa c_4(y - c_2) = 0$*

(2) in case (II) *is (a part of) a "quadric" (more precisely, a "hyperbolic paraboloid") of equation $z = \frac{c_4}{\kappa}(x - c_1) + c_1(y - c_2) + (x - c_1)(y - c_2) + c_3$;*

(3) in case (III) *is (a part of) a "plane" of equation $z = (c_1 + c_4)(c_2 - y) + c_3$.*

Moreover, any CMC *surface M of H_3 described Theorem (4.3), is (a part of) a "Lorentzian cylinder" of equation*

$$(x - c_1)^2 + \kappa^2(y - c_2)^2 = \kappa^2 m^2.$$

By the results of [16] and of Theorems 4.3 and 5.2, we reach the following general conclusion concerning totally umbilical surfaces of the Heisenberg group.

Theorem 5.4. *With the obvious exception of the flat Lorentzian metric g_3, there exist no totally umbilical (in particular, no totally geodesic) surfaces for the Heisenberg group H_3, equipped with a (either Riemannian or Lorentzian) left-invariant metric.*

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