Lecture Notes on General Relativity

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BIBLIOGRAPHY

We present a basic introduction to General Relativity prepared for a four-month, graduate-level course in the EPFL (14 weeks). The students are assumed to be familiar with Classical Mechanics (Lagrangian formulation, conservation laws, etc...) and Special Relativity, without any prior knowledge of Classical Field Theory.

The notes are thought to be pedagogical and physically oriented. Forget about things like pullbacks or bundles on the tangent space. The ubiquitous tensors will be introduced in the familiar context of classical mechanics and electromagnetism. The transition to curved spacetime will be motivated by physical concepts such as the equivalence principle or the existence of tidal forces.

Even though these notes are intended to be self-contained and comprehensible they are just a short guide to the main aspects treated in the course and do not aim to replace any the many outstanding textbooks/notes on the subject. In particular, I do not make any claim of originality. Most of the material presented here follows the treatment of the following books, that you are "kindly" encouraged to use, the more the merrier ☺ (The signs in parenthesis are the metric signature)

Textbooks

- **Gravitation and Cosmology** by S. Weinberg. A timeless classic. Excellent book. Great insight. Physical understanding of General Relativity is put ahead of its mathematical formulation, which is only introduced when needed. No exercises. $(+---)$

- **Spacetime and Geometry: An Introduction to General Relativity** by Sean M. Carroll. Very readable and up-to-date book. Certainly a good complement to Weinberg's book. Mathematical tools are presented in detail. Good discussion on black holes. Many exercises. Some lectures notes by the same author can be found in http://arxiv.org/abs/gr-qc/9712019. Strongly recommended . $(-+++)$

- **Einstein theory in a Nutshell** by A. Zee. Probably some of you know the fantastic book in QFT with a very similar title. This book about General Relativity shares the same fresh, irreverent and conversational style putting a lot on emphasis on the physical aspects. Pedagogical and ideal for someone studying General Relativity for the very first time. $(-+++)$

- **Gravity, An introduction to Einstein's General Relativity** by J.B. Hartle. An introductory course which assumes that the reader is completely new to the subject and proceeds step by step. Useful sets of downloadable Mathematica notebooks to calculate Christoffels, Riemann tensor,... for any metric you fill in http://web.physics.ucsb.edu/~gravitybook/mathematica.html. $(-+++)$

- **A first course in general relativity** by Bernard F. Schutz. Another classic. Intuitive presentation. Mathematics are presented in an easy-to-follow way. Lots of worked-out examples throughout the text. $(-+++)$

- **General Relativity, An introduction for physicists** by M.P. Hobson, G. Efstathiou and A.N. Lasenby. Similar to Schutz but slightly less geometrical. Pros: Important ideas are presented as simply as possible. Simple notation and quite pedagogical. Cons: A bit imprecise. Some notation problems. Lots of typos. $(+---)$

- **Gravitation: Foundations and Frontiers** by T. Padmanabhan. A modern approach based on Classical Field Theory with many topics not usually covered in other books. Around 200 (highly original) exercises. Some videos of the lectures by Prof. Padmanabhan can be found in http://gr-lectures-paddy.blogspot.ch $(-+++)$.

- **Advanced mechanics and General Relativity** by Joel Franklin. Very didactic book based on Lagrangian mechanics. Geometric concepts are introduced early, using examples from Newtonian mechanics and Special Relativity . The structure and predictions of General Relativity are developed in analogy with familiar physical systems. Plenty of worked-out examples and well-chosen exercises. $(-+++)$.

- **The Classical Theory of Fields** by L. D. Landau and E. F. Lifshitz. A unique book. Gradual introduction to electromagnetism and General Relativity. Good exercises. $(+---)$

- **General Relativity** by N.M.J. Woodhouse. Excellent introductory course. Short, well organized, exceptionally clear and mathematically precise. $(+---)$

- **Gravitation** by C. W. Misner, K.S. Thorne and J. A. Wheeler. Another classic. Called the "gravitating black brick" by many students due to its extension. Very complete, full of boxes, tables and citations. Multiple perspectives on General Relativity. Probably a bit overwhelming as a textbook for a first course in General Relativity, but recommended as a supplementary text. $(-+++)$

- **General Relativity** by Robert. M. Wald. More focused on the mathematical formalism than on developing the physical insight. Careful discussions of tensor formalism, the basic singularity, stability, uniqueness theorems and black hole thermodynamics. A superb treatment of some advanced topics, like ADM formulation, is presented in the appendices. Beyond the scope of this course. $(-+++)$.

- **'t Hooft's lecture notes: Introduction to General Relativity** $(-+++)$. http://www.staff.science.uu.nl/~hooft101/GtH_lectures.html

- **Jose L-F: Barbón's lecture notes: Notes on Gravitation**. Very original and different approach. As Padmanabhan's book, it can be good complement for those of you familiar with Classical Field Theory. $(-+++)$
http://members.ift.uam-csic.es/jfbarbon/Teaching_files/AG12.pdf

Problem books

You will find many exercises in the main text of these lecture notes. They are marked with the sign and are thought to be quick exercises that you should solve while studying; just to make sure that you understood what is written in the text. More complicated problems are listed at the end of each chapter and will be solved during the problem sessions. If that training is not enough and you still feel the need of learning and testing your skills,you should have a look to the following books

- **Problem book in Relativity and Gravitation** by A. R. Lightman, W. H. Press, R.H. Price and S. A. Teukolsky .
- **Cosmology and Astrophysics through Problems** by T. Padmanabhan.

NOTATION

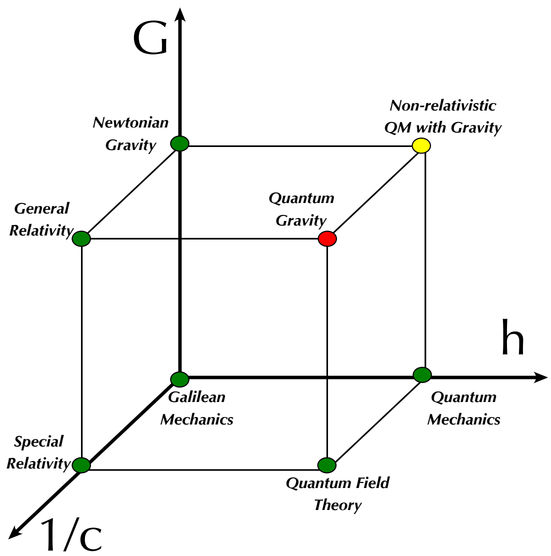
One of the first things one realizes when studying General Relativity (and when writing these lecture notes) is that there are almost so many sign conventions and notations as authors writing books about it. I will try to stick in these notes to one of the most extended choices, that of Misner, Thorne and Wheeler or Hartle. Don't worry if you don't understand what follows right now. All the concepts will appear in the future; here we are just trying to summarize our choices.

- Unless otherwise stated, Greek indices refer to spacetime coordinates, ranging from 0 to 4 to spatial coordinates
- Repeated indices are generally summed.
- The signature of the metric is chosen to be $(-,+,+,+)$,
- General metrics are denoted by $g_{\mu\nu}$
- The metric determinant is denoted by g and is negative
- The Christoffel symbols are given by $\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right)$
- Colon and semicolon denote respectively ordinary and covariant differentiation.
- The Riemann tensor is defined as $R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} + \Gamma^{\mu}_{\alpha\rho} \Gamma^{\alpha}_{\nu\sigma} - (\rho \leftrightarrow \sigma)$
- The Ricci tensor is $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$
- The Ricci scalar is $R = g^{\mu\nu} R_{\mu\nu}$
- The energy momentum tensor is defined through $\delta S_M = -\frac{1}{2} \int d^4x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu}$
- The Einstein's equations take the form $G_{\mu\nu} = +8\pi G T_{\mu\nu}$

What is General Relativity?

General Relativity is an indisputable elegant edifice of *pure geometry* that replaces the Newtonian theory and, as its name suggests, takes into account Special Relativity. It plays a central role in the description of many astrophysical objects such as black holes or pulsars as well as in our *understanding of the Universe* in its entirety. From an anthropocentric point of view, it is also a necessary ingredient for the operation of the Global Positioning System (yes, Google maps *also* knows relativity).

The relation between General Relativity and other physical theories is summarized in the following figure. For small velocities and weak gravitational fields General Relativity reduces to Newtonian gravity. On the other hand, Special Relativity is recovered in the presence of weak gravitational fields, or within sufficiently localized regions in strong gravitational fields.



NATURAL UNITS

The traditional systems of units were established by environmental, historical and physiological factors: their choice was completely arbitrary. In those standards, all the physical quantities are usually expressed in terms of the fundamental units of length, mass, time and temperature. Note however that one could take any 4 independent quantities as the fundamental units in which the physical objects are measured. For instance we could replace the units of length $[L]$, mass $[M]$ and time $[T]$ by the units of action $[S]$, velocity $[v]$ and energy $[E]$

$$[S] = [M][L]^2[T]^{-1}\,, \qquad [v] = [L][T]^{-1}\,, \qquad [E] = [M][L]^2[T]^{-2}\,. \tag{1}$$

When relativity or quantum mechanics are important, the speed of light

$$c = 299\,792\,458 \ \mathrm{m\,s^{-1}}\,, \tag{2}$$

and the reduced Planck's constant

$$\hbar = 1.054\,571\,596 \times 10^{-34} \ \mathrm{J\ s}\,, \tag{3}$$

set respectively the natural scales for action and velocity. In this case, a given physical quantity with units $[Q] = [M]^a[L]^b[T]^c$ becomes

$$[Q] = [M]^a[L]^b[T]^c = E^\alpha \hbar^\beta c^\gamma\,, \tag{4}$$

where the relations between the different coefficients are given by

$$\alpha = a - b - c\,, \qquad \beta = b + c\,, \qquad \gamma = b - 2a\,. \tag{5}$$

Using the previous expressions we obtain

$$[M] = [E]c^{-2}\,, \qquad [T] = [E]^{-1}\hbar\,, \qquad [L] = [E]^{-1}\hbar c\,. \tag{6}$$

Setting $c = \hbar = 1$, the dimensional pattern of the fundamental physical magnitudes in natural units becomes

$$[E] = [M] = [L]^{-1} = [T]^{-1}\,. \tag{7}$$

Masses, time and distances can be measured in units of energy!

Exercise

Consider an arbitrary mechanical system. Which is the dimension of the Lagrangian in natural units?

In some particular situations, as in the study of the Early Universe, it is also conventional to set the value of the Boltzmann's constant k_B to one in order to measure also temperatures in units of energy. In high energy physics and cosmology the unit of energy is usually taken to be a multiple of electron-volt[1] (eV). For instance, the masses of the electron, the proton and the neutron in natural units ($\hbar = c = 1$) become

$$m_{\rm e} = 0.511 \text{ MeV} \qquad m_{\rm p} \simeq 938.3 \text{ MeV} \qquad m_{\rm n} \simeq 939.6 \text{ MeV}\,. \tag{8}$$

Exercise

- What is the numerical value of the Compton length $l_c = \frac{\hbar}{m_e c}$ of an electron in natural units?
- Which is the typical velocity of the electron in the hydrogen atom?

Let us finally note that the numerical value of a given quantity in the traditional systems of units can be always recovered by restoring the missing $\hbar$ and c factors. Two useful relations for this kind of calculations are

$$\hbar c = 197.327053(59) \text{ MeV} \cdot \text{fm}\,, \qquad \hbar = 6.5821220(20) \times 10^{-22} \text{ MeV} \cdot \text{s}\,. \tag{9}$$

Exercise

- Derive these relations.
- The total cross section for scattering in classical electrodynamics (Thomson scattering) is given, in natural units, by

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2}\,, \tag{10}$$

where α is the fine structure constant and m_e is the electron mass. Restore the powers of $\hbar$ and c and determine the numerical value of the Thomson cross section in barns.

Hint: 1 barn= 10^{-24} cm^2.

[1] 1 eV is defined as the energy acquired by an electron after passing a difference of potential of 1 Volt.

CHAPTER 1

EUCLIDEAN SPACETIME AND NEWTONIAN PHYSICS

> Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external . . .
>
> Isaac Newton
> Scholium of the Principia

The purpose of this chapter is to remind you the basic features of the Galilean spacetime and its symmetries, which are closely related to the form taken by Newton's laws as seen by inertial observers. Although ideas presented in this chapter will be all familiar to you, the way of looking at them will be probably new. We will introduce some tensorial notation that will be useful in the future. Indeed, local differential geometry can be understood as a refinement of the tensorial methods presented here.

1.1 Galilean Relativity

Newtonian mechanics is based in two basic axioms:

1. *Principle of Relativity*: The laws of physics are the same in all the inertial frames: No experiment can measure the absolute velocity of an observer; the results of any experiment do not depend on the speed of the observer relative to other observers not involved in the experiment.

2. There exists an absolute time, which is the same for any observer.

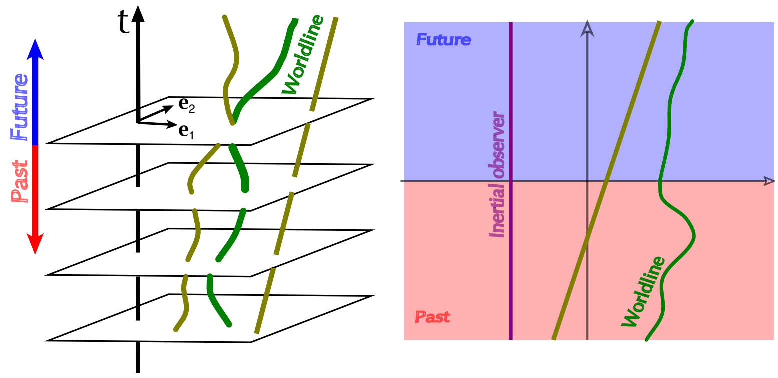


Figure 1.1: Galilean spacetime.

1.2 Euclidean spacetime: old wine in a new bottle

When formulating mechanics in an axiomatic form, Newton, based on everyday experience[1], assumed the spacetime to be Euclidean $\mathbb{E}^1 \times \mathbb{E}^3$, i.e. an intrinsically flat and orientable metric space with trivial topology and well-defined distances and angles. A physical process in this spacetime (such as the collision of two particles) is called an *event* and it is independent of the particular choice of coordinates used for its description. The spatial location of the event can be specified in Cartesian coordinates (x, y, z), in spherical coordinates (r, θ, ϕ), or making use of any 3 independent numbers obtained by a well-defined coordinate transformation. However, among all the coordinates systems that can be used in Newtonian physics, the inertial coordinate systems are privileged (and least for Newton and Galileo). An inertial frame is a frame moving freely in spacetime, free of any force, which carries *ideal clocks* and measuring rods forming an *orthonormal Cartesian coordinate system*. In such a frame, a particular event $\mathcal{P}$ is characterized by 4 coordinates: its position[2]

$$\{x^i\} = \{x, y, z\} = \{x^1, x^2, x^3\}\,, \tag{1.1}$$

and the time t at which it happens.

Time

Physical time is absolute (up to affine changes, see below) and it is used to characterize particle trajectories $x^i(t)$. The temporal separation dt between two events is well-defined, independently of their spatial separation (see below). Simultaneous events are characterized by equal time surfaces separating the future and the past of the events. Any event may cause any *simultaneous* or later event.

[1] This is, at velocities much smaller than the velocity of light.

[2] We emphasize here that we do not consider $\{x^i\}$ to be a vector since the homogeneity of space makes the choice of an origin completely arbitrary. *The distance between points is the only significant quantity.* On top of that, coordinates will no longer behave as a vector in the presence of gravity.

Space

For each spatial coordinate we define a set of orthonormal basis vectors along the x^i coordinate direction

$$\mathbf{e}_i = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}\,, \qquad \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}\,, \tag{1.2}$$

with δ_{ij} the 3 dimensional Kronecker delta

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{diag}(1,1,1)\,. \tag{1.3}$$

The infinitesimal displacement vector $d\mathbf{X}$ between two points (as any other vector in $\mathbb{E}^3$) can be expanded in terms of the basis vectors $\mathbf{e}_i$ as

$$d\mathbf{X} = \sum_{i=1}^{3} dx^i\,\mathbf{e}_i = dx^1\,\mathbf{e}_1 + dx^2\,\mathbf{e}_2 + dx^3\,\mathbf{e}_3\,, \tag{1.4}$$

with dx^i the so-called *contravariant components of the vector* in that orthonormal basis.

Einstein summation convention

Note the way in which we have located the indices in the previous equation. From now on, an index appearing twice in a product (in a superscript-subscript combination) will be understood to be automatically summed on or *contracted*. A quantity with no free tensor indices is said to be *fully contracted* The name of the pair of *contracted* indices (Latin indices $(i, j, k, \ldots)$ for the spatial coordinates or Greek ones $(\mu, \nu, \rho, \ldots)$ in 3+1 spacetime dimensions) is completely arbitrary and can be changed at will. For this reason, these indices are called *dummy indices*. Expressions with more than two repeated indices should never occur, being necessary in some cases to relabel them in order to avoid ambiguities. Non repeated indices are called *free indices* and must appear at the same level at both sides of the equations, for each independent term. As you will see, these rules are very useful, since they will allow us to reconstruct equations without any memorization, just by properly setting the indices up or down in the equation. On top of that, we will save a lot of time when writing expressions in General Relativity, which typically contain lots of indices. Using this convention, Eqs. (1.4) can be written as

$$d\mathbf{X} = \sum_{i=1}^{3} dx^i\,\mathbf{e}_i \qquad \rightarrow \qquad d\mathbf{X} = dx^i\,\mathbf{e}_i\,. \tag{1.5}$$

Exercise

Which of the following expressions do not make sense or are ambiguous according to the previous rules? Why? Restore the sums on dummy indices in the rest of equations.

$$x^i = A^i{}_j B^j{}_k x^k\,, \qquad x^i = A^j{}_k B^k{}_l x^l\,, \qquad D^i{}_j = A^i{}_k B^k{}_k C^k{}_j\,, \qquad D^i{}_j = A^i{}_k B^k{}_l C^l{}_j\,,$$

$$x^i = A^i{}_j x^j + B^i{}_k x^k\,, \qquad x^i = A^i{}_j x^j + B^i{}_j x^j\,, \qquad D^i{}_j = A^i{}_k B^k{}_l C^l{}_i\,.$$

The orthonormality of the basis vectors allows us to compute the contravariant components dx^i as the scalar product of the vector $d\mathbf{X}$ and the corresponding basis vector $\mathbf{e}_i$

$$d\mathbf{X}\cdot\mathbf{e}_i = \left(dx^j\mathbf{e}_j\right)\cdot\mathbf{e}_i = dx^j\left(\mathbf{e}_j\cdot\mathbf{e}_i\right) = dx^j\delta_{ji} \equiv dx_i\,, \tag{1.6}$$

where in the last step we have defined the so-called *covariant components* dx_i

$$dx_i \equiv \delta_{ij}dx^j\,. \tag{1.7}$$

The 3 dimensional Kronecker delta δ_{ij} allows therefore to lower (or raise) spatial indices. The definition of covariant vectors is done only for notational brevity, there is nothing deep on it. The location of the indices in Euclidean space is just a clever way of keeping into account the summation convention and does not give rise to any change in the numerical value of the different components

$$dx^i = +dx_i\,. \tag{1.8}$$

As we will see in the next chapters, this is not the general case in a non-Cartesian reference frame or in other spacetimes with undefined metric, such as the Minkowski spacetime, where the distinction between the temporal components of a *covariant* and *contravariant* vector becomes important.

The square of the infinitesimal spatial distance between two points in $\mathbb{E}^3$ is given by

$$|d\mathbf{X}|^2 \equiv dX^2 = \delta_{ij}dx^idx^j = dx^idx_i = dx^2 + dy^2 + dz^2\,, \tag{1.9}$$

where δ_{ij} plays the role of a metric in $\mathbb{E}^3$, for an orthonormal basis. The line element dX^2 is positive-definite.

1.3 Euclidean space isometry group

Requiring coordinate transformations between two inertial frames to leave the spatial (dX^2) and temporal (dt^2) distances unchanged, uniquely determines the form of these transformations. The coordinates in different frames will be distinguished by a bar over the kernel, i.e $\bar{x}^k$. Let us start by showing that the transformation must be linear. Using the chain rule, we have

$$d\bar{x}^k = \frac{\partial\bar{x}^k}{\partial x^i}dx^i\,, \tag{1.10}$$

which, imposing the invariance of line element $d\bar{X}^2 = dX^2$, implies

$$\delta_{ij} = \frac{\partial\bar{x}^k}{\partial x^i}\frac{\partial\bar{x}^l}{\partial x^j}\delta_{kl}\,. \tag{1.11}$$

Differentiating the previous expression with respect to x^p, and taking into account that δ_{ij} is a constant matrix, we get

$$\delta_{kl}\left(\frac{\partial^2\bar{x}^k}{\partial x^i\partial x^p}\frac{\partial\bar{x}^l}{\partial x^j} + \frac{\partial\bar{x}^k}{\partial x^i}\frac{\partial^2\bar{x}^l}{\partial x^p\partial x^j}\right) = 0\,. \tag{1.12}$$

Permuting ipj to pji and jip we obtain two equations

$$\delta_{kl}\left(\frac{\partial^2\bar{x}^k}{\partial x^p\partial x^j}\frac{\partial\bar{x}^l}{\partial x^i}+\frac{\partial\bar{x}^k}{\partial x^p}\frac{\partial^2\bar{x}^l}{\partial x^j\partial x^i}\right)=0\,, \tag{1.13}$$

$$\delta_{kl}\left(\frac{\partial^2\bar{x}^k}{\partial x^j\partial x^i}\frac{\partial\bar{x}^l}{\partial x^p}+\frac{\partial\bar{x}^k}{\partial x^j}\frac{\partial^2\bar{x}^l}{\partial x^i\partial x^p}\right)=0\,. \tag{1.14}$$

Subtracting (1.13) from (1.12), adding (1.14), and taking into account the symmetry of the metric and the fact that the usual derivatives commute, we get

$$\frac{\partial R^k{}_i}{\partial x^p}\delta_{kl}R^l{}_j=0\,, \tag{1.15}$$

where we have defined the matrix[3]

$$R^i{}_j\equiv\frac{\partial\bar{x}^i}{\partial x^j}\,. \tag{1.16}$$

Since the transformation $R^i{}_j$ is required to be have an inverse[4], we must conclude that

$$\frac{\partial R^k{}_i}{\partial x^p}=0\,, \tag{1.17}$$

which implies that the transformation must be linear

$$\bar{x}^i=R^i{}_jx^j+d^i\,, \tag{1.18}$$

with d^i some real and arbitrary integration constants and $R^i{}_j$ *independent of the coordinates*. Substituting back Eq.(1.18) into (1.11) we obtain the *similarity transformation*

$$\delta_{ij}=R^k{}_jR^l{}_j\delta_{kl}\,. \tag{1.19}$$

which is nothing else than the indexed version of the orthogonality condition $R^T\mathbb{I}R=R^TR=\mathbb{I}$ for a 3×3 matrix. $R^i{}_j$ is an $O(3)$ matrix! (as you probably expected). Taking the determinant at both sides of the orthogonality condition, we conclude that the determinant of an orthogonal matrix can take two different values, namely $\det R=\pm1$. Since we will be interested in rotations connected with the identity, we will restrict ourselves to *proper* rotations with determinant $\det R=+1$, i.e orientation preserving transformations

$$SO(3)=\{R|\;R^T\mathbb{I}R=\mathbb{I}\;,\det R=1\}\,. \tag{1.20}$$

Rotations with $\det R=-1$ can be obtained by applying a parity transformation $P^i{}_j=-\mathbb{I}$ in $\mathbb{E}^3$, which is also an orthogonal matrix $P^TP=\mathbb{I}$.

The laws of Newtonian mechanics are required to be *covariant*, i.e. to have the same form in each inertial frame of reference. In order to achieve so, we will make use of *tensors*, in this case *Cartesian tensors*, which have well defined transformation properties from frame to frame. As you will realize soon, these objects are the cornerstone of modern physics theories, such as Special or General Relativity. We will use them repeatedly in this course, so pay attention! We will start our trip using a concrete and familiar context for the introduction of the tensor notions: rotations $\bar{x}^i=R^i{}_jx^j$ in Euclidean space.

[3]The first index in $R^i{}_j$ labels rows and the second one labels columns.
[4]The system $\bar{\mathbf{x}}$ is not at all privileged.

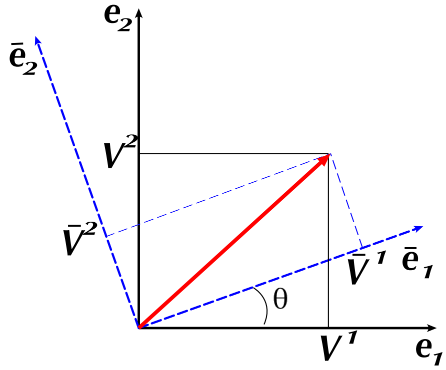


Figure 1.2: A rotation: transformation of the basis vectors and components.

1.4 Tensors in Euclidean space

1.4.1 Scalars

A scalar is single number that does not transform under a coordinate transformations (in this case rotations). Some particular examples of Galilean scalars are the spatial line element (dX), the temporal line element (dt), the 3-volume $d^3x \equiv |dx\, dy\, dz|$, the Lagrangian, the mass of a particle, its charge or any numerical constant.

Exercise
Show that the 3-volume is indeed a scalar under rotations.

If we can associate a number to all the points in some spacetime region, as for instance happens with the value of the temperature in the different points of the Earth, we say that we are dealing with a scalar field. Under coordinate transformations, it transforms as

$$\bar{\phi}\,(t, \bar{\mathbf{x}}) = \phi\,(t, \mathbf{x})\ , \qquad \text{or} \qquad \bar{\phi}\,(t, \mathbf{x}) = \phi\left(t, R^{-1}\mathbf{x}\right)\ . \tag{1.21}$$

1.4.2 Vectors

What is a vector? A vector $\mathbf{V}$ (in this case Cartesian) is an absolute geometrical object with a particular length and direction which does not depend on the choice of coordinates. The same happens with the rules of vector calculus. Concepts as the angle between two vectors can be defined independently of the coordinates. Even though there is no need of introducing the concept of *components of a vector in a given basis*, doing it is sometimes useful. Let us see what happens when we do it. Consider two orthonormal frames related for instance by a rotation of angle θ around the z axis, as illustrated in Fig 1.2. The vector $\mathbf{V}$ can be expanded in terms of the two set of basis vectors associated to this coordinate systems. In terms of the basis $\mathbf{e}_i$, the vector $\mathbf{V}$ has components V^i

$$\mathbf{V} = V^i\,\mathbf{e}_i = V^1\,\mathbf{e}_1 + V^2\,\mathbf{e}_2 + V^3\mathbf{e}_3\,, \tag{1.22}$$

while, in terms of the rotated basis $\bar{\mathbf{e}}_i$, it has different components $\bar{V}^i$

$$\mathbf{V} = \bar{V}^i\,\bar{\mathbf{e}}_i = \bar{V}^1\,\bar{\mathbf{e}}_1 + \bar{V}^2\bar{\mathbf{e}}_2 + \bar{V}^3\bar{\mathbf{e}}_3\,, \tag{1.23}$$

but the vector itself ($\mathbf{V}$) does not change. The relation between the basis vector $\bar{\mathbf{e}}_i$ and $\mathbf{e}_i$ can be easily read from the figure to get

$$\begin{pmatrix} \bar{\mathbf{e}}_1 \\ \bar{\mathbf{e}}_2 \\ \bar{\mathbf{e}}_3 \end{pmatrix}^T = \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}^T \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.24}$$

Using this relation, it is easy to write $\bar{V}$ in terms of the original basis vectors $\mathbf{e}_i$ and identify from there the transformation of the components. We obtain

$$\begin{pmatrix} \bar{V}_1 \\ \bar{V}_2 \\ \bar{V}_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \tag{1.25}$$

The previous exercise can be easily generalized to an arbitrary rotation, giving rise to the following transformation rules[5]

$$\bar{V}^i = R^i{}_j V^j\,, \qquad\qquad \bar{\mathbf{e}}_i = \left(R^{-1}\right)^j{}_i\,\mathbf{e}_j\,. \tag{1.27}$$

which, in a much powerful notation, can be written as

$$\bar{V}^i = \frac{\partial \bar{x}^i}{\partial x^j} V^j\,, \qquad\qquad \bar{\mathbf{e}}_i = \frac{\partial x^i}{\partial \bar{x}^j}\mathbf{e}_j\,. \tag{1.28}$$

In conclusion, a vector $\mathbf{V}$ remains unchanged under (in this case) rotations due to the simultaneous and opposite change of its components V^i and the basis $\mathbf{e}_i$

$$\mathbf{V} = \bar{V}^i\bar{\mathbf{e}}_i = \left(\frac{\partial \bar{x}^i}{\partial x^j}V^j\right)\left(\frac{\partial x^k}{\partial \bar{x}^i}\mathbf{e}_k\right) = V^j\delta^k_j\mathbf{e}_k = V^k\mathbf{e}_k = \mathbf{V}\,. \tag{1.29}$$

From now on, and in a clear abuse of language, we will frequently employ a standard shorthand and will refer to V^i as *a vector* instead of saying *the components of a vector* $\mathbf{V}$. A *vector* is said to be *contravariant* if it transforms as the displacement vector dx^i (cf. Eq. (1.10))

$$\bar{V}^i = \frac{\partial \bar{x}^i}{\partial x^j}V^j\,. \tag{1.30}$$

On the other hand, a *vector* is said to be *covariant* if it transforms as the basis vectors $\mathbf{e}_i$ (cf. Eq. (1.29))

$$\bar{V}_i = \frac{\partial x^i}{\partial \bar{x}^j}V_j\,. \tag{1.31}$$

[5] The example has been presented using the *passive viewpoint*, in which the same vector ends up with different components when the reference frame is changed. The expression

$$\bar{V}^i = R^i{}_j V^j\,, \tag{1.26}$$

can also describe the *active viewpoint* in which a given vector is mapped to a different vector under the same basis choice.

A particular example of an object with the previous transformation properties is the gradient of a scalar function

$$\frac{\partial f}{\partial \bar{x}^i} = \frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial f}{\partial x_j} \tag{1.32}$$

The gradient is the difference of the function per unit distance in the direction of the basis vector. When the basis vector "shrink" the gradient must "shrink" too.

You maybe think that I am being a bit pedantic here. For you the gradient was, till now, a re-gular vector, as good as the displacement vector. Now I am giving them two different names and two 'different" transformation rules! Indeed... you are right... I am being quite pedantic... but just to prepare the notation for the future. Note the matrix $\left(R^{-1}\right)^j{}_i$ is just an index notation for $(R^{-1})^T$, which for the particular case of an orthogonal matrix, is equal to the transformation matrix R itself. As we already said in Section (1.2), there is no clear difference between covariant and contravariant components as long as one transforms between Euclidean orthonormal basis. However, this is not the case in general coordinate systems (such as polar coordinates) or in Special Relativity. Be patient.

Exercise

Show that

- the 3-divergence of a vector field $\partial_i V^i$ transforms as a scalar field.
- the Laplacian operator $\nabla^2 = \partial_i \partial^i$ transforms as a Galilean scalar operator.

1.4.3 Tensors: linear machines

The previous examples are just particular cases of a general class of quantities that transform with a linear and homogeneous transformation law under coordinate transformation: *tensors.* In order to get some intuition, let us start by considering in detail the transformation laws of rank-2 tensors. In the same way that a vector $\mathbf{V}$ can be expanded in terms on the basis $\mathbf{e}_i$, a *geometric* Cartesian tensor $\mathcal{T}$ can be expanded as

$$\mathcal{T} = T^{ij}\mathbf{e}_i \otimes \mathbf{e}_j \,. \tag{1.33}$$

where $\otimes$ denotes the direct product. The transformation property of the different components T^{ij} under a rotation follows immediately from the previous expresion: T^{ij} transform as the product of two contravariant vectors A^i and B^j

$$\bar{A}^i\bar{B}^j = R^i{}_k R^j{}_l A^k A^l \qquad \longrightarrow \qquad \bar{T}^{ij} = R^i{}_k R^j{}_l T^{kl} \,. \tag{1.34}$$

Rotations	$\frac{\partial \bar{x}^i}{\partial x^j} \equiv R^i{}_j$ are constants!
Scalar	$\bar{\phi} = \phi$
Contravariant vector	$\bar{V}^i = \frac{\partial \bar{x}^i}{\partial x^j} V^j$
Covariant vector	$\bar{V}_i = \frac{\partial x^j}{\partial \bar{x}^i} V_j$
Contravariant rank-2 tensor	$\bar{T}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} T^{kl}$
Covariant rank-2 tensor	$\bar{T}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} T_{kl}$
Mixed rank-2 tensor	$\bar{T}^i{}_j = \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^j} T^k{}_l$

Table 1.1

As we did in the previous section, we can define the covariant tensor T_{ij}, which transform as the product of two covariant vectors

$$\bar{A}_i \bar{B}_j = \left(R^{-1}\right)^k{}_i \left(R^{-1}\right)^l{}_j A_k A_l \qquad \longrightarrow \qquad \bar{T}_{ij} = \left(R^{-1}\right)^k{}_i \left(R^{-1}\right)^l{}_j T_{kl} \,. \tag{1.35}$$

As before, in a clear abuse of language, we will refer to these *tensor components as tensors.* Particular examples of rank-2 Cartesian tensor are the inertia tensor

$$I^{ij} = \int d^3x \; \rho(\mathbf{x}) \left(r^2 \delta^{ij} - x^i x^j\right) \tag{1.36}$$

or the quadrupole tensor (1.57) (cf. Section 1.5.1).

Exercise

Show that the inertia tensor I^{ij} is indeed a rank-(2,0) tensor.

Generalizing the transformation laws (1.34) and (1.35) we can define the transformations properties for arbitrary mixed tensors of contravariant rank m and covariant rank n

$$\bar{T}^{i_1 \ldots i_m}{}_{j_1 \ldots j_n} = \left(\prod_{p=1}^{m} \frac{\partial \bar{x}^{i_p}}{\partial x^{k_p}} \prod_{q=1}^{n} \frac{\partial x^{l_q}}{\partial \bar{x}^{j_q}} \right) T^{k_1 \ldots k_m}{}_{l_1 \ldots l_n} \tag{1.37}$$

$$= \left[R^{i_1}{}_{k_1} \ldots R^{i_m}{}_{k_m}\right] \left[(R^{-1})^{l_1}{}_{j_1} \ldots (R^{-1})^{l_n}{}_{j_n}\right] T^{k_1 \ldots k_m}{}_{l_1 \ldots l_n} \,. \tag{1.38}$$

Tensors (components) are objects with any number of indices. They share the same transformation properties as vectors and can be classified according to the number of upper or lower indices. For instance, we say that a scalar is a rank-0 tensor and a contravariant (or covariant) vector is a contravariant (or covariant) rank-1 tensor. In general, a tensor with m upper indices and n lower indices is called a rank-(m, n) tensor.

A tensor is *not* just a quantity carrying indices. It is the transformation law what defines a tensor (see below). Not all quantities with indices are tensors.

1.4.4 Some useful properties

Let me present some useful properties and definitions regarding tensors:

1. The sum (or difference) of two like-tensors is a tensor of the same kind. The proof of this is straightforward. Imagine we take sum or difference of two general tensors $T^{i_1\ldots i_m}{}_{j_1\ldots j_n}$ and $R^{i_1\ldots i_m}{}_{j_1\ldots j_n}$ and apply the transformation rule (1.37), we will get

$$\begin{aligned}
\bar{S}^{i_1\ldots i_m}{}_{j_1\ldots j_n} &\equiv \bar{T}^{i_1\ldots i_m}{}_{j_1\ldots j_n} \pm \bar{R}^{i_1\ldots i_m}{}_{j_1\ldots j_n} \\
&= \left(\prod_{p=1}^{m}\frac{\partial \bar{x}^{i_p}}{\partial x^{k_p}}\prod_{q=1}^{n}\frac{\partial x^{l_q}}{\partial \bar{x}^{j_q}}\right) T^{k_1\ldots k_m}{}_{l_1\ldots l_n} \pm \left(\prod_{p=1}^{m}\frac{\partial \bar{x}^{i_p}}{\partial x^{k_p}}\prod_{q=1}^{n}\frac{\partial x^{l_q}}{\partial \bar{x}^{j_q}}\right) R^{k_1\ldots k_m}{}_{l_1\ldots l_n} \\
&= \left(\prod_{p=1}^{m}\frac{\partial \bar{x}^{i_p}}{\partial x^{k_p}}\prod_{q=1}^{n}\frac{\partial x^{l_q}}{\partial \bar{x}^{j_q}}\right)\left(T^{k_1\ldots k_m}{}_{l_1\ldots l_n} \pm R^{k_1\ldots k_m}{}_{l_1\ldots l_n}\right) \\
&= \left(\prod_{p=1}^{m}\frac{\partial \bar{x}^{i_p}}{\partial x^{k_p}}\prod_{q=1}^{n}\frac{\partial x^{l_q}}{\partial \bar{x}^{j_q}}\right) S^{i_1\ldots i_m}{}_{j_1\ldots j_n}\,. \qquad (1.39)
\end{aligned}$$

2. Given two tensors of rank s and t, the product transforms as a tensor of rank $(s+t)$.

3. If the expression $T^{\cdots}{}_{\ldots} = R^{\cdots}{}_{\ldots} S^{\cdots}{}_{\ldots}$ is invariant under coordinate transformations and $T^{\cdots}{}_{\ldots}$ and $R^{\cdots}{}_{\ldots}$ are tensors, then $S^{\cdots}{}_{\ldots}$ is a tensor.

   Exercise
   Prove this for the particular case $T_i = R_j S^j{}_i$.

4. A *tensor contraction* occurs when one of a tensor's *free covariant indices* is set equal to one of its *free contravariant* indices[6]. A sum is understood to be performed on the now repeated indices. For instance, $T_{ij}{}^j$ is a contraction on the second and third indices of the tensor $T_{ij}{}^k$.

5. The contraction of a rank-2 tensor is a scalar (its trace) whose value is independent of the coordinate system chosen.

[6]Note the words covariant and contravariant. A contraction is never done between two covariant or two contravariant indices.

If all the components of a Cartesian tensor $T^{i_1 \dots i_m}{}_{j_1 \dots j_n}$ in a given inertial reference frame are zero, they will be zero in any other inertial reference frame.

1.4.5 Symmetric and antisymmetric tensors

An arbitrary rank-2 tensor can be decomposed into a completely symmetric and a completely antisymmetric part

$$T_{ij} = T_{(ij)} + T_{[ij]} \,, \tag{1.40}$$

where we have used the common notation $(,)$ and $[,]$ to denote respectively symmetrization and antisymmetrization over the indices included inside, i.e.

$$T_{(ij)} \equiv \frac{1}{2}\left(T_{ij} + T_{ji}\right) \,, \qquad T_{[ij]} \equiv \frac{1}{2}\left(T_{ij} - T_{ji}\right) \,. \tag{1.41}$$

Completely symmetric and antisymmetric rank-2 tensors satisfy $T_{ij} = \pm T_{ji}$, where the plus sign stands for the symmetric and the minus sign for the antisymmetric one. Particular examples of symmetric tensors are the inertia tensor (1.36) or the quadrupole tensor (1.57) (cf. Section 1.5.1).

Exercise

Prove that the trace of a tensor is invariant under rotations. Show that a tensor T_{ij} in n dimensions has three separately invariant parts

$$T_{ij} = \frac{1}{n} T^k{}_k \delta_{ij} + T_{(ij)} + \left(T_{[ij]} - \frac{1}{n} T^k{}_k \delta_{ij} \right) \,. \tag{1.42}$$

Exercise

Write down the explicit expressions for the completely symmetric and antisymmetric parts of a rank-3 tensor T_{ijk}.

1.4.6 Permutation tensor

The Levi-Civita or permutation tensor[7] of rank 3

$$\epsilon_{ijk} = \epsilon^{ijk} \begin{cases} +1, & \text{if } ijk \text{ is an even permutation of 123} \\ +1, & \text{if } ijk \text{ is an odd permutation of 123} \\ 0, & \text{otherwise} \end{cases} \tag{1.43}$$

[7] Technically, I should say that it is a *psedotensor*, but we are not interested in introducing this concept here. We will only deal with rotations.

flips the sign upon the interchange of any pair of indices and vanishes when two of the indices are equal. Most of the basic identities of vector algebra and vector calculus can be easily proved by using an important relation between the metric tensor δ_{ij} and ϵ_{ijk}, the *contracted epsilon identity*[8]

$$\epsilon_{ijk}\epsilon^{i}{}_{lm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \,. \tag{1.44}$$

You will deal with this expression in the exercises.

1.5 Covariance and Classical Mechanics

The main property of tensors is that their transformation law is linear and homogeneous. Each component of a tensor, in this case Cartesian, is a linear combination of the components of the tensor in the original frame, namely

$$\bar{T}^{i_1\dots i_m}{}_{j_1\dots j_n} = \left(\prod_{p=1}^{m}\frac{\partial \bar{x}^{i_p}}{\partial x^{k_p}}\prod_{q=1}^{n}\frac{\partial x^{l_q}}{\partial \bar{x}^{j_q}}\right) T^{k_1\dots k_m}{}_{l_1\dots l_n} \,. \tag{1.45}$$

In order to ensure that fundamental equations satisfy the Galilean Principle of Relativity the only thing we have to do is to write *tensorial equations*. For instance, if two quantities $S^{ij}{}_k$ and $T^{ij}{}_k$ transform as rank-$(2,1)$ Cartesian tensors, a fundamental law of the kind

$$S^{ij}{}_k = T^{ij}{}_k \,, \tag{1.46}$$

will retain its form in any inertial reference frame, since both sides of the equation transform in the same way under coordinate transformations (in this case rotations). The fundamental equation (1.46) is then said to be covariant and the transformation is said to be a symmetry of the physical theory.

1.5.1 Newton's theory of gravity

A physical example of the previous discussion is the Newtonian theory of gravity published by Netwon in 1687 within the *Philosophiae Naturalis Principia Mathematica*. In such a theory, the gravitational force F_i exerted on a *gravitational test mass* m_G

$$F_i = -m_G\, \partial_i \Phi \,. \tag{1.47}$$

[8] In most of the books you will find this expression with all indices down. Remember that the index convention we chose is just a way of keeping track of the sums that can be easily extended to the Minkowski case. For Cartesian tensors the position of the indices makes no difference.

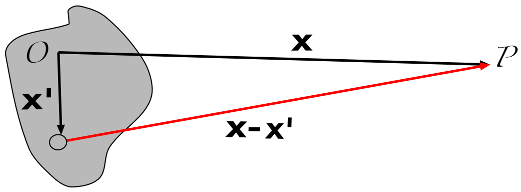


Figure 1.3: Multipolar expansion.

is determined by a single function[9], the *gravitational potential* Φ, which depends on the matter distribution through the so-called *Poisson equation*[10]

$$\nabla^2 \Phi(t, x^i) = 4\pi G \rho(t, x^i) \,. \tag{1.49}$$

Eqs. (1.47) and (1.49) are respectively a *vector* and a *scalar covariant equation.* If they are valid in a given inertial frame, they will be automatically valid in any inertial frame, since their form will be preserved under rotations and translations.

Exercise: Cosmological constant

Galilean invariance allows for an additional constant Λ in the Poisson equation, which becomes

$$\nabla^2 \Phi(t, \mathbf{x}) + \Lambda = 4\pi G \rho(t, \mathbf{x}) \,. \tag{1.50}$$

Observations of galaxies with typical masses of $10^{30} M_\odot$, and intergalactic separations of order 1 Mly do not show any significant deviation from Newton's inverse square. law. Assuming this deviation to be smaller than 1%, determine an upper bound on the magnitude of Λ.

The solution of the Poisson equation can be worked out in the same way that you did for the electromagnetic potential in your Classical Electrodynamics course. The only difference (albeit fundamental) is the sign of the matter distribution. A formal solution of the Poisson's equation for an arbitrary mass distribution can be obtained by applying the superposition principle or using Green functions to obtain

$$\Phi(\mathbf{x}) = -G \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' \,, \tag{1.51}$$

[9]Eqs. (1.47) and (1.49) are left unaltered by the addition to Φ of an arbitrary function of time $f(t)$, namely

$$\Phi(t, \mathbf{x}) \to \Phi(t, \mathbf{x}) + f(t) \,. \tag{1.48}$$

Since the transformation affects only the field Φ and not the coordinates, the invariance of Eqs. (1.47) and (1.49) under (1.48) is referred as an *internal or gauge symmetry.* The gravitational field $\Phi(t, \mathbf{x})$ has no dynamical degrees of freedom. Eq. (1.49) is not a dynamical equation for the determination of the potential, but rather a constraint on the initial spatial distribution of the potential, which must apply at all times.

[10]No value of the proportionality *Newton's gravitational constant* G was available to Newton. Its numerical value was firstly determined by Cavendish in 1797 using a torsion balance, being the result reasonably close to present laboratory measurements, $G = 6.673(10) \times 10^{-11} \mathrm{N\,m^2/\,kg^2}$. The gravitational constant remains the most uncertain of all the fundamental constants of physics.

where $\mathbf{x} = x^i \mathbf{e}_i$ is the radius vector of the point at which the gravitational potential is computed, and $\mathbf{x}' = x'^i \mathbf{e}_i$ is an arbitrary point in the matter distribution. Note that the Newtonian potential is negative, as expected for an attractive force.

Exercise: Green's functions (*)

Use the Green's function method to prove Eq.(1.51).

The previous expression becomes the usual $-GM/r$ only for a spherical mass distribution. The general result for a non-spherical distribution is slightly more complicated. As any distribution function, the essential features of the matter distribution can be be characterized by its moments. For an observer sufficiently far away from the object we can perform a Taylor expansion around $\mathbf{x}' = 0$ to obtain

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = e^{-\mathbf{x}' \cdot \nabla} \frac{1}{r} = \frac{1}{r} - \left(\mathbf{x}' \cdot \nabla\right) \frac{1}{r} + \frac{1}{2}(\mathbf{x}' \cdot \nabla)^2 \frac{1}{r} + \ldots + \frac{(-1)^n}{n!}(\mathbf{x}' \cdot \nabla)^n \frac{1}{r} + \ldots \quad (1.52)$$

$$= \frac{1}{r} + \frac{x'^k x_k}{r^3} + \frac{\left(3x'^k x'^l - r'^2 \delta^{kl}\right) x_k x_l}{2r^5} + \ldots\,, \quad (1.53)$$

where we have used the standard expression for the exponential $e^x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$ and defined the distance $r^2 = x^k x_k$. Inserted back in Eq.(1.51), we realize that the potential created by the matter distribution

$$\Phi(\mathbf{x}) = -G \left[\frac{M}{r} + \frac{D^k x_k}{r^3} + \frac{Q^{kl} x_k x_l}{2r^5} + \ldots\right], \quad (1.54)$$

can be organized in a series whose individual terms contain information on the spatial structure at an increasing level of detail while decaying the more rapidly in space the higher the information content is. The quantities

$$M = \int \rho(\mathbf{x}')\, d^3 x'\,, \quad (1.55)$$

$$D^k = \int \rho(\mathbf{x}') x'^k d^3 x'\,, \quad (1.56)$$

and

$$Q^{kl} = \int \rho(\mathbf{x}') \left(3x'^k x'^l - r'^2 \delta^{kl}\right) d^3 x'\,, \quad (1.57)$$

are respectively the total mass of the system, the mass dipole moment and the mass quadrupole moment tensor. The dipole moment can be eliminated by simply choosing the origin of coordinates of the center of mass. The quadrupole moment is the second moment of the mass distribution with its trace removed. It is proportional to $1/r^3$, which gives rise to a deviation from the inverse square law of the form $1/r^4$.

Exercise: Multipole expansion

- Prove Eq. (1.52).
- Prove that the quadrupole tensor for a spherical distribution vanishes.
- Prove that a change of the origin modifies the quadrupole tensor by only adding a constant.

CHAPTER 2

MINKOWSKI SPACETIME AND SPECIAL RELATIVITY

> Scarcely anyone who truly understand relativity theory can escape this magic.
>
> A. Einstein

In the previous chapter we saw that tensors are a very good tool for writing covariant equations in 3-dimensional Euclidean space. In this chapter we will generalize the tensor concept to the framework of the Special Theory of Relativity, the Minkowski spacetime. I will assume the reader to be familiar at least with the rudiments of Special Relativity, avoiding therefore any kind of historical introduction to the theory.

2.1 Einstein's Relativity

Special Relativity is based on two basic axioms, formulated by Einstein in 1905[1]:

1. *Principle of Relativity (Galileo)*: The laws of physics are the same in all the inertial frames: No experiment can measure the absolute velocity of an observer; the results of any experiment do not depend on the speed of the observer relative to other observers not involved in the experiment.

2. *Invariance of the speed of light*: The speed of light in vacuum is the same in all the inertial frames.

Instantaneous action at a distance is inconsistent with the second postulate and must be replaced by *retarded action at a distance*. Absolute simultaneity will only apply as an approximation at low velocities for nearby events.

[1] *On the electrodynamics of moving bodies.*

2.2 Minkowski spacetime: new wine in a old bottle

The framework of Special Relativity is a 4-dimensional manifold called Minkowski (or pseudo-Euclidean) space-time. The differential and topological structures of the Newtonian and Minkowskian spacetimes coincide[2], but they differ in the metrical structure, i.e. in the definition of distances. While in Newtonian spacetime the spatial and temporal distances are independent, in Minskowskian spacetime *space and time by themselves, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality*[3]. Space and time are distinguished only by a sign, which will play however a central role. For any inertial frame of reference in Minkowski spacetime there is a set of coordinates[4]

$$\{x^\mu\} = \{t, x, y, z\} = \{x^0, x^1, x^2, x^3\} = \{x^0, x^i\}\,, \tag{2.1}$$

and a set of orthonormal basis vectors

$$\{\mathbf{e}_\mu\} = \{\mathbf{e}_t, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{\mathbf{e}_0, \mathbf{e}_i\}\,, \tag{2.2}$$

satisfying the *Lorentz orthonormality condition*[5]

$$\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \eta_{\mu\nu}\,, \tag{2.3}$$

with

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{diag}(-1, 1, 1, 1)\,. \tag{2.4}$$

The inverse of (2.4) is traditionally denoted by $\eta^{\mu\nu}$ and satisfies[6]

$$\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu{}_\rho\,, \tag{2.5}$$

where the 4-dimensional Kronecker delta $\delta^\mu{}_\rho$ is the indexed version of the identity matrix, i.e. $\delta^\mu{}_\rho = 1$ if $\mu = \rho$ and zero otherwise. Note that $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ are numerically equivalent.

Exercise

Which is the value of $\delta^\mu{}_\mu$?

[2] Both of them are smooth, continuous, homogeneous, isotropic, orientable,...

[3] Minkowski, 1908.

[4] As in the previous chapter, we will not consider the quadruplet $\{x^\mu\}$ to be a vector in Minkowski spacetime. Coordinate indices will be always upper indices.

[5] Note that the time coordinate is 4-dimensionally orthogonal to the spatial coordinates.

[6] Einstein's summation convention is used.

In terms of the basis vectors, the infinitesimal displacement $d\mathbf{S}$ between two points in *space-time* can be expressed as

$$d\mathbf{S} = dx^\mu \mathbf{e}_\mu \,, \tag{2.6}$$

where dx^μ are the so-called *contravariant* components. These are computed via the scalar product of the vector $d\mathbf{S}$ and the corresponding basis vector $\mathbf{e}_\mu$.

$$d\mathbf{S}\cdot\mathbf{e}_\mu = (dx^\nu \mathbf{e}_\nu)\cdot\mathbf{e}_\mu = dx^\nu\,(\mathbf{e}_\nu\cdot\mathbf{e}_\mu) = dx^\nu \eta_{\nu\mu} = dx_\mu \,, \tag{2.7}$$

where we have defined the *covariant* or dual components by lowering the index of the contravariant components with the metric

$$dx_\mu \equiv \eta_{\mu\nu} dx^\nu \,. \tag{2.8}$$

Exercise

Starting with a covariant vector defined by Eq.(2.8) , show that the inverse of the metric $\eta^{\mu\nu}$ can be used to raise indices

$$\eta^{\mu\nu} dx_\nu = dx^\mu \,. \tag{2.9}$$

As in the Euclidean case, contravariant and covariant vectors are just an appropriate way of simplifying the notation and taking into account the summation convention. Note however that in the present case lowering or raising indices changes the sign of the temporal component while keeping intact the spatial ones

$$dx^0 = -dx_0 \,, \qquad\qquad dx^i = +dx_i \,. \tag{2.10}$$

The upper and lower index notation automatically keeps track of the minus signs associated to the temporal component. The indefiniteness of the metric is automatically incorporated in the notation!

In some old-fashioned books and in 't Hooft's lecture notes you will find a fourth coordinate $x^4 = it$, instead of the coordinate $x^0 = t$ appearing before. Written in terms of x^4 the Minkowski spacetime has the appearance of a positive-definite 4 dimensional Euclidean space

$$\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \delta_{\mu\nu} \tag{2.11}$$

and there is no difference between lower and upper indices. This notation is however confusing since it hides the non-positive definite character of the metric.

The square of the infinitesimal distance between two events in Minkowskian spacetime is given by

$$|d\mathbf{S}|^2 \equiv ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dx^\mu dx_\mu = -dt^2 + dX^2 \,. \tag{2.12}$$

where $\eta_{\mu\nu}$ is the *Minskowski metric* and $dX^2 \equiv dx^2 + dy^2 + dz^2$ denotes the spatial interval.

> Note that we have arbitrarily chosen a $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ *spacelike convention* for the metric signature, which keeps intact the notation used for Cartesian tensors in Euclidean spacetime. Some books use a different *timelike convention* for the signature of the metric, taking $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Although the physics is independent of the convention used, the signs appearing in the formulas in those books may differ from those in the expressions presented here. For instance, using the convention $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, Eq.(2.40) would change to
>
> $$p_\mu p^\mu = m^2 \,. \tag{2.13}$$
>
> Note however that in both cases you recover $E^2 = p^2 + m^2$ after expanding the expression into the different components.

Note that, contrary to the Newtonian case, the metric $\eta_{\mu\nu}$ is not positive-definite. Given the Lorentzian signature $(-+++)$, the interval (2.12) can be positive, zero, or negative

- If $ds^2 = 0$, $dX/dt = 1$ and the interval corresponds to the trajectory of a light ray. This interval is called *null* or *lightlike* interval. The set of all lightlike wordlines leaving or arriving to a given point x^μ spans the future or past *lightcone* of the event. There is a lightcone associated to each point in spacetime.
- If $ds^2 < 0$ the interval is said to be *timelike*. It corresponds to the wordline of a particle with nonzero rest mass moving with a velocity smaller than light, $dX/dt < 1$. Two events separated by such an interval are both inside the lightcone and can be in causal contact. There will exist a frame in which the two events happen at same position but at different times.
- If $ds^2 > 0$ the interval is termed *spacelike*. There will exist a frame in which the two events happen at the same time but at different places, without any causal relation between them.

The different concepts are summarized in Fig.2.1.

2.3 Minkowski spacetime isometry group

The transformations of Special Relativity are defined as those that do not change the Minkowski line element (2.12) (*not* the spatial or temporal intervals separately!). Following the procedure outlined in the previous chapter, and taking into account that $\eta_{\mu\nu}$ is also a constant

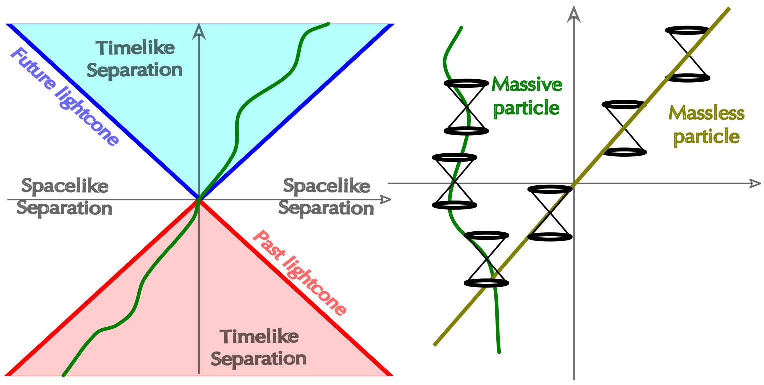


Figure 2.1: Minkowski spacetime.

metric, the requirement $ds^2 = d\bar{s}^2$ gives rise to the following condition

$$\frac{\partial \Lambda^\rho{}_\mu}{\partial x^\pi}\eta_{\rho\sigma}\Lambda^\sigma{}_\nu = 0 \qquad \rightarrow \qquad \frac{\partial \Lambda^\rho{}_\mu}{\partial x^\pi} = 0\,, \tag{2.14}$$

where we have defined

$$\Lambda^\mu{}_\nu \equiv \frac{\partial \bar{x}^\mu}{\partial x^\nu}\,. \tag{2.15}$$

As before, the transformation relating the two reference frames must be linear[7]. This set of transformations constitute the so-called *inhomogeneous Lorentz group or the Poincare group*, which is a combination of *translations*

$$x^\mu \to x^\mu + a^\mu \tag{2.16}$$

and linear *homogeneous Lorentz transformations*

$$x^\mu = \Lambda^\mu{}_\nu\, x^\nu \ , \tag{2.17}$$

with Λ a 4×4 matrix, *independent of the coordinates*. The first (upper) index in $\Lambda^\mu{}_\nu$ labels rows, while the second (lower) one labels columns.

In order to preserve the line element (2.12) the constant matrices $\Lambda^\mu{}_\nu$ are required to satisfy the pseudo-orthogonality condition

$$\eta_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu \ , \tag{2.18}$$

which, in matrix notation, becomes

$$\eta = \Lambda^{\rm T}\eta\Lambda\,, \tag{2.19}$$

with $^{\rm T}$ denoting matrix transpose. Eq. (2.18) is the relativistic analogue of the orthogonality condition (1.19). The determinant of the Λ matrices is also ± 1. As we did in the previous chapter, we will not consider the full Lorentz group[8] (which is neither connected nor compact)

$$O(3,1) = L_+ \cup PL_+ \cup TL_+ \cup PTL_+\,, \tag{2.20}$$

[7]Note that this is basically due to the fact that we are dealing with *constant* metrics.

[8]Note that now, not only the determinant of Λ, but also the element $\Lambda^0{}_0$, plays a special role in the splitting (2.20). .

with P and T the parity $P^\mu{}_\nu = \text{diag}(+1, -\mathbb{I})$ and time reversal $T^\mu{}_\nu = \text{diag}(-1, +\mathbb{I})$ operations. We will restrict ourselves to the continuous Lorentz transformations connected with the identity (the *proper Lorentz group*)

$$L_+ \equiv SO(3,1) = \{\Lambda | \Lambda^T \eta \Lambda = \eta\,,\; \Lambda^0{}_0 \geq 0\,, \det\Lambda = 1\} \tag{2.21}$$

with S denoting special or reflection-free. These transformations are the relativistic analog of *proper rotations* in Euclidean spacetime.

Exercise

Verify that the restricted set of Lorentz transformations (2.21) forms a group :

- Closure: The product of any two Lorentz transformations is another Lorentz transformation.
- There is an identity transformation.
- Every Lorentz transformation has an inverse.
- The product of Lorentz transformations is associative.

The fact that $\eta^2 = \mathbb{I}_4$, with $\mathbb{I}_4$ the identity matrix, allows us to easily compute the inverse Lorentz transformation

$$\eta^2 = \eta\left(\Lambda^{\mathrm{T}}\eta\Lambda\right) = \left(\eta\,\Lambda^{\mathrm{T}}\eta\right)\Lambda = \mathbb{I}_4 \qquad \to \qquad \Lambda^{-1} = \eta\Lambda^{\mathrm{T}}\eta\,, \tag{2.22}$$

which, writing explicitly the components, becomes

$$\left(\Lambda^{-1}\right)^\mu{}_\nu = \eta^{\mu\lambda}\Lambda^\rho{}_\lambda \eta_{\rho\nu} = \Lambda_\nu{}^\mu\,. \tag{2.23}$$

The position of the indices is important $\Lambda^\mu{}_\nu \neq \Lambda_\nu{}^\mu$!!

How many Lorentz transformation are there? Each Lorentz transformation is represented by a 4×4 matrix, which makes a total of 16 components. The pseudo-orthogonality condition (2.18) imposes however some constraints. Indeed, taking the transpose of such a equation leaves it unchanged. The independent components are just the diagonal elements plus half the off-original elements. We are left therefore with $16-10=6$ independent Lorentz transformations. There are two different kinds of homogeneous Lorentz transformations. The most obvious one are spatial rotations

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 \\ 0 & R^i{}_j \end{pmatrix}\,, \tag{2.24}$$

where $R^i{}_j$ is a 3×3 orthogonal matrix $\delta_{kl} = \delta_{ij} R^i{}_k R^j_l$, with i, j running only in spatial directions. There are three independent rotations matrices, one per spatial direction. For instance, a rotation of angle θ around the z-axis will take the form

$$R^i{}_j = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{2.25}$$

The difference with Euclidean transformations arises when considering the so-called *boosts*, which mix the spatial and temporal components. There are three of them, each one associated to the mixing of a particular spatial component with time. As an example consider a boost of *rapidity* $\eta = \tanh^{-1} v$ along the x direction

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{2.26}$$

where we have defined the parameter $\gamma = 1/\sqrt{1-v^2}$, with v the 3-velocity. After the boost, the temporal and spatial coordinates are a linear and homogeneous combination of the spatial and temporal coordinates in the old frame.

Exercise

Verify that Eq. (2.26) gives rise to the standard Lorentz transformation

$$t' = \gamma(t - vx) , \qquad x' = \gamma(x - vt) . \tag{2.27}$$

For doing so, assume $\Lambda^\mu{}_\nu$ to be the transformation from the rest frame of a given inertial observer to the frame of a second initial observer moving with speed v along the x axis and determine the relation between that velocity and η .

The convenience of using the rapidity parameter η instead of the velocity v resides in the fact that η combines additively. In fact, if we consider two consecutive boosts in the same direction, we have

$$\Lambda(\eta_1)\Lambda(\eta_2) = \Lambda(\eta_1 + \eta_2) . \tag{2.28}$$

Exercise

Consider the composition of 2 boosts with velocities v_1 and v_2 along the x direction. Show that $\Lambda_1\Lambda_2$ gives rise to a boost with 3-velocity

$$v = \frac{v_1 + v_2}{1 + v_1 v_2} . \tag{2.29}$$

What happens in the limit $v_1, v_2 \ll 1$? Generalize the previous to an arbitrary direction. Is the general result symmetric under the interchange of the two velocities?. Note that

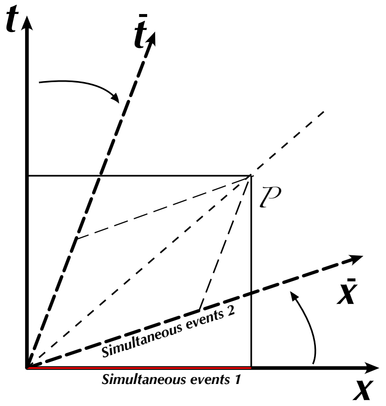


Figure 2.2: A boost transformation

- all the expressions are written in natural units.

The form of Eq. (2.26)

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos i\eta & \sin i\eta & 0 & 0 \\ \sin i\eta & \cos i\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.30}$$

and the property (2.28) closely resemble those of spatial rotations (2.25). The main difference is the change of trigonometric functions by their hyperbolic analogue, reflecting the relative sign of the temporal direction with respect to the spatial directions. Note however two important differences between boosts and ordinary rotations

- The rotation parameter θ in Eq.(2.25) runs between 0 and 2π, with both points included. The rapidity parameter η is non-compact and can take whatever value in $\mathbb{R}$.

- The boost matrix (2.26) is symmetric, which is not the case for ordinary rotations, cf. Eq. (2.25):

Although we should not take the analogy between rotations and boosts too seriously, it is instructive to look at the action of a Lorentz transformation on a spacetime diagram. As shown in Fig. 2.2, a Lorenzt boost rotates time and space by the same angle $\eta = \tanh^{-1} v$, but in opposite directions! In Special Relativity the simultaneity of two events depends on the observer, the hyperplanes of constant coordinate time do not have an invariant meaning. Note however that the light cone (i.e the dashed line at 45 degrees in the diagram) is invariant under Lorentz transformations.

Exercise

Use the diagram Fig.2.2 to derive the well-known effects of *space contraction*

$$\bar{L} = L\sqrt{1-v^2} \qquad \rightarrow \qquad \bar{L} < L\,, \tag{2.31}$$

and *time dilatation*

$$\bar{T} = \frac{T}{\sqrt{1-v^2}} \qquad \rightarrow \qquad \bar{T} > T\,. \tag{2.32}$$

Hint: Don't be confused by drawing just a slice of Minkowski spacetime in an Euclidean paper. Use the Minkowski metric!

2.4 Tensors in Minkowski spacetime

The Einstein's *Principle of Relativity* introduced at the beginning of this chapter implies that all the laws of physics must retain their mathematical form in all the inertial frames, i.e. they must be covariant under Lorentz transformations. As we learnt in the previous chapter, tensorial equations automatically satisfy this requirement. Since the required discussion of tensors in Minkowski spacetime closely follows that in Section 1.4 for Euclidean spacetime[9], we will simply summarize the results in the Table 2.1.

Lorentz transformations	$\frac{\partial \bar{x}^\mu}{\partial x^\nu} \equiv \Lambda^\mu{}_\nu$ are constants!
Scalar	$\bar{\phi} = \phi$
Contravariant vector	$\bar{V}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} V^\nu$
Covariant vector	$\bar{V}_\mu = \frac{\partial x^\nu}{\partial \bar{x}^\mu} V_\nu$
Contravariant rank-2 tensor	$\bar{T}^{\mu\nu} = \frac{\partial \bar{x}^\mu}{\partial x^\rho} \frac{\partial \bar{x}^\nu}{\partial x^\sigma} T^{\rho\sigma}$
Covariant rank-2 tensor	$\bar{T}_{\mu\nu} = \frac{\partial x^\rho}{\partial \bar{x}^\mu} \frac{\partial x^\sigma}{\partial \bar{x}^\nu} T_{\rho\sigma}$
Mixed rank-2 tensor	$\bar{T}^\mu{}_\nu = \frac{\partial \bar{x}^\mu}{\partial x^\rho} \frac{\partial x^\sigma}{\partial \bar{x}^\nu} T^\rho{}_\sigma$

Table 2.1

[9]Note indeed that the only conceptual difference is the replacement of rotations by Lorentz transformations, the replacement of Latin indices by Greek indices, and the use of the Minkowski metric, instead of the Euclidean one, for lowering and raising indices.

Exercise

How does the volume element $d^4x = dx^0 dx^1 dx^2 dx^3$ transforms under Lorentz transformations?

2.5 Covariance and Relativistic Mechanics

In Newtonian spacetime the trajectory of a particle is described by the position 3-vector as a function of time $x^i(t)$, with t an absolute element of the theory. In Special Relativity a particle of *mass* m follows timelike worldlines in Minkowski spacetime. The time depends on the chosen reference frame and it is just another coordinate at the same level of the spatial coordinates. The trajectory $x^\mu(\sigma)$ can be expressed in terms of a completely arbitrary parameter σ, which changes continuously along the wordline and does not need to have any particular physical interpretation. However, an interesting (and natural) possibility for the particular case of massive particles is to identify it with the *proper time* τ of the particle. This proper time τ is defined as the time measured by an observer in the particle's rest frame ($dX = 0$), which can always be achieved by performing a Lorentz transformation. The proper time interval $d\tau$ is therefore related to the Minkowski spacetime interval[10]

$$ds^2 = -d\tau^2 < 0\,. \tag{2.33}$$

The connection between the proper time and the measurement made in an inertial reference frame with coordinate time interval dt is given by

$$\gamma \equiv \frac{dt}{d\tau} = \left(1 - v^2\right)^{-1/2}\,. \tag{2.34}$$

Note that γ is a growing function of the 3-velocity $v^i = \frac{dx^i}{dt}$ and it is always bigger than one. The proper time goes by at a slower rate than the coordinate time t.

4-velocity

Given τ, and in clear analogy with the 3-dimensional case, the 4-velocity u^μ along the trajectory is given by the 4-vector

$$u^\mu \equiv \frac{dx^\mu}{d\tau}\,, \tag{2.35}$$

which is tangent to the worldline of the particle and automatically normalized

$$\eta_{\mu\nu} u^\mu u^\nu = u^\mu u_\mu = -1\,. \tag{2.36}$$

In terms of its components, the 4-velocity u^μ can be written as

$$\frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\left(1, v^i\right)^{\mathrm{T}} = \gamma\left(1, v^i\right)^{\mathrm{T}}\,. \tag{2.37}$$

[10] Note that the proper time is not a useful parametrization for the worldline of massless particles, such as photons, since these particles move on the light cone and can travel any distance in zero proper time $d\tau^2 = -ds^2 = 0$.

For an observer at rest $\gamma = 1$ and Eq.(2.37) becomes simply $u^\mu = (1,0,0,0)^T$. In the Newtonian limit $v \ll c$, $\mathrm{d}\tau \to dt$ and $u^i \to v^i$.

Let us consider the behaviour of u^μ with respect to Lorentz transformations. Since the proper time $d\tau$ is invariant under Lorentz transformations and dx^μ transforms as a contravariant tensor, we have

$$u^\mu = \Lambda^\mu{}_\nu u^\nu \,. \tag{2.38}$$

The 4-velocity is a timelike 4-vector.

4-momentum

Since the mass m of the particle is a scalar under Lorentz transformations, the 4-momentum

$$p^\mu = m u^\mu \tag{2.39}$$

is a Lorentz 4-vector with components $p^\mu = \left(E, p^i\right)^{\mathrm{T}} = \left(m\gamma, m\gamma v^i\right)^{\mathrm{T}}$.

In the instantaneous rest frame of the particle, $p^\mu = (m,0)^T$. This can be used to simplify many computations. We can compute things in this particular frame and then re-express the result in a form valid in any other inertial frame by appealing to covariance.

Using the normalization condition for the 4-velocities (2.36), the normalization condition for the 4-momentum becomes

$$p_\mu p^\mu = -m^2 \,, \tag{2.40}$$

which is nothing else than the well-known energy-momentum relation $E = \sqrt{p^2 + m^2}$ written in a covariant way[11]. In the Newtonian limit ($|p^i| \ll m$) this relation becomes the familiar expression of the Newtonian theory together with the energy equivalent of the mass mc^2, i.e. $E \simeq m + \frac{p^2}{2m}$. Note that the 4-momentum remains well defined even for massless particles, where it has zero square norm and becomes lightlike, $p_\mu p^\mu = 0$. We will take this as a definition of a massless classical particle. The indefiniteness of Minkowski metric allows for non-zero values of the temporal and spatial parts as long as they cancel out in $p_\mu p^\mu$. In particular, we can always find a frame in which $p^\mu = (E,0,0,E)^T$.

4-acceleration

It is important to remark that Special Relativity, as Newtonian mechanics, is concerned with the relation between inertial observers and not with the behavior of the objects that they

[11] Note that p is the three momentum p^i.

are studying. In particular, the observed objects can be accelerating. The 4-acceleration in Minkowski spacetime can be defined as[12]

$$a^\mu = \lim_{\epsilon\to 0} \frac{u^\mu(\tau+\epsilon) - u^\mu(\tau)}{\epsilon} \qquad \to \qquad a^\mu \equiv \frac{du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau}\,. \tag{2.42}$$

The 4-acceleration (2.42) transforms in the proper way under Lorentz transformations since $\Lambda^\mu{}_\nu$ is linear and depends only on the relative velocity between the two frames. This fact allows it to pass completely the $d^2/d\tau^2$. The acceleration a^μ is spacelike 4-vector

$$\eta_{\mu\nu}a^\mu a^\nu > 0 \tag{2.43}$$

orthogonal to the timelike 4-velocity

$$a^\mu u_\mu = 0\,, \tag{2.44}$$

as can be easily shown by computing the derivative of Eq. (2.36) with respect to the proper time.

Energy-momentum conservation

The objects defined above allow us to easily generalize the Newton's second law, which becomes

$$f^\mu = \frac{dp^\mu}{d\tau} = ma^\mu\,. \tag{2.45}$$

Note that Eq.(2.45) is a tensorial identity, which maintains its form under Lorentz transformations and automatically satisfies Einstein's principle of Relativity. The explicit expression of f^μ depends on the considered interaction (cf. the first exercise in Section 2.7). The components of the 4-vector f^μ in Eq. (2.45) are proportional to the Newtonian force F^i and to the work done by F^i per unit time, i.e. $f^\mu = \gamma\,(v_i F^i, F^i)^T$. Contrary to what happens in Newtonian physics, energy and momentum conservation laws are not independent. The conservation laws of Newtonian mechanics in a given collision between particles are replaced by a conservation law for the *total 4-momentum*[13]

$$\sum_{in} p^\mu_{in} = \sum_{out} p^\mu_{out}\,, \tag{2.46}$$

with the subscripts *in* and *out* denoting the incoming and outgoing particles. Note that the conservation law is Lorentz covariant and reduces to the Newtonian momentum and energy conservation for small velocities.

---

[12]You will probably wondering why I am making such a mess writing out the explicit definition in (2.42) instead of directly writing

$$a^\mu \equiv \frac{du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau}\,. \tag{2.41}$$

The point I want you to notice in that the two vectors $v^\mu(\tau+\epsilon)$ and $v^\mu(\tau)$ are located at different points in spacetime. What we are really doing when computing the acceleration in Special Relativity is *trivially moving* the two vectors to the same point in spacetime before subtracting then. This trivial operation of moving a vector from one point to another will turn out to be not so trivial in General Relativity. As you will see, we will need to introduce some extra machinery in order to do that... but let's move one step at a time...

[13]The interaction is assumed to be a contact interaction. Particles are free away from the interaction point.

Exercise
Show that a photon cannot spontaneously decay into an electron-positron pair.

2.6 Relativistic Lagrangian for free particles

The equation of motion for a free particle following from (2.45)

$$\frac{dp^\mu}{d\tau} = 0\,, \tag{2.47}$$

can be also be derived from a Lagrangian formulation where the role of the generalized coordinates is played by the space-time coordinates x^μ and the classical time t is replaced by an appropriate parameter σ. The simplest guess for the relativistic action would be a *naive* generalization of the Newtonian action, namely

$$S = \frac{1}{2} m \int d\tau \left(\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) . \tag{2.48}$$

Note however that the previous expression is not invariant under reparametrizations of the path[14] $\tau \to f(\tau)$. The dynamic of the particle seems to depend on the "internal coordinate" τ used in the description of the curve $x^\mu(\tau)$. Moreover, the action (2.48) does not contain any information about the lightcone. On top of that, t neither has a smooth massless limit.

To solve these problems, we will substitute the proper time by an arbitrary parameter σ and introduce a non-dynamical function[15] $e(\sigma)$, the so-called *einbein*. This quantity will be treated as an additional generalized coordinate during the intermediate computations and fixed to a particular value only at the end. To clarify the construction, we proceed in several steps. We start by replacing the problematic mass appearing in the action (2.48) by $e^{-1}(\sigma)$. This gives rise to the following structure

$$S \sim \frac{1}{2} \int d\sigma \left(\frac{1}{e(\sigma)} \eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} \right) . \tag{2.49}$$

In order for the previous action to be invariant under reparametrizations of the path[16] $\sigma \to f(\sigma)$, the einbein $e(\sigma)$ must be chosen to transform in the proper way. The transformation rule can be determined by inspection: the property $e(\sigma)d\sigma$ must remain invariant. In other words, the infinitesimal displacement $d\sigma$ and the *einbein* must transform in an opposite way

$$d\bar{\sigma} = \dot{f}(\sigma) d\sigma\,, \qquad \bar{e}(\bar{\sigma}) = \left(\dot{f}(\sigma)\right)^{-1} e(\sigma)\,. \tag{2.50}$$

With these transformations at hand, we proceed now to reintroduce the mass parameter m in the action (2.49). The form of the new term is essentially determined by pure dimensional

[14] The reparametrization invariance of the action should be understood as a gauge symmetry: a redundancy of the description, not a symmetry relating different solutions of the theory.

[15] i.e. no kinetic term for $e(\sigma)$ will be included.

[16] Or if you want invariant under 1D general coordinate transformations.

arguments, reparametrization invariance and the massless limit. In order for the new piece to be reparametrization invariant, the integration measure $d\sigma$ must come together with a factor $e(\sigma)$. This gives a term $d\sigma e(\sigma)$ with dimension $[E]^{-2}$, which must be compensated[17] by something with dimension $[E]^2$ and proportional to m. There you are: the new term is $d\sigma e(\sigma) m^2$. The resulting action is the so-called *einbein action*

$$S = \frac{1}{2}\int d\sigma \left(\frac{1}{e(\sigma)}\eta_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} - m^2 e(\sigma) \right) \tag{2.51}$$

and give rise to the following Euler-Lagrange equations for the generalized coordinates $x^\mu(\sigma)$ and $e(\sigma)$

$$\frac{d}{d\sigma}\left(e^{-1}(\sigma)\frac{dx^\mu}{d\sigma} \right) = 0 \qquad \text{and} \qquad \eta_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx\nu}{d\sigma} = -m^2 e^2(\sigma)\,. \tag{2.52}$$

The massive or massless character of particles is automatically incorporated in the second equation. Indeed, choosing $e(\sigma) = 1$ and taking the limit $m \to 0$, we obtain the equations of motion for a free massless particle

$$\frac{d^2 x^\mu}{d\sigma^2} = 0\,, \qquad \eta_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = 0\,. \tag{2.53}$$

On the other hand, the equations for massive particles can be obtained by choosing $e(\sigma) = 1/m$ and using the proper time as affine parameter ($\sigma = \tau$)

$$m\frac{d^2 x^\mu}{d\tau^2} = 0\,, \qquad \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1\,. \tag{2.54}$$

These kind of choices in which $e(\sigma) =$ constant are called *affine* and restrict the function $f(\sigma)$ to the form

$$\dot{f} = 1 \qquad \to \qquad f(\sigma) = \sigma + \text{constant}\,. \tag{2.55}$$

Exercise

Consider the massive case. Show that the action (2.51) is equivalent to the *geometrical action*

$$S = -m\int d\tau\,. \tag{2.56}$$

[17] Recall that, in natural units, the action is dimensionless.

2.7 Maxwell's equations

In their traditional form, Maxwell's equations are given by [18]

$$\boldsymbol{\nabla}\times\mathbf{E}+\partial_t\mathbf{B}=0\,, \tag{2.57}$$
$$\boldsymbol{\nabla}\cdot\mathbf{B}=0\,, \tag{2.58}$$
$$\boldsymbol{\nabla}\cdot\mathbf{E}=\rho\,, \tag{2.59}$$
$$\boldsymbol{\nabla}\times\mathbf{B}-\partial_t\mathbf{E}=\mathbf{J}\,, \tag{2.60}$$

with $\mathbf{E}$ and $\mathbf{B}$ the electric and magnetic fields, $\mathbf{J}$ the current density and ρ the charge density. They are 8 coupled linear differential equations, in which the boundary conditions are usually taken to be such that for infinite systems the fields $\mathbf{E}$ and $\mathbf{B}$ go to zero at infinity. Note also the symmetry $\mathbf{E}\leftrightarrow\mathbf{B}$ in the absence of sources.

The homogeneous Maxwell's equations (2.57) and (2.58) can be solved by introducing the so- called *electromagnetic potentials*: a scalar potential φ and a vector potential[19] $\mathbf{A}$ satisfying

$$\mathbf{E}=-\boldsymbol{\nabla}\varphi-\partial_t\mathbf{A}\,,\qquad \mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}\,. \tag{2.61}$$

Using them, the inhomogeneous Maxwell's equations become

$$\partial_t^2\varphi-\nabla^2\varphi=\rho\,,\qquad \partial_t^2\mathbf{A}-\nabla^2\mathbf{A}=\mathbf{J}\,. \tag{2.62}$$

Given the electromagnetic potentials φ and $\mathbf{A}$ in Eq. (2.61) the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ are completely determined, but no viceversa. $\mathbf{A}$ and ϕ are *gauge fields* (see the exercise below).

Familiarity with Maxwell's equations soon leads to the appreciation of the unified nature of the electromagnetic field and its relativistic nature. Although in their 19th century version Maxwell's equations (2.57)-(2.60) do not seem at all invariant under Lorentz transformations, they can be written in a more compact and elegant way, that makes explicit their covariant form. Introducing the 4-vector potential (gauge field) $A^\mu\equiv(\varphi,\mathbf{A})$ and the charge-current density 4-vector $J^\mu\equiv(\rho,\mathbf{J})$, we obtain[20]

$$\partial_\nu F^{\mu\nu}=J^\mu\,, \tag{2.63}$$
$$\epsilon_{\mu\nu\rho\sigma}\partial^\rho F^{\mu\nu}=0\,, \tag{2.64}$$

where the antisymmetric quantity $F^{\mu\nu}\equiv\partial^\mu A^\nu-\partial^\nu A^\mu$ is the gauge invariant *(Faraday) field strength* with components

$$F^{0i}=E^i\,,\qquad F^{ij}=\epsilon^{ijk}B_k\,, \tag{2.65}$$

and the different ϵ are totally antisymmetric tensors in the corresponding dimension[21] n

$$\epsilon^{\mu_1\mu_2\ldots\mu_n}=\begin{cases}+1, & \text{if } \mu_1\mu_2\ldots\mu_n \text{ is an even permutation of } 01\ldots(n-1)\,,\\ -1, & \text{if } \mu_1\mu_2\ldots\mu_n \text{ is an odd permutation of } 01\ldots(n-1)\,,\\ 0, & \text{otherwise}\,.\end{cases} \tag{2.66}$$

[18] Note that they are written using the Heaviside-Lorentz convention, in which no 4π factors appear.
[19] In the 3-dimensional sense!
[20] Eq. (2.64) is sometimes called a *Bianchi identity*. As you will see, this will not be the last time that we will find one of these identities.
[21] Note that some books use the opposite sign convention for (2.66).

In covariant notation, (2.62) becomes

$$\Box A^\mu = J^\mu \tag{2.67}$$

where we have defined the *d'Alambertian operator*

$$\Box \equiv \partial_\mu \partial^\mu = -\partial_t^2 + \partial_i^2 \,. \tag{2.68}$$

- The covariant components $\epsilon_{\mu\nu\rho\sigma}$ of the permutation tensor $\epsilon^{\mu\nu\rho\sigma}$ in Minkowski spacetime are defined by lowering each of the indices with the metric tensor $\eta_{\mu\nu}$

$$\epsilon_{\mu\nu\rho\sigma} = \eta_{\mu\lambda}\eta_{\nu\kappa}\eta_{\rho\pi}\eta_{\sigma\tau}\epsilon^{\lambda\kappa\pi\tau} \,, \tag{2.69}$$

  from which it follows that $\epsilon_{0123} = -\epsilon^{0123}$.

- Note that, whereas a cyclic permutation of the indices in the 3-dimensional permutation symbol leaves it unchanged ($\epsilon^{ijk} = \epsilon^{jki}$), a cyclic permutation of the 4-dimensional permutation symbol gives rise to a minus sign ($\epsilon^{\mu\nu\rho\sigma} = -\epsilon^{\nu\rho\sigma\mu}$).

Exercise

Just for those of you knowing Classical Field Theory. The previous equations of motion can be obtained from the following Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu \,, \tag{2.70}$$

where J^μ is treated as an external source.

- Check that with the Lagrangian density (2.70), the action $S = \int \mathcal{L}\, d^4x$ is invariant under Lorentz transformations

$$A^\mu = \Lambda^\mu{}_\nu A^\nu \,, \qquad F^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma} \,, \qquad J^\mu = \Lambda^\mu{}_\nu J^\nu \,, \tag{2.71}$$

  and *gauge transformations*

$$A^\mu \to A^\mu - \partial^\mu \chi \,, \tag{2.72}$$

  where $\chi = \chi(t, \mathbf{x})$ is an arbitrary function of space-time. The concept of purely electric or magnetic fields and that of a static charge distribution with zero current become meaningless, being a good description only in a particular reference frame.

- Which is the Lorentz invariant generalization of the equation of motion $\mathbf{F} = \mathbf{q}(\mathbf{E}+\mathbf{v}\times\mathbf{B})$? Can you guess the associated Lagrangian[a]? Check the consistency of the two results by computing the Euler-Lagrange equations (3.30) of the obtained Lagrangian .

[a]Hint: Which is the generalization of the cross product in the 4-dimensional case? Consider the Lagrangian of a charged particle in an electrostatic potential Φ.

Taking the 4-divergence of Eq. (2.63) we obtain

$$\partial_\mu J^\mu = \partial_\mu \partial_\nu F^{\mu\nu} = 0\,, \tag{2.73}$$

where in the last step we used the fact that $F^{\mu\nu}$ is an antisymmetric tensor, i.e. $F^{\mu\nu} = -F^{\nu\mu}$. Eq.(2.73) is nothing else than the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0\,. \tag{2.74}$$

The conservation of total charge $Q(t)$

$$\dot{Q}(t) = \int_{\mathbb{R}^3} \dot{\rho}(t, x^i) d^3x^i = -\int_{\mathbb{R}^3} \partial_k J^k(t, x^i) d^3x^i = 0\,, \tag{2.75}$$

is imposed by the field equations. If the charge is not conserved there is no solution!

Exercise

Prove that the product $S_{\mu\nu}A^{\mu\nu}$ of a symmetric tensor $S^{\mu\nu}$ and an antisymmetric tensor $A^{\mu\nu}$ is zero.

CHAPTER 3

"THE HAPPIEST THOUGHT" OF EINSTEIN'S LIFE

For an observer falling freely from the roof of a house there exists - at least in his immediate surroundings - no gravitational field [...]. The observer therefore has the right to interpret his state as 'at rest' (at least until he hits the ground!).

A. Einstein (1920)

The Poisson equation for the gravitational field

$$\nabla^2\Phi(t,x^i) = 4\pi G\rho(t,x^i)$$

is a linear partial differential equation of 2nd order which does not contain any explicit time dependence. The gravitational potential responds *instantaneously* to the changes in the matter distribution! This was awkward even for Newton

> *That one body may act upon another at a distance through a vacuum, without the mediation of anything else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity, that I believe no man, who has philosophical matters a competent faculty of thinking, can ever fall into it (Principia, p. 643, Ref. 395).*

and it is in clear contradiction with Special Relativity. The instinctive reaction of many physicist when facing this consistency problem was to apply the recipes used when writing the covariant version of Maxwell equations (promote the operator ∇ to $\Box$, introduce some kind of vector potential A^μ for the gravitational field, generalize the Newtonian force to some combinations of fields and 4-velocities, get retarded potentials, etc ...). None of the attempts was successful[1]. Einstein eventually concluded that a new approach to the problem must be

[1] We will be back to this point in the future.

taken. The purpose of this chapter is to present you Einstein's new look on gravity. As you will see, the *new look* turned out to be a real *old look* that went back to Galileo himself.

3.1 Inertial and gravitational masses

Two different masses enter in the Newton's theory of mechanics and gravitation. According to Newton's second law the acceleration $a^i = d^2x^i/dt^2$ experienced by an object is proportional to the exerted force divided by the *inertial mass* of the object

$$f^i = m_I a^i \,, \tag{3.1}$$

independently of the origin of the force. This mass m_I measures the resistance of an object to accelerations. On the other hand, we have the *gravitational mass*, which measures the strength of the gravity (in the same way that the electric charge measures the strength of the electric force). The force exerted on a *gravitational mass* m_G close to the surface of the Earth is given by

$$f^i = m_G \, g^i \,. \tag{3.2}$$

Comparing the expressions (3.1) and (3.2), we conclude that the *acceleration of gravity* should depend *a priori* on the ratio of the *gravitational mass* to the *inertial mass*.

$$a^i = \left(\frac{m_G}{m_I}\right) g^i \,. \tag{3.3}$$

Nevertheless, as verified by Galileo's ramp experiments[2], all bodies fall with the same acceleration in a gravitational field

$$a^i = g^i \,. \tag{3.4}$$

This observation implies the equality of the quantity controlling inertia (m_I) and that measuring the strength of gravity (m_G)

$$m_I = m_G \,, \tag{3.5}$$

for all materials, independently of its composition.

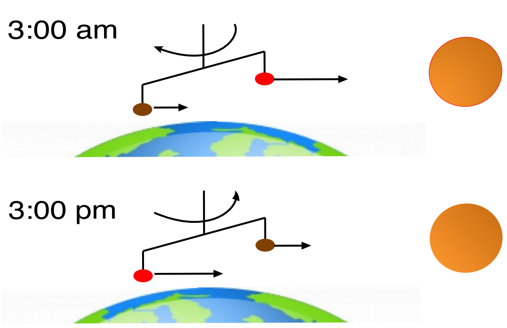
Exercise

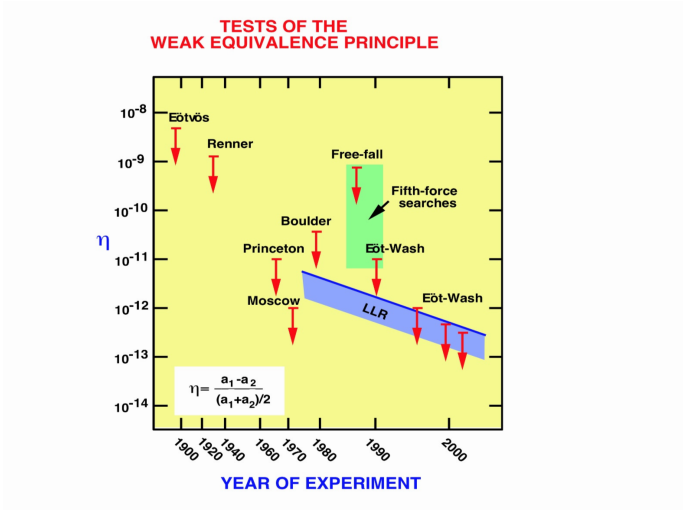
Consider the magnitude of the electrostatic interaction at a distance r between two particles of charges q_1, q_2 and *inertial masses* m_{1i},m_{2i}

$$F_e = \frac{q_1 q_2}{4\pi r^2} \,. \tag{3.6}$$

How does the magnitude of the acceleration felt by particle 2 depends on its properties?

[2] Yes, ramps and a water clock. The image of Galileo dropping balls from the leaning power of Pisa is just a widespread italian legend.





The results of Galileo's experiments were confirmed, among others, by Newton himself and by the Baron Eötvös de Vásárosnamény, who used respectively pendula and a torsion balance with different materials[3].

Exercise

How does the oscillation period of a simple pendulum depend on the ratio m_I/m_G?

The difference in the acceleration experienced by the two bodies is encoded in the so-called Eötvös parameter

$$\eta = \frac{\Delta a}{a} = \frac{2|a_1 - a_2|}{|a_1 + a_2|} = \sum_I \eta^I \left(\frac{E_1^I}{m_{I,1}c^2} - \frac{E_2^I}{m_{I,2}c^2} \right) , \tag{3.7}$$

where in the last step we have made explicit the contribution of the various energy forms E^I to the difference between inertial and gravitational masses

$$m_G - m_I \equiv \sum_I \eta^I \frac{E^I}{c^2} . \tag{3.8}$$

The experimental results are summarized in the following table.

[3] Eötvos located two test objects on the opposite ends of a dumbbell suspended from a torsion fiber. If the inertial and gravitational masses of those objects were different the centripetal effects associated with the rotation of the Earth would give rise to a torque (everywhere but at the poles) that could be measured with a delicate torsion balance pointing west-east. For a detailed description of the Eötvös' original experiments see for instance Weinberg's book.

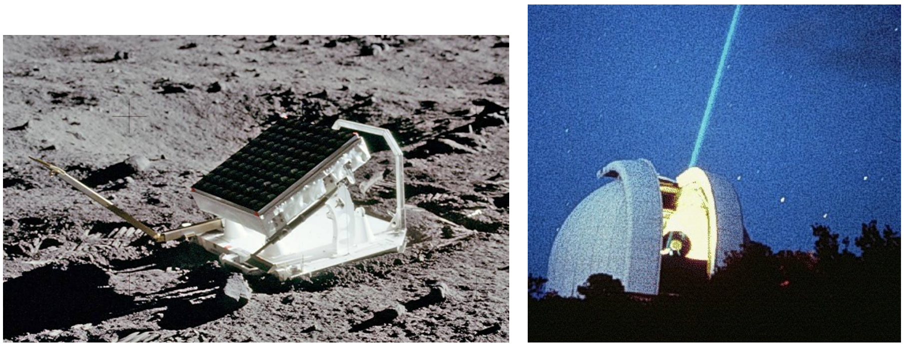


Figure 3.1: Lunar Laser Ranging (LLR) experiments

	$\frac{\|m_I - m_G\|}{m_I}$
Rest mass, proton and neutrons	$< 10^{-11}$
Rest mass, electrons	$< 2 \times 10^{-8}$
Electric fields in nucleus	$< 4 \times 10^{-10}$
Magnetic fields in nucleus	$< 2 \times 10^{-7}$
Strong fields in nucleus	$< 5 \times 10^{-10}$
Weak fields in nucleus	$< 10^{-2}$
Gravitational energy in Earth	$< 2 \times 10^{-3}$

Note that the displayed cases do not include the contribution of the gravitational self-interaction of the masses, which, for laboratory size experiments, is extremely small. Its contribution can be however tested via the so-called Nördvedt effect. If the gravitational self-energy did not follow Galileo's equivalence principle, the Earth and the Moon would fall at different rates towards the Sun, elongating the orbit of the Moon in the Sun direction. As shown by Lunar Laser Ranging experiments (LLR), which use reflectors that were located in the surface in the Moon during the Apollo 11 mission, cf. Fig. 3.1, the gravitational self-energy behaves as any other energy form, in perfect agreement with Eötvos' results. Indeed, LLR experiments provide the most accurate tests of the equivalence between inertial and gravitational masses. The constrains are really impressive

$$\eta = \frac{2|a_{\rm E} - a_{\rm M}|}{(a_{\rm E} + a_{\rm M})} = (-1 \pm 1.4) \times 10^{-13} \,. \tag{3.9}$$

Not only matter but also antimatter seems to follow the Galileo's result. Important constraints of the order 10^{-9} were obtained by the CPLEAR collaborations from neutral kaon systems[4].

[4]Under the assumption of exact CPT symmetry.

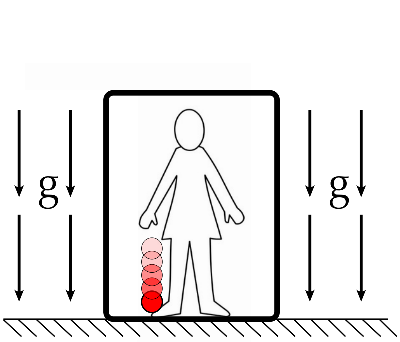


Figure 3.2: Einstein's *Gedankenexperiment*

3.2 The Equivalence Principle

The experimental equality of inertial and gravitational masses is a quite surprising and mysterious property, relating two completely different concepts and not required at all for the consistency of Newton's theory. For Galileo and Newton, this was just a coincidence. For Einstein, it would be the first stone in the impressive geometrical edifice of General Relativity. Einstein's theory will be constructed on top of something so simple that even Galileo could have discovered: the relation between gravity and inertia. To illustrate this equality let me consider one of the most famous Einstein's *Gedankenexperimente*. Imagine yourself dropping a ball in the surface of the Earth. You will see the ball falling with constant acceleration. "The effect of gravity", you would say. Now imagine yourself performing the same experiment inside a completely isolated rocket in outer space which moves with a constant acceleration $a = g$. You will observe the same: the ball falling with constant acceleration. Without knowing it, you would not be able to decide if you were in the true gravitational field of the Earth or in a rocket! This apparently trivial observation is summarized in the so-called *Weak Equivalence Principle*:

Einstein's Equivalence Principle

The trajectories of particles in the gravitational field are *locally* indistinguishable from the trajectories of free particles as viewed from an accelerated reference frame.

The gravitational interaction resembles also the pseudo forces resulting from the use of non-inertial reference frames. For example, if there is a frame of reference rotating with angular velocity ω with respect to an inertial reference frame, *all* bodies appear to accelerate spontaneously with the same acceleration in that rotating frame. It seems that there is a universal force acting on all bodies with *a magnitude proportional to their inertial masses*

$$\mathbf{F} = -m_I \left[\dot{\omega} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\right] . \tag{3.10}$$

Accelerated frames and local gravitational forces appear to be intimately related. Both of them act in the same way on all bodies, are proportional to mass and can be transformed away by changing to a suitable reference frame; a local free falling frame in the case of gravity.

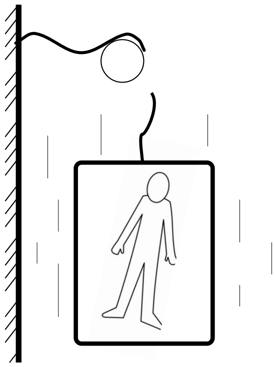


Figure 3.3: A local inertial reference frame.

Exercise

Einstein's toy: A version of the following device was constructed as a birthday present for Albert Einstein. The device consists of hollow broomstick with a cup at the top, together with a metal ball and an elastic string. When the broomstick is held vertical, the ball can rest in the cup. The ball is attached to one end of the elastic string, which passes through a hole in the bottom of the cup, and down the hollow centre of the broomstick to the bottom, where its other end is secured. You hold the broomstick vertical, with your hand at the bottom, the cup at the top, and with the ball out of the cup, suspended on its elastic string. The tension in the string is not enough to draw the ball back into the cup. The problem is to find an elegant way to get the ball back into the cup. (Inelegant ways are: using your hands or shaking the stick up and down).

3.3 Life in the rocket: Rindler spacetime

Since accelerated reference frames mimic the local effects of the gravitational field, the understanding of their properties seems to be a first step towards the correct description of gravity. Let me start by analyzing the rocket *Gedankenexperiment* presented above.

Note the slight change of notation below. From now on, we will reserve the first letters of the Greek alphabet $\alpha, \beta, \ldots$ for indices associated to *inertial* reference frames. Intermediate letters of the Greek alphabet $\mu, \nu, \ldots$ will stand for general (non-inertial, accelerated) reference frames.

Consider the movement of the accelerated rocket from the point of view of an inertial observer $\{\xi^\alpha\}$, *momentarily* at rest with respect to the rocket's trajectory. The orientation of the coordinate grid is such that the rocket moves along the ξ^3-direction. In that *instantaneous* inertial frame, the rocket is seen to undergo a constant acceleration g

$$u^\mu = (1,0,0,0)\,, \qquad a^\mu u_\mu = 0 \quad \to \quad a^\mu = (0,0,0,g)\,, \quad a^\mu a_\mu = g^2\,. \tag{3.11}$$

In order to determine the wordline $\xi^\alpha(\tau)$ of the rocket at later times, let us look for a general solution of the covariant equation $u^\alpha u_\alpha = -1$. Writing it explicitly

$$\eta_{\alpha\beta}u^\alpha u^\beta = -(u^0)^2 + (u^3)^2 = -1\,, \tag{3.12}$$

the solution becomes pretty obvious

$$u^0 = \cosh f(\tau)\,, \qquad u^3 = \sinh f(\tau)\,. \tag{3.13}$$

The unknown function $f(\tau)$ can be determined by taking the derivative of the last two equations

$$a^\alpha = \frac{du^\alpha}{d\tau} = \dot f(\tau)\,(\sinh f(\tau), 0, 0, \cosh f(\tau)) \tag{3.14}$$

and imposing the covariant condition $a^\mu a_\mu = g^2$. We get $g^2 = \dot f^2$, $f(\tau) = g\tau$ and

$$u^\alpha = (\cosh g\tau, 0, 0, \sinh g\tau)\,. \tag{3.15}$$

The work is almost done. Integrating (3.15) with the initial condition $\xi^\alpha(0) = (0,0,0,g^{-1})$, we get

$$\xi^\alpha(\tau) = g^{-1}(\sinh g\tau, 0, 0, \cosh g\tau)\,. \tag{3.16}$$

The "constantly accelerated" rocket describes an equilateral hyperbola with semi-major axis $1/g$

$$(\xi^3)^2 - (\xi^0)^2 = g^{-2}\,. \tag{3.17}$$

Let us now look at the problem from the point of view of an accelerated observer sitting in the rocket. Since the transformation from inertial to accelerated frames is not a Lorentz transformation, we should expect a change in the Minkowski line element ds^2. The natural coordinates for the accelerated observer are those adapted to its trajectory. Let's call them $(x^0, x^3) = (\eta, \rho)$. The transformation to this frame takes the form[5]

$$\xi^0(\eta,\rho) = \rho\sinh\eta\,, \qquad \xi^3(\eta,\rho) = \rho\cosh\eta\,. \tag{3.18}$$

In terms of the new coordinates η and ρ, the Minkowski line element $ds^2 = -(d\xi^0)^2 + (d\xi^3)^2$ becomes modified

$$ds^2 = -\rho^2 d\eta^2 + d\rho^2 \equiv g_{\mu\nu}dx^\mu dx^\nu\,. \tag{3.19}$$

The metric $g_{\mu\nu} = \mathrm{diag}(-\rho^2, 1)$ is now space-time dependent!

[5]The change of coordinates is just the Lorentzian analogue of polar coordinates. inspired on the wordline equation (3.16).

3.4 Beyond inertial observers

Let us formalize the concepts appearing in the previous example. The distance between two neighboring points, as measured in a local inertial frame at rest with respect to the particle[6], is given by the Minkowski metric $\eta_{\alpha\beta}$

$$ds^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta \,. \tag{3.20}$$

When performing a transformation to a general coordinate system[7] $x^\mu = x^\mu(\xi^\alpha)$, the line element becomes modified

$$ds^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu \frac{\partial \xi^\beta}{\partial x^\nu} dx^\nu = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu \,. \tag{3.21}$$

and distances are no longer determined by the Minkowski metric, but rather by a metric

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \,, \tag{3.22}$$

which generically depends on the spacetime coordinates. The inverse of the new metric is defined through the relation $g^{\mu\nu} g_{\nu\lambda} = \delta^\mu{}_\lambda$ and can be easily computed by taking into account the identity

$$\frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial \xi^\beta}{\partial x^\mu} = \delta^\beta{}_\alpha \,. \tag{3.23}$$

We get

$$g^{\mu\nu} = \eta^{\alpha\beta} \frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial x^\nu}{\partial \xi^\beta} \,. \tag{3.24}$$

Exercise

Prove the *similarity transformation* (3.24).

Note that the reference frame x^μ is not at all privileged. We could perfectly move now into another non-inertial reference frame $\bar{x}^\rho(\xi^\alpha)$ in which

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \frac{\partial x^\mu}{\partial \bar{x}^\rho} d\bar{x}^\rho \frac{\partial x^\nu}{\partial \bar{x}^\sigma} d\bar{x}^\sigma = g_{\mu\nu} \frac{\partial x^\mu}{\partial \bar{x}^\rho} \frac{\partial x^\nu}{\partial \bar{x}^\sigma} d\bar{x}^\rho d\bar{x}^\sigma = \bar{g}_{\rho\sigma} d\bar{x}^\rho d\bar{x}^\sigma \,, \tag{3.25}$$

with

$$\bar{g}_{\rho\sigma} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \bar{x}^\rho} \frac{\partial x^\nu}{\partial \bar{x}^\sigma} \,. \tag{3.26}$$

The results of this section are summarized in Figure 3.4.

[6] In the context of a particle in the gravitational field these frames are called *local free falling reference frames*.

[7] General means completely arbitrary. It can be a curvilinear coordinate system, an accelerated system, a rotating system ... whatever you want.

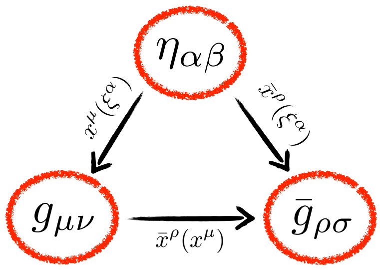


Figure 3.4

Exercise

Show that $g_{\mu\nu}$ must be symmetric, i.e. $g_{\mu\nu} = g_{\nu\mu}$.

3.5 The geodesic equation

The equation of motion for a free particle in an accelerated reference frame can be obtained by applying the transformation (3.22) to the Lagrangian of a free relativistic particle (2.51). We get

$$S = \frac{1}{2}\int d\sigma \left(e^{-1}(\sigma) g_{\mu\nu} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} - m^2 e(\sigma) \right) . \tag{3.27}$$

As in the Minkowski case, the action (3.27) is invariant under reparametrizations of the path $\sigma \to \sigma = f(\sigma)$ provided that we let $e(\sigma)$ transform in such a way that the quantity $e(\sigma)d\sigma$ is left invariant. Note also that the action (3.27) is invariant under general coordinate transformations[8], as can be easily seen by taking into account the similarity relation (3.26). The Euler-Lagrange equation $\frac{\partial L}{\partial e} = 0$ for the non-dynamical variable $e(\sigma)$

$$g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = -m^2 e(\sigma)^2 , \tag{3.28}$$

automatically incorporates the massive ($e(\sigma) = 1/m$) and massless ($e(\sigma) = 1, m \to 0$) cases we are interested in. In these two limits, the action (3.27) takes the form

$$S_{\text{massive}} = \frac{1}{2} m \int d\sigma \left(g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} - 1 \right) , \qquad S_{\text{massless}} = \frac{1}{2}\int d\sigma \left(g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} \right) , \tag{3.29}$$

with $\sigma = \tau$ for the massive case. S_{massive} and S_{massless} are very similar. Indeed, the computation of the equations of motion is formally equivalent in both cases[9]. Let us denote by a dot the derivative with respect to σ and forget in what follows about the irrelevant factors m and $m/2$. The equations of motion

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{x}^\rho} \right) - \frac{\partial L}{\partial x^\rho} = 0 \tag{3.30}$$

[8] This should be expected from pure Lagrangian mechanics.

[9] The only difference is the presence of a global factor m and a constant term $m/2$ which do not play any role in the variation of the action

for the generalized coordinates x^ρ can be computed as follows. The simplest part is the variation of $1/2 g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ with respect to x^ρ. All the dependence on the coordinates is hidden in the metric

$$\frac{\partial L}{\partial x^\rho} = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\rho}\dot{x}^\mu \dot{x}^\nu \,. \tag{3.31}$$

The variation with respect to $\dot{x}^\rho$ is slightly more involved, but can be however written in a very compact way by taking into account the properties

$$\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\rho} = \delta^\mu{}_\rho \qquad g_{\mu\nu}\delta^\nu{}_\rho = g_{\mu\rho}\,, \qquad g_{\mu\nu} = g_{\nu\mu}\,, \tag{3.32}$$

together with some simple index relabeling. We get

$$\frac{\partial}{\partial \dot{x}^\rho}\left(\frac{1}{2}g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right) = \frac{1}{2}\left(g_{\mu\nu}\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\rho}\dot{x}^\nu + g_{\mu\nu}\dot{x}^\mu \frac{\partial \dot{x}^\nu}{\partial \dot{x}^\rho}\right) = \frac{1}{2}\left(g_{\rho\nu}\dot{x}^\nu + g_{\mu\rho}\dot{x}^\mu\right) = g_{\rho\nu}\dot{x}^\nu \,. \tag{3.33}$$

Collecting the two pieces, the Euler-Lagrange equations (3.30) become

$$\begin{aligned}
\frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}^\rho} - \frac{\partial L}{\partial x^\rho} &= \frac{d}{d\sigma}\left(g_{\rho\nu}\dot{x}^\nu\right) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\rho}\dot{x}^\mu \dot{x}^\nu \\
&= \frac{\partial g_{\rho\nu}}{\partial x^\sigma}\dot{x}^\sigma \dot{x}^\nu + g_{\rho\nu}\ddot{x}^\nu - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\rho}\dot{x}^\mu \dot{x}^\nu \\
&= g_{\rho\nu}\ddot{x}^\nu + \dot{x}^\nu \dot{x}^\sigma \left(\frac{\partial g_{\rho\nu}}{\partial x^\sigma} - \frac{1}{2}\frac{\partial g_{\sigma\nu}}{\partial x^\rho}\right) \\
&= g_{\rho\nu}\ddot{x}^\nu + \frac{1}{2}\,\dot{x}^\nu \dot{x}^\sigma \left(\frac{\partial g_{\rho\nu}}{\partial x^\sigma} + \frac{\partial g_{\rho\sigma}}{\partial x^\nu} - \frac{\partial g_{\sigma\nu}}{\partial x^\rho}\right) \\
&= 0\,.
\end{aligned} \tag{3.34}$$

The work is done. Multiplying by the inverse metric and relabeling indices we obtain the equation we were looking for, the so-called *geodesic equation*

$$\frac{d^2 x^\mu}{d\sigma^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\sigma}\frac{dx^\rho}{d\sigma} = 0\,, \qquad \Gamma^\mu{}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(\partial_\rho g_{\sigma\nu} + \partial_\nu g_{\sigma\rho} - \partial_\sigma g_{\nu\rho}\right)\,. \tag{3.35}$$

Exercise

- Consider a reparametrization $\sigma \to f(\sigma)$. Show that the geodesic equation (3.35) retains its form only if $f(\sigma) = a\sigma + b$.
- Compute the geodesic equation associated to the Rindler metric (3.16).

The geodesic equation is automatically covariant since the Lagrangian from which it was derived was invariant under general coordinate transformations. The transformation of the so-called *Christoffel symbols* $\Gamma^\mu{}_{\nu\lambda}$ is however non-homogeneous

$$\bar{\Gamma}^{\mu'}{}_{\nu'\rho'} = \Gamma^\mu{}_{\nu\rho}\frac{\partial \bar{x}^{\mu'}}{\partial x^\mu}\frac{\partial x^\nu}{\partial \bar{x}^{\nu'}}\frac{\partial x^\rho}{\partial \bar{x}^{\rho'}} + \frac{\partial \bar{x}^{\mu'}}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial \bar{x}^{\nu'}\partial \bar{x}^{\rho'}}\,. \tag{3.36}$$

They are *not* a tensor.

Exercise

- Which is the form taken by Eq. (3.37) in a Cartesian coordinate system?
- Prove the transformation law (3.36) using the fact the the geodesic equation is covariant.
- How many independent components have the Christoffel symbols in four dimensions?

The Christoffel symbols encode the *local* aspects of the gravitational interaction as well as the *fictitious* forces (centrifugal, Coriolis, etc)

$$F^\mu \equiv \frac{d^2x^\mu}{d\sigma^2} = -\Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} \tag{3.37}$$

arising when using non-inertial reference frames. This kind of forces can always be eliminated by going to an inertial reference frame or to a *free-falling frame*. Note that this would not be the case if the Christoffel symbols were tensors.

Local free-falling reference frames

The geodesic equation allows for a precise definition of *local free-falling frames*. According to Equivalence Principle, in those frames the geodesic equation must become

$$\left.\frac{d^2\xi^\alpha}{d\sigma^2}\right|_P = 0\,. \tag{3.38}$$

A free-falling reference frame at P is therefore defined as[a]

$$g_{\mu\nu}(P) = \eta_{\mu\nu}\,, \qquad \partial_\sigma g_{\mu\nu}(P) = 0 \tag{3.39}$$

with the second condition being equivalent to the vanishing of the Christoffel symbols at that point, i.e. $\Gamma^\mu{}_{\nu\lambda}(P) = 0$.

[a] The existence of these frames is guaranteed by the so-called *Local flatness theorem*.

3.5.1 Massive particles don't go on diet

It is easy to show that the geodesic equation (3.35) is equivalent to a conservation equation

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\sigma}\frac{dx^\rho}{d\sigma} = 0 \quad \rightarrow \quad \frac{d}{d\sigma}\left(g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\right) = 0 \tag{3.40}$$

associated to translations in the parameter σ, $f(\sigma) = \sigma + c$. The physical meaning of Eq. (3.40) is quite obvious and should be expected; it simply states that massless/massive particles remain massless/massive along the geodesic. In other words, given an initial condition

$$\left. g_{\mu\nu}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma}\right|_{\sigma=0} = \begin{cases} -1 & \text{for massive particles} \\ 0 & \text{for massless particles} \end{cases}, \tag{3.41}$$

it will by satisfied for all values of σ[10]. We will rediscover this equation in Chapter 4, when dealing with the concept of parallel transport.

Exercise

Prove the relation (3.40).

3.5.2 Conserved quantities

Note that if the metric coefficients are independent of one coordinate[11] x^ν, the Lagrangian (3.27) will be also independent of such a coordinate. In such a case, the covariant component $\dot{x}_\nu$ is a conserved quantity along affinity parametrized geodesics

$$p_\nu = \frac{\partial L}{\partial \dot{x}^\nu} = g_{\mu\nu}\dot{x}^\mu = \dot{x}_\nu = \text{constant}\,. \tag{3.42}$$

We will make use of this important property in Chapter 8, when dealing with the Schwarzschild geometry.

3.6 The Newtonian limit

Let us see how the usual results of Newtonian gravity fit into the geometric picture. Of course, we cannot expect to link the relativistic formulation presented above with a non-relativistic theory of gravity without doing some assumptions. We will consider a massive particle moving at small velocity (with respect to the speed of light)

$$\frac{dx^i}{dt} \ll 1 \qquad \longrightarrow \qquad \frac{dx^i}{d\tau} \ll \frac{dt}{d\tau} \tag{3.43}$$

in a "weak"

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad |h_{\mu\nu}| \ll 1 \tag{3.44}$$

and stationary[12] gravitational field

$$\partial_0 g_{\mu\nu} = \partial_0 h_{\mu\nu} = 0\,. \tag{3.45}$$

The first two conditions (small velocities and weak fields) are quite natural from the point of view of a non-relativistic description. On the other hand, the stationarity condition (3.45) is just a good approximation for the particular cases we will be interested in in this section: the gravitational fields of the Sun and the Earth.

At first order in the small perturbation $h_{\mu\nu}$, the geodesic equation (3.35) takes the form

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0\,, \tag{3.46}$$

[10] Note that for the massive case we can identify $\sigma = \tau$

[11] The coordinate x^ν is then said to be *cyclic*.

[12] Or varying sufficiently slow over the scale probed by the particle.

where the Christoffel symbols Γ^μ_{00} are completely determined by the perturbation $h_{\mu\nu}$ around the Minkowsky metric[13]

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\rho} \left(\frac{\partial g_{0\rho}}{\partial x^0} + \frac{\partial g_{0\rho}}{\partial x^0} + \frac{\partial g_{00}}{\partial x^\rho} \right) = -\frac{1}{2} g^{\mu\rho} \frac{\partial g_{00}}{\partial x^\rho} = -\frac{1}{2} \eta^{\mu\rho} \frac{\partial h_{00}}{\partial x^\rho} \,. \tag{3.47}$$

Splitting the spatial and temporal components of Eq. (3.46) and using the stationarity condition (3.45), we obtain[14]

$$\frac{d^2 t}{d\tau^2} = 0 \,, \qquad \frac{d^2 x^i}{d\tau^2} = \frac{1}{2} c^2 \frac{\partial h_{00}}{\partial x^i} \,, \tag{3.48}$$

The first of these two equations allows us to identify the proper time τ with the coordinate time t and write

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2} c^2 \frac{\partial h_{00}}{\partial x^i} \,. \tag{3.49}$$

The value of the unknown function h_{00} can be determined by comparing Eq. (3.48) with the Newtonian expression for a particle in a gravitational field

$$\frac{d^2 x^i}{dt^2} = -\delta^{ij} \frac{\partial \Phi}{\partial x^j} \,. \tag{3.50}$$

The first *true* component of the gravitational metric tensor[15] comes directly from Newton's theory!

$$h_{00} = -\frac{2\Phi}{c^2} \qquad \longrightarrow \qquad g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right) \,. \tag{3.51}$$

Indeed... this is the *first* and the *last* component that we can expect to get from Newton ... Newtonian gravity involves just one scalar function: the gravitational potential Φ, nothing else. This observation naturally raises the question of how to compute the remaining components of the metric. Let's forget about this problem for a while and enjoy our findings. As you will see, we can learn a lot of new things without knowing the precise form of the other metric components. The correction to the Minkowski metric is proportional to the so-called *gravitational self-energy* Φ/c^2. This quantity can be understood as the ratio of the Newtonian potential energy to the relativistic energy. For an object of mass M and typical size R we have

$$\frac{|\Phi|}{c^2} = \frac{GM^2}{R} \cdot \frac{1}{Mc^2} = \frac{GM}{Rc^2} \,. \tag{3.52}$$

Some orders of magnitude for Φ can be found in Table 3.1. Note that, even for a white dwarf or a galaxy, the gravitational self-energy is very small; the weak field approximation used in the derivation of Eq. (3.51) is justified. The correction to the Minkowski metric is expected to be important only for very compact object such as a neutron star or a black hole.

[13] Note that, since we are interested only in first order terms, we can raise and lower indices with the Minkowski metric $\eta_{\mu\nu}$. For example $h^\mu{}_\nu = g^{\mu\lambda} h_{\lambda\nu} \simeq \eta^{\mu\lambda} h_{\lambda\nu} + \mathcal{O}(h^2_{\mu\nu})$.

[14] Note that we have restored the speed of light c for later convenience.

[15] Note that we could in principle allow for an extra constant C in Eq. (3.51), i.e $h_{00} = -\frac{2\Phi}{c^2} + C$, which should be fixed by requiring the metric to approach the flat Minkowski metric at infinity. For isolated mass distributions the gravitational potential Φ vanishes at infinity and therefore $C = 0$.

	Mass	Size	$\lvert\Phi\rvert/c^2$
Atome	10^{-26} Kg	10^{-10} m	10^{-43}
Human	10^{2} Kg	1 m	10^{-25}
Earth	10^{25} Kg	10^{7} m	10^{-9}
Sun	10^{30} Kg	10^{9} m	10^{-6}
Galaxy	10^{41} Kg	10^{21} m	10^{-7}
White Dwarf	10^{30} Kg	10^{7} m	10^{-4}
Neutron Star	10^{30} Kg	10^{4} m	0.1
Black Hole			1

Table 3.1: Gravitational self-energy: Orders of magnitude

3.7 The power of the equivalence principle

Let us consider the direct consequences of the previous results. In order to get some intuition let me go back for a moment to the constantly accelerated rocket and perform the following Gedankexperiment. Imagine two observers in the rocket, one on the base and one on a platform close to the ceiling. The observer at the bottom sends some light pulses with a frequency dictated by its proper time interval $\nu_1 = 1/\Delta\tau_1$. Due to the acceleration of the rocket, these pulses are received by the observer at the top at a lower rate $\nu_2 = 1/\Delta\tau_2$ than the rate at which they were emitted[16]. According to the Equivalence Principle the same phenomenon should happen in the presence of gravity. Yes, gravity must affect the flow of time!

Let us put our Gedankenexperiment into equations. Having a look to Eq. (3.51), we see that the interval in proper time $d\tau$ at a fixed point in the vicinity of a massive object differs from the interval in coordinate time dt

$$d\tau = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} = \sqrt{1+\frac{2\Phi(r)}{c^2}}dt\,. \tag{3.53}$$

A local measurement of this effect is nevertheless impossible since our measure instruments are affected by gravity in the same way that the timing of the objects we want to measure. Observable effects on the flow of time can only appear when we compare two different points in the gravitational potential Φ, as we did in the rocket *Gedankenexperiment.*

3.7.1 Gravity and the flow of time

Consider two observers in the weak gravitational field of a spherically symmetric and stationary mass distribution. Although the Newtonian limit developed in the Section 3.6 provides

[16] By the time at which the light arrives to the top the ceiling is moving faster than when the light was emitted.

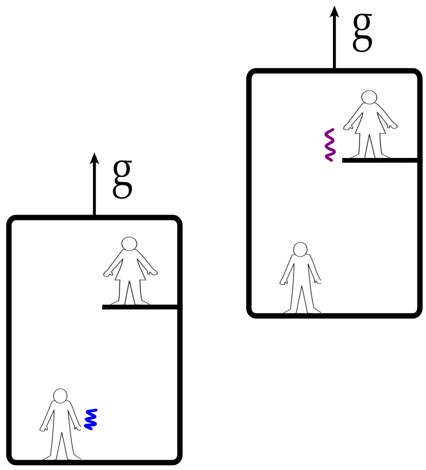


Figure 3.5: Gravitational redshift, *Gedankenexperiment*.

only information about the g_{00} element, the large symmetry of the problem severely constrains the form of the metric to be

$$ds^2 = -\left(1 + \frac{2\Phi(r)}{c^2}\right) dt^2 + g_{rr}(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \,, \tag{3.54}$$

with $g_{rr}(r)$ an undetermined function of the radial distance, whose explicit form will not be needed in what follows. In order to disentangle the effect of gravity from other velocity dependent Doppler-like effects and to make the analysis as clear as possible, we will require the observers to be at rest in a radial configuration with coordinates r_1 and r_2. Imagine the observer at r_1 sending pulses of light to the observer at r_2. The period of emitted pulses is the interval in proper time of the emitter

$$\Delta\tau_1 = \int \sqrt{-g_{00}(r_1)}dt = \sqrt{-g_{00}(r_1)} \int dt = \sqrt{-g_{00}(r_1)}\Delta t_1 \,. \tag{3.55}$$

On the other hand, the period of received pulses is the interval in proper time of the receiver

$$\Delta\tau_2 = \int \sqrt{-g_{00}(r_2)}dt = \sqrt{-g_{00}(r_2)} \int dt = \sqrt{-g_{00}(r_2)}\Delta t_2 \,. \tag{3.56}$$

The coordinate interval elapsed between the emission of two pulses Δt_1 is equal to the coordinate time interval elapsed between the reception on two pulses Δt_2, as can be easily seen by noting that the coordinate time interval needed to go from r_1 to r_2

$$ds^2 = -g_{00}dt^2 + g_{rr}dr^2 = 0 \qquad \longrightarrow \qquad \Delta t = \int_{r_1}^{r_2} dr \left(\frac{-g_{rr}(r)}{g_{00}(r)}\right) \tag{3.57}$$

is independent of the coordinate time t. Taking the ratio of Eqs. (3.56) and (3.55), we get the first prediction of the Equivalence Principle

$$\frac{\Delta\tau_2}{\Delta\tau_1} = \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}} = \sqrt{\frac{1 + 2\Phi(r_2)/c^2}{1 + 2\Phi(r_1)/c^2}} \,. \tag{3.58}$$

Time of flight	41.2 h	48.6 h
$\Delta\tau$ (ns)	Eastward	Westward
$\Delta\tau_{\rm G}$	144 ± 14	179 ± 18
$\Delta\tau_{\rm SR}$	-184 ± 18	96 ± 10
$\Delta\tau_{\rm tot}$	-40 ± 23	275 ± 21
$\Delta\tau_{\rm obs}$	-59 ± 10	273 ± 7

Table 3.2: Hafele Keating: Predictions and experimental results

For weak gravitational fields, the previous expression can be approximated by its binomial expansion[17]

$$\frac{\Delta\tau_2}{\Delta\tau_1} \simeq 1 + \frac{\Phi(r_2) - \Phi(r_1)}{c^2}\,, \tag{3.59}$$

which is usually quoted in terms of the ratio

$$\frac{\Delta\tau}{\tau} \equiv \frac{\Delta\tau_2 - \Delta\tau_1}{\Delta\tau_1} = \frac{\Phi(r_2) - \Phi(r_1)}{c^2}\,. \tag{3.60}$$

Time goes by slower

Clocks slow down in those places where the gravitational potential is larger (in magnitude). In particular, clocks at a distance r from the surface of a massive spherical body of mass M ($\Phi = -\frac{GM}{r}$) slow down by a factor $\sqrt{1 - \frac{2GM}{rc^2}}$ with respect to clocks at $r \to \infty$ ($\Phi = 0$).

The dilation of time was tested by Hafele and Keating in 1972 using cesium-beam atomic clocks transported on commercial flights around the Earth and compared on return to standard clocks in the US Naval Observatory. The net effect on the reading of the on-flight clocks is a combination of special relativistic effects and gravitational changes in the flow of time. The two contributions act in an opposite way. Special Relativity tends to decrease the rate of the clock in the plane with respect to the standard clock in the surface of the Earth[18]. On the other hand, gravity tends to speed up the clock in the plane with respect to the clock in the stronger gravitational field of the Earth. The experiment was performed twice, once flying towards the east and once flying towards the west. The results and their comparison with the predictions are summarized in Table 3.2. As you can see, the agreement between the theory and the theoretical prediction is notably good.

[17] $(1+x)^{1/2} = 1 + \frac{1}{2}x$.

[18] This is just a consequence of the well-known time dilation effect in Special Relativity.

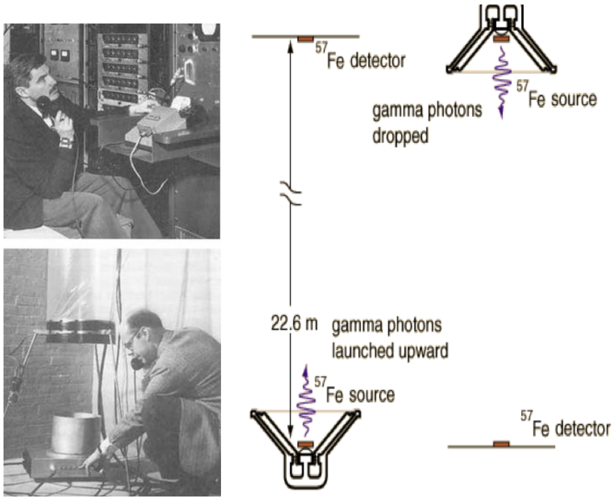


Figure 3.6: The highs and lows: Redka and Pound at the top and bottom of the tower.

Exercise
How older are the theorists of the upper floor of the Cubotron with respect to the experimentalists in the lower floor at the end of their academic life? Should this effect be taken into account by the Swiss pension system?

3.7.2 Gravitational shift of frequencies

An immediate consequence of the previous result is the gravitational frequency shift. Denoting by ν_1 the frequency of the light emitted at r_1 and by ν_2 the frequency of the light received at r_2, Eq. (3.58) can be rewritten as

$$\frac{\nu_1}{\nu_2} \equiv 1 + z = \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}} = \sqrt{\frac{1 + 2\Phi(r_2)/c^2}{1 + 2\Phi(r_1)/c^2}}\,, \tag{3.61}$$

where we have defined the so-called *redshift parameter*

$$z = \frac{\Delta\nu}{\nu} \equiv \frac{\nu_1 - \nu_2}{\nu_2}\,. \tag{3.62}$$

It $z > 0$ the received light is said to be *redshifted*, while if $z < 0$ the light is said to be *blueshifted*. For weak gravitational potentials, Eq. (3.61) can be approximated by

$$z = \frac{\Delta\nu}{\nu} \simeq \frac{\Phi(r_2) - \Phi(r_1)}{c^2}\,. \tag{3.63}$$

To get an estimate of the order of magnitude of this frequency shift, consider for instance the light from the Sun ($r = r_1$) received on Earth ($r = r_2$)[19]. Since $r_2 > r_1$, $|\Phi(r_1)| > |\Phi(r_2)|$ and

[19] $\Phi(r_1) = -GM_\odot/r_1$ and $\Phi(r_2) = -GM_\odot/r_2$ are small (cf. Table 3.1). The binomial expansion is justified. The gravitational field of the Earth is neglected.

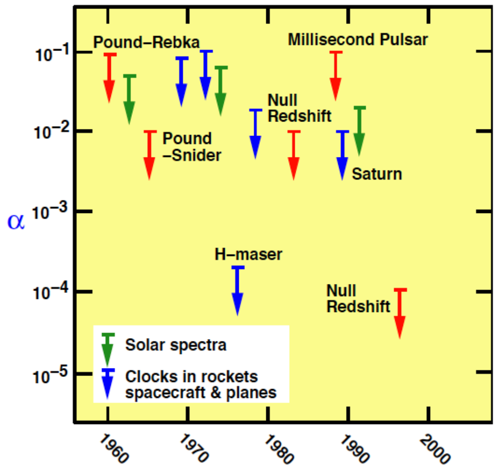


Figure 3.7: Different tests of the gravitational redshift. The parameter α parametrizes the deviations from the Equivalence Principle, $\Delta\nu/\nu = (1+\alpha)\Delta\Phi/c^2$.

therefore[20] $z > 0$. As in our *Gedankenexperiment*, the light redshifts ($\nu_1 > \nu_2$) as it climbs upwards in the gravitational potential!

The gravitational frequency shift is a test of the Equivalence Principle, not of the Einstein's theory of gravity in its full form. Note that the spatial part of the metric $g_{rr}(r)$ did not played any role in the previous developments.

Numerically, the gravitational redshift of the light emitted by the Sun is very small

$$\frac{\Delta\nu}{\nu} = 2.12 \times 10^{-6}\,, \tag{3.64}$$

and indeed very difficult to detect due to the broadening of spectral lines and to Doppler shifts associated to the convection currents in the solar atmosphere[21].

A more precise non-astronomical test of the gravitational frequency shift was performed by Pound and Redka in 1960 using gamma rays produced in a 14.4 keV atomic transition in ^{57}Fe. These gamma rays were emitted at the top of a tower of 22.6 meters in the Jefferson Physical Laboratory at Harvard university and directed down towards a similar sample of ^{57}Fe located at the bottom of the tower. The absorption of the gamma rays by the receiver is only efficient if the frequency at reception coincides with the frequency at emission (Mössbauer effect). Due to the gravitational shift of frequencies this was not the case. Pound and Rebka compensated the gravitational shift in a very clever way: a Doppler shift induced by the

[20] Remember that the gravitational potential is negative.

[21] The gravitational redshift (3.64) corresponds numerically to the Doppler shift associated to a velocity of 0.6 Km/h, which is easily exceed by the hot gases in the surface of the Sun.

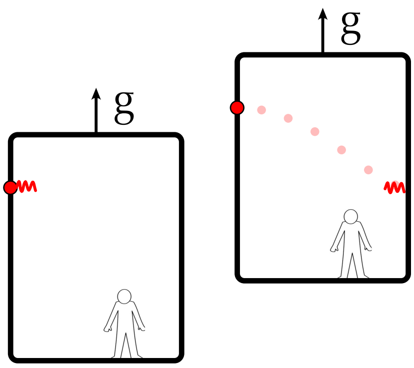


Figure 3.8: Deflection of light, *Gedankenexperiment*.

vertical motion of the source at the top of the tower. By looking for a resonance in the absorption they were able to obtain a direct measurement of the gravitational redshift. The result was in excellent agreement with the Equivalence Principle's prediction

$$\frac{(\Delta\nu/\nu)_{\rm exp}}{(\Delta\nu/\nu)_{\rm th}} = 1.05 \pm 0.10\,. \qquad (3.65)$$

Exercise

Determine the predicted value $\Delta\nu/\nu$ in the Pound-Rebka experiment. Are the gamma rays traveling down the tower blueshifted or redshifted?

3.8 The weakness of the Equivalence Principle

A second (and incomplete) prediction of the Equivalence Principle is the deflection of light. This effect can be easily understood with another *Gedankenexperiment*. Imagine an observer in the accelerated rocket. A pulse of light is emitted by some device from one of the walls in the transverse direction to the rocket motion. Due to the acceleration of the rocket, the pulse will hit the opposite wall at a height below that of the emission. Since the uniform acceleration of the rocket is locally indistinguishable from a gravitational field, we should expect the same *deflection of light* in a gravitational field.

The bending of light in a gravitational field was considered by Newton himself, but he didn't performed any proper computation. The first known result about the deflection of light was presented by the German astronomer Johann Georg von Soldner in 1804. Based on Newton's corpuscular theory of light, Soldner predicted a deflection angle of $0.87''$ for a ray of light grazing the surface of the Sun. Einstein, unaware of Soldner's computations and based on the Equivalence Principle, obtained the same number one hundred years later, in 1911. Let us reproduce his arguments and (wrong) results.

3.8.1 Einstein's 1911 (wrong) treatment

In Einstein's 1911 paper, the speed of light is considered as a *scalar* quantity which depends on the value of the gravitational field

$$ds^2 = -(1+2\Phi)dt^2 + dX^2 = 0 \qquad \rightarrow \qquad c = c_0\,(1+\Phi) \tag{3.66}$$

The Minkowski value $c = c_0$ is only recovered at long distances ($r \to \infty$), where the gravitational potential is negligible[22]. According to Huygens' principle the position of a wavefront at a time $t + \Delta t$ can be determined by considering each point of the wavefront at t as a source of spherical waves. The wavefront at $t + \Delta t$ is then given by the envelope of the multiple spherical wavefronts originated at t. Imagine a wave front in the vicinity of a matter distribution M. Consider two points P_1 and P_2 separated by a spatial distance δl at time t. The velocity of light at those points (c_1 and c_2) depends of the value of the gravitational field. Having a look to Fig. 3.9, we conclude that in a time δt the wavefront in refracted by an angle

$$\delta\alpha = \frac{(c_1 - c_2)\,\delta t}{\delta l} = \frac{\delta\Phi}{\delta l}\delta t\,, \tag{3.67}$$

with $\delta\Phi/\delta l$ the component of the gravitational acceleration along the wavefront. This infinitesimal refraction angle can be integrated along the full path to obtain the total deflection angle[23]

$$\alpha = \int d\alpha = \int \frac{d\Phi}{dl}dt\,. \tag{3.68}$$

Since the velocity of light along the path is nearly constant we can set $dt = ds$, with s measuring the distance along the path. Evaluating the integral (3.68) for an impact parameter b, we get

$$\alpha = \int \frac{d\Phi}{dl}ds = \int_{-\pi/2}^{\pi/2} \frac{GM}{r^2}\cos\theta ds = \frac{2GM}{b}\,, \tag{3.69}$$

which for the particular case of a photon grazing the surface of the Sun becomes[24]

$$\alpha = \frac{2GM_\odot}{c^2 R_\odot} \approx 0.875''\,. \tag{3.70}$$

[22] In "Relativity,The Special and General Theory", Einstein wrote:

> *[...] our result shows that, according to the general theory of relativity, the law of the constancy of the velocity of light in vacuo, which constitutes one of the two fundamental assumptions in the special theory of relativity and to which we have already frequently referred, cannot claim any unlimited validity. A curvature of rays of light can only take place when the velocity of propagation of light varies with position. Now we might think that as a consequence of this, the special theory of relativity and with it the whole theory of relativity would be laid in the dust. But in reality this is not the case. We can only conclude that the special theory of relativity cannot claim an unlimited domain of validity; its result hold only so long as we are able to disregard the influences of gravitational fields on the phenomena (e.g. of light).*

[23] This is a good approximation for small deflections angles, as is the case of the deflection of light by the Sun.

[24] Note that we have restored the powers of c.

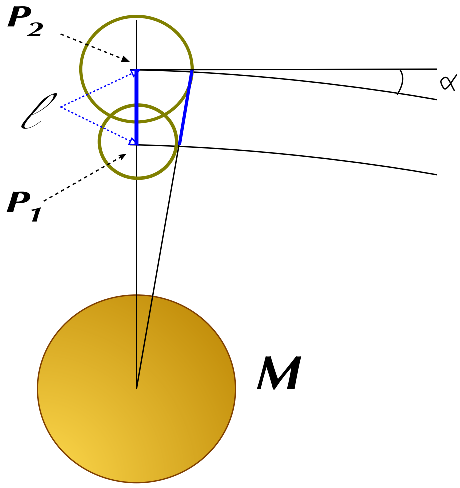


Figure 3.9: Deflection of light, Huygens's principle.

As Einstein stated in the original paper, since 'the fixed stars in the part of the sky near the sun are visible during a total eclipse of the sun, this consequence of the theory may be compared to experiment". He indeed "urgently wishers astronomers to take up this question" and measure the deflection of light during a solar eclipse. Fortunately for him . . . they didn't do it on time. Einstein's 1911 prediction based only in the equivalence principle was incomplete[25]. No measurement of the deflection angle was performed between 1911 and 1915, the moment at which he straightens out his result to

$$\alpha = 2 \times \frac{2GM_\odot}{c^2 R_\odot} \approx 2 \times 0.875'' \,. \tag{3.71}$$

Although different expeditions to observe solar eclipses were organized, all of them were cancelled, either for climatological or political reasons. One of the most interesting stories is that of the German astronomer and mathematician Erwin Finlay Freundlich, which, interested on testing Einstein's prediction, convinced the german armament manufacturer Krupp to finance a trip to Crimea on 21st August 1914. Unfortunately for him, and fortunately for Einstein, the German astronomer was arrested by the Russians as a suspected spy before being able to perform any measurement.

[25] We will see why later on.

CHAPTER 4

GENERAL COORDINATES

> No one can understand the new law of gravitation without a thorough knowledge of the theory of invariants and of the calculus of variations
>
> J. J. Thomson
> Royal Society, 1919

In Euclidean and Minkowski spacetimes we were dealing with global Cartesian coordinate systems. Our goal in this Chapter is to develop the mathematical tools needed to write physical equations in a way completely independent of the particular coordinate system we actually end up using.

4.1 General coordinate transformations

Consider a completely arbitrary set of coordinates $\{x^\mu\}$ and a set of natural basis vectors $\{\mathbf{e}_\mu\}$ tangent to the curves $x^\mu =$ constant in N dimensions. These basis vectors are allowed to change in magnitude and/or direction from point to point. In general, they will satisfy

$$\mathbf{e}_\mu(x) \cdot \mathbf{e}_\nu(x) = g_{\mu\nu}(x)\,. \tag{4.1}$$

with $g_{\mu\nu}$ generically depending on the coordinates. Once we have established a coordinate basis, we can define the components of a vector. Let us consider the infinitesimal displacement vector between two points

$$d\mathbf{S} = dx^\mu \mathbf{e}_\mu\,, \tag{4.2}$$

whose scalar product with itself defines the line element

$$|d\mathbf{S}|^2 \equiv ds^2 = \mathbf{e}_\mu(x) \cdot \mathbf{e}_\nu(x)\, dx^\mu dx^\nu = g_{\mu\nu}(x) dx^\mu dx^\nu\,. \tag{4.3}$$

The above expression represents a generalization of the Pythagorean theorem for an arbitrary coordinate system. As usual, the inverse of the metric $g^{\mu\nu}$ is defined through the relation $g^{\mu\nu} g_{\nu\lambda} = \delta^{\mu}{}_{\lambda}$. Note that both $g_{\mu\nu}(x)$ and $g^{\mu\nu}(x)$ are symmetric.

Exercise

- Prove that $g_{\mu\nu}$ is a symmetric matrix. Hint: Assume $g_{\mu\nu}$ is not symmetric and decompose it into a symmetric and an antisymmetric part.
- Which is the number of independent components of a general symmetric matrix in N dimensions?

Consider two different coordinate systems x^μ and $\bar{x}^\mu$ related to each other by an arbitrary coordinate transformation

$$\bar{x}^\mu = f^\mu(x^1, x^2, \ldots, x^N)\,, \qquad (\mu = 1, 2, \ldots, N)\,. \tag{4.4}$$

The N arbitrary real functions $f^\mu(x^1, x^2, \ldots, x^N)$ are assumed to be single valued, continuous and differentiable over the whole range of their arguments. By differentiating each of these functions with respect to the coordinates, we obtain a $N \times N$ transformation matrix

$$\left[\frac{\partial \bar{x}^\mu}{\partial x^\nu}\right] = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} & \cdots & \frac{\partial f^1}{\partial x^N} \\ \frac{\partial f^2}{\partial x^1} & \frac{\partial f^2}{\partial x^2} & \cdots & \frac{\partial f^2}{\partial x^N} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f^N}{\partial x^1} & \frac{\partial f^N}{\partial x^2} & \cdots & \frac{\partial f^N}{\partial x^N} \end{pmatrix}, \tag{4.5}$$

whose entries are, in general, functions of the coordinates. The determinant of the transformation matrix

$$J(x) = \left|\frac{\partial \bar{x}^\mu}{\partial x^\nu}\right|\,, \tag{4.6}$$

is called the *Jacobian* of the transformation. If the N arbitrary real functions are independent, the Jacobian is different from zero and the coordinate transformation (4.4) can be inverted to express x^μ in terms of $\bar{x}^\mu$

$$x^\mu = g^\mu(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)\,. \tag{4.7}$$

Exercise

The transformation matrix and the Jacobian associated to this inverse coordinate transformation (4.7) are given respectively by the inverse of the transformation matrix (4.5), $[\frac{\partial x^\mu}{\partial \bar{x}^\nu}]$, and the inverse of the Jacobian (4.6), $\bar{J} = J^{-1}$. Prove it.

As you may expect at this point, there is a simple relationship between the coordinate basis vectors in the two systems. This relation can be found by requiring the invariance of the line element ds^2, which is a purely geometrical quantity independent of the coordinate system used to describe it. We get

$$d\bar{x}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} dx^\nu \,, \qquad \bar{\mathbf{e}}_\mu = \frac{\partial x^\nu}{\partial \bar{x}^\mu} \mathbf{e}_\nu \,. \tag{4.8}$$

The previous expressions are equivalent to the *similarity relation* between the metrics in the two coordinate systems that we found in the previous chapter

$$\bar{g}_{\rho\sigma}(\bar{x}) = g_{\mu\nu}(x(\bar{x})) \frac{\partial x^\mu}{\partial \bar{x}^\rho} \frac{\partial x^\nu}{\partial \bar{x}^\sigma} \,. \tag{4.9}$$

Exercise

Taking into account Eq. (4.9), determine the number of independent components of the metric.

A worked-out example: Polar coordinates in the plane

Let us illustrate the previous results with the simplest example one can think of: polar coordinates in $\mathbb{R}^2$. These coordinates are defined by

$$x^1 = r\cos\theta\,, \qquad x^2 = r\sin\theta \qquad \leftrightarrow \qquad r = \sqrt{(x^1)^2 + (x^2)^2}\,, \qquad \theta = \arctan\frac{x^2}{x^1}\,. \tag{4.10}$$

The Jacobian matrices associated with this transformation

$$\frac{\partial x^\mu}{\partial \bar{x}^\nu} = \begin{pmatrix} \frac{\partial x^1}{\partial r} & \frac{\partial x^1}{\partial \theta} \\ \frac{\partial x^2}{\partial r} & \frac{\partial x^2}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}, \tag{4.11}$$

$$\frac{\partial \bar{x}^\mu}{\partial x^\nu} = \begin{pmatrix} \frac{\partial r}{\partial x^1} & \frac{\partial r}{\partial x^2} \\ \frac{\partial \theta}{\partial x^1} & \frac{\partial \theta}{\partial x^2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{pmatrix}, \tag{4.12}$$

are different from zero, i.e. non singular, except at $r = 0$. The polar coordinate system admits a pair of basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$, adapted to the coordinates and related to the Cartesian basis vectors by (cf. (4.8))

$$\mathbf{e}_r = \frac{\partial x^1}{\partial r}\,\mathbf{e}_1 + \frac{\partial x^2}{\partial r}\,\mathbf{e}_2 = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2\,, \tag{4.13}$$

$$\mathbf{e}_\theta = \frac{\partial x^1}{\partial \theta}\,\mathbf{e}_1 + \frac{\partial x^2}{\partial \theta}\,\mathbf{e}_2 = -r\sin\theta\,\mathbf{e}_1 + r\cos\theta\,\mathbf{e}_2\,. \tag{4.14}$$

Note that the resulting basis is not a unit basis

$$|\mathbf{e}_r|^2 = 1\,, \qquad |\mathbf{e}_\theta| = r^2\,. \tag{4.15}$$

On the other hand, the relation between the infinitesimal displacements in both coordinate system is given by (cf. (4.8))

$$dr = \frac{\partial r}{\partial x^1}dx^1 + \frac{\partial r}{\partial x^2}dx^2 = \cos\theta dx^1 + \sin\theta dx^2\,, \tag{4.16}$$

$$d\theta = \frac{\partial \theta}{\partial x^1}dx^1 + \frac{\partial \theta}{\partial x^2}dx^2 = -\frac{1}{r}\sin\theta dx^1 + \frac{1}{r}\cos\theta dx^2\,. \tag{4.17}$$

The components of the metric tensor and its inverse in this basis can be computed either through the definition of the metric (4.1)

$$g_{rr} = \mathbf{e}_r\cdot\mathbf{e}_r = 1\,, \qquad g_{\theta\theta} = \mathbf{e}_\theta\cdot\mathbf{e}_\theta = r^2\,, \qquad g_{r\theta} = g_{\theta r} = \mathbf{e}_r\cdot\mathbf{e}_\theta = 0\,, \tag{4.18}$$

or through its transformation properties (4.9)

$$g_{rr} = \frac{\partial x^i}{\partial r}\frac{\partial x^j}{\partial r}\delta_{ij} = \left(\frac{\partial x^1}{\partial r}\right)^2 + \left(\frac{\partial x^2}{\partial r}\right)^2 = \cos^2\theta + \sin^2\theta = 1\,, \tag{4.19}$$

$$g_{\theta\theta} = \frac{\partial x^i}{\partial \theta}\frac{\partial x^j}{\partial \theta}\delta_{ij} = \left(\frac{\partial x^1}{\partial \theta}\right)^2 + \left(\frac{\partial x^2}{\partial \theta}\right)^2 = (-r\sin\theta)^2 + (r\cos\theta)^2 = r^2\,. \tag{4.20}$$

In both cases, we get

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}. \tag{4.21}$$

for the metric and its inverse. The line element (4.3) written in polar coordinates becomes

$$ds^2 = dr^2 + r^2 d\theta^2\,, \tag{4.22}$$

which is what one usually writes down when asked for the metric of this coordinate system.

Exercise
Repeat the same exercise for spherical coordinates.

4.2 Tensors

The transformation laws of tensors under general coordinate transformations are just a generalization of those found in Chapters 1 and 2, the main difference being the replacement of the constant matrices $R^i{}_j$ and $\Lambda^\mu{}_\nu$ by the arbitrary transformation matrix (4.5) and the use of the metric $g_{\mu\nu}$ and its inverse for lowering and raising indices

$$V_\mu \equiv g_{\mu\nu}V^\nu\,, \qquad V^\mu \equiv g^{\mu\nu}V_\nu\,. \tag{4.23}$$

The simplest transformation rules are summarized in Table 4.1. For a general tensor with m contravariant indices and n covariant indices we have

$$\bar{T}^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} = \left(\prod_{p=1}^{m}\frac{\partial\bar{x}^{\mu_p}}{\partial x^{\rho_p}}\prod_{q=1}^{n}\frac{\partial x^{\sigma_q}}{\partial\bar{x}^{\nu_q}}\right)T^{\rho_1\ldots\rho_m}{}_{\sigma_1\ldots\sigma_n}\,. \tag{4.24}$$

4.3 Tensorial densities

In Special Relativity the volume element $d^4x = dx^0dx^1dx^2dx^3$ provides an invariant integration measure because of the unit value of the determinant of Lorentz transformations. When

General coord. transformations	$\frac{\partial \bar{x}^\mu}{\partial x^\nu}$ are arbitrary!
Scalar	$\bar{\phi} = \phi$
Contravariant vector	$\bar{V}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} V^\nu$
Covariant vector	$\bar{V}_\mu = \frac{\partial x^\nu}{\partial \bar{x}^\mu} V_\nu$
Contravariant rank-2 tensor	$\bar{T}^{\mu\nu} = \frac{\partial \bar{x}^\mu}{\partial x^\rho} \frac{\partial \bar{x}^\nu}{\partial x^\sigma} T^{\rho\sigma}$
Covariant rank-2 tensor	$\bar{T}_{\mu\nu} = \frac{\partial x^\rho}{\partial \bar{x}^\mu} \frac{\partial x^\sigma}{\partial \bar{x}^\nu} T_{\rho\sigma}$
Mixed rank-2 tensor	$\bar{T}^\mu{}_\nu = \frac{\partial \bar{x}^\mu}{\partial x^\rho} \frac{\partial x^\sigma}{\partial \bar{x}^\nu} T^\rho{}_\sigma$

Table 4.1

considering arbitrary coordinate transformations this is no longer true and the Jacobian of the transformation appears in the transformation law

$$d^4\bar{x} = \left| \frac{\partial \bar{x}^\mu}{\partial x^\nu} \right| d^4x = J d^4x \,. \tag{4.25}$$

A similar thing happens with the determinant of the metric. Taking the determinant of the matrix equation (4.9), we get

$$\bar{g} = J^{-2} g \,. \tag{4.26}$$

The previous expression can be combined with Eq. (4.25) to obtain a generally covariant quantity

$$\sqrt{|\bar{g}|} d^4\bar{x} = \sqrt{|g|} d^4x \,, \tag{4.27}$$

which we can use as an appropriate volume element in arbitrary dimensions. The absolute value in Eq. (4.27) is introduced to take into account the case of a metric with Lorentzian signature $(- + ++)$ and negative determinant.

A worked-out example: Polar coordinates in the plane

The volume element in polar coordinates is given by $dV = \sqrt{|g|} dr d\theta = r dr d\theta$.

The volume density d^4x and the determinant of the metric g are just particular cases of a general class of quantities called tensor densities. A tensor density transforms as a tensor except for the appearance of the Jacobian to a given power w called the weight of the tensor density, namely

$$\bar{\mathcal{D}}^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} = J^{-w} \left(\prod_{p=1}^{m} \frac{\partial \bar{x}^{\mu_p}}{\partial x^{\rho_p}} \prod_{q=1}^{n} \frac{\partial x^{\sigma_q}}{\partial \bar{x}^{\nu_q}} \right) \mathcal{D}^{\rho_1 \ldots \rho_m}{}_{\sigma_1 \ldots \sigma_n} \,. \tag{4.28}$$

Ordinary tensors can be therefore considered as tensor densities of weight zero. The determinant of the covariant rank-2 metric tensor is a *scalar density* of weight 2, while d^4x is a scalar density of weight -1. Eq. (4.27) can be easily generalized to obtain a rule for transforming tensorial densities into tensors

$$|\bar{g}|^{-\frac{w}{2}}\bar{\mathcal{D}}^{\mu_1\dots\mu_m}{}_{\nu_1\dots\nu_n} = \left(\prod_{p=1}^{m}\frac{\partial\bar{x}^{\mu_p}}{\partial x^{\rho_p}}\prod_{q=1}^{n}\frac{\partial x^{\sigma_q}}{\partial\bar{x}^{\nu_q}}\right)|g|^{-\frac{w}{2}}\mathcal{D}^{\rho_1\dots\rho_m}{}_{\sigma_1\dots\sigma_n}\,. \tag{4.29}$$

Exercise

- Show that the totally antisymmetric quantity

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1, & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } 0123\,, \\ -1, & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123\,, \\ 0, & \text{otherwise}\,. \end{cases} \tag{4.30}$$

  is a tensor density under general coordinate transformations. Determine its weight. Construct a tensor from it using the metric.

- Show that the components of $\epsilon_{\mu\nu\rho\sigma}$ remain unchanged under general coordinate transformations.

4.4 Covariant derivative

We saw that in Special relativity the derivative of a vector is a tensor under Lorentz transformations. However, when general coordinate transformations are taken into account, the *usual* derivative of the components of a tensor is *not a tensor*

$$\begin{aligned}\bar{\partial}_\nu\bar{V}^\mu &= \left(\frac{\partial x^\rho}{\partial\bar{x}^\nu}\partial_\rho\right)\left(\frac{\partial\bar{x}^\mu}{\partial x^\sigma}V^\sigma\right) \\ &= \frac{\partial x^\rho}{\partial\bar{x}^\nu}\frac{\partial\bar{x}^\mu}{\partial x^\sigma}\partial_\rho V^\sigma + \frac{\partial x^\rho}{\partial\bar{x}^\nu}\frac{\partial^2\bar{x}^\mu}{\partial x^\rho\partial x^\sigma}V^\sigma\end{aligned} \tag{4.31}$$

The second term spoils the tensorial property of the derivative for first-rank tensors. In order to determine why this happens, let's go back to geometrical, real coordinate independent objects. Consider for instance the expansion of a vector

$$\mathbf{V} = V^\mu\mathbf{e}_\mu(x)\,, \tag{4.32}$$

in terms of arbitrary basic vectors $\mathbf{e}_\mu(x)$, which, contrary to the Euclidean or Minkowski cases, generically depend on the coordinates. The derivative of such a vector contains two different contributions, one due to the intrinsic change of the vector field from place to place and one describing the variation of the basis vectors from place to place

$$\frac{\partial \mathbf{V}}{\partial x^\nu} = \frac{\partial V^\mu}{\partial x^\nu}\mathbf{e}_\mu + V^\mu \frac{\partial \mathbf{e}_\mu}{\partial x^\nu} . \tag{4.33}$$

The first term is present even in Cartesian coordinates and is a linear combination of the basis vectors, i.e. a vector. On the other hand, the second term involves the derivative of the basis vectors $\mathbf{e}_\mu$. The relation between these vectors and those in a *(local) inertial reference frame*[1] $\mathbf{e}_\alpha$ is given by

$$\mathbf{e}_\mu = \frac{\partial \xi^\alpha}{\partial x^\mu}\mathbf{e}_\alpha . \tag{4.34}$$

Taking the derivative of the previous expression and using the fact that the vectors $\mathbf{e}_\alpha$ are constant

$$\frac{\partial \mathbf{e}_\alpha}{\partial x^\nu} = 0 , \tag{4.35}$$

we get the following relation

$$\frac{\partial \mathbf{e}_\mu}{\partial x^\nu} = \left[\frac{\partial}{\partial x^\nu}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\right)\right]\mathbf{e}_\alpha . \tag{4.36}$$

The right-hand side of the previous equation is a linear combination of the *inertial* basis vectors $\mathbf{e}_\alpha$ and therefore is a vector. This allows us to rewrite $\frac{\partial \mathbf{e}_\mu}{\partial x^\nu}$ in terms of the *arbitrary* basis vectors $\mathbf{e}_\mu$

$$\frac{\partial \, \mathbf{e}_\mu}{\partial x^\nu} = \Gamma^\rho{}_{\mu\nu} \, \mathbf{e}_\rho . \tag{4.37}$$

For the time being, the so-called *affine connection* $\Gamma^\rho{}_{\mu\nu}$ is just a 3-index notation denoting the linear combination of *arbitrary* basis vectors $\mathbf{e}_\mu$. The index μ specifies the basis vector that is differentiated, ν the coordinate with respect to which it is differentiated and ρ the component of the resulting vector[2]. Inserting the definition (4.37) into Eq.(4.33) we obtain

$$\frac{\partial \mathbf{V}}{\partial x^\nu} = \frac{\partial V^\mu}{\partial x^\nu}\mathbf{e}_\mu + V^\mu \Gamma^\rho{}_{\mu\nu} \, \mathbf{e}_\rho , \tag{4.38}$$

where the basis vector in the right hand side can be factored out by simply relabeling the dummy indices μ and ρ

$$\frac{\partial \mathbf{V}}{\partial x^\nu} = \left(\frac{\partial V^\mu}{\partial x^\nu} + V^\rho \Gamma^\mu{}_{\rho\nu}\right)\mathbf{e}_\mu \equiv \left(\nabla_\nu V^\mu\right)\mathbf{e}_\mu . \tag{4.39}$$

The quantities in parenthesis are the components of a tensor, called the *covariant derivative*, which takes into account the variation of basis vectors from point to point. We will denote it in two alternative ways, either with the symbol ∇

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu{}_{\rho\nu} V^\rho . \tag{4.40}$$

[1] Note the use of the first letters of the Greek alphabet for denoting quantities in (local) inertial frames.

[2] Yes, I am deliberately using the same symbol I used for the Christoffel symbols. You will understand why in a while. Be patient.

or with a semicolon

$$V^\mu{}_{;\nu} = V^\mu{}_{,\nu} + \Gamma^\mu{}_{\rho\nu} V^\rho \,. \tag{4.41}$$

Note that the standard derivative ∂_μ has been denoted by a colon. The notation (4.41) is specially convenient for its brevity and for remembering the definition of the covariant derivative (the ν index appears in the last position of each term).

A worked-out example: Polar coordinates in the plane

Taking the derivatives of the basic vectors (4.13) and (4.14) and taking into account that the Cartesian vectors are constant, we have

$$\frac{\partial \mathbf{e}_r}{\partial r} = 0 \quad \rightarrow \Gamma^\mu{}_{rr} = 0 \,, \tag{4.42}$$

$$\frac{\partial \mathbf{e}_\theta}{\partial r} = \frac{1}{r}\mathbf{e}_\theta \rightarrow \Gamma^r{}_{\theta r} = 0 \,, \qquad \Gamma^\theta{}_{\theta r} = \frac{1}{r} \,, \tag{4.43}$$

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \frac{1}{r}\mathbf{e}_\theta \rightarrow \Gamma^r{}_{r\theta} = 0 \,, \qquad \Gamma^\theta{}_{r\theta} = \frac{1}{r} \,, \tag{4.44}$$

$$\frac{\partial \mathbf{e}_\theta}{\partial \theta} = -r\mathbf{e}_r \rightarrow \Gamma^r{}_{\theta\theta} = -r \,, \qquad \Gamma^\theta{}_{\theta\theta} = 0 \,, \tag{4.45}$$

and the associated covariant derivatives become

$$V^r{}_{;r} = V^r{}_{,r} \,, \qquad V^\theta{}_{;\theta} = V^\theta{}_{,\theta} + \frac{1}{r} V^r \,, \tag{4.46}$$

$$V^\theta{}_{;r} = V^\theta{}_{,r} + \frac{1}{r} V^\theta \,, \qquad V^r{}_{;\theta} = V^r{}_{,\theta} - r V^\theta \,. \tag{4.47}$$

Note that the final expressions do not involve any Cartesian tensors and allow you to directly derive the formulae for the divergence of a vector field

$$\boldsymbol{\nabla} \cdot \mathbf{V} = V^r{}_{;r} + V^\theta{}_{;\theta} = V^r{}_{,r} + V^\theta{}_{,\theta} + \frac{1}{r} V^r = \frac{1}{r}\frac{\partial}{\partial r}(r V^r) + \frac{\partial}{\partial \theta} V^\theta \,. \tag{4.48}$$

and the Laplacian of a scalar field[a]

$$\nabla \cdot \nabla \phi \equiv \nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \,, \tag{4.49}$$

in polar coordinates. You should recognize the result... The formulae appearing in your favorite electromagnetism books are just a consequence of the existence of non-vanishing connection coefficients in curvilinear coordinates!

[a] Note that $\nabla_\mu \phi = \partial_\mu \phi$.

For a covariant tensor the covariant derivative takes a slightly different form. Consider a scalar

$$\phi = V_\mu U^\mu ,. \tag{4.50}$$

Since a scalar does not depend on the basis vectors, its covariant derivative coincides with the standard derivative

$$\nabla_\nu \phi = \partial_\nu \phi = \frac{\partial V_\mu}{\partial x^\nu} U^\mu + V_\mu \frac{\partial U^\mu}{\partial x^\nu} \tag{4.51}$$

Using Eq. (4.40) for replacing $\partial_\nu U^\mu$ in favor of $\nabla_\nu U^\mu$ and relabeling dummy indices in the term containing the connection we get

$$\nabla_\nu \phi = \left(\frac{\partial V_\mu}{\partial x^\nu} - \Gamma^\rho{}_{\mu\nu} V_\rho \right) U^\mu + V_\mu \nabla_\nu U^\mu \,. \tag{4.52}$$

All the terms in the previous expressions, except the one in parenthesis are tensor components. Since the multiplication and addition of tensor components always gives rise to tensors, the quantity in parenthesis must be a tensor. The covariant derivative of V_μ becomes is then given by

$$\nabla_\nu V_\mu = \partial_\nu V_\mu - \Gamma^\rho{}_{\mu\nu} V_\rho \,, \tag{4.53}$$

or

$$V_{\mu;\nu} = V_{\mu,\nu} - \Gamma^\rho{}_{\mu\nu} V_\rho \,. \tag{4.54}$$

in the semicolon notation. Note the similarities and differences between (4.41) and (4.54). In both cases, the index with respect to which the covariant derivative is taken (ν in this case) is the last subscript of the connection Γ. The remaining indices can only be arranged in one way without raising and lowering them. For a covariant vector (su**p**erscript) the sign of the connection is **p**ositive; for a covariant vector (subscript) the connection carries a minus sign. These transformation rules can be generalized to extra covariant and contravariant indices by introducing a factor Γ for each index with the proper sign and index matching to obtain

$$T^{\mu\dots}_{\nu\dots;\rho} = T^{\mu\dots}_{\nu\dots,\rho} + \Gamma^\mu{}_{\lambda\rho} T^{\lambda\dots}_{\nu\dots} + \dots - \Gamma^\lambda{}_{\nu\rho} T^{\mu\dots}_{\lambda\dots} - \dots \quad . \tag{4.55}$$

A practical rule for covariant derivatives

- Write down the partial derivative of the tensor.
- The expression is to be corrected by a set of terms, one for each index of the tensor. Each term will be a product of a Γ and the original tensor.
- A term will have **p**ositive sign if the index we are correcting for is a su**p**erscript and a negative sign if the index is a subscript.
- The index with respect to which the covariant derivative is taken will be the last subscript of all the connections Γ.
- The index of the tensor that is being corrected will be replaced by a dummy index, that will be contracted with one of the indices of Γ.

- The remaining indices can be placed in an unique way.

Exercise
Write explicitly $T^{\mu}{}_{\nu;\rho}$ and $T^{\mu\nu}{}_{;\rho}$.

4.4.1 Relation between the connection and the metric tensor

Since the definition of the affine connection in (4.37) involves only basis vectors and their derivatives, it is clear that $\Gamma^{\rho}{}_{\mu\nu}$ must be somehow related to the metric and its derivatives. Note however that there is certain ambiguity in the way we introduced the connection coefficients $\Gamma^{\nu}{}_{\mu\rho}$. We could have perfectly written the same expression with a different ordering of indices, i.e $\Gamma^{\nu}{}_{\rho\mu}$ instead of $\Gamma^{\nu}{}_{\mu\rho}$. In the most general case, these two quantities are not necessarily equal to each other

$$T^{\nu}{}_{\mu\rho} \equiv \Gamma^{\nu}{}_{\mu\rho} - \Gamma^{\nu}{}_{\rho\mu} \neq 0 \,. \tag{4.56}$$

and the spacetime is said to have torsion[3]. In what follows, we will *require* our spacetime to be torsionless and will take

$$\Gamma^{\nu}{}_{\mu\rho} = \Gamma^{\nu}{}_{\rho\mu} \tag{4.57}$$

Under this assumption, the relation between the metric and the connection can be determined as follows. Starting with (4.1), and differentiating it with respect to x^{ρ} we obtain

$$\begin{aligned} \partial_{\rho} g_{\mu\nu} &= \partial_{\rho}\mathbf{e}_{\mu} \cdot \mathbf{e}_{\nu} + \mathbf{e}_{\mu} \cdot \partial_{\rho}\mathbf{e}_{\nu} \\ &= \Gamma^{\sigma}{}_{\mu\rho}\mathbf{e}_{\sigma} \cdot \mathbf{e}_{\nu} + \mathbf{e}_{\mu} \cdot \Gamma^{\sigma}{}_{\nu\rho}\mathbf{e}_{\sigma} \\ &= \Gamma^{\sigma}{}_{\mu\rho} g_{\sigma\nu} + \Gamma^{\sigma}{}_{\nu\rho} g_{\mu\sigma} \,, \end{aligned} \tag{4.58}$$

where in the last last step we have made use of the defining equation (4.37) and the metric definition (4.1). By cyclically permuting the indices we obtain the following two equivalent expressions

$$\partial_{\nu} g_{\rho\mu} = \Gamma^{\sigma}{}_{\rho\nu} g_{\sigma\mu} + \Gamma^{\sigma}{}_{\mu\nu} g_{\rho\sigma} \,, \tag{4.59}$$

$$\partial_{\mu} g_{\nu\rho} = \Gamma^{\sigma}{}_{\nu\mu} g_{\rho\sigma} + \Gamma^{\sigma}{}_{\rho\mu} g_{\nu\sigma} \,, \tag{4.60}$$

which, combined with Eq.(4.58) and using the *assumed* property $\Gamma^{\sigma}{}_{\mu\nu} = \Gamma^{\sigma}{}_{\nu\mu}$, allows us to form the following combination

$$\partial_{\rho} g_{\mu\nu} + \partial_{\nu} g_{\rho\mu} - \partial_{\mu} g_{\nu\rho} = 2\Gamma^{\sigma}{}_{\rho\nu} g_{\mu\sigma} \,. \tag{4.61}$$

Multiplying by the inverse metric $g^{\kappa\mu}$, using $g^{\kappa\mu} g_{\mu\sigma} = \delta^{\kappa}{}_{\sigma}$ and relabeling indices we obtain the result we were looking for[4]

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_{\nu} g_{\sigma\rho} + \partial_{\rho} g_{\nu\sigma} - \partial_{\sigma} g_{\nu\rho} \right) \,. \tag{4.62}$$

[3] As we will see below the connection Γ is not a tensor. The torsion $T^{\nu}{}_{\mu\rho}$, involving the difference of two connections, is however a tensor.

[4] Note that in the presence of torsion it would differ from the *affine connection* defined by (4.37).

A connection satisfying the previous property is called a *metric connection*, a *Christoffel connection*, a *Levi-Civita connection* or a *Riemannian connection*.

A worked-out example I: Polar coordinates in the plane

The only nonzero derivative of the covariant metric components is $g_{\theta\theta,r} = 2r$. Therefore, coming back to Eq. (4.62), the non-zero components are

$$\Gamma^{r}{}_{\theta\theta} = \frac{1}{2} g^{rr} \left(\partial_\theta g_{r\theta} + \partial_\theta g_{\theta r} - \partial_r g_{\theta\theta} \right) = -r \,, \tag{4.63}$$

$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{1}{2} g^{\theta\theta} \left(\partial_\theta g_{\theta r} + \partial_r g_{\theta\theta} - \partial_\theta g_{r\theta} \right) = \frac{1}{r} \,, \tag{4.64}$$

which coincide with the results (4.42)-(4.45) obtained by using the definition (4.37).

Remember that the metric connection (4.62) is *not* a tensor

$$\bar{\Gamma}^{\mu}{}_{\nu\rho} = \Gamma^{\lambda}{}_{\sigma\kappa} \frac{\partial \bar{x}^{\mu}}{\partial x^{\lambda}} \frac{\partial x^{\sigma}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\kappa}}{\partial \bar{x}^{\rho}} + \frac{\partial \bar{x}^{\mu}}{\partial x^{\lambda}} \frac{\partial^2 x^{\lambda}}{\partial \bar{x}^{\nu} \partial \bar{x}^{\rho}} \,. \tag{4.65}$$

Exercise

- Prove explicitly the relation (4.68) by using the relation (4.62).
- Show that the difference of two connections *is a tensor*.

4.4.2 Properties of the covariant derivative

i) Linearity: The covariant derivative of a linear combination of tensors with constant coefficients is the same as the linear combination of the tensors once the covariant differentiation has been carried out

$$\nabla_\rho \left(a U^{\mu}{}_{\nu} + b V^{\mu}{}_{\nu} \right) = a \nabla_\rho U^{\mu}{}_{\nu} + b \nabla_\rho V^{\mu}{}_{\nu} \,. \tag{4.66}$$

ii) Leibniz's or chain rule: The covariant derivative of outer and inner products of tensors obey the same rules as the usual derivative

$$\nabla_\rho \left(U^{\mu} V_{\nu} \right) = \left(\nabla_\rho U^{\mu} \right) V_{\nu} + U^{\mu} \left(\nabla_\rho V_{\nu} \right) \,. \tag{4.67}$$

iii) Metric compatibility: The covariant derivative of the metric tensor is zero

$$\nabla_\rho g_{\mu\nu} = 0 \,. \tag{4.68}$$

In other words, the metric tensor is not constant $\partial_\rho g_{\mu\nu} \neq 0$ but it is covariantly constant. The result follows immediately from comparing the general expression for the covariant derivative of a rank-2 covariant tensor with (4.58)[5].

iv) The raising and lowering of tensor indices is not affected by covariant differentiation. For example

$$\nabla_\nu V^\mu = \nabla_\nu \left(g^{\mu\sigma} V_\sigma\right) = g^{\mu\sigma} \nabla_\nu V_\sigma \,, \tag{4.69}$$

where we have made use of properties ii) and iii). Note that this would not be the case if our connection was not metric-compatible. We should be very careful about index placement in that case.

v) The covariant derivative of the Kronecker delta $\delta^\mu{}_\nu$ is zero

$$\nabla_\rho \delta^\mu{}_\nu = \partial_\rho \delta^\mu{}_\nu + \Gamma^\mu{}_{\sigma\rho}\delta^\sigma{}_\nu - \Gamma^\sigma{}_{\nu\rho}\delta^\mu{}_\sigma = \Gamma^\mu{}_{\nu\rho} - \Gamma^\mu{}_{\nu\rho} = 0. \tag{4.70}$$

vi) The covariant derivative commutes with the contraction of indices. For example

$$\nabla_\nu T^{\mu\rho}{}_\rho = \partial_\nu T^{\mu\rho}{}_\rho + \Gamma^\mu{}_{\lambda\nu} T^{\lambda\rho}{}_\rho + \Gamma^\rho{}_{\lambda\nu} T^{\mu\lambda}{}_\rho - \Gamma^\kappa{}_{\rho\nu} T^{\mu\rho}{}_\kappa = \partial_\nu T^{\mu\rho}{}_\rho + \Gamma^\mu{}_{\lambda\nu} T^{\lambda\rho}{}_\rho \,. \tag{4.71}$$

Exercise

- Consider a tensor $T^\mu{}_\nu = U^\mu V_\nu$. Use the Leibniz's rule (4.67) together with the expressions for the covariant derivatives of a covariant and a contravariant vector to compute $T^\mu{}_{\nu;\rho}$. Is the result consistent with the practical rules below Eq. (4.55) ?

- Verify Eq.(4.68) for the particular case of polar coordinates in the plane.

4.4.3 Some useful formulas

Let me present (without proving them) some useful formulae involving the connection and the covariant derivatives.

1. Contraction of the Christoffel symbols

$$\Gamma^\mu{}_{\mu\rho} = \frac{1}{2}\partial_\nu \log\sqrt{|g|} = \frac{1}{\sqrt{|g|}}\partial_\nu\sqrt{|g|} \,. \tag{4.72}$$

2. Divergence of a contravariant vector

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}V^\mu\right) = \partial_\mu V^\mu + \Gamma^\mu{}_{\mu\rho}V^\rho \,. \tag{4.73}$$

[5] Indeed, we have implicitly assumed that the affine connection was metric-compatible in our derivation of Eq. (4.58).

3. Covariant form of the Gauss theorem

$$\int d^4x\sqrt{|g|}\nabla_\mu V^\mu = 0\,. \tag{4.74}$$

4. Covariant Laplacian[6]

$$\nabla^2\phi \equiv \nabla_\mu\nabla^\mu\phi = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\partial^\mu\phi\right)\,. \tag{4.75}$$

5. Covariant divergence of a rank-2 tensor

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}T^{\mu\nu}\right) + \Gamma^\nu{}_{\mu\rho}T^{\mu\rho}\,. \tag{4.76}$$

6. Covariant divergence of a rank-2 symmetric tensor

$$\nabla_\mu S^\mu{}_\nu = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}S^\mu{}_\nu\right) - \frac{1}{2}\left(\partial_\nu g_{\mu\rho}\right)S^{\mu\rho}\,. \tag{4.77}$$

7. Covariant divergence of a rank-2 antisymmetric tensor

$$\nabla_\mu A^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}A^{\mu\nu}\right)\,. \tag{4.78}$$

Eqs. (4.73), (4.77) and (4.78) are particularly useful, since they allow us to compute the covariant derivative of an object without having to compute the Christoffel symbols.

Exercise

- Derive all the expressions in this section.
- Use Eq. (4.75) to re-derive the expression for the Laplacian in polar coordinates.

4.5 An application: Maxwell equations in arbitrary coordinates

As shown at the end of Chapter 2, the electromagnetic field equations in Cartesian coordinates take the form

$$\partial_\beta F^{\alpha\beta} = J^\alpha\,, \qquad \partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} = 0\,, \tag{4.79}$$

[6]The symbol used for the Laplacian operator depends on the dimension of the spacetime considered. The three-sided symbol ∇^2 in (4.75) is the most common notation in arbitrary dimension. The 4-dimensional case is sometimes singled-out. It has a special name, *D'Alambertian* and its own four-sided symbol, $\Box$.

with $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. Let us see which is the form taken by those equations in an arbitrary coordinate system x^μ. In order to do that, let us write the field strength and the current in terms of $x^\mu(\xi^\alpha)$

$$F^{\mu\nu} \equiv \frac{\partial x^\mu}{\partial \xi^\alpha}\frac{\partial x^\nu}{\partial \xi^\beta} F^{\alpha\beta}\,, \qquad J^\mu \equiv \frac{\partial x^\mu}{\partial \xi^\alpha} J^\alpha \tag{4.80}$$

and replace the partial derivatives ∂_μ by the covariant derivatives ∇_μ to take into account the fact that the basis vector are in general not constant. We obtain

$$\nabla_\nu F^{\mu\nu} = J^\mu\,, \qquad \nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} = 0\,, \tag{4.81}$$

with $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$. The resulting equations are *fully covariant*, i.e. if they are valid in an arbitrary coordinate system they will be valid in all coordinate systems. Taking into account the antisymmetricity of $F^{\mu\nu}$ together with the property (4.78), the first Eq. in (4.81) can be written in a very convenient form[7]

$$\partial_\mu\left(\sqrt{|g|}F^{\mu\nu}\right) = \sqrt{|g|}J^\mu\,, \tag{4.82}$$

which, taking into account (4.73), allows as to easily compute the expression for the continuity equation (2.73) in arbitrary coordinate systems

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}J^\mu\right) = 0. \tag{4.83}$$

Exercise

Fill the steps in the derivation of Eqs. (4.82) and (4.83).

4.6 Parallel transport and geodesics

In Cartesian coordinates, a vector field will be considered as constant if its components with respect to the coordinates are the same everywhere. In that case, the vector field can be thought as parallel transported to itself at every point in space. In other words, it moves parallel to itself when it moves along a curve. Within the framework of Cartesian coordinates this is a consistent definition since the Cartesian basis vectors are parallel to the axes everywhere. The previous property gives rise to an alternative definition for a geodesic: *It is the curve for which the tangent vector always points in the same direction.* Indeed, the equation of motion for a free particle

$$\frac{du^\mu}{d\tau} = 0 \tag{4.84}$$

[7] The second equation also becomes simpler when taking into account the antisymmetry of the field strength tensor, $F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0$.

can be interpreted as a straight line in spacetime (i.e. the shortest distance between two points, a geodesic) or as the fact that the vector u^μ remains constant along the line parametrized by τ.

On the other hand, if we consider a non-Cartesian set of coordinates, such as polar coordinates in the plane, the notion of parallel transport is more difficult to define since as we saw in the previous sections the basis vectors change from point to point. The covariant derivative can be nevertheless used to provide a natural definition for parallel transport in an arbitrary spacetime. To see this, consider the derivative of a vector $\mathbf{V} = V^\mu \mathbf{e}_\mu$ along a curve parametrized by an affine parameter σ

$$\begin{aligned}\frac{d\mathbf{V}}{d\sigma} &= \frac{dV^\mu}{d\sigma}\mathbf{e}_\mu + V^\mu \frac{d\mathbf{e}_\mu}{d\sigma} \\ &= \frac{dV^\mu}{d\sigma}\mathbf{e}_\mu + V^\mu \frac{d\mathbf{e}_\mu}{dx^\rho}\frac{\partial x^\rho}{d\sigma} \\ &= \frac{dV^\mu}{d\sigma}\mathbf{e}_\mu + \Gamma^\nu{}_{\mu\rho} V^\mu \frac{\partial x^\rho}{d\sigma}\mathbf{e}_\nu \,. \end{aligned} \tag{4.85}$$

Relabelling indices and factoring out the basis vector, we get

$$\frac{d\mathbf{V}}{d\sigma} = \left(\frac{dV^\mu}{d\sigma} + \Gamma^\mu{}_{\nu\rho} V^\nu \frac{\partial x^\rho}{d\sigma}\right)\mathbf{e}_\mu \equiv \frac{DV^\mu}{d\sigma}\mathbf{e}_\mu \,. \tag{4.86}$$

where we have defined the components of the *intrinsic derivative* $\frac{DV^\mu}{d\sigma}$ of a contravariant vector as

$$\frac{DV^\mu}{d\sigma} \equiv \frac{dV^\mu}{d\sigma} + \Gamma^\mu{}_{\nu\rho} V^\nu \frac{\partial x^\rho}{d\sigma} \,. \tag{4.87}$$

A similar condition can be found for the components of the intrinsic derivative of a covariant vector

$$\frac{DV_\mu}{d\sigma} \equiv \frac{dV_\mu}{d\sigma} - \Gamma^\nu{}_{\mu\rho} V_\nu \frac{\partial x^\rho}{d\sigma} \,. \tag{4.88}$$

The condition of parallel transport, $d\mathbf{V}/d\sigma = 0$, implies

$$\frac{DV^\mu}{d\sigma} \equiv \frac{dV^\mu}{d\sigma} + \Gamma^\mu{}_{\nu\rho} \frac{\partial x^\rho}{d\sigma} V^\nu = 0 \,, \tag{4.89}$$

Geodesics and parallel transport

A direct application of (4.89) to the vector $u^\nu = \frac{dx^\nu}{d\sigma}$ tangent to a given trajectory $x^\mu(\sigma)$ gives rise to

$$\frac{d^2 x^\mu}{d^2\sigma} + \Gamma^\mu{}_{\nu\rho} \frac{\partial x^\rho}{d\sigma}\frac{dx^\nu}{d\sigma} = 0 \,. \tag{4.90}$$

Voilá! A curve is a geodesic if it parallel transports its own tangent vector!

The concept of intrinsic derivative and parallel transport can be generalized to objects with more indices. The parallel transport of a tensor T along the path $x^\mu(\lambda)$ is defined by the requirement

$$\frac{DT^{\mu\cdots}{}_{\nu\dots}}{d\sigma} \equiv \frac{dx^\rho}{d\sigma}\nabla_\rho T^{\mu\cdots}{}_{\nu\dots} = 0\ . \tag{4.91}$$

Applying this to the metric and taking into account (4.68) we conclude that

$$\frac{D}{d\sigma}g_{\mu\nu} = \frac{dx^\rho}{d\sigma}\nabla_\rho g_{\mu\nu} = 0\,, \tag{4.92}$$

which gives rise to an important property of parallel transport: it conserves the direct product of two parallel transported vectors

$$\frac{D}{D\sigma}\left(V_\mu U^\mu\right) = \frac{D}{D\sigma}\left(g_{\mu\nu}V^\mu U^\nu\right) = \left(\frac{Dg_{\mu\nu}}{D\sigma}\right)V^\mu U^\nu + g_{\mu\nu}\left(\frac{DV^\mu}{D\sigma}U^\mu + V^\mu\frac{DU^\mu}{D\sigma}\right) = 0\,, \tag{4.93}$$

and therefore their norm, orthogonality, etc ....

4.7 Summary

The general procedure for converting an equation which is valid in Cartesian inertial coordinates to an equation valid in arbitrary coordinate systems is:

- Write the equations in a Lorentz invariant form.
- Replace the Minkowski metric $\eta_{\mu\nu}$ by $g_{\mu\nu}$.
- Replace partial derivatives by covariant derivatives (*colon-goes-to semicolon rule*).
- Replace ordinary derivatives along curves by intrinsic derivatives.

This prescription is known as *minimal coupling* prescription, since it does not introduce any terms apart from those already present.

Exercise

The equation of motion of a charged particle in an electromagnetic field in Cartesian coordinates takes the form

$$m\frac{du^\alpha}{d\tau} = qF^\alpha{}_\beta u^\beta\,. \tag{4.94}$$

Which is the form taken by the previous expression in an arbitrary coordinate system? Do you recognize the left-hand side?

CHAPTER 5

TIDAL FORCES AND CURVATURE

> What are the differential laws which determine the Riemann metric (i.e. $g_{\mu\nu}$) itself?... The solution obviously needed invariant differential systems of the second order taken from $g_{\mu\nu}$. We soon saw that these had been already established by Riemann.
>
> A. EINSTEIN

In both Newtonian mechanics in the absence of gravity and Einstein's theory of Relativity, inertial frames are characterized by the absence of accelerations, which are absolute elements of the theory. If particles move in straight lines at constant speed the system is inertial. On the other hand, if the trajectory in spacetime is not a straight line the system must be accelerating. The situation is slightly different when gravity is taken into account. The equality between inertial and gravitational masses does not allow to *locally* distinguish the acceleration of a given reference frame from purely gravitational effects. Gravity can be *locally switched off* by properly choosing a local inertial frame associated to an observer in free-fall in the gravitational field. The word *locally* is fundamental, since the *global* behaviours of accelerations and gravity are completely different: while the true gravitational field vanishes at large distances, the apparent gravitational field in an accelerating frame takes a nonzero constant value at infinity. Real and apparent gravity can be distinguished by tracking the relative acceleration of nearby local inertial observers that appears due to the non-homogeneity of the gravitational field!

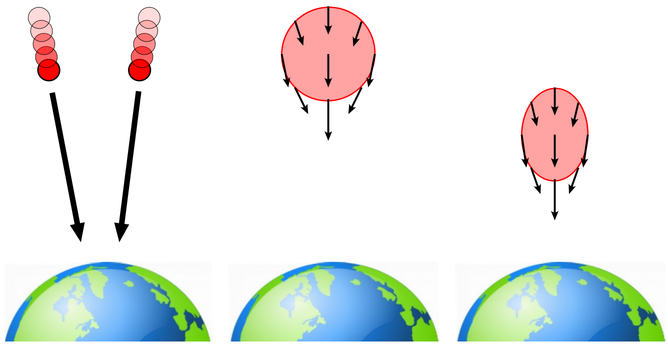


Figure 5.1: The effect of tidal forces.

5.1 Gravity is a central force: Tides

Non-uniform gravitational fields are observable. Consider for instance two non-interacting particles falling towards the surface of the Earth (cf. Fig.5.1). Since the Earth is spherical in shape, both particles move towards the center of the Earth in such a way the separation between them decreases as they fall. The central character of the gravitational field gives rise to *tidal forces*. Let's put this into equations.

In an inertial frame the equations of motion for the particles are given by the usual Newtonian expressions, namely

$$\frac{d^2x^i}{dt^2} = -\delta^{ik}\frac{\partial\Phi(x^j)}{\partial x^k}\,, \tag{5.1}$$

$$\frac{d^2(x^i+\xi^i)}{dt^2} = -\delta^{ik}\frac{\partial\Phi(x^j+\xi^j)}{\partial x^k}\,, \tag{5.2}$$

with ξ^i the separation vector between the two particles. For sufficiently small separations Eq. (5.2) can be Taylor expanded to linear order in ξ^i to obtain

$$\frac{d^2(x^i+\xi^i)}{dt^2} = -\delta^{ik}\left(\frac{\partial\Phi(x^i)}{\partial x^k}+\frac{\partial}{\partial x^j}\left(\frac{\partial\Phi(x^i)}{\partial x^k}\right)\xi^j+\dots\right)\,. \tag{5.3}$$

The *Newtonian deviation equation* for the separation vector ξ^i becomes therefore

$$\frac{d^2\xi^i}{dt^2} = -\delta^{ik}\left(\frac{\partial^2\Phi}{\partial x^k\partial x^j}\right)\xi^j\,. \tag{5.4}$$

The non-relativisitic *tidal tensor*

$$E^i{}_j \equiv \delta^{ik}\frac{\partial^2\Phi}{\partial x^k\partial x^j}\,, \tag{5.5}$$

determines the *tidal forces*, which tend to bring the particles together. This is the fundamental object for the description of gravity and not their individual accelerations $g_i = \partial_i\Phi$!

Exercise
Assume the tidal tensor $E^i{}_j$ to be reduced to diagonal form, as in the example below. Show that the components of that tensor cannot all have the same sign.

As a particular example, that will be useful in the future, consider two particles in the gravitational field of a spherically symmetric distribution of mass M, i.e $\Phi = -GM/r$. The tidal tensor (5.5) in this case becomes

$$E_{ij} = (\delta_{ij} - 3n_i n_j)\frac{GM}{r^3}\,, \tag{5.6}$$

where $n^i \equiv x^i/r$ are the components of the unit vector in the radial direction. Writing explicitly the different components in polar coordinates we obtain

$$\frac{d^2\xi^r}{dt^2} = +\frac{2GM}{r^3}\xi^r\,, \qquad \frac{d^2\xi^\theta}{dt^2} = -\frac{GM}{r^3}\xi^\theta\,, \qquad \frac{d^2\xi^\phi}{dt^2} = -\frac{GM}{r^3}\xi^\phi\,. \tag{5.7}$$

Note the different signs: the object is stretched in the radial direction and compressed in the transverse directions. Tidal forces squeeze a sphere into an ellipsoid (cf. Fig.5.1).

Exercise
Assuming the water in the oceans to be in static equilibrium and taking into account the results of the previous example, estimate the height of the tides generated by the Moon.

Using the tidal tensor (5.5) we can write the equations governing the structure of Newtonian gravity in the following suggestive way

$$E^i{}_i = 4\pi G\rho \qquad \text{Poisson's equation} \tag{5.8}$$

$$\frac{d^2\xi^i}{dt^2} = -E^i{}_j\,\xi^j \qquad \text{Geodesic Deviation} \tag{5.9}$$

$$\left.\begin{array}{r} E_{ij} = E_{ji} \\ E^i{}_{[j,l]} = 0 \end{array}\right\} \qquad \text{Bianchi Identities} \tag{5.10}$$

where the symbol $[j,l]$ stands for antisymmetrization in the corresponding indices, i.e.

$$E^i{}_{[j,l]} \equiv \frac{1}{2}\left(E^i{}_{j,l} - E^i{}_{l,j}\right)\,. \tag{5.11}$$

5.2 Geodesic deviation

Let us now study this issue taking into account the things that we learned in the previous chapter. Consider a bunch of geodesics $x^\mu(\sigma,\lambda)$ classified by the value of some parameter λ (cf. Fig. 5.2). Which is the requirement for having tidal forces? To answer this question, let

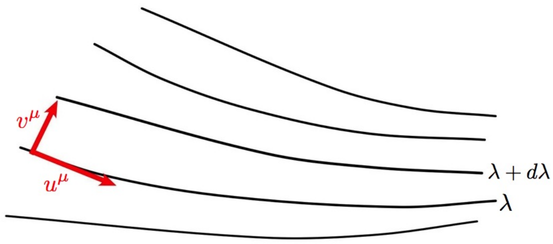


Figure 5.2: Bunch of geodesics classified by the value of λ.

me define two kinds of vectors (cf. Fig. 5.2): the tangent vector to the trajectory, $\frac{\partial x^\mu(\sigma,\lambda)}{\partial\sigma}$, that we will shortly denote by $u^\mu(\sigma,\lambda)$, and the derivative in the λ direction, $\frac{\partial x^\mu(\sigma,\lambda)}{\partial\lambda}$, that we will shortly denote by v^μ.

Taking Newtonian gravity as a guide, we expect the motion of the particles to be described by a second order differential equation involving the change of the separation vector v^μ along the path

$$\frac{D^2 v^\mu}{d\sigma^2} = u^\sigma \nabla_\sigma \left(u^\rho \nabla_\rho v^\mu\right) . \tag{5.12}$$

The right hand-side of this equation should contain the information about the true gravitational field. Using the relation[1]

$$v^\rho \nabla_\rho u^\mu = u^\rho \nabla_\rho v^\mu \,, \tag{5.13}$$

between the covariant derivatives of u^μ and v^μ, we get two pieces

$$\frac{D^2 v^\mu}{d\sigma^2} = u^\sigma \nabla_\sigma \left(u^\rho \nabla_\rho v^\mu\right) = u^\sigma \nabla_\sigma \left(v^\rho \nabla_\rho u^\mu\right) = u^\sigma \left(\nabla_\sigma v^\rho\right)\left(\nabla_\rho u^\mu\right) + u^\sigma v^\rho \nabla_\sigma \nabla_\rho u^\mu \,. \tag{5.14}$$

Changing the order of the covariant derivatives appearing in the first piece and using back Eq. (5.13) in the second piece, we obtain

$$\begin{aligned}
\frac{D^2 v^\mu}{d\sigma^2} &= \underbrace{u^\sigma \left(\nabla_\sigma v^\rho\right)}_{v^\sigma(\nabla_\sigma u^\rho)} \left(\nabla_\rho u^\mu\right) + u^\sigma v^\rho \underbrace{\nabla_\sigma \nabla_\rho}_{\nabla_\rho\nabla_\sigma + [\nabla_\sigma,\nabla_\rho]} u^\mu \\
&= \underbrace{v^\sigma \left(\nabla_\sigma u^\rho\right)\left(\nabla_\rho u^\mu\right)}_{\sigma\leftrightarrow\rho} + u^\sigma v^\rho \nabla_\rho \nabla_\sigma u^\mu + u^\sigma v^\rho \left[\nabla_\sigma, \nabla_\rho\right] u^\mu \\
&= v^\rho \left(\nabla_\rho u^\sigma\right)\left(\nabla_\sigma u^\mu\right) + u^\sigma v^\rho \nabla_\rho \nabla_\sigma u^\mu + u^\sigma v^\rho \left[\nabla_\sigma, \nabla_\rho\right] u^\mu \\
&= v^\rho \nabla_\rho \left(u^\sigma \nabla_\sigma u^\mu\right) + u^\sigma v^\rho \left[\nabla_\sigma, \nabla_\rho\right] u^\mu \,,
\end{aligned} \tag{5.15}$$

where in the last steps we have simply performed some index relabelings and collected terms. The first term in the last line of (5.15) vanishes since, as we show in Section 4.6, the tangent vector to the trajectory is parallel transported along the geodesic, $u^\sigma \nabla_\sigma u^\mu = 0$. We are left therefore with a very compact expression

$$\frac{D^2 v^\mu}{d\sigma^2} = u^\sigma v^\rho \left[\nabla_\sigma, \nabla_\rho\right] u^\mu \,, \tag{5.16}$$

[1] It follows directly from the definition of the covariant derivatives and the relation $\partial u^\mu/\partial\lambda = \partial v^\mu/\partial\sigma$.

which hides however a big amount of work inside the commutator of the two covariant derivatives.

What we should expect

Before proceeding to the explicit computation of this commutator, let me anticipate what is gonna happen. Note that the commutator of two covariant derivatives acting on a scalar ϕ

$$[\nabla_\sigma, \nabla_\rho]\phi = \nabla_\sigma \partial_\rho \phi - \nabla_\rho \partial_\sigma \phi = \left(\Gamma^\kappa{}_{\sigma\rho} - \Gamma^\kappa{}_{\rho\sigma}\right) \partial_\kappa \phi \;, \tag{5.17}$$

vanishes for a symmetric connection $\Gamma^\kappa{}_{\rho\sigma} = \Gamma^\kappa{}_{\sigma\rho}$, like the metric connection we are working with (cf. Eq. (4.62)). Taking this into account, let me compute the quantity

$$[\nabla_\sigma, \nabla_\rho]\,(\phi u^\mu) = ([\nabla_\sigma, \nabla_\rho]\,\phi)\, u^\mu + \phi\,[\nabla_\sigma, \nabla_\rho]\, u^\mu = \phi\,[\nabla_\sigma, \nabla_\rho]\, u^\mu \,. \tag{5.18}$$

The final result has important consequences. In particular, it tells us that $[\nabla_\sigma, \nabla_\rho]\, u^\mu$ cannot depend on the derivatives of u^ρ because in that case it would also have to depend on the derivatives of the scalar field ϕ. As the dependence on the vector u^μ is linear, we are left with an expression of the form

$$[\nabla_\sigma, \nabla_\rho]\, u^\mu = R^\mu{}_{\nu\sigma\rho} u^\nu \;, \tag{5.19}$$

with $R^\mu{}_{\nu\rho\sigma}$ some unknown coefficients. Although the particular combination of connections inside these coefficients cannot be determined without performing the full computation, it is nice to have an idea of the final result before computing it, right?

Let us start the explicit computation of the commutator $[\nabla_\sigma, \nabla_\rho]\, u^\mu$ from the definition of the covariant derivative

$$\nabla_\rho u^\mu = \partial_\rho u^\mu + \Gamma^\mu{}_{\kappa\rho} u^\kappa \,. \tag{5.20}$$

Differentiating with respect to x^σ we obtain

$$\begin{aligned} \nabla_\sigma \nabla_\rho u^\mu &= \partial_\sigma\left(\nabla_\rho u^\mu\right) + \Gamma^\mu{}_{\lambda\sigma} \nabla_\rho u^\lambda - \Gamma^\kappa{}_{\rho\sigma} \nabla_\kappa u^\mu \\ &= \partial_\sigma \partial_\rho u^\mu + \partial_\sigma\left(\Gamma^\mu{}_{\kappa\rho} u^\kappa\right) + \Gamma^\mu{}_{\lambda\sigma}\left(\partial_\rho u^\lambda + \Gamma^\lambda{}_{\kappa\rho} u^\kappa\right) - \Gamma^\kappa{}_{\rho\sigma}\left(\partial_\kappa u^\mu + \Gamma^\mu{}_{\lambda\kappa} u^\lambda\right) \;, \end{aligned} \tag{5.21}$$

where we have treated $\nabla_\rho u^\mu$ as a second rank tensor. Computing the difference $\nabla_\sigma \nabla_\rho u^\mu - \nabla_\rho \nabla_\sigma u^\mu$ we find that the terms involving first derivatives of u^μ vanish, as expected. The covariant derivatives of vectors do not commute by a value that depends only on the vector field at the point in question

$$[\nabla_\sigma, \nabla_\rho]\, u^\mu = -\left(\partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\kappa\rho}\Gamma^\kappa{}_{\nu\sigma} - \Gamma^\mu{}_{\kappa\sigma}\Gamma^\kappa{}_{\nu\rho}\right) u^\nu \equiv -R^\mu{}_{\nu\rho\sigma} u^\nu = R^\mu{}_{\nu\sigma\rho} u^\nu \,. \tag{5.22}$$

Ambiguities

Note that the non-commutation of covariant derivatives gives rise to some ambiguities in the *minimal coupling* prescription (*colon-goes-to-semicolon*) introduced in the previous Chapter. To illustrate this, consider for instance a physical law which in an inertial frame takes the form

$$U^\mu \partial_\mu \partial_\nu V^\nu = U^\mu \partial_\nu \partial_\mu V^\nu = 0\,, \tag{5.23}$$

with U^μ and V^ν some vector fields. Which should be the covariant generalization of this law? Should we write something like

$$U^\mu \nabla_\mu \nabla_\nu V^\nu = 0\,, \tag{5.24}$$

or rather something like

$$U^\mu \nabla_\nu \nabla_\mu V^\nu = 0\,? \tag{5.25}$$

According to (5.22), these two equations are not equal; they differ by a factor proportional $R^\mu{}_{\nu\rho\sigma}$, which is not necessarily zero. The *colon-goes-to-semicolon* prescription is ambiguous. This is reminiscent of the problem of ordering operators in quantum mechanics: the minimal prescription does not say anything about how to order the operators. The correct way of adapting the laws of physics to spaces with non-vanishing $R^\mu{}_{\nu\rho\sigma}$ can be only determined by experiments.

The n^4 quantities

$$R^\mu{}_{\nu\rho\sigma} \equiv \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\kappa\rho}\Gamma^\kappa{}_{\nu\sigma} - \Gamma^\mu{}_{\kappa\sigma}\Gamma^\kappa{}_{\nu\rho} \tag{5.26}$$

are the components of a tensor, as can be easily seen by applying the quotient theorem[2] to Eq. (5.22). This tensor is called the *curvature or Riemann tensor* and it is defined in terms of the metric and its first and second derivatives.

Exercise:

- Which is the value of $R^\mu{}_{\nu\rho\sigma}$ for a 2 dimensional Euclidean metric written in Cartesian coordinates? And if the metric is written in polar coordinates?
- Derive the action of the commutator of two covariant derivatives on a covariant vector. Hint: This should be a fast exercise. Remember the metric compatibility.
- Use the previous result to determine the action of the commutator of covariant derivatives on an arbitrary rank-(r,s) tensor.

Substituting (5.22) into Eq. (5.16) we obtain the so-called *geodesic deviation* equation

$$\frac{D^2 v^\mu}{d\sigma^2} = -R^\mu{}_{\nu\rho\sigma} u^\nu u^\sigma v^\rho\,. \tag{5.27}$$

[2] cf. property 3 in Section 1.4.4

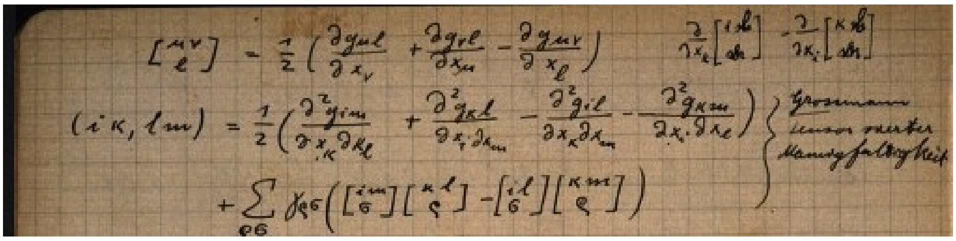


Figure 5.3: First appearance of the Riemann tensor in Einstein's Zurich notebooks. The Riemann tensor is written in the old-fashioned notation (ik, lm). According to some urban legends, Einstein learned the methods of Ricci and Levi-Civita through his school friend Marcel Grossmann. It was Grossmann the one who went to the library searching for methods to deal with arbitrary coordinate systems and discovered the Ricci and Levi-Civita's 1901 paper. The annotation "Grossmann tensor fourth rank" that you can find in the right hand side of the formula suggests indeed that Grossmann conveyed the Riemann tensor formula to Einstein.

The term in the right-hand side is the sought-after effect of gravity that cannot be removed by going to a free falling frame: the tidal acceleration. In the non-relativistic limit, the intrinsic derivative on the left hand side becomes d^2/dt^2 and $u^\mu \approx \delta^\mu{}_0$, in such a way that

$$\frac{d^2 v^\mu}{dt^2} = -R^\mu{}_{0\rho 0} v^\rho \,. \tag{5.28}$$

Taking into account Eqs. (5.4) and (5.5) we can to identify $R^\mu{}_{0\rho 0}$ with the non-relativistic tidal tensor [3]

$$E^i{}_j = R^i{}_{0j0} \,. \tag{5.30}$$

Exercise:

Compute the Christoffel symbols and the curvature tensor to the lowest order for the line element

$$ds^2 = -\left(1 + 2\phi\right) dt^2 + \delta_{ij} dx^i dx^j \,. \tag{5.31}$$

Interpret the result.

[3] Note that the deviation between two neighboring geodesics parametrized by the values λ and $\lambda + d\lambda$ is given by

$$\xi^\mu = \frac{dx^\mu}{\partial \lambda} \delta\lambda = v^\mu \delta\lambda \,. \tag{5.29}$$

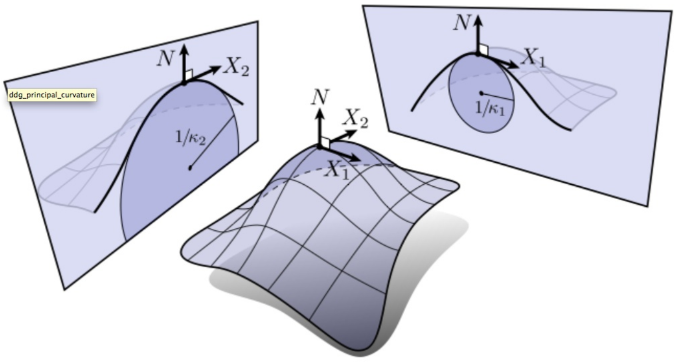


Figure 5.4: Principals curvatures of a surface.

5.3 Flat versus curved: A dirty and quick introduction to curvature.

The geodesic equation is a clear manifestation of the geometrical character of the Einstein's theory of gravity: it is a theory of curved spacetimes. To understand this, let me start with a basic and dirty introduction to the theory of surfaces and the concept of *curvature*. When I say curvature I mean what you understand by curvature in your everyday experience; objects such as eggshells, donuts, tennis balls, etc... are curved. A two dimensional surface can be though as embedded in the usual 3-dimensional Euclidean space[4]. At any given point P on the 2-dimensional surface, we can introduce a tangent plane with Cartesian coordinates (X_1, X_2) (cf. Fig. 5.4). This Euclidean space is called the *tangent space* to the surface at P. The deviation $z(X_1, X_2)$ of the curved surface from the tangent plane describes the local properties of our geometry. Since curvature effects arise only through the second derivatives of $z(x, y)$, it is convenient to use a quadratic function

$$z(X_1, X_2) = \frac{1}{2}\mathbf{X}^T M \mathbf{X}\,, \tag{5.32}$$

with

$$M = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \qquad \mathbf{X} \equiv (X_1, X_2)^T\,, \tag{5.33}$$

[4]We do this just for visualization purposes; that is why I said that my introduction is somehow dirty. There is no need to choose a particular embedding for studying the geometry of the surface; the geometry can be completely determined by measuring angles and distances on the surface. This is indeed a theorem, known as *Gauss' Egregium Theorem.* It words of Gauss himself, it reads

> *Formula itaque art[iculi] praec[edentis] sponte perducit ad egregium Theorema. Si superficies curva in quamcunque aliam superficiem explicatur, mensura curvaturae in singulis punctis invariata manet,*

which, for those of you not knowing latin means

> *Thus the formula of the preceding article leads itself to the remarkable Theorem. If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged.*

and a, b and c quantities with dimensions of inverse length[5]. Eq. (5.32) can be recast in a diagonal form by rotating the coordinates, $\bar{\mathbf{X}} = R\mathbf{X}$, and accordingly transforming the matrix M, $\bar{M} = R^{-1}MR$. In the new coordinate basis (ξ, η),we obtain

$$z(\xi,\eta) = \frac{1}{2}\left(\kappa_1\xi^2 + \kappa_2\eta^2\right) \equiv \frac{1}{2}\left(\frac{\xi^2}{\rho_1} + \frac{\eta^2}{\rho_2}\right) , \tag{5.34}$$

where we have defined the so-called *principal curvatures* κ_1 and κ_2 and the *principal radii* of curvature ρ_1 and ρ_2.

The result is quite intuitive. It simply states that any surface is locally the sum of two parabolas in the ξ and η directions and with radius of curvature ρ_1 and ρ_2 respectively (cf. Fig. 5.4).

Exercise

Expand a circle of radius ρ around some point. Comment on the result.

The square of the distance between two nearby points with coordinates[6] (x, y) and $(x + dx, y + dy)$ is given by

$$ds^2 = d\xi^2 + d\eta^2 + dz^2 = (\kappa_1\xi d\xi + \kappa_2\eta d\eta)^2 + \left(d\xi^2 + d\eta^2\right) \equiv \gamma_{\mu\nu}dx^\mu dx^\nu . \tag{5.35}$$

Since the measure of the surface curvature cannot depend on the set of coordinates used, it must be related to the basis-independent attributes of the matrix M. These attributes are its eigenvalues, or equivalently, its determinant and trace. The determinant $K = \det M = \kappa_1\kappa_2$ is called *intrinsic or Gaussian curvature* and can be expressed entirely in terms of intrinsic measurements on the surface, without any reference to the external embedding space. Starting from a point P on the surface and proceeding along a geodesic on the surface for a proper distance ϵ, we arrive to a point Q_1. Repeating this process with geodesics starting off in different directions, we obtain a set of points $Q_1, Q_2, \ldots$, all of them sitting at the circumference $C(\epsilon)$ of a geodesic disc centered at P (cf. Fig. 5.5). A simple computation using the metric (5.35) shows that the quantity[7]

$$\lim_{\epsilon\to 0^+} \frac{3}{\pi\epsilon^3}\left(2\pi\epsilon - C(\epsilon)\right) , \tag{5.37}$$

measuring the difference between the circumference $C(\epsilon)$ of our geodesic disc and a circumference in the plane, corresponds precisely to the value of the Gaussian curvature K at P

$$K = \kappa_1\kappa_2 = \frac{1}{\rho_1\rho_2} = \lim_{\epsilon\to 0^+} \frac{3}{\pi\epsilon^3}\left(2\pi\epsilon - C(\epsilon)\right) . \tag{5.38}$$

[5] A *local* region is defined for values of X_1 and X_2 much smaller than a^{-1}, b^{-1}, c^{-1}

[6] Note that although M is diagonal, the metric is not.

[7] There is not an absolute scale for Gaussian curvature, neither a unique choice of the normalization factor $3/\pi\epsilon^3$ appearing in Eq. (5.37). People have just agreed on the convention that the curvature of the unit sphere should be equal to 1 (although there are some natural motivations for it). For a small geodesic disc on the unit sphere of radius ϵ we have

$$C(\epsilon) \sim 2\pi\left(\epsilon - \frac{1}{6}\epsilon^3\right) , \tag{5.36}$$

which explains the proportionality factor $3/\pi\epsilon^3$.

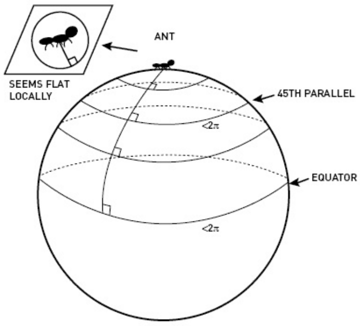


Figure 5.5: A clever ant determining the curvature of a sphere via the Bertrand-Diquet-Puiseux formula.

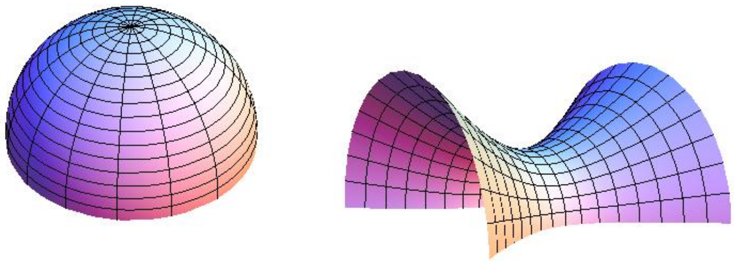


Figure 5.6: Positive ($K > 0$) and negatively curved ($K < 0$) spaces.

This expresion, relating the Gaussian curvature of a surface to the circumference of a geodesic circle, is known as the *Bertrand-Diquet-Puiseux formula*, and is closely related to the Gauss-Bonnet theorem that we will discuss below. Spaces with $K = 0$ everywhere are said to be *flat or developable*, since they can be "developed" or flattened out into a plane without stretching or tearing them (cf. Fig. 5.7). On the other hand, spaces with $K > 0$ everywhere are said to be *positively curved*, while spaces with $K < 0$ everywhere are said to be *negatively curved or saddle like.* For someone living on a given point of a space embedded in a higher dimensional space, the curvature at that point will be positive if the space curves away in the same way in any direction, while it will be negative if the space curves away in a different way when moving in different directions (cf. Fig. 5.6).

A worked-out example: a *truly* curved space.

As a direct application of the Bertrand-Diquet-Puiseux formula, consider the metric of the 2-dimensional sphere of unit radius

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2\,, \tag{5.39}$$

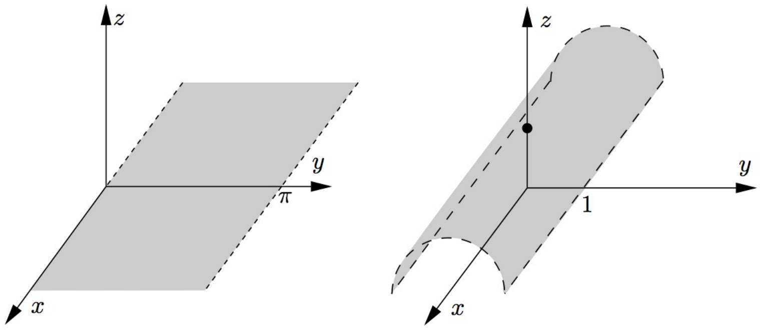


Figure 5.7: A plane sheet of paper ($\kappa_1 = \kappa_2 = 0$) rolled in the form of a cylinder of radius r ($\kappa_1 = 1/r$ and $\kappa_2 = 0$). The extrinsic curvature *changes* from 0 to $\kappa_1 + \kappa_2 = 1/r$.

and take P to be the origin. The distance from the origin to the point (ϵ, θ) is given by

$$\int_0^{\epsilon} ds = \epsilon. \tag{5.40}$$

The set of points with coordinates (ϵ, θ) form a disc whose circumference is given by

$$\int d\theta \sin\epsilon = 2\pi \sin\epsilon\,. \tag{5.41}$$

Applying (5.38), we get

$$\lim_{\epsilon\to 0} \frac{6}{\epsilon^2}\left(1 - \frac{\sin\epsilon}{\epsilon}\right) = 1\,. \tag{5.42}$$

The sphere (5.39) is a positively curved space.

On the other hand, the *extrinsic curvature*[8] is defined through the trace of M, namely $\kappa_1+\kappa_2$. The difference between the two can be easily understood by considering, for instance, a plane sheet of paper ($\kappa_1 = \kappa_2 = 0$) rolled in the form of a cylinder of radius r which will look like a curved 2-dimensional surface embedded in a 3-dimensional Euclidean space (cf. Fig 5.7). For the cylindrical surface we have $\kappa_1 = 1/r$ and $\kappa_2 = 0$. The intrinsic curvature retains the value of the flat sheet of paper. On the other hand, the extrinsic curvature changes from 0 to $\kappa_1 + \kappa_2 = 1/r$.

Exercise: Coodinates should not be trusted

Is the 2-dimensional space $ds^2 = \cos^2\phi d\phi^2 + \sin^2\phi d\theta^2$ curved or flat?

[8] In some books, the extrinsic curvature is normalized as $(\kappa_1 + \kappa_2)/2$ and called *mean curvature*.

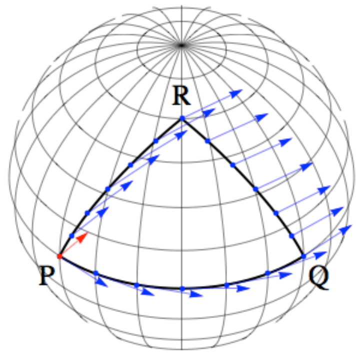


Figure 5.8: Parallel transport of a vector around a closed path on the sphere.

A worked-out space: a *fake* curved space

Consider the metric

$$ds^2 = \cos^2\phi d\phi^2 + \sin^2\phi d\theta^2 \,. \tag{5.43}$$

The distance from the origin to the point (ϵ, θ) is given by

$$\int_0^\epsilon ds = \int_0^\epsilon \cos\phi d\phi = \sin\epsilon. \tag{5.44}$$

and therefore $R = 0$, at you should have been expected: the space (5.43) is just a weird way of writing the flat metric $ds^2 = dx^2 + dy^2$ in polar coordinates (r, θ) with $r = \sin\phi$.

5.4 Parallel transport around a closed path

Consider the sum of the angles of a triangle, let's call them α, β and γ. As you know this sum is equal to π rad in flat space. What happens in a curved surface? When a surface is curved the sum of the angles in the triangle[9] is in general different from π. The more curved the surface is, the larger is the difference with respect to the flat result. The quantified version of this rather intuitive result is the result of so-called *Gauss-Bonnet theorem*[10]:

$$\int_S K dS = \alpha + \beta + \gamma - \pi \tag{5.45}$$

with K the Gauss curvature and S the area inside the triangle. To generalize this form of curvature, note that when the tangent vector at the PQ side is parallel transported from P to Q (cf. Fig. 5.8), it forms an angle $\pi - \beta$ with the tangent vector of the next side of the

[9] We are implicitly assuming that the sides of the triangle are geodesics, the curved analog of Euclidean straight lines.

[10] The standard presentation of the theory of surfaces is usually based on Gauss' Egregium Theorem and finishes with the derivation of the Gauss-Bonnet theorem. This sequence is however not chronological. Gauss deduced the Egregium Theorem starting from the Gauss-Bonnet theorem.

triangle. The same happens in the other vertices. This means that if we make a parallel transport around the whole close path, we obtain an angle $\pi - \beta + \pi - \gamma + \pi - \alpha$, which, forgetting about 2π multiples and writing the appropriate sign is given by $\alpha + \beta + \gamma - \pi$. The Gauss curvature measures the variation, in relation with the area, of parallel transported vectors around closed paths.

Ways of determining curvature

- Make distance measurements in different directions to construct the metric and then use it to find the curvature
- Take a vector and go around two different paths.

Note that in both cases, we don't make any reference to the higher-dimensional space in which we are embedded.

Although the intuitive reasoning presented above was bidimensional, it can be easily generalized to arbitrary dimension. To do that consider the parallel transport equation

$$\frac{dv^\mu}{d\sigma} = -\Gamma^\mu{}_{\nu\rho} v^\nu \frac{dx^\rho}{d\sigma} \tag{5.46}$$

and apply it to the case in which v^μ is parallel-transported along a small curve $\mathcal{C}$ from some initial point P. The value of the vector at any other point σ along this curve is given by

$$v^\mu(\sigma) = v^\mu_P - \int_o^\sigma \Gamma^\mu{}_{\nu\rho} v^\nu \frac{dx^\rho}{d\sigma} d\sigma \,. \tag{5.47}$$

Let us assume the loop $\mathcal{C}$ to be infinitesimally small. In that case, the quantities in the integrand of previous expression can be Taylor expanded around the point P to get

$$\Gamma^\mu{}_{\nu\rho}(\sigma) = \Gamma^\mu{}_{\nu\rho}|_P + \partial_\lambda \Gamma^\mu{}_{\nu\rho}\Big|_P \Delta x^\lambda + \dots \tag{5.48}$$

$$v^\mu(\sigma) = v^\mu_P - \Gamma^\mu{}_{\nu\rho}\Big|_P v^\nu_P \Delta x^\rho + \dots \tag{5.49}$$

with $\Delta x^\lambda \equiv x^\lambda(\sigma) - x^\lambda_P$. Plugging back these expressions into (5.47) and retaining only those terms up to first order in Δx^λ, we obtain

$$v^\mu(\sigma) = v^\mu_P - \Gamma^\mu{}_{\nu\rho}\Big|_P v^\nu_P \int_0^\sigma \frac{dx^\rho}{d\sigma} d\sigma - \left(\partial_\lambda \Gamma^\mu{}_{\nu\rho} - \Gamma^\mu{}_{\kappa\rho}\Gamma^\kappa{}_{\nu\lambda}\right)\Big|_P v^\nu_P \int_0^\sigma \left(x^\lambda - x^\lambda_P\right) \frac{dx^\rho}{d\sigma} d\sigma \tag{5.50}$$

The second and the last term (the part associated to x^λ_P) vanish for a closed path ($\oint dx^\rho = 0$) . We are left therefore with a net change

$$\Delta v^\mu = -\left(\partial_\lambda \Gamma^\mu{}_{\nu\rho} - \Gamma^\mu{}_{\kappa\rho}\Gamma^\kappa{}_{\nu\lambda}\right)\Big|_P v^\nu_P \oint x^\lambda dx^\rho. \tag{5.51}$$

This effect can be written in a more meaningful form by adding the result of interchanging the dummy indices ρ and λ. Doing this, and taking into account that

$$\oint d(x^\rho x^\lambda) = \oint \left(x^\rho dx^\lambda + x^\lambda dx^\rho\right) = 0\,, \tag{5.52}$$

we get

$$\Delta v^\mu = -\frac{1}{2}\left(\partial_\rho \Gamma^\mu{}_{\nu\lambda} - \partial_\lambda \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\kappa\rho}\Gamma^\kappa{}_{\nu\lambda} - \Gamma^\mu{}_{\kappa\lambda}\Gamma^\kappa{}_{\nu\rho}\right)\Big|_P v^\nu_P \oint x^\rho dx^\lambda \,. \tag{5.53}$$

Denoting by

$$A^{\rho\lambda} \equiv \oint x^\rho dx^\lambda \tag{5.54}$$

the total area enclosed by the loop $\mathcal{C}$ and taking into account Eq. (5.26), we finally obtain

$$\Delta v^\mu = -\frac{1}{2} R^\mu{}_{\nu\rho\lambda} v^\nu_P A^{\sigma\lambda} \,. \tag{5.55}$$

The change of the vector when it moves along a closed path is proportional to the Riemann tensor and to the area enclosed by the loop[11]! $R^\mu{}_{\nu\rho\sigma}$ is the generalization[12] of the Gauss curvature K. The components of a vector v^μ will remain unchanged after parallel transport *if and only if* the curvature tensor vanishes. In that happens, the spacetime is actually flat. Any apparent dependence of the metric on the coordinates will be just an illusion due to the use of some weird coordinate system and we will be able to find a *global* coordinate system in which the metric takes a Cartesian form.

Exercise:
Determine the Gauss curvature of a spherical surface of radius R through the Gauss-Bonnet theorem. Hint: Apply it, for instance, to the triangle determine by the $1/8$ part of the sphere.

5.5 Properties of the Riemann tensor

Eq. (5.26) provides a way of computing the 256 components of the Riemann tensor directly from the line element. This is usually a rather tedious process, even for Einstein (cf. Fig. 5.9). Fortunately, the covariant form of the Riemann tensor $R_{\mu\nu\rho\sigma} \equiv g_{\mu\lambda} R^\lambda{}_{\nu\rho\sigma}$ shows many

[11]Note that although our derivation was performed under the assumption of having an infinitesimal loop, it can be easily extended to larger closed curves. A given surface A bounded by a curve $\mathcal{C}$ can be understood as the sum of many small areas bounded by closed curves $\mathcal{C}_N$. Since the changes in Δv^μ around any of the interior curves cancel and only the outer edges contribute, we can express the change in the components v^μ along $\mathcal{C}$ as the sum of the changes around the small curves, namely

$$\Delta v^\mu = \sum_N (\Delta v^\mu)_N \,. \tag{5.56}$$

[12]Indeed the geodesic deviation equation (5.27) is nothing else than the generalization of the Jacobi equation

$$\frac{d^2 y}{d\sigma^2} + Ky = 0 \tag{5.57}$$

between two geodesics in a two dimensional surface.

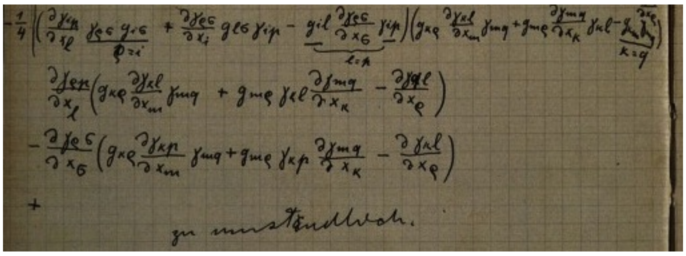


Figure 5.9: Einstein's manipulations of the Riemann tensor (Zurich notebook). The computation is abandoned, "zu umstaendlich" (too involved).

interesting symmetries in its indices that will simplify our life. Writing it explicitly in terms of the metric and Christoffel symbols we get

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\nu\partial_\rho g_{\mu\sigma} + \partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\nu\partial_\sigma g_{\mu\rho} - \partial_\mu\partial_\rho g_{\nu\sigma}\right) + g_{\lambda\kappa}\left(\Gamma^\lambda{}_{\nu\rho}\Gamma^\kappa{}_{\mu\sigma} - \Gamma^\lambda{}_{\nu\sigma}\Gamma^\kappa{}_{\mu\rho}\right) . \quad (5.58)$$

Using this expression we can derive the following properties:

- **Symmetry**: The Riemann tensor $R_{\rho\sigma\mu\nu}$ is symmetric under the interchange of the first *pair* of indices with the second *pair* of indices

$$R_{\mu\nu\rho\sigma} = +R_{\rho\sigma\mu\nu} . \quad (5.59)$$

- **Antisymmetry**: The Riemann tensor $R_{\rho\sigma\mu\nu}$ is antisymmetric under the interchange of either the first two indices or the second two indices

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} = -R_{\nu\mu\rho\sigma} = R_{\nu\mu\sigma\rho} . \quad (5.60)$$

  This is a direct consequence of the definition of the Riemann tensor (the operator $[\nabla_\sigma, \nabla_\rho]$ is antisymmetric) and the metric compatibility

$$[\nabla_\sigma, \nabla_\rho]g_{\mu\nu} = 0 \qquad \longrightarrow \qquad R^\kappa{}_{\mu\rho\sigma}g_{\kappa\nu} + R^\kappa{}_{\nu\rho\sigma}g_{\mu\kappa} = (R_{\nu\mu\sigma\rho} + R_{\mu\nu\rho\sigma}) = 0 . \quad (5.61)$$

- **1st Bianchi identity**: The cyclic sum of the last three indices is zero

$$3R_{\mu[\nu\rho\sigma]} \equiv R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0 . \quad (5.62)$$

  This can be easily understood by applying the operator $[\nabla_\rho, \nabla_\sigma]$ to the gradient $\nabla_\nu\phi$ of a scalar field. For any scalar $\nabla_{[\rho}\nabla_\sigma\nabla_{\nu]}\phi = 0$, which implies

$$R^\kappa{}_{[\nu\rho\sigma]}\nabla_\kappa\phi = 0 . \quad (5.63)$$

  Since the resulting expression is valid for all gradients, Eq. (5.62) follows immediately. Note that the result is non-trivial only when the three indices $\nu\rho\sigma$ are different. When two of these indices are equal one of the terms drop and the remaining terms just express the antisymmetry in the last two indices of the curvature tensor.

- **2nd Bianchi identity**: The Riemann tensor satisfies the differential identity[13]

$$\nabla_\kappa R^\mu{}_{\nu\rho\sigma} + \nabla_\sigma R^\mu{}_{\nu\kappa\rho} + \nabla_\rho R^\mu{}_{\nu\sigma\kappa} = 0\,. \tag{5.65}$$

The proof is left as an exercise.

> **Exercise**
> Prove Eq. (5.65) Hint: Use a local inertial frame.

- **Ricci tensor and Ricci scalar**: There are two important contractions of the Riemann tensor[14]. The first one is a second rank tensor obtained from contracting a pair of indices. Since $R_{\mu\nu\rho\sigma}$ is antisymmetric in $\mu\nu$ and $\rho\sigma$, the only non-trivial contraction is between μ and ρ or between μ and σ. These two contractions differ only by a change of sign. Taking the first contraction, we obtain the so-called *Ricci tensor*

$$R_{\nu\sigma} \equiv g^{\mu\rho} R_{\mu\nu\rho\sigma} = R^\mu{}_{\nu\mu\sigma} = \partial_\mu \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\mu} + \Gamma^\mu{}_{\kappa\mu}\Gamma^\kappa{}_{\nu\sigma} - \Gamma^\mu{}_{\kappa\sigma}\Gamma^\kappa{}_{\nu\mu}\,, \tag{5.66}$$

which is symmetric, as can be easily seen by taking into account the relation (4.72)

$$\partial_\sigma \Gamma^\mu{}_{\nu\mu} = \partial_\sigma \left(\frac{1}{\sqrt{|g|}} \partial_\nu \sqrt{|g|} \right) = -\frac{1}{|g|} \partial_\sigma \sqrt{|g|} \partial_\nu \sqrt{|g|} + \frac{1}{\sqrt{|g|}} \partial_\nu \partial_\sigma \sqrt{|g|}\,. \tag{5.67}$$

The second contraction is the so-called *Ricci scalar or Ricci curvature*

$$R \equiv R^\nu{}_\nu = g^{\nu\sigma} R_{\nu\sigma} = g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu\rho\sigma}\,. \tag{5.68}$$

That's all. There are no more non-vanishing contractions. The result (5.68) is quite remarkable. Among the 20 independent components of the Riemann tensor that transform into linear combinations of each other under general coordinate transformations, there is *one* which remains unchanged. R is the only scalar involving the metric and two derivatives.

> **Exercise:**
>
> Among the different ways of constructing a scalar from the Riemann tensor discussed above, why did I not discuss the contraction $\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$?

[13] This identity is related to the Jacobi identity

$$[[\nabla_\mu, \nabla_\nu], \nabla_\rho] + [[\nabla_\nu, \nabla_\rho], \nabla_\mu] + [[\nabla_\rho, \nabla_\mu], \nabla_\nu] = 0\,. \tag{5.64}$$

[14] We will only discuss the contractions at the lower order in the curvature tensor. Higher order contractions such as R^2, $R_{\mu\nu}R^{\mu\nu}$ or the square of the Riemann tensor, the so-called *Kretschmann* scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, will be introduced at its due time.

- **Contracted Bianchi identities**: Note the important result that follows from the Bianchi identity (5.65) and the definition of the Ricci scalar. Contracting the indices $\mu\rho$ in (5.65) we get

$$\nabla_\kappa R^\rho{}_{\nu\rho\sigma} + \nabla_\sigma R^\rho{}_{\nu\kappa\rho} + \nabla_\rho R^\rho{}_{\nu\sigma\kappa} = \nabla_\kappa R_{\nu\sigma} - \nabla_\sigma R_{\nu\kappa} + \nabla_\rho R^\rho{}_{\nu\sigma\kappa} = 0\,, \quad (5.69)$$

where we have made use of the antisymmetry property (5.60). Multiplying by the metric $g^{\nu\sigma}$, contracting the indices ν and σ and taking into account that $\nabla_\rho R^{\rho\sigma}{}_{\sigma\kappa} = -\nabla_\rho R^{\sigma\rho}{}_{\sigma\kappa} = -\nabla_\rho R^\rho{}_\kappa$, Eq. (5.69) becomes

$$\nabla_\kappa R - \nabla_\sigma R^\sigma{}_\kappa - \nabla_\rho R^\rho{}_\kappa = 0\,. \quad (5.70)$$

The previous expression can be written in a much more enlightening way

$$\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0\,. \quad (5.71)$$

The divergence of the so-called *Einstein tensor*

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad (5.72)$$

vanishes by construction[15]! The symmetry of the Einstein tensor under the interchange of its indices follows directly from the symmetries of the Ricci tensor and the metric. Which is the geometrical meaning of this tensor? To answer this, consider an observer moving with 4-velocity u^μ and compute the spatial components of the Riemann tensor in the instantaneous rest frame of such an observer[16]

$$\mathcal{R}_{\gamma\epsilon\lambda\kappa} = h^\mu{}_\gamma h^\nu{}_\epsilon h^\rho{}_\lambda h^\sigma{}_\kappa R_{\mu\nu\rho\sigma} \quad (5.73)$$

where we have made used of the projection operator $h^\mu{}_\nu = \delta^\mu{}_\nu + u^\mu u_\nu$. Contracting the indices γ and λ and the indices ϵ and κ in the previous expression, we get the scalar

$$\mathcal{R} = h^{\mu\rho} h^{\nu\sigma} R_{\mu\nu\rho\sigma} = (g^{\mu\rho} + u^\mu u^\rho)(g^{\nu\sigma} + u^\nu u^\sigma) R_{\mu\nu\rho\sigma} = R + 2u^\mu u^\rho R_{\mu\rho}\,. \quad (5.74)$$

which, comparing with the definition (5.72) of the Einstein tensor , can be written as

$$\mathcal{R} = 2u^\mu u^\rho G_{\mu\rho}\,. \quad (5.75)$$

$G_{\mu\nu} u^\mu u^\nu$ measures the local scalar curvature of the spatially projected curvature tensor.

A final warning

There are several sign conventions involved in the definition of the Riemann tensor and its contractions. Be careful when taking results from different books or articles. Our convention is that of Misner, Thorne and Wheeler. A very useful reference sheet taken precisely from this book can be found in the Moodle.

[15] Remember this, we will made use of it very soon.

[16] $\mathcal{R}_{\mu\nu\rho\sigma}$ is *not* the curvature of the 3-space orthogonal to u^μ, ${}^{(3)}R_{\mu\nu\rho\sigma}$!

5.6 Independent components of the Riemann tensor

How many independent components has the Riemann tensor $R_{\mu\nu\rho\sigma}$ in n dimensions? As a 4-indexed object in n dimensions we have a priori n^4 independent components, but the symmetries (5.59)-(5.62) will significantly reduce this number. In order to see this, consider the Riemman tensor $R_{\mu\nu\rho\sigma}$ as the expression of a symmetric $m \times m$ matrix[17] $R_{AB} = R_{BA}$ with indices $A = \{\mu\nu\}$ and $B = \{\rho\sigma\}$. This matrix has $\frac{1}{2}m(m+1)$ independent components. The value of m is determined by the number of choices that we have for A and B, which, taking into account Eq.(5.60), have the same content as a $n \times n$ antysimmetric matrix. We have therefore $m = \frac{1}{2}n(n-1)$ possible choices of A and B. The total number of components so far is

$$\frac{m(m+1)}{2} = \frac{1}{2}\left(\frac{n(n-1)}{2}\right)\left(\frac{n\,(n-1)}{2}+1\right) = \frac{\left(n^4 - 2n^3 + 3n^2 - 2n\right)}{8}\,, \tag{5.76}$$

but we have still to substract the constraints imposed by Eq.(5.62). To determine the number of extra constraints, notice that if one sets any two components equal (for instance $\mu = \nu$) we get identically zero (one term goes away by antisymmetry and the other two cancel). Only if the 4 indices are different we get a constraint. The number of independent constraints is the same as the number of combinations of 4 objects that can be chosen from n objects

$$\binom{n}{4} = \frac{n!}{4!(n-4)!} = \frac{n(n-1)(n-2)(n-3)}{24}\,. \tag{5.77}$$

The final number of independent components of the Riemann tensor becomes

$$C_R = \frac{m(m+1)}{2} - \frac{n!}{4!(n-4)!} = \frac{n^2(n^2-1)}{12}\,. \tag{5.78}$$

Evaluating this for different dimensions we get

Number of dimensions	1	2	3	4	5
Total components of ${R^\mu}_{\nu\rho\sigma}$	1	16	81	256	625
Independent components of ${R^\mu}_{\nu\rho\sigma}$	0	1	6	20	50

The number of independent components in 4 dimensions has been reduced from 256 to 20! The fact that the number is still quite large is reasonable, since we need a lot of numbers to specify how the space curves in many different directions.. As we will see in the next Section, these are precisely the degrees of freedom in the second derivatives of the metric that we cannot set to zero by performing a change of coordinates.

[17] This is sometimes called the *Petrov notation*.

Exercise

- In one dimension the Riemann tensor is always identically zero. Explain why.

  Hint: Remember the geometrical interpretation of the Riemann tensor.

- How many components have the Ricci tensor and the Ricci scalar in $2, 3$ and 4 dimensions? And the Einstein tensor? Is there any dimension in which the Riemann and the Ricci tensors haves the same number of independent components?

5.6.1 Local versus global flatness: A counting exercise

The Equivalence Principle is based on the existence of locally inertial (or freely falling) reference frames

$$g_{\mu\nu}(P) = \eta_{\mu\nu}\,, \qquad \partial_\sigma g_{\mu\nu}(P) = 0\,, \tag{5.79}$$

in which gravity can be transformed away. So, one of the things that we will like to verify is that this kind of coordinate systems exist in the context of Riemannian geometry, i.e., if we can always introduce a free falling frame (5.79) at an arbitrary point for an arbitrary metric $g_{\mu\nu}$. For doing that, consider a coordinate transformation from the coordinates x^μ to some coordinates ξ^α in the neighborhood of some point P. Performing a Taylor expansion around P, we get

$$\xi^\alpha(x) = \xi^\alpha(P) + A^\alpha{}_\mu \Delta x^\mu + B^\alpha_{\mu\nu}\Delta x^\mu \Delta x^\nu + C^\alpha{}_{\mu\nu\rho}\Delta x^\mu \Delta x^\nu \Delta x^\rho + \dots\,, \tag{5.80}$$

with $\Delta x^\mu \equiv x^\mu - P^\mu$ and[18]

$$A^\alpha{}_\mu = \frac{\partial \xi^\alpha}{\partial x^\mu}\Big|_P\,, \qquad B^\alpha{}_{\mu\nu} = \frac{1}{2}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\Big|_P\,, \qquad D^\alpha{}_{\mu\nu\rho} = \frac{1}{6}\frac{\partial^3 \xi^\alpha}{\partial x^\mu \partial x^\nu \partial x^\rho}\Big|_P\,. \tag{5.81}$$

Let us see if we can generically choose the values of the coefficients $A^\alpha{}_\mu, B^\alpha{}_{\mu\nu}, D^\alpha_{\mu\nu\rho}\dots$ in such a way that the conditions (5.79) are satisfied[19]. In four dimensions, the matrix $A^\alpha{}_\mu$ has $4^2 = 16$ independent components. Since we need only 10 conditions to impose $g_{\mu\nu}(P) = \eta_{\mu\nu}$, we are left with 6 components to spare, precisely the number of Lorentz transformations and rotations that we can make without modifying the form of metric in the Minkowski metric $\eta_{\mu\nu}$! The requirement $\partial_\sigma g_{\mu\nu}(P) = 0$ give rise to $4 \times 4(4+1)/2 = 40$ conditions, which are precisely the number of components of the symmetric quantity $B^\alpha{}_{\mu\nu}$. We have just proven that one can always choose coordinates in such a way that the metric reduces to the inertial form (5.79) in an *infinitesimal region* around a point P. In the mathematical literature, this is known as the *local flatness theorem*.

But, what happens with the other coefficients? Can we make also put the second derivatives of the metric to zero by simply performing coordinates transformations? The answer is no. The second derivatives of the metric, $\partial_\sigma \partial_\rho g_{\mu\nu}$, have $10 \times 10 = 100$ independent components, while $D^\alpha_{\mu\nu\rho}$ has only $4^2 \times (5 \times 6)/6 = 80$ components. This means that among the 100

[18] Note that, in spite of the appearances, the coefficients in the previous expression are not tensors, because they only transform as such under *global linear coordinate transformations.*

[19] Note that the coefficients $B^\alpha{}_{\mu\nu}, C^\alpha_{\mu\nu\rho}\dots$ are completely symmetric in the lower indices.

components of the metric second derivatives only 80 can be set to zero at P via coordinate transformations. Precisely the number of independent components of the Riemann tensor in 4 dimensions! Indeed, it is not difficult to prove that, at quadratic order in the coordinates, we can write

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3}\left(R_{\mu\rho\nu\sigma} + R_{\nu\rho\mu\sigma}\right)\Delta x^\rho \Delta x^\sigma \tag{5.82}$$

The second derivatives of the metric (or if you want the first derivative of the Christoffel symbols) encode the information about the *true* gravitational field $R_{\mu\nu\rho\sigma}$!. A free falling observer can pretend that he/she is not in the presence of a gravitational field, but the tidal forces cannot be eliminated!

Exercise

Repeat this exercise in arbitrary dimensions. What happens?

5.6.2 The Weyl tensor

In 4 dimensions, the Riemann tensor has 20 independent components, while the Ricci tensor and the scalar of curvature can only account for $10+1$ of those components. This should be somehow expected, since the Ricci tensor and the scalar curvature contain the information about the "traces" of the Riemann tensor, and not of it as a whole. The 20 independent components of the Riemann curvature tensor in 4 dimensions can be written in terms of three irreducible pieces: the scalar curvature R, the tracefree part of Ricci tensor

$$S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R\,, \tag{5.83}$$

and the so-called *Weyl tensor*

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{1}{3} R g_{\mu[\rho} g_{\sigma]\nu}\,. \tag{5.84}$$

The Weyl tensor is a linear rank-(0,4) tensor in $R_{\mu\nu\rho\sigma}$ with no dependence on the derivatives of the metric except through $R_{\mu\nu\rho\sigma}$. It has indeed the same symmetry properties as the Riemann tensor, and therefore the same number of potential components. Note however that the Weyl tensor is traceless

$$C^\mu{}_{\nu\mu\sigma} = g^{\mu\rho} C_{\mu\nu\rho\sigma} = 0\,. \tag{5.85}$$

which, taking into account the symmetry in the indices ν and σ, leaves as with $20-10=10$ independent components, which together with the $10-1=9$ independent components of the trace free part of the Ricci tensor $S_{\mu\nu}$, and the single component of the curvature scalar R, makes the 20 components of the Riemann tensor. Note that no new quantities can be obtained by contracting the indices of the above *irreducible components.*

An important property of the Weyl tensor is its behaviour under *conformal transformations.* A conformal transformation can be understood as a local dilatation, in which the line element changes from ds^2 to $\Omega^2(x) ds^2$, with $\Omega^2(x)$ an arbitrary and non-vanishing function called *conformal factor*[20]. Through a trivial, but quite involved computation, one can verify that when we perform one of these conformal transformations

$$g_{\mu\nu} \longrightarrow \Omega^2(x) g_{\mu\nu} \,, \tag{5.87}$$

the totally covariant Weyl tensor transforms accordingly

$$C_{\mu\nu\rho\sigma} = \Omega^2(x) C_{\mu\nu\rho\sigma} \,, \tag{5.88}$$

and therefore[21] $C^{\mu}{}_{\nu\rho\sigma}$ is conformally invariant[22]. This has an interesting consequence: in those case in which the metric can be written as the result of the conformal transformation of a flat spacetime, $g_{\mu\nu} = f(x)\delta_{\mu\nu}$ or $g_{\mu\nu} = f(x)\eta_{\mu\nu}$, the Weyl tensor is zero and the Riemann tensor can be entirely expressed in terms of the Ricci tensor $R_{\mu\nu}$ and the scalar of curvature R.

Exercise

Prove that the Weyl tensor (5.84) is indeed traceless.

5.7 A laboratory for Riemannian geometry: 2 dimensional manifolds

In two dimensions the covariant Riemann tensor $R_{\mu\nu\rho\sigma}$ has only one independent component. Since the indices can take only two different values, say 1 and 2, and $R_{\mu\nu\rho\sigma}$ is antisymmetric in μ and ν and ρ and σ, and symmetric in the interchange of the combinations $\mu\nu$ and $\rho\sigma$ as a whole, we are left with an expression of the form R_{1212}. Let us see how this component is related to the Ricci scalar. In order to do that, let me express the Riemann tensor as a linear combination of two tensors

$$S_{\mu\nu\rho\sigma} = g_{\mu\rho} g_{\nu\sigma} \,, \qquad T_{\mu\nu\rho\sigma} = g_{\mu\sigma} g_{\nu\rho} \,, \tag{5.89}$$

depending only in the metric and respecting the symmetries of the Riemann tensor[23]

$$R_{\mu\nu\rho\sigma} = A \left(S_{\mu\nu\rho\sigma} - T_{\mu\nu\rho\sigma} \right) \,. \tag{5.90}$$

Contracting the previous expression to obtain the Ricci scalar in the left-hand side we get

$$R = A g^{\mu\rho} g^{\nu\sigma} \left(S_{\mu\nu\rho\sigma} - T_{\mu\nu\rho\sigma} \right) = A \left(g^{\mu\rho} g_{\mu\rho} g^{\nu\sigma} g_{\nu\sigma} - g^{\mu\rho} g_{\mu\sigma} g^{\nu\sigma} g_{\nu\rho} \right) = (4-2)A = 2A \,, \tag{5.91}$$

[20] Note that this kind of transformations conserve the angle between vectors

$$\cos\left(U, V\right) = \frac{U_\mu V^\mu}{\sqrt{\left(U_\nu U^\nu\right)\left(V_\rho V^\rho\right)}} \,. \tag{5.86}$$

[21] Note the position of the indices

[22] This is true in any dimension

[23] The combination $S - T$ is antisymmetric under $\mu \leftrightarrow \nu$

which allows as to identify the unknown factor A in Eq. (5.90) and write the fully covariant expression[24]

$$R_{\mu\nu\rho\sigma} = K\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\mu\rho}\right)\,, \tag{5.93}$$

where we have defined the Gaussian curvature as $K = R/2$.

5.7.1 A worked-out example: 2 dimensional sphere

Let us go trough the whole process of computing the Ricci scalar. This kind of computations are usually involved, but with a bit of practice and care they are quite tractable[25]. The line element on the surface of a sphere of radius a can be obtained by substituting the coordinate transformations

$$x = a\sin\theta\cos\phi\,, \qquad y = a\sin\theta\sin\phi\,, \qquad z = a\cos\theta\,,$$

into the Euclidean line element $ds^2 = dx^2 + dy^2 + dz^2$. We obtain

$$ds^2 = a^2 d\theta^2 + a^2\sin^2\theta d\phi^2 \qquad \longrightarrow \qquad g_{\mu\nu} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix}\,. \tag{5.94}$$

The Christoffel symbols can be computed in many different ways, being the most practical one the Lagrangian method. The only non-vanishing terms are

$$\Gamma^{\theta}_{\phi\phi} = -\cos\theta\sin\theta\,, \qquad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta\,. \tag{5.95}$$

The $\mu = \theta$ component of the Riemann tensor is given by

$$R^{\theta}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\theta}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\theta}{}_{\nu\rho} + \Gamma^{\theta}{}_{\lambda\rho}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\theta}{}_{\lambda\sigma}\Gamma^{\lambda}{}_{\nu\rho}\,. \tag{5.96}$$

Among the two possibles values of the indices appearing in the $\Gamma\Gamma$ pieces, only the $\lambda = \rho = \phi$ choice contributes, so we can expand the sum over λ in the last two terms

$$R^{\theta}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\theta}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\theta}{}_{\nu\rho} + \Gamma^{\theta}{}_{\phi\rho}\Gamma^{\phi}{}_{\nu\sigma} - \Gamma^{\theta}{}_{\phi\sigma}\Gamma^{\phi}{}_{\nu\rho}\,. \tag{5.97}$$

Since the Riemann tensor is antisymmetric in ρ and σ, we cannot have $\rho = \sigma$. Let's set therefore $\rho = \phi$ and $\sigma = \theta$ (keeping in mind that the alternative choice, $\rho = \theta$ and $\sigma = \phi$, just gives rise to a relative minus sign). We have

$$R^{\theta}{}_{\nu\phi\theta} = \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\nu\theta} - \partial_\theta \Gamma^{\theta}{}_{\nu\phi} = 0\,, \tag{5.98}$$

[24] This is the particular expression of a much more general relation

$$R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\mu\rho}\right)\,. \tag{5.92}$$

for a *maximally symmetric spacetime* with constant R is arbitrary dimension. Unfortunately, I don't have the time to go trough it. The interested reader can have a look to this subject in Weinberg's book.

[25] Since this is the first non-trivial computation of the Ricci scalar that we perform, I will do it in great detail. Although I could directly compute R_{1212} (we are dealing with a 2-dimensional metric) I prefer not to do so in order to teach you some general tricks related to the symmetries of the Riemann tensor that will be useful when dealing with more complicated metrics.

from which we get, potentially, two terms

$$R^{\theta}{}_{\theta\phi\theta} = \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\theta\theta} - \partial_\theta \Gamma^{\theta}{}_{\theta\phi} = 0 \tag{5.99}$$

$$\begin{aligned} R^{\theta}{}_{\phi\phi\theta} &= \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\phi\theta} - \partial_\theta \Gamma^{\theta}{}_{\phi\phi} \\ &= (-\cos\theta\sin\theta)(\cot\theta) - \sin^2\theta + \cos^2\theta = -\sin^2\theta \\ &= -R^{\theta}{}_{\phi\theta\phi}\,. \end{aligned} \tag{5.100}$$

For $\mu = \phi$, the Riemann tensor becomes

$$R^{\phi}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\phi}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\phi}{}_{\nu\rho} + \Gamma^{\phi}{}_{\lambda\rho}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\phi}{}_{\lambda\sigma}\Gamma^{\lambda}{}_{\nu\rho}\,. \tag{5.101}$$

As before, the only option is $\rho = \phi$ and $\sigma = \theta$

$$R^{\phi}{}_{\nu\phi\theta} = \partial_\phi \Gamma^{\phi}{}_{\nu\theta} - \partial_\theta \Gamma^{\phi}{}_{\nu\phi} + \Gamma^{\phi}{}_{\lambda\phi}\Gamma^{\lambda}{}_{\nu\theta} - \Gamma^{\phi}{}_{\lambda\theta}\Gamma^{\lambda}{}_{\nu\phi}\,. \tag{5.102}$$

Taking into account that the metric does not depend on ϕ, the previous expression reduces to

$$R^{\phi}{}_{\nu\phi\theta} = -\partial_\theta \Gamma^{\theta}{}_{\nu\phi} - \Gamma^{\phi}{}_{\phi\theta}\Gamma^{\phi}{}_{\nu\phi}\,, \tag{5.103}$$

which is different from zero only if $\nu = \theta$

$$R^{\phi}{}_{\theta\phi\theta} = -\partial_\theta \Gamma^{\theta}{}_{\theta\phi} - \Gamma^{\phi}{}_{\phi\theta}\Gamma^{\phi}{}_{\theta\phi} = \frac{1}{\sin^2\theta} - \cot^2\theta = 1\,. \tag{5.104}$$

The Ricci tensor is obtained by contracting the upper and second lower index. In matrix notation we have

$$R_{\mu\nu} = \begin{pmatrix} R^{\theta}{}_{\theta\theta\theta} + R^{\phi}{}_{\theta\phi\theta} & R^{\theta}{}_{\theta\theta\phi} + R^{\phi}{}_{\theta\phi\phi} \\ R^{\theta}{}_{\phi\theta\theta} + R^{\phi}{}_{\phi\phi\theta} & R^{\theta}{}_{\phi\theta\phi} + R^{\phi}{}_{\phi\phi\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \tag{5.105}$$

The Ricci scalar is

$$R = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = \frac{1}{a^2} + \frac{1}{a^2\sin^2\theta}\sin^2\theta = \frac{2}{a^2}\,. \tag{5.106}$$

The Gaussian curvature

$$K \equiv \frac{R}{2} = \frac{1}{a^2} \tag{5.107}$$

is positive and constant, as expected, and coincides with the result (5.42) obtained by directly applying Bertrand-Diquet-Puiseux formula (5.38).

Remember: This was quite an explicit computation to show how to use the symmetries to rapidly derive the final result. In two dimensional cases it is better two remember that the Riemann tensor has only one independent component, directly compute the R_{1212} component

$$R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta \qquad \longrightarrow \qquad R_{\theta\phi\theta\phi} = g_{\theta\theta}R^{\theta}{}_{\phi\theta\phi} = a^2\sin^2\theta \tag{5.108}$$

and contract it with the inverse metric to obtain the scalar of curvature

$$R = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = \frac{2}{a^2} = \frac{2R_{\theta\phi\theta\phi}}{|g|}\,. \tag{5.109}$$

Note that the result

$$R = \frac{2R_{\theta\phi\theta\phi}}{|g|} \tag{5.110}$$

is just a particular version of Eq. (5.93).

Exercise

Compute the intrinsic curvature of the two-dimensional cone in Cartesian and polar coordinates. Interpret the result.

CHAPTER 6

EINSTEIN EQUATIONS

> You will be convinced of the general theory of relativity once you have studied it. Therefore I am not going to defend it with a single word.
>
> A. EINSTEIN

6.1 The energy-momentum tensor

Having decided that our description of the motion of test particles and light in a gravitational field should be based on the idea of curved space times with a metric, we must now complete the theory by postulating a law to say how the sources of the gravitational field determine the metric. To construct this gravitational field equation we must first find a covariant way of expressing the source term ρ in the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho\,. \tag{6.1}$$

It is clear that the relativistic generalization of Eq. (6.1) cannot simply involve ρ as the source of the relativistic gravitational field, since ρ is the energy density measured by only one observer, that at rest with respect to the fluid element. It is not the first time that we find this kind of situation. The relativistic formulation of Maxwell equations needed the combination of the charge density ρ_e and the charge current J^i into a 4-vector $J^\mu = (\rho, J^i)$ with the right transformation properties. Can we do something similar here? The most naive trial would be a combination of the energy density ρ with some energy flux s^i into a 4-vector, let's say $s^\mu = \left(\rho, s^i\right)$. However, the total energy in this case, $E = \int \rho\, d^3x$, is *not* a Lorentz invariant quantity[1], due to its combination with the linear 3-momentum p^i into

[1] Note that in the electromagnetic case, the total electric charge $Q = \int \rho_e d^3x$ *is* Lorentz invariant.

the 4-momentum $p^\mu = \left(E, p^i\right)$. We are forced therefore to look for a higher rank object encoding the relation among the energy density, the energy flux, the momentum density and the momentum flux or stress. This quantity is the so-called *energy-momentum-stress* tensor. Let's construct it.

6.1.1 Newtonian fluids

While point particles are characterized by their energy and momentum, the motion of continuous matter is usually characterized by two quantities: the mass density $\rho(t, x^i)$ and the velocity of the fluid $v(t, x^i)$, which generally depend on space and time. The evolution of a continuous system is determined by two equations:

i) A continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \left(\rho v^j\right)}{\partial x^j} = 0\,, \tag{6.2}$$

reflecting the fact that mass is neither created or destroyed in Classical Mechanics (the flowing of mass out from a volume is equal to the loss of mass in it).

ii) A Newton's 2nd law for fluids

$$f^i = \rho a^i = \rho \left(\frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} \right)\,, \tag{6.3}$$

with

$$a^i = \lim_{\Delta t \to 0} \frac{v^i(t + \Delta t, x + \Delta x) - v^i(t, x)}{\Delta t}\,, \tag{6.4}$$

and $f^i = f^i(t, x)$ the total force per unit volume around a point x at time t. The so-called *total derivative* of the velocity field

$$\frac{Dv^i}{dt} \equiv \frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} \tag{6.5}$$

contains two pieces, the local derivative $\partial \mathbf{v}/\partial t$, which gives the change of the velocity $\mathbf{v}$ as a function of time at a given point in space, and the so-called *convective derivative*, $(\mathbf{v} \cdot \nabla)\,\mathbf{v}$, which represents the change of $\mathbf{v}$ for a moving fluid particle due to the inhomogeneity of the fluid vector field.

If we assume that there are not other forces apart from those exerted by the fluid on itself, we are left with internal forces like pressure or friction acting only between neighboring regions of matter. Consider a infinitesimal volume dV with surface area dA centered at a point x at time t. Let us denate by n_j the normal vector to the surface. In a perfect fluid[2], the force F^i exerted by the matter on the area is proportional to the area itself $F^i = p(t, x)\delta^{ij} n_j dA$, with $p(t, x)$ the pressure at that point at time t. In the most general case, we will also have *shear forces*

$$F^i(t, x) = T^{ij}(t, x) n_j dA\,, \tag{6.6}$$

[2] A perfect fluid is defined as one for which there are no forces between the particles, no heat conduction and no viscosity.

due to the tendency of fluid elements moving with different velocities to drag adjacent matter. The coefficients T^{ij} are the components of the so-called *stress tensor*, which must be symmetric, $T^{ij} = T^{ji}$.

Exercise

Consider the 3-component of the torque acting on an infinitesimal cube of a material of density ρ and side length L. Compare it with the moment of inertia of the cube $I = \frac{1}{6}\rho L^5$. What happens if $T^{ij} \neq T^{ji}$ in the limit $L \to 0$?

The total force exerted per unit area in a given direction[3] can be transformed into a total force by unit volume via the Gauss' theorem

$$-\int_A T^{ij} n_j dA = -\int_V \left(\partial_j T^{ij}\right) dV \qquad \longrightarrow \qquad f^i = -\frac{\partial T^{ij}}{\partial x^j}\,. \tag{6.7}$$

Plugging in this result into the Newton 2nd law (cf. Eq. (6.3))

$$\rho\left(\frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j}\right) + \frac{\partial T^{ij}}{\partial x^j} = 0\,, \tag{6.8}$$

and using the continuity equation to write

$$\rho \frac{\partial v^i}{\partial t} = \frac{\partial \left(\rho v^i\right)}{\partial t} - v^i \frac{\partial \rho}{\partial t} = \frac{\partial \left(\rho v^i\right)}{\partial t} + v^i \frac{\partial \left(\rho v^j\right)}{\partial x^j} \tag{6.9}$$

The previous result and the continuity equation (6.2), the Newton's 2nd law (6.3) for this particular case ($f^i = -\partial_j T^{ij}$) can be written as

$$\frac{\partial \left(\rho v^i\right)}{\partial x^0} + \frac{\partial}{\partial x^j}\left(\rho v^i v^j + T^{ij}\right) = 0\,, \tag{6.10}$$

which is the so-called *Euler equation*.

6.1.2 Relativistic fluids

Eqs. (6.2) and (6.10) can be unified into a single equation in the framework of Special Relativity. To see this, note that the 3-velocity v^i is contained in the relativistic 4-velocity $u^\mu = \left(u^0, u^i\right) = \left(\gamma, \gamma v^i\right)$. Taking into account the non-relativistic limit of this relation, $u^\mu = \left(1, v^i\right)$, we can rewrite (6.2) and (6.10) as

$$\frac{\partial \left(\rho u^0 u^0\right)}{\partial x^0} + \frac{\partial \left(\rho u^0 u^j\right)}{\partial x^j} = 0\,, \qquad \frac{\partial \left(\rho u^i u^0\right)}{\partial x^0} + \frac{\partial}{\partial x^j}\left(\rho u^i u^j + T^{ij}\right) = 0\,, \tag{6.11}$$

which can be considered as parts of the single equation

$$\partial_\nu T^{\mu\nu} = 0\,, \qquad T^{\mu\nu} = \rho u^\mu u^\nu + t^{\mu\nu}\,, \tag{6.12}$$

[3] The minus sign appears because we are considering the force exerted on matter *inside* the volume by the matter *outside*

with $t^{\mu\nu} = \text{diag}\left(0, T^{ij}\right)$. The quantity $T^{\mu\nu}$ is the so-called *energy-momentum-stress tensor* or in a shorter version the *energy-momentum tensor*[4] or the *stress-energy tensor*. It is a rank-2 symmetric tensor encoding all the information about energy density, momentum density, stress, pressure .... The ten components of this tensor have the following interpretation:

- T^{00} is the local energy density, including any potential contribution from forces between particles and their kinetic energy.
- T^{0i} is the energy flux in the i direction. This includes not only the bulk motion but also any other processes giving rise to transfers of energy, as for instance heat conduction.
- T^{i0} is the density of the momentum component in the i direction, i.e. the 3−momentum density. As the previous case, it also takes into account the changes in momentum associated to heat conduction.
- T^{ij} is the 3-momentum flux or stress tensor, i.e the rate of flow of the i momentum component per unit area in the plane orthogonal to the j-direction. The component T^{ii} encodes the isotropic pressure in the i direction while the components T^{ij} with $i \neq j$ refer to the *viscous stresses* of the fluid.

6.1.3 Relativistic perfect fluids

A relativistic perfect fluid is defined to be one in which the $t^{\mu\nu}$ part of the stress-energy tensor $T^{\mu\nu}$, as seen in a local reference frame moving along with the fluid, has same form as the non-relativistic perfect fluid

$$t^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} . \tag{6.13}$$

Heat conduction, viscosity or any other transport or dissipative processes in this case are negligible. The form of Eq. (6.13) in an arbitrary inertial frame can be obtained by performing a general Lorentz transformation

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma v^i \\ \gamma v^i & \delta^{ij} + v^i v^j (\gamma - 1)/v^2 \end{pmatrix} = \begin{pmatrix} u^0 & u^i \\ u^i & \delta^{ij} + u^i u^j/(1+\gamma) \end{pmatrix} , \tag{6.14}$$

moving from the rest frame $u^\mu = (1, \mathbf{0})$ to one in which the fluid moves with 3-velocity v^i. We get

$$\bar{t}^{\mu\nu} = \Lambda^\nu{}_\rho \Lambda^\nu{}_\sigma t^{\rho\sigma} = p\left(\eta^{\mu\nu} + u^\mu u^\nu\right) , \tag{6.15}$$

with u^μ the 4-velocity vector field tangent to the worldines of the fluid particles. Taking into account this result, the full stress-energy tensor (6.12) takes the form

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p\eta^{\mu\nu} . \tag{6.16}$$

[4] This name can be sometimes misleading as it can be confused with the energy-momentum *4-vector* p^μ in sentences including things like "the energy-momentum conservation equation...". The difference should be always clear from the context.

The resulting equation is manifestly covariant and can be easily generalized to arbitrary coordinate systems or curved spacetimes by simply replacing the local metric $\eta^{\mu\nu}$ by a general metric $g^{\mu\nu}$

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu} \,. \tag{6.17}$$

The conservation law $\partial_\nu T^{\mu\nu} = 0$ in Eq. (6.12) becomes a *local conservation law*

$$\nabla_\nu T^{\mu\nu} = 0 \tag{6.18}$$

in which the standard derivative ∂_μ is replaced by the covariant derivative ∇_μ. The word *local* is, as always in this course, important. Eq. (6.18) is *not* a conservation law, nor should it be. As we will see, energy is not conserved in the presence of dynamical spacetime curvature but rather changes in response to it.

Exercise
Prove Eq. (6.15).

6.2 The microscopic description

The relation between ρ and p is usually characterized by an equation of state $p = p(\rho)$ which depends on the microscopic particles involved in the fluid. In order to get some insight about the possible equations of state, let me consider a macroscopic collection of N structureless point particles interacting through spatially localized collisions. The energy density associated to any of them is given by

$$T^{00}_n = E_n\, \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n(t)\right) = m_n \gamma_n \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n(t)\right) \,, \tag{6.19}$$

with $\gamma_n = 1/\sqrt{1 - v_n^2}$ and $n = 1, \dots, N$ a label selecting the particular particle we are referring to. Taking into account the identity

$$\begin{aligned}\int_{-\infty}^{+\infty} d\tau\, \delta^{(4)}\left(x - x(\tau)\right) &= \int_{-\infty}^{+\infty} d\tau\, \delta\left(t - t(\tau)\right) \delta^{(3)}\left(\mathbf{x} - \mathbf{x}(\tau)\right) \\ &= \frac{d\tau}{dt} \delta^{(3)}\left(\mathbf{x} - \mathbf{x}(t)\right) = \frac{1}{\gamma} \delta^{(3)}(\mathbf{x} - \mathbf{x}(t)) \,, \end{aligned} \tag{6.20}$$

the non-Lorentz invariant 3-dimensional Dirac delta appearing in Eq. (6.19) can be transformed into a Lorentz invariant 4-dimensional Dirac delta[5]

$$T^{00}_n = m_n \int_{-\infty}^{+\infty} d\tau_n\, u^0_n u^0_n \delta^{(4)}\left(x - x_n(\tau_n)\right) \,. \tag{6.21}$$

[5] The fact that the 4-Dimensional Dirac delta $\delta^{(4)}(x)$ is Lorentz invariant follows directly from the definition $\int d^4x \delta^{(4)}(x) = 1$ and the fact that the volume element d^4x is Lorentz invariant.

The same procedure can be applied to the spatial momentum density (or energy current) of the particle

$$T_n^{0i} = p_n^i\,\delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n(t)\right) = m_n\gamma_n v_n^i\,\delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n(t)\right) = E_n v_n^i\,\delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n(t)\right) \,, \tag{6.22}$$

and to the flux of the i momentum component in the j direction (or viceversa)

$$T_n^{ij} = p_n^i v_n^j\,\delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n(t)\right) = p_n^j v_n^i\,\delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n(t)\right) \,. \tag{6.23}$$

We obtain

$$T_n^{0i} = m_n \int_{-\infty}^{+\infty} d\tau_n\, u_n^0 u_n^i \delta^{(4)}\left(x - x_n(\tau_n)\right) \,, \qquad T_n^{ij} = m_n \int_{-\infty}^{+\infty} d\tau_n\, u_n^i u_n^j \delta^{(4)}\left(x - x_n(\tau_n)\right) \,. \tag{6.24}$$

Eqs. (6.21) and (6.24) can be rewritten in a very compact way in terms of the stress-energy-momentum tensor $T^{\mu\nu}$

$$T_n^{\mu\nu} = m_n \int_{-\infty}^{+\infty} d\tau_n\, u_n^\mu u_n^\nu \delta^{(4)}\left(x - x_n(\tau_n)\right) = \int_{-\infty}^{+\infty} d\tau_n\, \frac{p_n^\mu p_n^\nu}{m_n} \delta^{(4)}\left(x - x_n(\tau_n)\right) \,, \tag{6.25}$$

which is manifestly symmetric and Lorentz invariant since $u_n^\mu u_n^\nu$ is a tensor under Lorentz transformations and both m_n and $d\tau_n\,\delta^{(4)}\left(x - x_n(\tau_n)\right)$ are Lorentz scalars. The total energy density of the whole system of particles can be written as the sum of the individual contributions, namely

$$T^{\mu\nu} = \sum_{n=1}^{N} T_n^{\mu\nu} \,. \tag{6.26}$$

6.2.1 Energy-momentum tensor conservation and geodesics

Let us see under which conditions the total energy momentum tensor (6.26) is conserved. Taking the derivative with respect to the coordinates we get

$$\partial_\mu T^{\mu\nu} = \sum_{n=1}^{N} m_n \int_{-\infty}^{+\infty} d\tau_n\, u_n^\mu u_n^\nu \partial_\mu \delta^{(4)}\left(x - x_n(\tau_n)\right) \,, \tag{6.27}$$

which using

$$u_n^\mu \partial_\mu \delta^{(4)}\left(x - x_n(\tau_n)\right) = \frac{dx_n^\mu}{d\tau_n}\frac{\partial}{\partial x^\mu}\delta^{(4)}\left(x - x_n(\tau_n)\right) = -d/d\tau_n \delta^{(4)}\left(x - x_n(\tau_n)\right) \,, \tag{6.28}$$

can be written as

$$\partial_\mu T^{\mu\nu} = -\sum_{n=1}^{N} m_n \int_{-\infty}^{+\infty} d\tau_n\, \frac{d}{d\tau_n}\left(u_n^\nu \delta^{(4)}\left(x - x_n(\tau_n)\right)\right) + \sum_{n=1}^{N} m_n \int_{-\infty}^{+\infty} d\tau_n\, \dot{u}_n^\nu \delta^{(4)}\left(x - x_n(\tau_n)\right) \,.$$

The first term in the right hand side of the previous expression disappears in the particles are stable, i.e. if the orbits are closed or come from negative infinite time and disappear into positive infinite time. We are left then with the second term, which can be written as

$$\partial_\mu T^{\mu\nu} = \sum_{n=1}^{N} \int_{-\infty}^{+\infty} d\tau_n\, \frac{dp_n^\nu}{d\tau_n} \delta^{(4)}\left(x - x_n(\tau_n)\right) = \sum_{n=1}^{N} \frac{dp_n^\nu}{dt} \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n\right) \,, \tag{6.29}$$

with $p^\nu_n = m_n u^\nu_n$ the 4-momentum of the individual particles. The *local energy momentum conservation* $\partial_\mu T^{\mu\nu} = 0$ requires the particles to be free. Or in other words, the condition $\partial_\mu T^{\mu\nu} = 0$ is equivalent to the geodesic equation in Minkowski space-time, $dp^\mu_n/d\tau = 0$. This will be also the case in curved spacetime.

6.2.2 The fluid limit

On distances d much larger than the typical mean free path a, the number of particles is large and the statistical fluctuations about the mean properties of the fluid are expected to be small[6]. Imagine a comoving observer exploring distances $d \gg a$. If the fluid is isotropic[7], the average value of the $T^{0i} \propto u^0 u^i$ component measured by this observer will be zero since the vector u^i points in all possible directions. In this case, the fluid can be characterized in terms of two quantities: its mean density and pressure over the volume $\Delta V = d^3$

$$\rho = \left\langle \sum_n E_n \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n\right) \right\rangle_{\Delta V}, \qquad p = \frac{1}{3}\sum_i \left\langle \sum_n p^i_n v^i_n \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n\right) \right\rangle_{\Delta V}. \tag{6.30}$$

A simple inspection of Eqs. (6.30) reveals that, for standard matter, $0 \leq p \leq \rho/3$. In any other reference frame, the energy-momentum tensor for the perfect fluid reads

$$T^{\mu\nu}(x) = (\rho(x) + p(x))u^\mu(x)u^\nu(x) + p(x)\eta^{\mu\nu}, \tag{6.31}$$

with $u^\mu(x)$ denoting now the average value of the 4-velocities u^μ_i of the individual particles N_R inside the volume[8]. The perfect fluid form (6.31) can be used to model very different physical situations that often fall into one of the following categories:

1. **Non-relativistic matter**: For small velocities the dispersion relation $E_n = \sqrt{m_n^2 + \mathbf{p}_n^2}$ can be approximated by $E_n \simeq m_n + \mathbf{p}_n^2/2m_n$, which plugged back into (6.30) gives rise to $\rho \simeq m_n n + \frac{3}{2}p$. Taking into account that the statistical definition of temperature T is twice the energy possessed by each degree of freedom and assuming a monoatomic gas with 3 kinetic degrees of freedom, we can write $T = (2/3) \times \mathbf{p}_n^2/2m_n$ and therefore $\rho \simeq m_n n + \frac{3}{2}T$.

2. **Dust**: A perfect fluid with zero pressure. $p = 0$, $t^{\mu\nu} = 0$, $T^{\mu\nu} = \rho\,\mathrm{diag}\,(1, 0, 0, 0)$.

3. **Radiation**: A perfect highly relativistic fluid. In this case $E_n \simeq |\mathbf{p}_n| \gg m_n$ and therefore[9] $\rho \simeq 3p$. The energy momentum for radiation is traceless, $T = T^\mu{}_\mu = \eta_{\mu\nu}T^{\mu\nu} = -\rho + 3p = 0$.

[6] Remember that, when we later apply the Equivalence Principle, we will have another scale into play: the scale L at which the gravitational effects start to be important. If this scale happens to be much larger than the scale d ($L \gg d \gg a$), the mean properties of the fluid can be safely considered as constant over the region.

[7] i.e if the fluid is *perfect*.

[8] Note that, when writing $u^\mu(x), \rho(x)$ and $p(x)$ we are explicitly taking into account that the averages can vary from one region to another.

[9] Note that the quantity $\sum_i p^i_n v^i_n$ in Eq. (6.30) can be written as $\sum_i p^i_n v^i_n = \sum_i m\gamma_n v^i_n v^i_n = \frac{|\mathbf{p}_n|^2}{E_n}$, which goes to $|\mathbf{p}_n|$ when $E_n \simeq |\mathbf{p}_n|$.

A worked-out example: The electromagnetic field

The paradigmatic case of a fluid with a radiation equation of state is the electromagnetic field. To see this explicitly, consider the energy density of the electromagnetic field

$$T_{00} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) , \tag{6.32}$$

and write it in the way seen by an observer moving with 4-velocity u^μ. The electron field seen by that observer is given by

$$E_\mu = F_{\mu\nu} u^\nu . \tag{6.33}$$

Using this expression we get the following covariant expression for the square of the electric field

$$\mathbf{E}^2 = F_{\mu\nu} u^\nu F^\mu{}_\rho u^\rho . \tag{6.34}$$

A similar expression for the magnetic field square can be obtained from the explicit expression for the square of the electromagnetic field strength tensor

$$F_{\mu\nu} F^{\mu\nu} = -2\left(E^2 - B^2\right) . \tag{6.35}$$

We obtain

$$\mathbf{B}^2 = F_{\mu\nu} u^\nu F^\mu{}_\rho u^\rho + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} . \tag{6.36}$$

Putting Eqs. (6.34) and (6.36) together, the covariant generalization of the energy density (6.32) becomes

$$\rho = \left(F_{\rho\mu} F^\rho{}_\nu - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} \eta_{\mu\nu}\right) u^\mu u^\nu , \tag{6.37}$$

where we have inserted a factor $u_\mu u^\mu = -1$. The work is basically done. The quantity in parenthesis is the sought-for energy-momentum tensor for the electromagnetic field!

$$T_{\mu\nu} = F_{\rho\mu} F^\rho{}_\nu - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} \eta_{\mu\nu} . \tag{6.38}$$

Exercise

- Compute the T^{0i} in terms of the electric and magnetic fields. Do you recognize the result?
- Prove that the electromagnetic energy-momentum is symmetric $T_{\mu\nu} = T_{\nu\mu}$ and traceless, $T^\mu{}_\mu = 0$. The electromagnetic field behaves as a fluid with equation of state $p = 1/3\rho$.

6.3 Einstein equations: Heuristic derivation

We have finally all the tools needed to derive the Einstein field equations for the gravitational field. In the Poisson equation, the gravitational field is determined by the matter distribution. The relativistic version of the matter distribution[10], the energy-momentum tensor $T^{\rm M}_{\mu\nu}$, must be somehow equated[11] to some tensor $K_{\mu\nu}$ depending of the metric $g_{\mu\nu}$ and its first and second derivatives[12]

$$K_{\mu\nu} = \kappa^2 T^{\rm M}_{\mu\nu}\,, \tag{6.39}$$

with κ^2 a proportionality constant to be determined. But, what tensor? Einstein got the answer to this question trough a complicated process of intuition, trial and error; *superhuman exertions* in his own words. As claimed above, the left-hand side of Eq.(6.39) should contain a second order differential operator acting on the metric. We already found some quantities with this property in the previous chapter: the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ and its contractions. The most natural tentative for $K_{\mu\nu}$ would be the Ricci tensor $R_{\mu\nu}$, since this is the contraction appearing in the Newtonian limit of the geodesic deviation equation ($R^i{}_{0i0} = E^i{}_i$). This was also one of the first *trial and error* choices of Einstein

$$R_{\mu\nu} \approx \kappa^2 T^{\rm M}_{\mu\nu}\,. \tag{6.40}$$

Note however that this choice is inconsistent, since the divergence $\nabla^\nu R_{\mu\nu}$ of the Ricci tensor is, in general, different from zero and, according to our minimal coupling prescription, the energy-momentum should be locally conserved, $\nabla^\nu T^{\rm M}_{\mu\nu} = 0$. Indeed, making use of the Bianchi identity (5.71) we can write $\nabla^\mu R_{\mu\nu} = 1/2\nabla_\nu R$, which together with the trace of Eq. (6.40), $R = \kappa^2 g^{\mu\nu} T^{\rm M}_{\mu\nu} = \kappa^2 T^{\rm M}$, implies the condition $\nabla_\mu T^{\rm M} = 0$. Since the covariant derivative of the scalar quantity $T^{\rm M}$ is just the partial derivative, we should necessarily have a constant $T^{\rm M}$ throughout the whole spacetime, which is highly implausible, since, as we know, $T^{\rm M} = 0$ for the electromagnetic field and $T^{\rm M} > 0$ for standard matter. On top of that, Eq. (6.40) hides 10 differential equations for 6 physical unknowns: the components of the metric that cannot be freely changed by performing coordinates transformations in the 4 coordinates. We have to try harder.

The most general combination of symmetric tensors involving up to two derivatives of the metric is

$$K_{\mu\nu} = R_{\mu\nu} + a g_{\mu\nu} R + \Lambda g_{\mu\nu} \tag{6.41}$$

with a and Λ some unknown constants to be determined[13]. Imposing the local conservation of the energy-momentum tensor $\nabla^\mu T^{\rm M}_{\mu\nu}$=0 in Eq. (6.39) we get

$$\nabla^\mu K_{\mu\nu} = \nabla^\mu \left(R_{\mu\nu} + a g_{\mu\nu} R\right) = 0\,, \tag{6.42}$$

[10] *Matter* should be understood in a broad sense, meaning really matter, radiation etc...

[11] A relativistic generalization should take the form of an equation between tensors.

[12] The requirement of having derivatives only up to second order is certainly reasonable. If this were not the case, one would have to specify for the *Cauchy problem* not only the value of the metric and its first derivative, but also higher derivatives on a spacelike surface.

[13] A possible proportionality constant in front of $R_{\mu\nu}$ has been factored out and incorporated in the still unknown factors κ and *Lambda* in the right hand side of Eq. (6.39).

where we have taken into account that the covariant derivative ∇_μ is metric compatible and therefore $\nabla_\mu (\Lambda g^{\mu\nu}) = 0$. Our situation now is much better than that of Einstein, we are aware of the contracted form of the Bianchi identities[14] (5.71) and know the precise value of a that satisfies Eq. (6.42), namely $a = 1/2$. Taking this into account, we can rewrite Eq. (6.39) as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T^{\rm M}_{\mu\nu}\,, \tag{6.43}$$

with

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\,, \tag{6.44}$$

the Einstein tensor defined in previous chapter (cf. Eq. (5.72)) and Λ the famous *cosmological constant* term. Writing this cosmological constant term in the right hand side of the equation, we can interpret it as the energy-momentum tensor of a fluid with a weird equation of state $p = -\rho$

$$G_{\mu\nu} = \kappa^2 \left(T^{\rm M}_{\mu\nu} + T^{\Lambda}_{\mu\nu}\right)\,, \qquad T^{\Lambda}_{\mu\nu} = -\frac{\Lambda}{\kappa^2} g_{\mu\nu}\,. \tag{6.45}$$

Defining $T_{\mu\nu} \equiv T^{\rm M}_{\mu\nu} + T^{\Lambda}_{\mu\nu}$, we can write

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu}\,. \tag{6.46}$$

Even though our derivation was quite heuristic, the solution that we have obtained is unique (Lovelock theorem). The resulting tensorial equation is a set of ten differential equations[15] for the metric $g_{\mu\nu}(x)$ given the energy-momentum tensor $T_{\mu\nu}(x)$. However, due to the existence of the Bianchi identities, not all the components are longer independent. There are only 6 independent equations to determine 6 independent components of the metric tensor.

As differential equations they are very complicated, even in vacuum. Both the Ricci scalar and the scalar curvature involve derivatives and products of Christoffel symbols, which in turn involve derivatives of the metric tensor. There is also some dependence on the metric hidden in the energy-momentum tensor. On top of that, the equations are not linear, as it should be expected, since, according to the Equivalence Principle, every form of energy, including the *gravitational self-energy*, must be a source of the gravitational field[16]. The non-linearity of the equation forbids us to apply the superposition principle, given two known solutions they cannot be combined to get a new one.

The Einstein equation in words

The physical meaning of Einstein equations can be clarified by considering an observer with velocity u^μ. The energy density as measured in the energy frame of such an observer is given by $\rho = T_{\mu\nu} u^\mu u^\nu$. Taking this into account, together the interpretation of the Einstein tensor that we developed in the previous Chapter, the physical content

[14] He wasn't.

[15] Both sides of the equation are symmetric rank-2 tensors.

[16] Note however that they have a well-posed initial-value structure, i.e. they determine the future values of $g_{\mu\nu}$ from given initial data. This consideration is of key importance for the study of systems evolving in time from some initial state, as for instance, gravitational waves.

of (6.43) can be summarized as

$$\left(G_{\mu\nu} - \kappa^2 T_{\mu\nu}\right) u^\mu u^\nu = 0\,, \tag{6.47}$$

which in words reads

$$\left[\begin{array}{c}\text{Scalar curvature of the spatial}\\ \text{sections measured by an}\\ \text{observer with velocity } u^\mu\end{array}\right] = 2\kappa^2 \left[\begin{array}{c}\text{Energy density measured by}\\ \text{an observer with 4-velocity } u^\mu\end{array}\right].$$

6.4 The linearized theory of gravity

Equation (6.46) looks very promising but we have still to prove that it is able to reproduce the Newtonian theory of gravity and determine the value of the unknown constants κ and Λ. The fastest way to obtain the Newtonian limit is to use the assumptions discussed in Section 3.6. Let me however relax these assumptions and obtain the general expression for the Einstein equation in the so-called *weak field limit*. This limit is defined by the condition

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad \text{with} \qquad |h_{\mu\nu}| \ll 1\,. \tag{6.48}$$

The quantity $h_{\mu\nu}$ is then understood as a small perturbation on top of the Minkowski background. Consistently with this point of view, we will raise and lower its indices with the flat Minkowski metric $\eta_{\mu\nu}$, namely $h^{\mu}{}_{\sigma} = \eta^{\mu\rho}h_{\rho\sigma}\,, h^{\mu\nu} = \eta^{\nu\sigma}h^{\mu}{}_{\sigma}\,.$

In order to compute the expression for the Einstein tensor $G_{\mu\nu}$ at the lowest order in perturbation theory we must first determine the linearized version of the Ricci tensor and the scalar curvature, which are functions of the metric connection $\Gamma^{\mu}{}_{\nu\rho}$. Inserting the expansion (6.48) into the definition of the metric connection, we get

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}\left(\partial_\nu h_{\sigma\rho} + \partial_\rho h_{\sigma\nu} - \partial_\sigma h_{\rho\nu}\right) + \mathcal{O}(h^2_{\mu\nu})\,. \tag{6.49}$$

The next step is to compute the 4 pieces of Riemman tensor, which, written in a very schematic way, have the structure $R^{\cdot}{}_{\cdot\cdot\cdot} \sim \partial\Gamma - \partial\Gamma + \Gamma\Gamma + \Gamma\Gamma$. Taking into account (6.49), we realize that only the first two terms ($\sim \partial\Gamma$) give a contribution to the leading order

$$R^{\mu}{}_{\nu\rho\sigma} = \frac{1}{2}\partial_\rho\left(\partial_\sigma h^{\mu}{}_{\nu} + \partial_\nu h^{\mu}{}_{\sigma} - \partial^{\mu}h_{\nu\sigma}\right) - (\rho \leftrightarrow \sigma) = \frac{1}{2}\left(\partial_\nu\partial_\rho h^{\mu}{}_{\sigma} + \partial_\sigma\partial^{\mu}h_{\nu\rho} - (\rho \leftrightarrow \sigma)\right) \tag{6.50}$$

The linearized version of the Ricci tensor and the scalar of curvature can be computed by simply performing contractions in the previous expression. Denoting respectively by $h \equiv h^{\mu}{}_{\mu}$ and $\Box = \partial^{\mu}\partial_{\mu}$ the trace of the perturbation tensor and the d'Alambertian operator and contracting the indices μ and σ in Eq. (6.50), we get[17]

$$R_{\nu\rho} = -\frac{1}{2}\left(\Box h_{\nu\rho} + \partial_\nu\partial_\rho h - \partial_\nu\partial_\sigma h^{\sigma}{}_{\rho} - \partial_\rho\partial_\sigma h^{\sigma}{}_{\nu}\right)\,, \tag{6.51}$$

which can be further contracted in the indices ν and ρ to obtain

$$R = R^{\nu}{}_{\nu} = \eta^{\nu\rho}R_{\nu\rho} = -\Box h + \partial_\nu\partial_\sigma h^{\nu\sigma}\,. \tag{6.52}$$

Collecting all the terms and inserting them into the definition of the Einstein tensor (6.44), we get

$$\begin{aligned} G_{\nu\rho} &= -\frac{1}{2}\left(\partial_\nu\partial_\rho h + \Box h_{\nu\rho} - \partial_\nu\partial_\sigma h^{\sigma}{}_{\rho} - \partial_\rho\partial_\sigma h^{\sigma}{}_{\nu} - \eta_{\nu\rho}\Box h + \eta_{\nu\rho}\partial_\mu\partial_\sigma h^{\mu\sigma}\right) \\ &= -\frac{1}{2}\left(\Box\tilde{h}_{\nu\rho} + \eta_{\nu\rho}\partial_\mu\partial_\sigma\tilde{h}^{\mu\sigma} - \partial_\nu\partial_\sigma\tilde{h}^{\sigma}{}_{\rho} - \partial_\rho\partial_\sigma\tilde{h}^{\sigma}{}_{\nu}\right)\,, \end{aligned} \tag{6.53}$$

[17] The global minus sign comes from the permutation of the last two indices two construct the Ricci scalar.

Newton	Einstein
Newton 2nd law $\frac{d^2x^i}{dt^2} = -\delta^{ij}\frac{\partial\Phi}{\partial x^j}$	Geodesic equation $\frac{d^2x^\mu}{d^2\sigma} = -\Gamma^\mu{}_{\nu\rho}\frac{\partial x^\rho}{d\sigma}\frac{dx^\nu}{d\sigma}$
Tidal deviation $\frac{d^2\xi^i}{dt^2} = -E^i{}_j\,\xi^j$	Geodesic deviation $\frac{D^2\xi^\mu}{d\sigma^2} = -R^\mu{}_{\nu\rho\sigma}u^\nu u^\sigma \xi^\rho$
1st Bianchi identity $E_{ij} = E_{ji}$	1st Bianchi identitiy $R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0$
2nd Bianchi identity $E^i{}_{[j,l]} = 0$	2nd Bianchi identitiy $\nabla_\kappa R^\mu{}_{\nu\rho\sigma} + \nabla_\sigma R^\mu{}_{\nu\kappa\rho} + \nabla_\rho R^\mu{}_{\nu\sigma\kappa} = 0$
mass density ρ	Energy-momentum tensor $T_{\mu\nu}$
Poisson equation $E^i{}_i = 4\pi G\rho$ single elliptic equation boundary data required	Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ 10 coupled equations 4 elliptic and 6 hyperbolic initial and boundary data required

Table 6.1: Newtonian vs Einstenian description of gravity.

where in the last step we have defined the so-called *trace reverse*

$$\tilde{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \,, \qquad\qquad h_{\mu\nu} = \tilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tilde{h} \,, \tag{6.54}$$

which keeps track of the extra terms obtained when passing from $R_{\nu\rho}$ to $G_{\nu\rho}$. The name trace reverse comes from the property $\tilde{h} \equiv \tilde{h}^{\mu}{}_{\mu} = -h$. Note also the useful properties

$$\tilde{\tilde{h}}_{\mu\nu} = h_{\mu\nu} \,, \qquad G_{\mu\nu} = \tilde{R}_{\mu\nu} \,. \tag{6.55}$$

The linearized Einstein equations becomes finally

$$\left(\Box \tilde{h}_{\nu\rho} + \eta_{\nu\rho}\partial_\mu\partial_\sigma \tilde{h}^{\mu\sigma} - \partial_\nu\partial_\sigma \tilde{h}^{\sigma}{}_{\rho} - \partial_\rho\partial_\sigma \tilde{h}^{\sigma}{}_{\nu}\right) = -2\kappa^2 T_{\nu\rho} \,. \tag{6.56}$$

The resulting expression is rather involved, but fortunately we still have some freedom to play with: the gauge freedom.

Gauge fixing

Eqs. (6.50) and (6.53), and therefore (6.56), are invariant under the transformation

$$h_{\nu\rho} \longrightarrow h_{\nu\rho} - \partial_\nu \xi_\rho - \partial_\rho \xi_\nu \,, \tag{6.57}$$

as can be easily verified be performing the explicit computation. This kind of change is called a *gauge transformation*, due to the strong analogy with the *gauge transformations* in the electromagnetic theory. The simplest way to understand this *gauge freedom* is to trace it back to the transformation of the full metric $g_{\mu\nu}$. Consider an infinitesimal transformation $x^\mu \to \bar{x}^\mu = x^\mu + \xi^\mu$. Under such a transformation the metric changes to

$$\begin{aligned} \bar{g}^{\mu\nu}\left(x^\rho + \xi^\rho\right) &= g^{\rho\sigma}(x^\rho)\frac{\partial \bar{x}^\mu}{\partial x^\rho}\frac{\partial \bar{x}^\nu}{\partial x^\sigma} \\ &= g^{\rho\sigma}\left(\delta^\mu{}_\rho + \partial_\rho \xi^\mu\right)\left(\delta^\nu{}_\sigma + \partial_\sigma \xi^\nu\right) \\ &= g^{\mu\nu}(x^\rho) + g^{\mu\sigma}\partial_\sigma \xi^\nu + g^{\nu\rho}\partial_\rho \xi^\mu \,. \end{aligned} \tag{6.58}$$

Expanding the left-hand side of this equation in a Taylor series in ξ^ρ and retaining only the terms up to linear order, we get

$$\bar{g}^{\mu\nu}(x^\rho) = g^{\mu\nu}(x^\rho) + \delta g^{\mu\nu} \,, \tag{6.59}$$

with

$$\delta g^{\mu\nu} \equiv -\xi^\rho \partial_\rho g^{\mu\nu} + g^{\mu\rho}\partial_\rho \xi^\nu + g^{\nu\rho}\partial_\rho \xi^\mu = \nabla^\nu \xi^\mu + \nabla^\mu \xi^\nu \,. \tag{6.60}$$

In the particular case in which the perturbation is performed around the Minkowski background, $g_{\mu\nu} = h_{\mu\nu} + \eta_{\mu\nu}$, the covariant derivatives in (6.59) become standard derivatives and we recover the transformation law (6.57). The linearized theory is invariant under (6.57) because the full nonlinear theory is invariant under general coordinate transformations! This is extremely interesting, since it allows us to further simplify the linearized version of the Einstein tensor by simply performing infinitesimal coordinates transformations, or in other words, changes from a splitting $g_{\mu\nu} = \eta_{\mu\nu} + h_{\rm old}$ to a different splitting $g_{\mu\nu} = \eta_{\mu\nu} + h_{\rm new}$. A simple inspection of Eq. (6.56) reveals that an interesting condition to be satisfied by the trace reverse tensor in the new coordinate system would be the tensor analog of the Lorenz gauge $\partial_\mu A^\mu = 0$ in the electromagnetic theory[a], namely

$$\partial_\rho \tilde{h}^{\nu\rho}_{\rm new} = 0\,. \tag{6.61}$$

Let us see if we are allowed to choose such a gauge. The change in the trace reverse tensor $\tilde{h}_{\mu\nu}$ follows directly from Eqs. (6.54) and (6.57)

$$\tilde{h}^{\nu\rho}_{\rm new} = \tilde{h}^{\nu\rho}_{\rm old} - \partial^\nu \xi^\rho - \partial^\rho \xi^\nu + \eta^{\nu\rho}\partial_\mu \xi^\mu\,. \tag{6.62}$$

Taking the derivative of this equation we get

$$\partial_\rho \tilde{h}^{\nu\rho}_{\rm new} = \partial_\rho \tilde{h}^{\nu\rho}_{\rm old} - \Box \xi^\nu\,. \tag{6.63}$$

In order to satisfy the gauge fixing (6.61), ξ^ν must be a solution of the inhomogeneous wave equation

$$\Box \xi^\nu = \partial_\rho \tilde{h}^{\nu\rho}_{\rm old}\,. \tag{6.64}$$

The existence of a solution transforming from an arbitrary $h_{\mu\nu}$ to the so-called *Lorenz gauge* $\partial_\rho \tilde{h}^{\nu\rho}_{\rm new} = 0$ is guaranteed[b] for sufficiently well behaved $\partial_\rho \tilde{h}^{\nu\rho}_{\rm old}$. In fact, the choice is not unique since we can always add to it any solution of the homogeneous wave equation $\Box \xi^\nu_H = 0$ and the result will still obey $\Box\left(\xi^\nu + \xi^\nu_H\right) = \partial_\rho \tilde{h}^{\nu\rho}_{\rm old}$. The Lorenz gauge $\partial_\rho \tilde{h}^{\nu\rho}_{\rm new} = 0$ is actually a set of gauges.

[a] It "kills" three of the four terms in (6.53).

[b] As you learnt in your electrodynamic course, the solution of this equation can be obtained by means of the retarded Green functions of the d'Alambertian operator.

In view of the previous discussion, we realize that most of the terms in the left-hand side of Eq. (6.56) merely serve to maintain gauge invariance. When the *Hilbert gauge condition*[18] $\partial_\rho \tilde{h}^{\nu\rho} = 0$ is imposed, the linearized version of the Einstein equation simplifies dramatically

$$\Box \tilde{h}_{\mu\nu} = -2\kappa^2 T_{\mu\nu}\,. \tag{6.65}$$

This equation is formally identical to the Maxwell equations in the Lorenz gauge and can be solved by using the Green's function method.

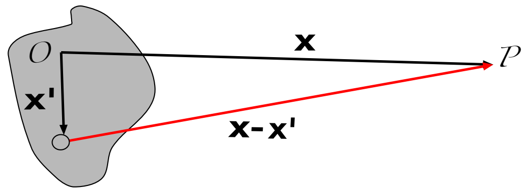
Green's functions

Consider a differential wave equation of the form

$$\Box f(t,\mathbf{x}) = s(t,\mathbf{x})\,, \tag{6.66}$$

with $f(t,\mathbf{x})$ a radiation field and $s(t,\mathbf{x})$ a source term. A Green's function $G(t,\mathbf{x};t',\mathbf{x}')$

[18] This gauge is also called Einstein gauge, harmonic gauge, de Donder gauge, Fock gauge or, in analogy with electromagnetism, Lorenz gauge.



is defined as the field generated at the point $(t, \mathbf{x})$ by a delta function source at $(t', \mathbf{x}')$. i.e.

$$\Box G(t, \mathbf{x}; t', \mathbf{x}') = \delta(t - t')\delta(\mathbf{x} - \mathbf{x}') \, . \tag{6.67}$$

The field due the actual source $s(t, \mathbf{x})$ can be obtained by integrating the Green's function against $s(t, \mathbf{x})$:

$$f(t, \mathbf{x}) = \int dt' d^3x' \, G(t, \mathbf{x}; t', \mathbf{x}') \, s(t', \mathbf{x}') \quad . \tag{6.68}$$

Physically the Green's function approach merely reflects the fact that (6.66) is a linear equation. The full solution of the equation can be obtained by solving for a point source and adding the resulting waves from each point inside the source.

The Green's function associated with the wave operator $\Box$ is very well known (see for instance the Jackson's book on electrodynamics.):

$$G(t, \mathbf{x}; t', \mathbf{x}') = -\frac{\delta(t' - [t - |\mathbf{x} - \mathbf{x}'|])}{4\pi|\mathbf{x} - \mathbf{x}'|} \, . \tag{6.69}$$

Exercise

Derive this equation in case you haven't done it before.

Using (6.69) into (6.65), we get[19]

$$\tilde{h}_{\mu\nu} = \frac{\kappa^2}{2\pi} \int \frac{T_{\mu\nu}\left(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}'\right)}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' \, , \tag{6.70}$$

which is analogous to the relation between the vector potential A_μ and the current J_μ in electromagnetism. Note the argument $t - |\mathbf{x} - \mathbf{x}'| = t - |\mathbf{x} - \mathbf{x}'|/c$. Eq. (6.70) is a *retarded solution*[20], taking into account the lag associated with the propagation of information from

[19] Note that we can always add to this particular solution an arbitrary solution of the homogeneous wave equation (vacuum). As in electromagnetism, the metric perturbation consists of the field generated by the source plus wave-like vacuum solutions propagating at the speed of light.

[20] The retarded solution is obtained by imposing the Kirchoff-Sommerfeld "no-incoming radiation" boundary condition at past null infinity

$$\lim_{t \to \infty} \left(\partial_r + \partial_t\right) \left(r\bar{h}_{\mu\nu}\right) = 0 \, , \tag{6.71}$$

with the limit taken along any surface with $ct + r =$constant, together with the condition that $r\tilde{h}_{\mu\nu}$ and $r\partial_\rho\tilde{h}_{\mu\nu}$ are bounded in this limit.

events at $\mathbf{x}$ to position $\mathbf{x}'$. Gravitational influences propagate at the finite speed of light. Action at a distance is gone forever! We will be back to this point at the next chapter , but before let me finish our main task: determining the value of the constants κ^2 and Λ. For doing that let me consider the case we know better: the gravitational field created by a static spherical mass distribution of total mass M. The energy-momentum tensor for such a system has only one non-vanishing component (cf. Eq. (6.45))

$$T^{00} = \left(\rho + \frac{\Lambda}{\kappa^2}\right) \operatorname{diag}(1,0,0,0) \,. \tag{6.72}$$

Plugging this into the time independent version of Eq. (6.70), we get

$$\tilde{h}_{00} = \frac{\kappa^2}{2\pi}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3\mathbf{x}' + \frac{1}{2\pi}\int \frac{\Lambda}{|\mathbf{x}-\mathbf{x}'|} d^3\mathbf{x}' \,, \qquad \tilde{h}_{0i} = 0\,, \qquad \tilde{h}_{ij} = 0\,. \tag{6.73}$$

If the mass distribution is concentrated around the origin ($\mathbf{x}' = 0$), the component h_{00} evaluated at a distance $r = |\mathbf{x}-\mathbf{x}'|$ becomes[21]

$$\tilde{h}_{00} = \frac{\kappa^2}{2\pi}\int \frac{\rho(\mathbf{x}')}{r} d^3\mathbf{x}' + \frac{1}{2\pi}\int \frac{\Lambda}{r} d^3\mathbf{x}' = \frac{\kappa^2}{2\pi}\frac{M}{r} + \frac{2}{3}\Lambda r^2 \tag{6.74}$$

with

$$M = \int \rho(\mathbf{x}')\, d^3\mathbf{x}' \tag{6.75}$$

the total mass of our spherical distribution. Taking now into account that $\tilde{h} = \eta^{\mu\nu}\tilde{h}_{\mu\nu} = -\tilde{h}_{00}$ and using the definition (6.54) we get

$$h_{00} = h_{11} = h_{22} = h_{33} = \frac{\kappa^2 M}{4\pi r} + \frac{1}{3}\Lambda r^2 \,. \tag{6.76}$$

Comparing this result with that obtained by performing the weak field limit of the geodesic equation in the $\Lambda = 0$ case, $h_{00}^{\Lambda=0} = -2\Phi = 2GM/r$, allows us to identify the sought-for proportionality constant

$$\kappa^2 = 8\pi G\,. \tag{6.77}$$

When $\Lambda \neq 0$, the Newtonian potential becomes modified at long distances

$$\Phi = -\frac{GM}{r} - \frac{\Lambda}{6} r^2 \tag{6.78}$$

and line element takes the form

$$ds^2 = -\left(1 - \frac{2GM}{r} - \frac{1}{3}\Lambda r^2\right) dt^2 + \left(1 + \frac{2GM}{r} + \frac{1}{3}\Lambda r^2\right) dX^2 \,, \tag{6.79}$$

with $dX^2 \equiv dx^2 + dy^2 + dz^2$. In Newtonian terms, a positive cosmological constant ($\Lambda > 0$) gives rise to a repulsive force per unit mass whose strength increases linearly with the distance

$$\mathbf{f} = -\frac{GM}{r^2}\mathbf{u}_r + \frac{\Lambda}{3} r\, \mathbf{u}_r \,, \tag{6.80}$$

[21] Note that the integral in over the prime variables!

Cosmological constant

If $\Lambda \neq 0$, it must be at least very small, $\rho^{\Lambda} \ll \rho^{\text{matter}}$, to avoid any observational effect in those situations in which the Newton's theory of gravity successfully explains the observations. Taking into account, for instance, that we do not see any modification of the Newtonian theory of gravity within the solar system, we can set the limit

$$|\rho_\Lambda| = \frac{|\Lambda|}{8\pi G} \leq \rho_{\text{Solar}} \qquad \longrightarrow \qquad |\rho_\Lambda| \leq \frac{3M_\odot}{4\pi R^3_{\text{Pluto}}} \simeq 10^{-29}\,\text{GeV}^4 \tag{6.81}$$

which, as assumed, makes the contribution of Λ completely negligible on the scale of the systems we will be interested in in this course[a] .

[a] It will play however a fundamental role at larger scales, as those you will considered in your Cosmology course.

	Linearized Gravity	Electromagnetism
Field equation	Einstein equation with $g_{\mu\nu} = h_{\mu\nu} + h_{\mu\nu}$	Maxwell equations
Basic potentials	Linearized metric $h_{\mu\nu}(x)$	4-vector potential $A^\mu = (\Phi, \mathbf{A})$
Sources	Energy-momentum tensor $T^{\mu\nu}$	4-vector current $J^\mu = (\rho, \mathbf{J})$
Lorenz gauge	$\partial_\mu \tilde{h}^{\mu\nu} = 0$ $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\text{h}$	$\partial_\mu A^\mu = 0$
Sourced wave equation	$\Box \tilde{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$	$\Box A_\mu = J_\mu$
Solution	$\tilde{h}_{\mu\nu} = 4G \int \frac{[T_{\mu\nu}]_{\text{ret}}}{\lvert\mathbf{x}-\mathbf{x}'\rvert} d^3\mathbf{x}'$	$\tilde{A}_\mu = \frac{1}{4\pi} \int \frac{[J_\mu]_{\text{ret}}}{\lvert\mathbf{x}-\mathbf{x}'\rvert} d^3\mathbf{x}'$

Table 6.2: Linearized Einstein equations vs Maxwell equations.

CHAPTER 7

GRAVITATIONAL WAVES

x

x

One of the most fascinating predictions of General Relativity is the existence of gravitational waves. Einstein theory of gravity abandons the Newtonian conception of space and time as a rigid structure in which the particles move. Spacetime is now alive and can curve, move and vibrate!

7.1 A bunch of questions

In this chapter we will try to answer the following questions

- How are gravitational waves generated?
- How do they propagate?
- Can we detect them? How?
- Why are they interesting?

Let me start by answering the simplest question, the second one.

7.2 Propagation in vacuum

The starting point for any discussion on gravitational waves is the time-dependent version of the linearized Einstein equations in the Lorenz gauge[1]

$$\Box\tilde{h}_{\mu\nu} = -16\pi G T_{\mu\nu}\,. \tag{7.1}$$

Consider the propagation of the perturbation $h_{\mu\nu}$ far away from the generating source. In this case, the energy-momentum tensor in Eq. (7.1) can be set to zero and we are left with the homogenous equation

$$\Box\tilde{h}_{\mu\nu} = 0\,. \tag{7.2}$$

The resulting vacuum case is quite particular since it still contains a *residual* gauge freedom on top the Lorenz condition

$$\partial^\nu\tilde{h}_{\mu\nu} = 0\,. \tag{7.3}$$

Having a look to Eqs. (6.62) and (6.63)

$$\tilde{h}^{\rm new}_{\mu\nu} = \tilde{h}^{\rm old}_{\mu\nu} - \left(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho\right)\,, \tag{7.4}$$

$$\partial^\nu\tilde{h}^{\rm new}_{\mu\nu} = \partial^\nu\tilde{h}^{\rm old}_{\mu\nu} - \Box\xi_\mu\,, \tag{7.5}$$

we realize that we can still make an infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ with $\Box\xi_\mu = 0$ without modifying the gauge condition (7.3). Indeed, if $\Box\xi_\mu$=0, we automatically have[2]

$$\Box\underbrace{\left(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho\right)}_{\xi_{\mu\nu}} = 0\,, \tag{7.6}$$

meaning that we can always subtract the combination $\xi_{\mu\nu}$ from $\tilde{h}_{\mu\nu}$ in Eq. (7.2). The quantity $\xi_{\mu\nu}$ depends of 4-arbitrary functions ξ_μ, which can be chosen at will to impose 4 extra conditions on the perturbation $\tilde{h}_{\mu\nu}$. In particular, we can take ξ_0 and ξ_i in such a way that $\tilde{h} = 0$ and $\tilde{h}_{0i} = 0$. The condition of vanishing trace $\tilde{h} = 0$ erases the distinction between the perturbation and its trace reverse

$$\tilde{h}_{\mu\nu} = h_{\mu\nu}\,. \tag{7.7}$$

On the other hand, the condition $\tilde{h}_{0i} = h_{0i} = 0$ applied the $\mu = 0$ component of the Lorenz gauge $\partial^\nu\tilde{h}_{\mu\nu} = 0$ implies that h_{00} is constant in time

$$\partial^0\tilde{h}_{00} + \partial^i\tilde{h}_{0i} = \partial^0 h_{00} + \partial^i h_{0i} = 0\,, \qquad \longrightarrow \qquad \partial^0 h_{00} = 0\,. \tag{7.8}$$

This component corresponds to the static part of the gravitational interaction, i.e to the Newtonian potential of the source which gave rise to the gravitational wave. The gravitational

[1]The derivation of this equation was performed around a Minkowski background. A more general treatment for perturbations around an arbitrary background $g^{(0)}_{\mu\nu}$ exists. The so-called *Isaacson shortwave approximation* can be still applied in those cases in which the perturbative scale of the waves $h_{\mu\nu}$ is much smaller than the curvature scale of the background $g^{(0)}_{\mu\nu}$.

[2]The flat d'Alambertian $\Box$ commutes with ∂^μ.

wave itself is the time-dependent part. As far as gravitational waves are concerned, the condition $\partial^0 h_{00} = 0$ really means $h_{00} = 0$.

The discussion presented above defines the so-called *transverse-traceless* (TT) or *radiation gauge*

$$h_{0\mu} = 0\,, \qquad h^i{}_i = 0\,, \qquad \partial^j h_{ij} = 0\,, \tag{7.9}$$

which completely fix all the local ambiguities and leaves us with $10-4-4=2$ degrees of freedom, the physical ones. The existence of such a gauge is guaranteed as long as there are no sources. Although, inside the source we are still allowed to perform a coordinate transformation with $\Box\xi_\mu = 0$ (or equivalently $\Box\xi_{\mu\nu}$=0) on top of the Lorenz gauge, we cannot set to zero any further component in $\tilde{h}_{\mu\nu}$, since $\Box\tilde{h}_{\mu\nu} \neq 0$. The situation is completely analogue to what happens in Classical Electrodynamics. Maxwell equations can be always reduced to the form $\Box A^\mu = J^\mu$ by imposing the Lorenz gauge condition $\partial_\mu A^\mu = 0$. Once there, we have still the freedom to implement a residual gauge transformation $A_\mu \longrightarrow A_\mu + \partial_\mu \xi$ with ξ satisfying the condition $\Box\xi = 0$. In the absence of sources, the function ξ can be used to get rid of one of the components in A^μ, let's say A^0. The Lorenz gauge reduces in this case to a transversality condition on A^i, namely $\partial_i A^i = 0$ and we are left with $4-1-1=2$ polarizations. If instead $j^0 \neq 0$, we have $\Box A^0 \neq 0$ and there is no choice of ξ able to satisfy simultaneously $\Box\xi = 0$ and $A^0 = 0$.

7.2.1 Plane wave solutions

Eq. (7.2) admits a planar wave solution[3] of the form[4]

$$\tilde{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\sigma x^\sigma} = A_{\mu\nu} e^{ik_i x^i} e^{-i\omega t}\,, \tag{7.10}$$

with $A_{\mu\nu}$ a symmetric rank-2 tensor called *polarization tensor* and $k^\mu = (\omega, \mathbf{k})$ a wave 4-vector satisfying the normalization condition[5]

$$\Box\tilde{h}_{\mu\nu} = \eta^{\rho\sigma}\partial_\rho\partial_\sigma\tilde{h}_{\mu\nu} = -k_\sigma k^\sigma \tilde{h}_{\mu\nu} = 0\,, \qquad \longrightarrow \qquad k_\sigma k^\sigma = 0\,. \tag{7.11}$$

Since k^σ is a null 4-vector, the dispersion relation takes the form $\omega = \mathbf{k}$; and gravitational perturbations propagate at the speed of light. The tranverse-traceless gauge (7.2) translates into the following restrictions on the components of the symmetric rank-2 tensor $A_{\mu\nu}$

$$A_{0i} = 0\,, \qquad A^i{}_i = 0\,, \qquad k^j A_{ij} = 0\,. \tag{7.12}$$

To clarify our findings, let me consider a particular. Imagine a wave propagating in the z-direction. In this case, $k^\mu = \left(\omega, 0, 0, k^3\right) = (\omega, 0, 0, \omega)$ and $A_{3i} = 0$, leaving as with only 4 non-vanishing components, namely $A_{11}, A_{12}, A_{21}, A_{22}$. Since A_{ij} is also symmetric and traceless, these components must satisfy $A_{11} = -A_{22} \equiv h_+$ and $A_{12} = A_{21} \equiv h_\times$, with

[3] This is just the paradigmatic case. In the linear theory, we are always allowed to build an arbitrary wave-like solution by simply considering a superposition of these plane waves.

[4] The real part of the complex-valued expression (7.10) is assumed to be taken at the end of the computation, as usual.

[5] The components $A_{\mu\nu}$ are assumed to be constant.

h_+ and $h_\times$ the so-called *"plus" and "cross" polarizations.* Written in matricial form the coefficient $A_{\mu\nu}$ in the tranverse-traceless gauge takes the form

$$A^{\rm TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{7.13}$$

	Linearized Gravity		Electromagnetism
Plane wave solution	$\tilde{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\sigma x^\sigma}$		$A_\mu = a_\mu e^{ik_\sigma x^\sigma}$
Lorenz gauge	$k_\sigma k^\sigma = 0$		$k_\sigma k^\sigma = 0$
	$k^\mu A_{\mu\nu} = 0$		$k^\mu a_\mu = 0$
	$h_{00} = 0$	$h_{i0} = 0$	$A_0 = 0$
TT gauge	$\partial_i h^{ij} = 0$	$h^i{}_i = 0$	$\partial_i A^i = 0$
	$h_{ij} = h^{\rm TT}_{ij}$		$A_i = A^T_i$
	symmetric, transverse and traceless		transverse

7.3 Interaction of gravitational waves with matter

Once we have learned how to describe the propagation of gravitational waves, the next step is to discuss their interactions with matter, or in others words, the way of detecting them. Although it might seem natural to think that we can learn something interesting by considering the geodesic equation

$$\frac{du^\mu}{d\tau} + \Gamma^\rho_{\mu\nu} u^\mu u^\nu = 0 \tag{7.14}$$

for a test particle in the gravitational field of the wave, this is not the case. To see this, consider our test particle to be at rest, $u^\mu = (1, 0, 0, 0)$, at an initial time, let's say, $\tau = \tau_0$. Evaluating the geodesic equation (7.14) at this time we get

$$\frac{du^\mu}{d\tau} = -\Gamma^\mu_{00}\Big|_{\tau=\tau_0} = \frac{1}{2}\left(2\partial_0 h^{\rm TT}_{0i} - \partial_i h^{\rm TT}_{00}\right)\Big|_{\tau=\tau_0} , \tag{7.15}$$

which is identically zero since both h_{0i} and h_{00} are zero in the transverse-traceless gauge. The particle does not seem to experience any acceleration, it completely ignores the wave! Does this mean that gravitational waves have no effect in matter? Certainly not! It simply reflects the fact the Riemannian spacetime is locally flat at any given point.

Gauge freedom in General Relativity

In General Relativity, *gauge freedom* means *freedom to choose the coordinates.* The transverse-traceless gauge is a choice of frame which *moves with the particle* at the lowest order of approximation. The coordinates stretch themselves, in response to the arrival of the wave, in such a way that the position of the free test mass initially at rest does not change.

To detect gravitational waves we must go beyond a single point in spacetime and explore its neighborhood. Consider the wave (7.13) passing through a ring of test particles in $x-y$ plane. Let's denote by v^μ the distance of a test particles to the center of the ring and use the geodesic deviation equation

$$\frac{D^2 v^\mu}{d\tau^2} = \eta^{\mu\lambda} R_{\lambda\nu\rho\sigma} u^\nu u^\rho v^\sigma \, . \tag{7.16}$$

The linearized Riemann tensor

$$R_{\lambda\nu\rho\sigma} = \frac{1}{2}\left(\partial_\nu \partial_\rho h_{\lambda\sigma} + \partial_\sigma \partial_\lambda h_{\nu\rho} - (\rho \leftrightarrow \sigma)\right) \, , \tag{7.17}$$

generated by the crossing gravitational wave is a gauge invariant quantity, meaning that we can compute it in any frame without affecting the result. Clearly the best choice is the TT gauge since the form of $h_{\mu\nu}$ in this frame is extremely simple. Assuming the particles to be moving slowly, $U^\nu \approx (1,0,0,0)$, Eq. (7.16) becomes[6]

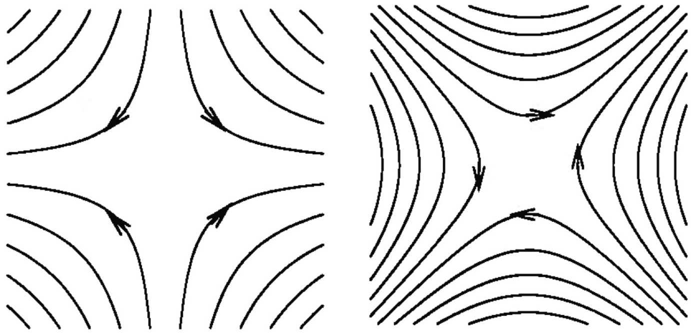
$$\frac{d^2 v^i}{dt^2} = R^i{}_{00j} v^j = \frac{1}{2}\frac{d^2 h_{ij}^{\mathrm{TT}}}{dt^2} v^j \, . \tag{7.18}$$

The resulting equation is extremely simple. The response of the particles can be understood in purely Newtonian terms, without any further reference to General Relativity. Since h_{ij}^{TT} is traceless, the effective Newtonian force per unit mass

$$F_i \equiv \frac{1}{2}\frac{d^2 h_{ij}^{\mathrm{TT}}}{dt^2} v^j \, , \tag{7.19}$$

is divergence free, $\partial^i F_i = 0$, meaning that there are no sources or sinks for the gravitational lines. Note also that, as in the electromagnetic case, only the transverse directions (v^x and v^y) to the wave propagation are affected (cf. Eq. (7.13)). If a particle is initially at $z=0$, it will remain at $z=0$. A pictorial representation of F^i can be obtained by drawing the lines of force the "plus" and "cross" polarizations

[6] Note that, at leading order in $h_{\mu\nu}$, $\tau = t$.



These lines are defined in such a way that at each point (x, y) they go in the direction of the force with a density proportional to the modulus of the force[7]. The effect of the components h_+ and $h_\times$ in the ring of particles is in clear agreement with the quadrupolar pattern displayed in the previous figure:

- **"Plus" polarization:** $h_+ \neq 0$ **and** $h_\times = 0$:

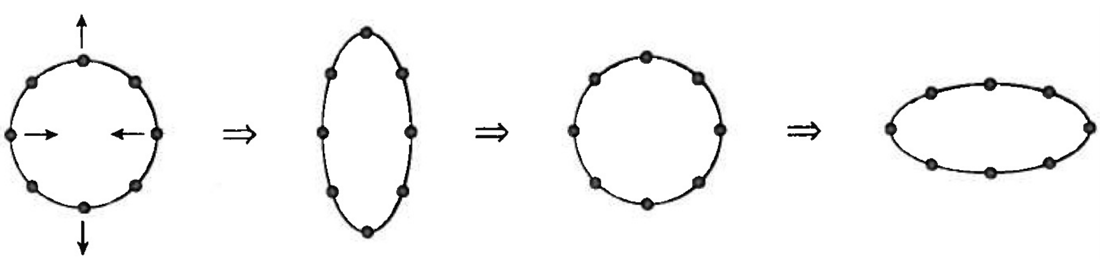
  In this case,

$$\frac{d^2 v^x}{dt^2} = \frac{v^x}{2}\frac{d^2}{dt^2}(h_+ e^{ik_\sigma x^\sigma}), \qquad \frac{d^2 v^y}{dt^2} = -\frac{v^y}{2}\frac{d^2}{dt^2}(h_+ e^{ik_\sigma x^\sigma}), \tag{7.20}$$

  whose solution, to lowest order of accuracy, can be written as

$$v^x = v_0^x + \frac{1}{2}h_+ e^{ik_\sigma x^\sigma} v_0^x \qquad v^y = v_0^y - \frac{1}{2}h_+ e^{ik_\sigma x^\sigma} v_0^y\ . \tag{7.21}$$

  with v_0^x and v_0^y staying for the initial separation of the particles in the x and y directions. A "+-polarized" wave makes the particles initially located in v_0^x and v_0^y bounce back and for in the x and y directions respectively. This fact, together with the 180° phase difference associated to the minus sign in Eq. (7.21), gives rise to the following pattern in the ring of particles



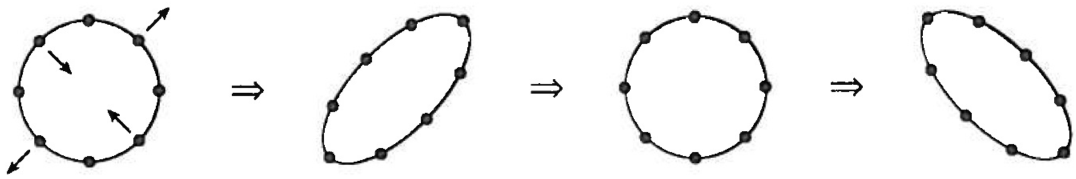
- **"Cross" polarization:** $h_\times \neq 0$ **and** $h_+ = 0$:

  In this case, the separation vector in a given direction depends also on the initial separation vector in the orthogonal direction

$$v^x = v_0^x + \frac{1}{2}h_\times e^{ik_\sigma x^\sigma} v_0^y, \qquad v^y = v_0^y + \frac{1}{2}h_\times e^{ik_\sigma x^\sigma} v_0^x\ . \tag{7.22}$$

[7] Observe that the second figure can be obtained by rotating the first one 45°.

A "×-polarized" wave gives rise to a stretching and a squeezing along the $\left(2^{-1/2}, 2^{-1/2}, 0\right)$ and $\left(-2^{-1/2}, 2^{-1/2}, 0\right)$ directions. The ring of particles bounces back and for describing a cross shape.

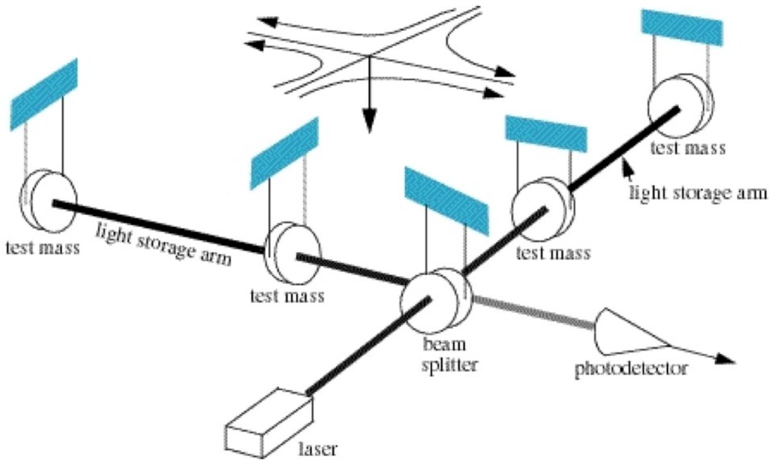


The components h_+ and $h_\times$ constitute the two independent linear polarizations of the gravitational wave and play a similar role to the vertical and horizontal polarization in electromagnetic waves. Different superpositions of these two modes can be always considered within the linear theory.

	Linearized Gravity	Electromagnetism
	$k^\mu = (\omega, 0, 0, \omega)$	$k^\mu = (\omega, 0, 0, \omega)$
Polarization	$A_{11} = -A_{22} \equiv h_+ \neq 0$	$a_1 \neq 0$
modes	$A_{12} = A_{21} \equiv h_\times \neq 0$	$a_2 \neq 0$
	$h_{\mathrm{R,L}} = \frac{1}{\sqrt{2}}\left(h_+ \pm i h_\times\right)$	$a_{\mathrm{R,L}} = \frac{1}{\sqrt{2}}\left(a_1 \pm i a_2\right)$

7.3.1 Laser interferometers

The most extended gravitational wave detectors are *laser based interferometric detectors*, whose basic operation can be summarized as follows: A laser beam is sent through a beam splitter and directed towards two very long resonant cavities. The light is reflected on mirrors at the end of the cavities and sent back to the beam splitter, which transmits half of each beam and reflects the other half. One part of each beam goes then back to the laser, while the other half-parts are combined to reach a photodetector in which the interference pattern is monitored. If a gravitational wave of amplitude h came out to pass trough the detector, its arm length will be periodically shorten in one direction and lengthen in the other, giving rise to a change in the interference pattern.



The total difference in length between the two arms can be derived[8] from Eq. (7.18)

$$\frac{\Delta L}{L} \sim h\,. \tag{7.23}$$

It is interesting to put some numbers. If we consider for instance the typical amplitude of gravitational waves emitted by a rotating binary system[9], $h \simeq 10^{-21}$, and a typical detector such as LIGO or Virgo with arm lengths of $3-4$ km, we get a change $\Delta L \simeq 10^{-16}$ cm.

7.4 The helicity of the graviton

In an hypothetical quantum theory of gravity, the gravitational waves presented in this Chapter would be quantized into particles satisfying the relativistic wave equation of a massless particle. The spin of the graviton can be inferred from the transformations properties of the classical field of the particle under rotations.

[8]We are implicitly assuming that the wave propagates orthogonally to the plane of the detector. In the general case, we get some angular coefficients of order 1.

[9]You will determine this number in the exercise session.

The Poincaré group has two physically interesting representations:

- **Massive representation**: These representations are characterized by the mass $m^2 = -p_\mu p^\mu$ and the spin s, which can take integer or half-integer values $s = 0, 1/2, 1, \ldots$. The representation with spin s has dimension $2s+1$. Example: A massive spin-1 particle has three-degrees of freedom.

- **Massless representation**: These representations are characterized by $p_\mu p^\mu = 0$ and a definite value of the *helicity*, which is defined as the projection of the total angular momentum (or the spin) in the direction of motion[a]

$$h = \mathbf{J} \cdot \mathbf{n} = (\mathbf{L} + \mathbf{S}) \cdot \mathbf{n} = \mathbf{S} \cdot \mathbf{n}\,. \tag{7.24}$$

Under a rotation of angle θ around that direction a helicity eigenstate $|h\rangle$ transforms as

$$|h\rangle \longrightarrow e^{ih\theta}|\phi\rangle\,. \tag{7.25}$$

There are always two helicity states $h = \pm s$, corresponding to the alignment or counteralignment of the spin and the momentum.

[a] Unfortunately the traditional symbol h for the helicity coincides with some of the notations used in this chapter.

Applying a global rotation of angle θ

$$R_{ij} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.26}$$

to our plane wave (7.13), we get

$$A_{ij}^{\mathrm{TT}} = \left(R^{-1}\right)^k{}_i \left(R^{-1}\right)^l{}_j A_{kl}^{\mathrm{TT}} \longrightarrow \begin{pmatrix} h_+ \cos 2\theta + h_\times \sin 2\theta & h_\times \cos 2\theta - h_+ \sin 2\theta & 0 \\ h_\times \cos 2\theta - h_+ \sin 2\theta & -h_+ \cos 2\theta - h_\times \sin 2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.27}$$

In the quantum theory the two polarization amplitudes h_+ and $h_\times$ become annihilation operators of gravitons and the circular polarization operator, $h_{\mathrm{R,L}} \equiv (h_+ \pm i h_\times)$, will transform as

$$h_{\mathrm{R,L}} \longrightarrow U h_{\mathrm{R,L}} U^\dagger \tag{7.28}$$

with $U = e^{iJ_3\theta/\hbar}$. Thus

$$h_{\mathrm{R,L}} \longrightarrow e^{\mp 2i\theta} h_{\mathrm{R,L}} \tag{7.29}$$

showing the spin of the graviton is 2, in unit of $\hbar$.

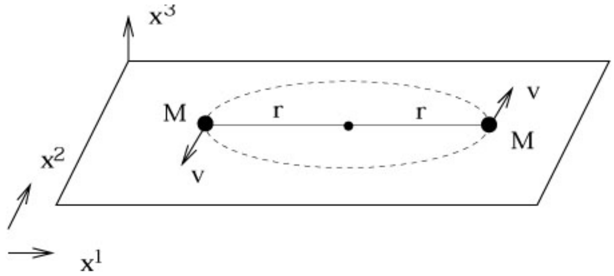


Figure 7.1: Hulse-Taylor binary pulsar.

	Linearized Gravity	Electromagnetism
Helicity	$A_{ij} = \left(R^{-1}\right)^k{}_i \left(R^{-1}\right)^l{}_j A_{kl}$ $h_{\mathrm{R,L}} \longrightarrow e^{\mp 2i\theta} h_{\mathrm{R,L}}$	$a_i = \left(R^{-1}\right)^j{}_i a_j$ $a_{\mathrm{R,L}} \longrightarrow e^{\mp i\theta} a_{\mathrm{R,L}}$

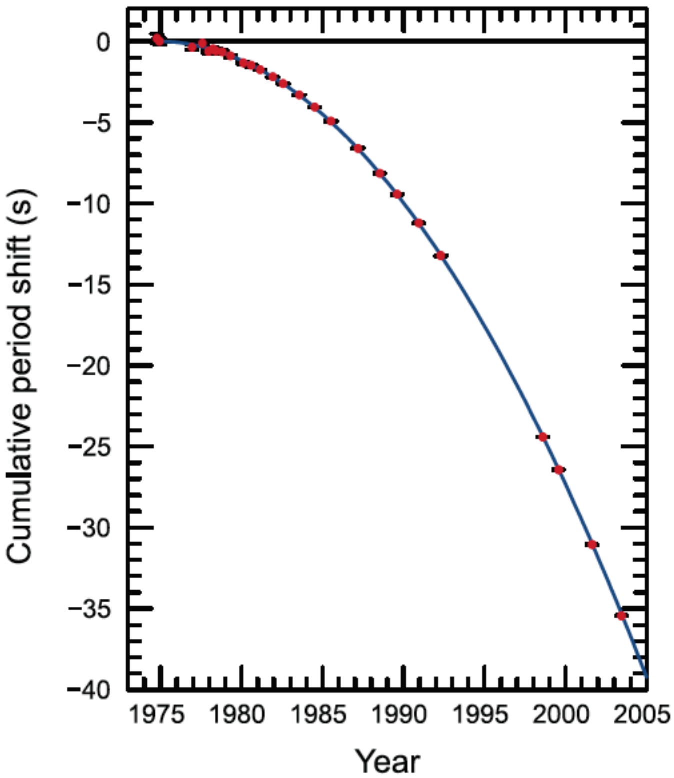


Figure 7.2: Hulse-Taylor binary pulsar.

CHAPTER 8

THE SCHWARZSCHILD-DROSTE SOLUTION

> As you see, the war treated me kindly enough, in spite of the heavy gunfire, to allow me to get away from it all and take this walk in the land of your ideas
>
> Schwarzschild's letter to Einstein during World War I

Most of the work done till now has been related to weak-field solutions of the Einstein equations. In this Chapter, we go a step forward an look for exact solutions. Given the non-linearity of the field equations and the associated difficulty in finding analytical solutions for arbitrary matter distributions, we will restrict ourselves to *vacuum solutions*. To determine our starting point, let me rewrite the Einstein equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T^M_{\mu\nu}$ in a much more convenient form. Multiplying by the inverse metric and taking the trace we obtain a relation between the Ricci scalar, the cosmological constant and the trace $T^M \equiv g^{\mu\nu} T^M_{\mu\nu}$ of the locally conserved energy-momentum tensor $T^M_{\mu\nu}$, namely

$$R^\mu{}_\mu - \frac{1}{2} R \delta^\mu{}_\mu + \Lambda \delta^\mu{}_\mu = \kappa^2 T^M \qquad \longrightarrow \qquad R = -\kappa^2 T^M + 4\Lambda \,. \tag{8.1}$$

Substituting back this result into the original Einstein equations we realize that they can be written as

$$R_{\mu\nu} = \kappa^2 \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) + \Lambda g_{\mu\nu} \,. \tag{8.2}$$

Vacuum solutions ($T^M_{\mu\nu} = \Lambda = 0$) correspond then to solutions of the equation

$$R_{\mu\nu} = 0 \,, \tag{8.3}$$

rather that to solutions of $G_{\mu\nu}$=0.

Vacuum solutions are not necessarily flat

Eq. (8.3) does *not* imply the vanishing of the Riemann tensor $R^{\mu}{}_{\nu\rho\sigma}$, which contains extra components.

The problem of finding a solution of this equation is further simplified in those cases in which the problem is highly symmetric. In what follows, we will look for *spherically symmetric solutions*.

8.1 A spherically symmetric ansatz

Consider the spacetime *outside* a spherically symmetric mass distribution, which can be static or not. A spacetime is said to posses a particular symmetry if the functional form of the metric under the action of such a symmetry is maintained. In particular, a spherically symmetric spacetime is a spacetime whose line element is invariant under rotations (or, if you want. a spacetime "with the symmetries of the sphere"). The only rotational invariants of the spacelike coordinates $\mathbf{x} = x^i$ and their differential are

$$\mathbf{x}\cdot\mathbf{x} \equiv r^2\,, \qquad d\mathbf{x}\cdot d\mathbf{x}\,, \qquad \mathbf{x}\cdot d\mathbf{x}\,. \tag{8.4}$$

The most general spatially isotropic metric that can be constructed with these elements takes the form

$$ds^2 = -a(t,r)dt^2 - 2b(t,r)dt\,(\mathbf{x}\cdot d\mathbf{x}) + c(t,r)\,(\mathbf{x}\cdot d\mathbf{x})^2 + d(t,r)d\mathbf{x}\cdot d\mathbf{x}\,, \tag{8.5}$$

with a, b, c and d some *arbitrary* functions of t and r. The required invariance under rotations suggests the use of spherical coordinates $\{r,\theta,\phi\}$. Performing the change of variables we realize that all the angular dependence in (8.5) is isolated in the $d\mathbf{x}\cdot d\mathbf{x}$ part

$$\mathbf{x}\cdot\mathbf{x} = r^2\,, \qquad \mathbf{x}\cdot d\mathbf{x} = r dr\,, \qquad d\mathbf{x}\cdot d\mathbf{x} = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\,. \tag{8.6}$$

Substituting these expressions into (8.5) we arrive to the equivalent form

$$ds^2 = -a(t,r)dt^2 - 2b(t,r)r dt dr + c(t,r)r^2 dr^2 + d(t,r)\left(dr^2 + r^2 d\Omega^2\right)\,, \tag{8.7}$$

where we have defined $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Collecting terms together and defining some, still arbitrary, functions

$$A(t,r) \equiv a(t,r)\,, \qquad B(t,r) \equiv r b(t,r)\,, \qquad C(t,r) \equiv r^2 c(t,r) + d(t,r)\,, \qquad D(t,r) \equiv r^2 d(t,r)\,,$$

to take into account the extra factors of r in Eq. (8.7), we are left with

$$ds^2 = -A(t,r)dt^2 - 2B(t,r)dtdr + C(t,r)dr^2 + D(t,r)d\Omega^2\,. \tag{8.8}$$

The resulting metric can be further simplified by using the freedom in the choice of coordinates. For instance, we can define a new radial coordinate $\bar{r}^2 \equiv D(t,r)$ and eliminate r and

dr in terms of $\bar{r}, t, d\bar{r}$ and dt. This gives rise to a big mess that changes the explicit form of the coefficients A, B, C to some new, but still arbitrary, coefficients A', B', C'

$$ds^2 = -A'(t,\bar{r})dt^2 - 2B'(t,\bar{r})dtd\bar{r} + C'(t,\bar{r})d\bar{r}^2 + \bar{r}^2 d\Omega^2 \,. \tag{8.9}$$

The next thing we can do is to find some new coordinate time $\bar{t}(t,\bar{r})$ to get rid of the nasty term $dtdr$. To do that, let me define this new time as

$$d\bar{t} = \mu(t,\bar{r}) \left[A'(t,\bar{r})dt + B'(t,\bar{r})d\bar{r}\right] = \partial_t \Psi(t,r)dt + \partial_r \Psi(t,r)dr \,, \tag{8.10}$$

where the new unknown *integrating factor* μ is determined by the condition that the second equality holds for some Ψ. In other words, we require $\mu(t,\bar{r})\,[A'(t,\bar{r})dt + B'(t,\bar{r})d\bar{r}]$ to be a total differential, so that the first equality makes sense.

Squaring Eq.(8.10)

$$d\bar{t}^2 = \mu^2 \left(A'^2 dt^2 + 2A'B'dtd\bar{r} + B'^2 d\bar{r}^2\right) \tag{8.11}$$

and isolating the terms related to dt^2 and $dtd\bar{r}$, we get

$$A'dt^2 + 2B'dtd\bar{r} = \frac{1}{A'\mu^2}d\bar{t}^2 - \frac{B'^2}{A'}d\bar{r}^2 \,. \tag{8.12}$$

In terms of the new temporal coordinate $\bar{t}$ the cross term disappears and the antsatz (8.9) becomes diagonal

$$ds^2 = -\frac{1}{A'\mu^2}d\bar{t}^2 + \left(C' + \frac{B'^2}{A}\right)d\bar{r}^2 + \bar{r}^2 d\Omega^2 \,. \tag{8.13}$$

Since the functions of $\bar{t}$ and $\bar{r}$ in this expression are arbitrary we can collect them into some arbitrary new functions[1]

$$e^{2\alpha} \equiv \frac{1}{A'\mu^2} \,, \qquad e^{2\beta} \equiv C' + \frac{B'^2}{A'} \,, \tag{8.14}$$

and write

$$ds^2 = -e^{2\alpha(\bar{t},\bar{r})}d\bar{t}^2 + e^{2\beta(\bar{t},\bar{r})}d\bar{r}^2 + \bar{r}^2 d\Omega^2 \,. \tag{8.15}$$

Dropping the bars to maintain the notation as light as possible, we arrive to our first important result

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}d\bar{r}^2 + r^2 d\Omega^2 \,. \tag{8.16}$$

Just by using spherical symmetry and our freedom to change coordinates, we have been able to reduce the 10 functions in $g_{\mu\nu}$ to two functions of only two variables! Rather impressive.

[1] This exponential form is specially useful for writing compact expressions for the components of the metric connections and the Riemann tensor.

8.2 Spherical symmetry and staticity

The unknown functions α and β can be determined by inserting the antsatz (8.16) into the vacuum Einstein equations (8.3). The first step in this procedure is to compute the metric connection $\Gamma^{\mu}{}_{\nu\sigma}$. The job is conceptually straightforward but rather tedious. Whatever the way you do it[2], you should obtain 12 non-vanishing components out of 40, namely

$$\begin{aligned}
&\Gamma^{t}{}_{tt} = \partial_t\alpha\,, && \Gamma^{t}{}_{tr} = \Gamma^{t}_{rt} = \partial_r\alpha\,, && \Gamma^{t}{}_{rr} = e^{2(\beta-\alpha)}\partial_t\beta\,,\\
&\Gamma^{r}{}_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha\,, && \Gamma^{r}{}_{tr} = \Gamma^{r}{}_{rt} = \partial_t\beta\,, && \Gamma^{r}{}_{rr} = \partial_r\beta\,,\\
&\Gamma^{r}{}_{\theta\theta} = -re^{-2\beta}\,, && \Gamma^{r}{}_{\phi\phi} = \sin^2\theta\,\Gamma^{r}_{\theta\theta}\,, && \Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}_{\theta r} = 1/r\,,\\
&\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta\,, && \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}_{\theta\phi} = \cot\theta\,, && \Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = 1/r\,.
\end{aligned} \tag{8.17}$$

The non-vanishing components of the Riemann tensor associated to these Christoffel symbols are given by

$$\begin{aligned}
&R^{t}{}_{rtr} = e^{2(\beta-\alpha)}\left[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right] + \left[\partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2\right]\,,\\
&R^{t}{}_{\theta t\theta} = -re^{-2\beta}\partial_r\alpha\,, \qquad R^{t}{}_{\phi t\phi} = -re^{-2\beta}\sin^2\theta\;\partial_r\alpha\,, \qquad R^{t}{}_{\theta r\theta} = -re^{-2\alpha}\partial_t\beta\,,\\
&R^{t}{}_{\phi r\phi} = -re^{-2\alpha}\sin^2\theta\;\partial_t\beta\,, \qquad R^{r}{}_{\theta r\theta} = re^{-2\beta}\partial_r\beta\,, \qquad R^{r}{}_{\phi r\phi} = re^{-2\beta}\sin^2\theta\;\partial_r\beta\,,\\
&R^{\theta}{}_{\phi\theta\phi} = (1-e^{-2\beta})\sin^2\theta\,. \qquad .
\end{aligned} \tag{8.18}$$

which, contracted, provide us with the non-vanishing components of the Ricci tensor

$$\begin{aligned}
&R_{tt} = \left[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right] + \left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \tfrac{2}{r}\partial_r\alpha\right]e^{2(\alpha-\beta)}\,,\\
&R_{rr} = \left[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right]e^{2(\beta-\alpha)} - \left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \tfrac{2}{r}\partial_r\beta\right]\,,\\
&R_{tr} = \frac{2}{r}\partial_t\beta\,, \qquad R_{\theta\theta} = 1 + e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1]\,, \qquad R_{\phi\phi} = R_{\theta\theta}\sin^2\theta\,.
\end{aligned} \tag{8.19}$$

Understanding the result

The result (8.19) can be easily understood from simple symmetry considerations. Consider for instance the $R_{r\theta}$ component and note that the metric (8.16) is invariant under "reflections" in the θ and ϕ coordinates, i.e. $\theta \to -\theta$ and $\phi \to -\phi$. When $\theta \to -\theta$, the sign of $R_{r\theta}$ changes and we are force to have $R_{r\theta} = 0$. The same kind of argument can be applied to many components to get

$$R_{r\theta} = R_{r\phi} = R_{t\theta} = R_{t\phi} = R_{\theta\phi} = 0\,. \tag{8.20}$$

[2] The quicker way to get $\Gamma^{\mu}{}_{\nu\sigma}$ is by using the Lagrangian procedure for geodesics, but you can also use the brute force method and compute them via Eq. (4.62).

The relation between $R_{\phi\phi}$ and $R_{\theta\theta}$ can be also derived without performing the explicit computation. To see this, consider the coordinate transformation $(\theta,\phi)\to(\bar\theta,\bar\phi)$ and write the expression for the angular part of the line element in both coordinate systems

$$d\theta^2+\sin^2\theta d\phi^2=\left[\left(\frac{\partial\theta}{\partial\theta'}\right)^2+\sin^2\theta\left(\frac{\partial\phi}{\partial\theta'}\right)^2\right]d\theta'^2+\dots \tag{8.21}$$

The invariance of the line element under rotations implies the equality

$$\left(\frac{\partial\theta}{\partial\theta'}\right)^2+\sin^2\theta\left(\frac{\partial\phi}{\partial\theta'}\right)^2=1\,. \tag{8.22}$$

Substituting this into the transformation law for the $R_{\theta\theta}$ component

$$R_{\theta'\theta'}=\left(\frac{\partial\theta}{\partial\theta'}\right)^2R_{\theta\theta}+\left(\frac{\partial\phi}{\partial\theta'}\right)^2R_{\phi\phi}\qquad\longrightarrow\qquad R_{\theta\theta}=\left(1-\sin^2\theta\left(\frac{\partial\phi}{\partial\theta'}\right)^2\right)+\left(\frac{\partial\phi}{\partial\theta'}\right)^2R_{\phi\phi}$$

and demanding $R_{\theta'\theta'}=R_{\theta\theta}$, we get the sought-for relation $R_{\phi\phi}=\sin^2R_{\theta\theta}$.

The empty-space field equations are obtained by setting each of the components (8.19) equal to zero. These gives rise to 5 equations among which only 4 are useful since the $R_{\phi\phi}$ component simply repeats the information of the $R_{\theta\theta}$ component. Among these 4 equations, the simplest one is that associated to R_{tr}. A simple inspection of this equation reveals a very interesting property: the function β must be independent of time

$$R_{tr}=0\qquad\longrightarrow\qquad\partial_t\beta=0\qquad\longrightarrow\qquad\beta=\beta(r)\,. \tag{8.23}$$

Taking into account this result and performing the time derivative of the vacuum equation $R_{\theta\theta}=0$, we get

$$\partial_tR_{\theta\theta}=0\qquad\longrightarrow\qquad\partial_t\partial_r\alpha=0\qquad\longrightarrow\qquad\alpha=\gamma(r)+\kappa(t)\,. \tag{8.24}$$

The coefficient $e^{2\alpha(r,t)}$ can be then splited into two pieces $e^{2\alpha(r,t)}=e^{2\gamma(r)}e^{2\kappa(t)}$. This allows us to perform an extra coordinate redefinition

$$\mathrm{d}t\to e^{-\kappa(t)}\mathrm{d}t\,,\qquad\gamma(r)\equiv\alpha(r)\,, \tag{8.25}$$

in Eq. (8.16) to obtain a much simpler line element

$$ds^2=-e^{2\alpha(r)}dt^2+e^{2\beta(r)}dr^2+r^2d\Omega^2\,, \tag{8.26}$$

specified by only two *time-independent* functions $\alpha(r)$ and $\beta(r)$. The resulting metric is static[3] even though we did not impose any requirement on the source apart from being spherically

[3] A static spacetime is one in which

i) The components of $g_{\mu\nu}$ are independent of the timelike component x^0.

ii) The line element is invariant under the transformation $x^0\to -x^0$.

If the second condition is not satisfied the spacetime is rather said to be *stationary*. A particular example of stationary metric is the one generated by a rotating star, where the change $x^0\to -x^0$ changes the sense of rotation.

symmetric. The source could be as dynamical as a collapsing or a pulsating star and the metric outside the matter distribution would still take the form (8.26), as long as the collapse is symmetric. This result is in perfect agreement with our discussion on gravitational waves: if a spherically symmetric body undergoes pure radial pulsations, there is no quadrupole and there is no emission of gravitational waves.

All vacuum solutions of the Einstein equations with $SO(3)$ symmetry are necessarily static.

8.3 The Schwarzschild-Droste solution

Thanks to symmetry, we are left with 3 equations of a single variable r for two unknows α and β. Let me rewrite them as

$$R_{tt} = +\left(\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2\alpha'}{r}\right) e^{2(\alpha-\beta)} = 0\,, \tag{8.27}$$

$$R_{rr} = -\left(\alpha'' + \alpha'^2 - \alpha'\beta' - \frac{2\beta'}{r}\right) = 0\,, \tag{8.28}$$

$$R_{\theta\theta} = 1 - e^{-2\beta}\left(1 + r\alpha' - r\beta'\right) = 0\,, \tag{8.29}$$

with the prime denoting derivatives with respect to r. Note that the first two equations are rather similar. Multiplying the first one by $e^{-2(\alpha-\beta)}$ and adding it to the second we get

$$e^{2(\beta-\alpha)}R_{tt} + R_{rr} = \frac{2}{r}\left(\alpha' + \beta'\right) = 0 \quad \longrightarrow \quad \alpha' + \beta' = 0 \quad \longrightarrow \quad \alpha(r) + \beta(r) = \text{constant}\,. \tag{8.30}$$

The integration constant appearing in the previous expression can be always set to zero by simply performing a coordinate redefinition, allowing us to set $\alpha = -\beta$. Inserting this result into Eq. (8.29) we get

$$R_{\theta\theta} = 0 \quad \longrightarrow \quad \left(1 + 2r\alpha'\right) e^{2\alpha} = 1 \quad \longrightarrow \quad \left(re^{2\alpha}\right)' = 1\,, \tag{8.31}$$

which can be easily integrated to obtain

$$re^{2\alpha} = r + C \quad \rightarrow \quad e^{2\alpha} = e^{-2\beta} = 1 + \frac{C}{r}\,, \tag{8.32}$$

or equivalently

$$ds^2 = -\left(1 + \frac{C}{r}\right) dt^2 + \left(1 + \frac{C}{r}\right)^{-1} dr^2 + r^2 d\Omega^2\,. \tag{8.33}$$

The obtained metric is *asymptotically flat*: it tends to the Minkowski metric when $r \to \infty$.

Birkhoff's theorem

Any solution of the vacuum Einstein equations with $SO(3)$ symmetry must be static and asymptotically flat.

The only thing left is to associate the constant C to some physical parameter. The most important use of a spherically symmetric vacuum solution is to represent the spacetime outside stars or planets. In that case, we would expect to recover the Newtonian limit

$$g_{00} = -\left(1 - \frac{2GM}{r}\right), \qquad g_{rr} = \left(1 + \frac{2GM}{r}\right), \tag{8.34}$$

at large r values. Comparing (8.34) with the $r \to \infty$ limit of the metric (8.32)

$$g_{00} = \left(1 + \frac{C}{r}\right), \qquad g_{rr} = \left(1 - \frac{C}{r}\right), \tag{8.35}$$

we get $C = -2GM$, which allows us to write the final and traditional expression for the so-called *Schwarzschild-Droste* metric[4]

$$ds^2 = -\left(1 - \frac{R_S}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{R_S}{r}\right)^{-1} \mathrm{d}r^2 + r^2 d\Omega^2 \tag{8.36}$$

with

$$R_S \equiv \frac{2GM}{c^2} \simeq 3\text{km}\left(\frac{M}{M_\odot}\right), \tag{8.37}$$

the *Schwarzschild radius*[5].

Exercise

- Verify that (8.36) satisfies Eq. (8.29). Explain why this is guaranteed to happen even though we initially had three equations for two unknowns.
- Show that the Schwarzschild metric can be written in a form that makes explicit its isotropic character, namely

$$ds^2 = -\frac{\left(1 - \frac{R_S}{4\rho}\right)^2}{\left(1 + \frac{R_S}{4\rho}\right)^2} dt^2 + \left(1 + \frac{R_S}{4\rho}\right)^4 \left(dx^2 + dy^2 + dz^2\right), \tag{8.38}$$

[4] Karl Schwarzschild found this exact solution in 1915 while serving in the German army on the Russian front during the World War I and died a year later from pemphigus, a painful autoimmune disease. An alternative derivation of this solution based on the Weyl method was presented by Droste around the same time but for some reason the physics community completely ignored it.

[5] We have momentarily restored the factors of c.

with

$$\rho = \frac{1}{2}\left(r - GM + \sqrt{r^2 - 2GMr}\right) . \tag{8.39}$$

8.4 Measuring distances and times

Which is the physical meaning of the coordinates (t, r, θ, ϕ) appearing in the Schwarzschild-Droste solution? Although they provide a global reference frame for an observer making measurements at an infinite distance from the source (asymptotic flatness), not all of them represent physical quantities measured by arbitrary observers. While θ and ϕ have the same interpretation than the spherical angular coordinates in flat spacetime, the coordinate radius r and the coordinate time t cannot be generically interpreted as the physical radius or the physical time measured by a clock.

Physical quantities must be computed from the metric. The physical interval in the radial direction measured by an arbitrary local observer is given by the proper distance ($dt = d\theta = d\phi$)

$$ds = \left(1 - \frac{R_S}{r}\right)^{-1/2} dr\,, \tag{8.40}$$

while the time measured by an stationary clock at r ($dr = d\theta = d\phi = 0$) is given by the proper time interval

$$d\tau = \left(1 - \frac{R_S}{r}\right)^{1/2} dt\,. \tag{8.41}$$

Understanding the result

In the Schwarzschild metric, space is foliated by spheres S^2 of area $4\pi r^2$ separated by a proper distance $\left(1 - \frac{R_S}{r}\right)^{-1/2} dr$.

8.5 Visualizing Schwarzschild spacetime

A mental image of the Schwarzschild-Droste spacetime can be obtained by embedding a subset of it into a higher dimensional spacetime[6]. Since our solution is static and spherically symmetric, we can, without loss of generality, fix t =constant and $\theta = \pi/2$. This leaves us with a 2-dimensional surface

$$dX^2 = \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\phi^2 = f(r)^{-1} dr^2 + r^2 d\phi^2\,, \tag{8.42}$$

[6]Remember that such embedding diagrams can be misleading. For instance, a 2-dimensional cylinder embedded in 3-dimensional Euclidean space can seem to be curved even though it is *intrinsically* flat, $K = \kappa_1\kappa_2 = 0$.

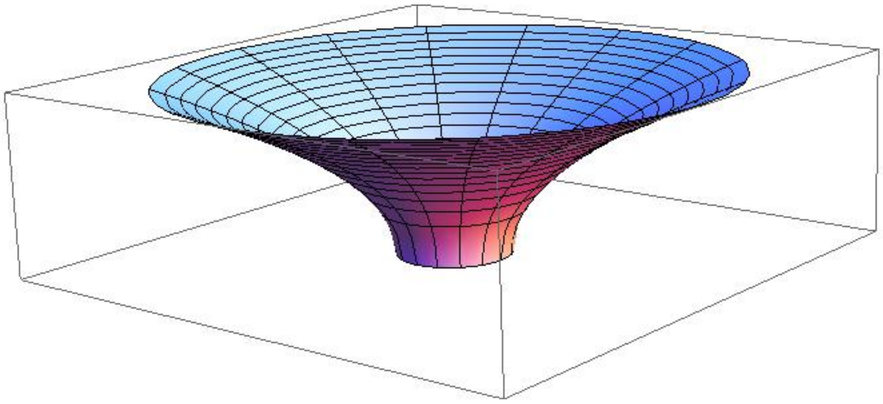


Figure 8.1: Embedding diagram for the Schwarzschild $(r - \phi)$ plane: Flamm's paraboloid

which can be easily embeded into the ordinary 3-dimensional Euclidean space

$$dX^2 = dz^2 + dr^2 + d\phi^2 = \left[1 + \left(\frac{dz(r)}{dr}\right)^2\right] dr^2 + r^2 d\phi^2 \,. \tag{8.43}$$

The function $z(r)$ can be obtained by simply comparing (8.42) and (8.43)

$$1 + \left(\frac{dz(r)}{dr}\right)^2 = f(r)^{-1} \,, \tag{8.44}$$

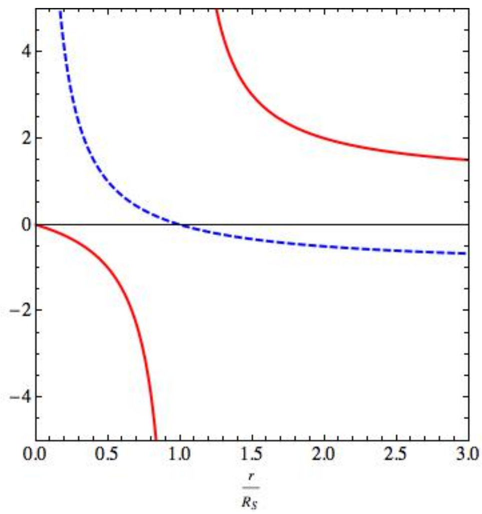
and performing a trivial integration

$$z(r) = \int_0^r dr' \sqrt{\frac{1 - f(r')}{f(r')}} = 2\sqrt{R_S(r - R_S)} + \text{constant} \,. \tag{8.45}$$

The resulting embedding diagram is the Flamm's paraboloid shown in Fig. 8.1. The distances between circles on this surface are larger than just Δr, in clear agreement with the discussion presented in the previous section.

8.6 Apparent singularity

The line element (8.36) appears to contain two singularities, one at $r = 0$ coming from the g_{tt} component (blue dashed line) and another at $r = R_S$ coming from g_{rr} (red line).



Is this a problem? Not necessarily. In most of the astrophysical applications the typical size R of the source is much larger than the Schwarzschild radius (8.37)

$$\left.\frac{R_S}{R}\right|_{\oplus} \approx 10^{-9}\,, \qquad \left.\frac{R_S}{R}\right|_{\odot} \approx 10^{-6}\,, \qquad \left.\frac{R_S}{R}\right|_{\rm NS} \approx 10^{-1}\,. \tag{8.46}$$

This fact makes the singularities at $r = 0$ and $r = R_s$ completely irrelevant in most of the cases, since they lie in the interior of objects where the exterior Schwarzschild solution does not apply. Indeed, the problem disappears when one consider realistic interior solutions of the Einstein equations

$$ds^2 = -\left(1 - \frac{2GM(r)}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM(r)}{r}\right)^{-1} \mathrm{d}r^2 + r^2 d\Omega^2\,, \tag{8.47}$$

since the function $M(r)$ decreases faster than r and effectively kills all the above singularities. We should worry and speculate about the singularities only in those cases in which the size of the object is such that the Schwarzschild-Droste solution applies all the way down to $r = R_S$. This kind of objects are called *black holes*. Even in that case the two singularities described above are not on equal footing. The metric coefficients in the line element (8.36) depend on the choice of a *particular coordinate system* and you should not extract any conclusion from them alone. Let me present an illustrative example.

A worked-out examples: Coordinates should not be trusted

Consider the completely regular and singularity-free Euclidean space in two dimensions

$$dX^2 = dx^2 + dy^2\,, \tag{8.48}$$

and perform a general coordinate transformation to a new variable ρ defined through

$x = 2\sqrt{\rho}$ to get

$$ds^2 = \frac{1}{\rho^2} d\rho^2 + dy^2 \,. \tag{8.49}$$

The metric appears to blow up at $\rho = 0$ even though we know that our space is, by construction, flat and free of singularities. The apparent singularity is a breakdown of our coordinate system at the point in which ρ becomes negative. It has nothing to do with a breakdown of the underlaying manifold!

In order to determine if we are dealing with some artifice of our coordinate system or with a true physical singularity, we cannot neither look to the curvature tensors alone, since their components are coordinate-dependent[7]. We should rather construct scalars out of the curvature tensors. If any the scalar blows up in a particular coordinate system, it will do in all of them. The simplest possibility would be to consider the Ricci scalar, R but we can also construct higher order scalars such as $R_{\mu\nu}R^{\mu\nu}$ $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$. For the particular case of the Schwarzschild-Droste metric, the first two quantities are not useful since are identically equal to zero. We are forced then to consider the square of the Riemann tensor, the so-called *Kretschmann scalar*. Taking into account the non-vanishing components of the Riemann tensor (8.18), we obtain

$$\mathcal{K} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{12R_S^2}{r^6} \,, \tag{8.50}$$

which is a perfectly regular quantity at the Schwarzschild radius, but becomes infinity at $r = 0$. This last point is a real physical singularity! The singularity at $r = R_S$ is, on the other hand, just a pathology of the specific coordinate system used.

8.7 Geodesics in Schwarzschild metric

Let us study the motion of pointlike objects in our recently found Schwarzschild solution. To do that, let me consider the reparametrization invariant action (3.27)

$$S = \int L\, d\sigma = \frac{1}{2}\int d\sigma \left(e^{-1}(\sigma) g_{\mu\nu} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} - m^2 e(\sigma) \right) \,, \tag{8.51}$$

in the massive ($e(\sigma) = 1/m$) and massless ($e(\sigma) = 1, m \to 0$) cases[8]

$$S_{\text{massive}} = \frac{1}{2} m \int d\sigma \left(g_{\mu\nu} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} - 1 \right) \,, \qquad S_{\text{massless}} = \frac{1}{2}\int d\sigma \left(g_{\mu\nu} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} \right) . \tag{8.52}$$

The geodesic equations for both actions can be directly written by taking into account the non-vanishing Christoffel symbols (8.17). Let's denote the derivatives with respect to the affine parameter σ by a dot. The explicit form[9] obtained by following this procedure turns

[7] They can catch singularities when going from one coordinate system to another through the transformation matrices $\partial \bar{x}^\mu / \partial x^\nu$.

[8] Remember that $\sigma = \tau$ in the massive case.

[9] Up to a global factor m in the massive case.

out to be not very useful since the resulting equations are coupled

$$\ddot{t} = -\frac{R_S}{r(r-R_S)}\dot{r}\dot{t}\,, \tag{8.53}$$

$$\ddot{r} = -\frac{R_S(r-R_S)}{2r^3}\dot{t}^2 + \frac{R_S}{2r(r-R_S)}\dot{r}^2 - (r-R_S)\left(\dot{\theta} + \sin^2\theta\dot{\phi}^2\right)\,, \tag{8.54}$$

$$\ddot{\theta} = -\frac{2}{r}\dot{\theta}\dot{r} + \sin\theta\cos\theta\dot{\phi}^2\,, \tag{8.55}$$

$$\ddot{\phi} = -\frac{2}{r}\dot{\phi}\dot{r} - 2\cot\theta\,\dot{\theta}\dot{\phi}\,. \tag{8.56}$$

Fortunately, our task can be greatly simplified by considering the symmetries of the Schwarzschild-Droste metric. Since (8.36) does not depend on the coordinates t and θ (they are *cyclic* coordinates in (8.52)), we have two conservation laws

$$\partial_t L = 0 \qquad \longrightarrow \qquad E = \left(1 - \frac{R_S}{r}\right)\dot{t} = \text{constant}\,, \tag{8.57}$$

$$\partial_\phi L = 0 \qquad \longrightarrow \qquad h = r^2\sin^2\theta\dot{\phi} = \text{constant}\,, \tag{8.58}$$

with a clear physical interpretation. In the massless case, E and h are the relativistic energy and angular momentum that the particle would have at $r = \infty$. In the massive case, they are the relativistic energy and angular momentum *per unit mass*.

Exercise

Check this by taking the non-relativistic limit of (8.57) and (8.58) at the equatorial plane $\theta = \pi/2$.

Conservation of angular momentum means that the particle moves in a plane, which we can set to be the equatorial plane $\theta = \frac{\pi}{2}$ without loss of generality. Indeed, a simple inspection of Eq. (8.55) shows that if we consider a geodesic passing through a point on the equator $\theta = \frac{\pi}{2}$ and tangent to the equatorial plane $\dot{\theta} = 0$, we will always have $\ddot{\theta} = 0$ and $\dot{\theta} = 0$.

On top of the above symmetries, we have still a generic conservation law associated to the invariance of the action (8.51) under reparametrizations of the path $\sigma \to \sigma = f(\sigma)$ (cf. Section 3.5.1). This reads

$$\frac{d}{d\sigma}\left(g_{\mu\nu}u^\mu u^\nu\right) = 0 \qquad \longrightarrow \qquad g_{\mu\nu}u^\mu u^\nu = -\epsilon\,, \tag{8.59}$$

with $\epsilon = 1$ and 0 for massive and massless particles respectively. Expanding this equation[10]

$$-\left(1 - \frac{R_S}{r}\right)\dot{t}^2 + \left(1 - \frac{R_S}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = -\epsilon \tag{8.60}$$

[10]Remember that $\theta = \pi/2$.

and plugging (8.57) and (8.58) we obtain a single equation for $r(\sigma)$

$$\frac{1}{2}\left(\frac{dr}{d\sigma}\right)^2 + V(r) = \mathcal{E}\,, \tag{8.61}$$

with

$$V(r) \equiv -\epsilon\frac{GM}{r} + \frac{h^2}{2r^2} - \frac{GMh^2}{r^3} \tag{8.62}$$

playing the role of an *exact* effective potential and

$$\mathcal{E} \equiv \frac{1}{2}\left(E^2 - \epsilon\right)\,. \tag{8.63}$$

Eq. (8.61) is structurally equivalent to that of a particle of unit mass and energy[11] $\mathcal{E}$ moving in an effective potential $V(r)$. It is interesting to compare the obtained potential with the Newtonian result

$$V_N(r) = -\frac{GM}{r} + \frac{h^2}{2r^2} \tag{8.64}$$

The first two terms in Eq. (8.62) are just the universal gravitational attraction and the centrifugal barrier that were already present in Newton's theory of gravity. The third term is new.

At sufficiently long distances, the extra contribution is rather small and does not significantly modify the Newtonian effective potential[12](cf. Fig. 8.2). The situation is completely different at short distances. The new term eventually dominates over the centrifugal barrier for small r and drives the potential to $-\infty$[13]. Let me analyze the massive and massless case separately.

Massive particles, $\epsilon = 1$, $\sigma = \tau$:

We can distinguish two cases:

- If $h^2 > 3R_S^2$, the potential displays both a maximum and a minimum at

$$\left.\frac{dV(r)}{dr}\right|_{\epsilon=1} = 0 \qquad \longrightarrow \qquad r_{\text{max,min}} = \frac{h^2}{R_S}\left[1 \pm \sqrt{1 - 3\left(\frac{R_S}{h}\right)^2}\,\right]\,, \tag{8.65}$$

  We have then four possibilities depending of the relation between the effective energy of the particle and the potential (cf. Fig. 8.3):

  1. Circular orbits: If $\mathcal{E} = V(r_{\text{max}})$ or $\mathcal{E} = V(r_{\text{min}})$ the particle describes an unstable or stable orbit respectively.
  2. Bound precessing orbits: If $0 > \mathcal{E} > V(r_{\text{min}})$ the particle is trapped into the potential and describes an elliptical orbit with shifting perihelion (see below).

[11] The true energy per unit mass in E but the effective potential for r rather responds to $\mathcal{E}$.
[12] The small correction will play however a central role! See next section.
[13] Note that the potential is always zero at $r = R_S$.

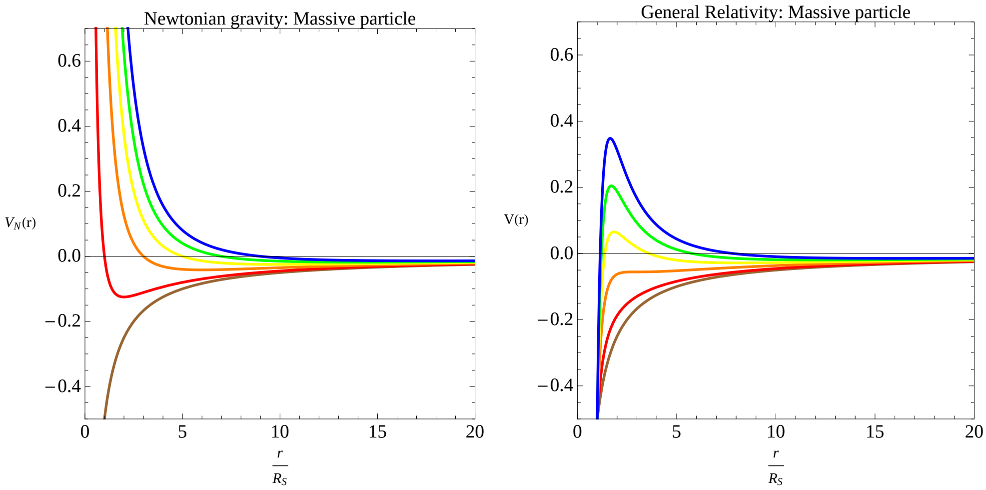


Figure 8.2: Effective potentials in Newtonian gravity and General Relativity for massive particles. Different lines correspond to $h^2/R_S^2 = 0, 1, 3, 5, 7, 9$ (from brown to blue). Note the change in the potential at the critical value $h^2/R_S^2 = 3$.

3. Scattering orbits: If $V(r_{\rm max}) > \mathcal{E} > 0$ the particle bumps in the potential and retreats back to infinity.
4. Plunging orbits: If $\mathcal{E} > V(r_{\rm max})$ the particle sails over the top of the potential to finally spiral into the black hole.

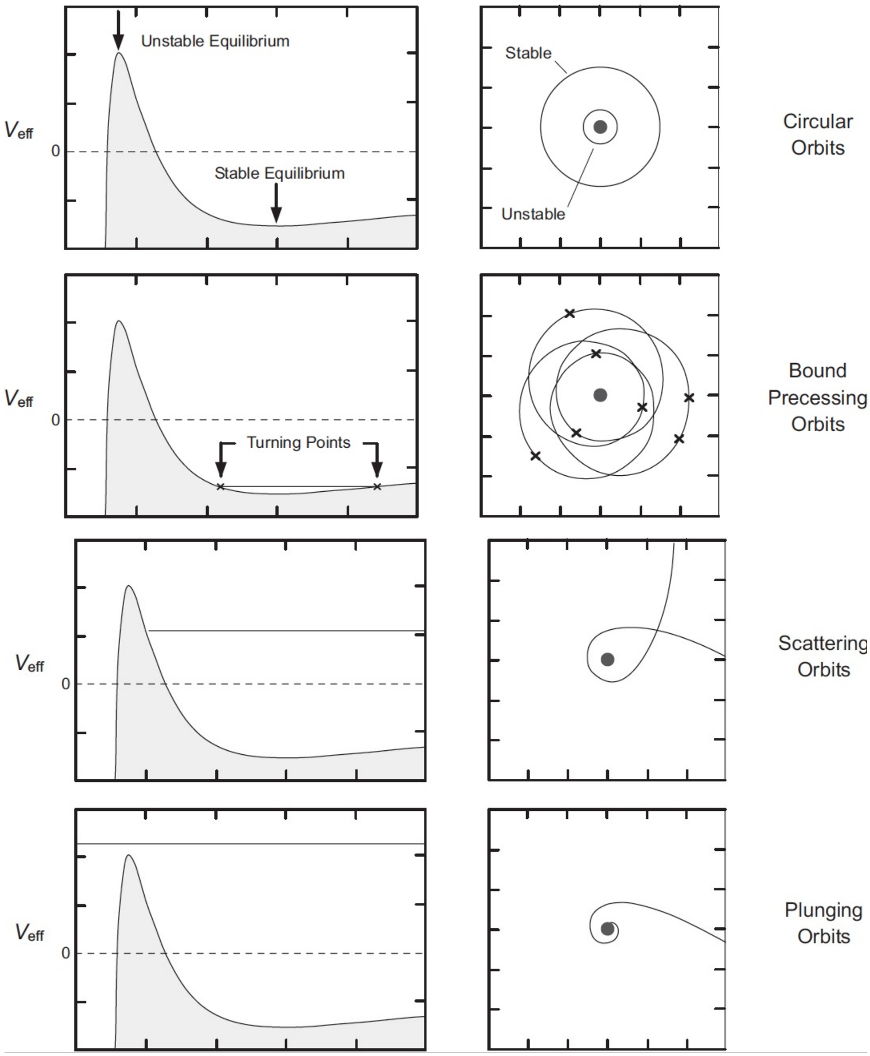


Figure 8.3: Orbits for massive particles in Schwarzschild-Droste geometry

Exercise
What happens with $r_{\rm max}$ and $r_{\rm min}$ when $h \to 0$? And when h decreases? Which is the minimal value of h and r allowing for a stable circular orbit?

- If $h^2 < 3R_S^2$ the centrifugal barrier disappears and the particle has no other option but to spiral into the singularity. Consider for clarity the limiting case $h = 0$ in which the particle follows a radial trajectory. In this case, the radial equation of motion (8.61) becomes[14]

$$\frac{dr}{d\tau} = \pm \left(\frac{R_S}{r}\right)^{1/2} \qquad \to \qquad \int \sqrt{r}dr = -R_S^{1/2} \int d\tau \,. \tag{8.66}$$

[14] Among the two signs in the square root we take the negative one, in such a way that we fall toward $r \to 0$

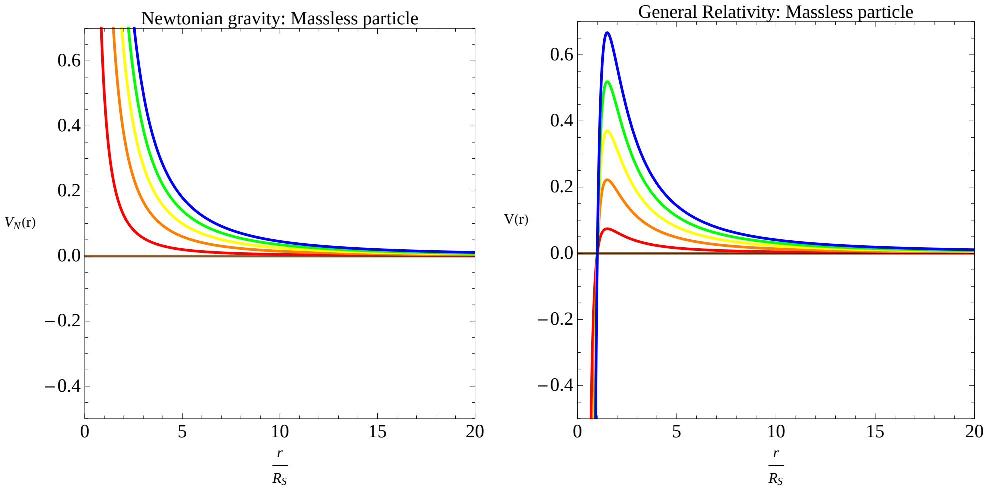


Figure 8.4: Effective potentials in Newtonian gravity and General Relativity for massless particles. Different lines correspond to $h^2/R_S^2 = 0, 1, 3, 5, 7, 9$ (from brown to blue).

Integrating the previous equation we get

$$\tau(r) = \frac{2}{3\sqrt{R_S}}\left(r_0^{3/2} - r^{3/2}\right) , \tag{8.67}$$

with $r_0 > r$ an integration constant fixing the initial value of the proper time to zero. The particle reaches the Schwarzschild radius in a finite proper time τ. The interval measured by an observer at rest at spatial infinite is however quite different. Indeed, it is infinite, as can be easily seen by evaluating

$$\frac{dr}{dt} = \frac{d\tau}{dt}\frac{dr}{d\tau} = -\left(1 - \frac{R_S}{r}\right)\left(\frac{R_S}{r}\right)^{1/2} \qquad \rightarrow \qquad \int dt = -R_S^{-1/2}\int \frac{r^{3/2}dr}{r - R_S} \tag{8.68}$$

at $r = R_S$. For the observer at infinite the particle appears to approach but never quite cross the horizon! This is just another indication that the Schwarzschild coordinates are flawed near $R = R_S$.

Exercise

What happens with t when the observer crosses the horizon?

Massless particles, $\epsilon = 0$:

The potential (8.62) with $\epsilon = 0$ displays a unique maximum for all values of h at

$$r_{\max} = \frac{3}{2}R_S . \tag{8.69}$$

Thus, the motion of massless particles can be divided into three cases:

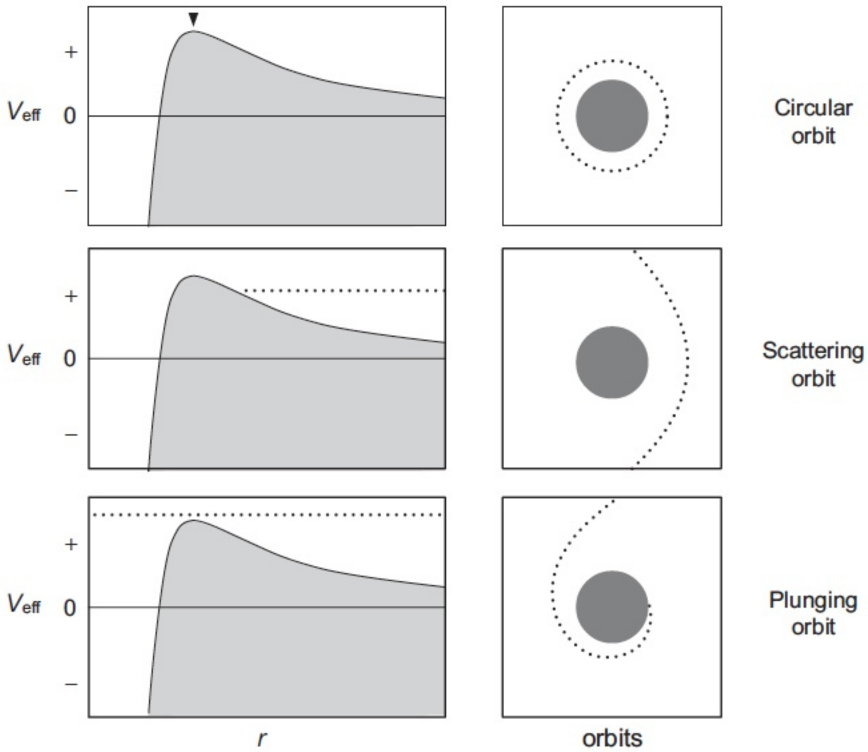


Figure 8.5: Orbits for massless particles in Schwarzschild-Droste geometry

1. Circular orbit: If $\mathcal{E} = V(r_{\rm max})$ the particle describes an unstable circular orbit.
2. Scattering orbits: If $V(r_{\rm max}) > \mathcal{E}$ the particle bumps in the potential and retreats back to infinity (deflection of light).
3. Plunging orbits: If $\mathcal{E} > V(r_{\rm max})$ the particle sails over the top of the potential to finally spiral into the black hole.

8.8 Solving the radial equation

Let us determine the equation for the orbits described in the previous section. For doing that, we make use of Eq. (8.58) with $\theta = \pi/2$ and change the derivatives with respect to the affine parameter in Eq. (8.61) to derivatives with respect the angular variable ϕ

$$\frac{dr}{d\sigma} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{h}{r^2}\frac{dr}{d\phi}\,, \tag{8.70}$$

to obtain

$$\left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^2} = \epsilon\frac{2GM}{r} + \frac{2GMh^2}{r^3} + 2\mathcal{E}\,. \tag{8.71}$$

The tricks to solve this kind of equation are well known. Let's perform a change of variable $u \equiv 1/r$ in (8.71)

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \epsilon\frac{2GMu}{h^2} + 2GMu^3 + \frac{2\mathcal{E}}{h^2}\,, \tag{8.72}$$

and derive the result with respect to ϕ. This gives rise to a second order differential equation of the form

$$\frac{d^2u}{d\phi^2} + u = \epsilon\frac{GM}{h^2} + 3GMu^2\,. \tag{8.73}$$

8.8.1 The massive case: Perihelion advance of Mercury

In the massive case $\epsilon = 1$ and (8.73) becomes

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + 3GMu^2 \tag{8.74}$$

The resulting equation is extremely similar to the Newtonian equation of motion of a particle of mass m in the equatorial plane

$$\frac{d^2u_0}{d\phi^2} + u_0 = \frac{GM}{h^2} \tag{8.75}$$

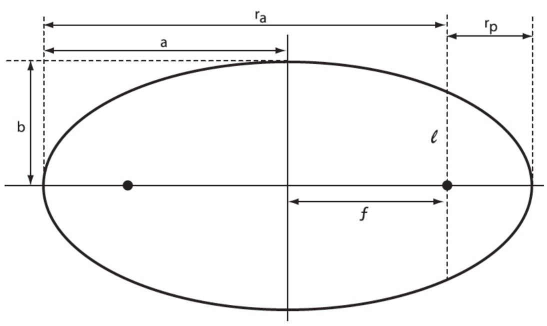
even though the interpretation of the radial variable r is completely different[15]. As you probably remember from your Classical Mechanics course, the general solution of (8.75) is a conic

$$u_0 = \frac{GM}{h^2}(1 + e\cos\phi) \qquad \longrightarrow \qquad r_0 = \frac{a(1-e^2)}{(1+e\cos\phi)} \tag{8.76}$$

with

$$a(1-e^2) = \frac{h^2}{GM}\,. \tag{8.77}$$

Orbital Mumbo Jumbo



[15] In Newtonian gravity r is the radial *distance* from the mass while in the relativistic it is just a radial *coordinate* that can be only related to a distance through the metric.

- **a=semi-major axis**: 1/2 of the long axis of the ellipse.
- **b=semi-minor axis**: 1/2 of the short axis of the ellipse.
- **e=eccentricity** : It characterizes the deviation of the ellipse from circular. When $e = 0$ the ellipse is a circle, when $e = 1$ the ellipse is a parabola. It is defined in terms of the semi major and semi minor axes a and b as

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2} \, . \tag{8.78}$$

- **f=focus**: The point over the semi-major axis at a distance $f = ae$ from the geometric center of the ellipse.
- **l=semi-latus rectum**: The distance $l = \frac{b^2}{a}$ from the focus to the ellipse along a line parallel to the semi-minor axis.
- r_p**=periapsis**: The distance $r_p = a(1-e)$ from the focus to the nearest point of approach of the ellipse.
- r_a**=apoapsis**: The distance $r_a = a(1+e)$ from the focus to the furthest point of approach of the ellipse.
- **The equation of the orbit:** It gives the distance to the orbiting body from the focus of the orbit as a function of the polar angle θ

$$r(\theta) = \frac{a(1-e^2)}{1 + e\cos\theta} \, . \tag{8.79}$$

If the gravitational field is sufficiently weak, Newtonian gravity alone is expected to provide a good approximation to the motion of massive particles in General Relativity. This suggest to treat to extra term $3GMu^2$ as a perturbation of top of the solution of Eq. (8.75). The perturbative solution of Eq. (8.74) can be determined by considering the antsatz

$$u = u_0 + \Delta u \, , \tag{8.80}$$

with u_0 given by (8.76). Inserting this into (8.74) we get

$$\frac{d^2\Delta u}{d\phi^2} + \Delta u = A\left[\left(1 + \frac{e^2}{2}\right) + 2e\cos\phi + \frac{1}{2}e^2\cos 2\phi\right] \tag{8.81}$$

with

$$A = \frac{3\,(GM)^3}{h^4} \tag{8.82}$$

a tiny parameter. To solve this equation let me notice two identities

$$\frac{d^2\phi}{d\phi^2}\,(\phi\sin\phi) + \phi\sin\phi = 2\cos\phi \, , \tag{8.83}$$

$$\frac{d^2}{d\phi^2}(\cos 2\phi) + \cos 2\phi = -3\cos 2\phi\,. \tag{8.84}$$

A direct comparison of (8.83) and (8.84) with (8.81) suggests the solution

$$\Delta u = A\left[\left(1+\frac{e^2}{2}\right) - \frac{1}{6}\cos 2\phi + e\phi\sin\phi\right]\,, \tag{8.85}$$

which can be checked by direct differentiation. The three terms in the square bracket are rather different. The first and the second one are just a constant and an oscillatory term around zero, both of them very small due to the constant A in front. The third one is different since it accumulates over successive orbits and gradually grows with time. Retaining only this last term we get

$$u = \frac{GM}{h^2}\left[1 + e\left(\cos\phi + \alpha\phi\sin\phi\right)\right] \tag{8.86}$$

which can be written in a more enlightening way

$$u \approx \frac{GM}{h^2}\left[1 + e\cos\left(1-\alpha\right)\phi\right] \tag{8.87}$$

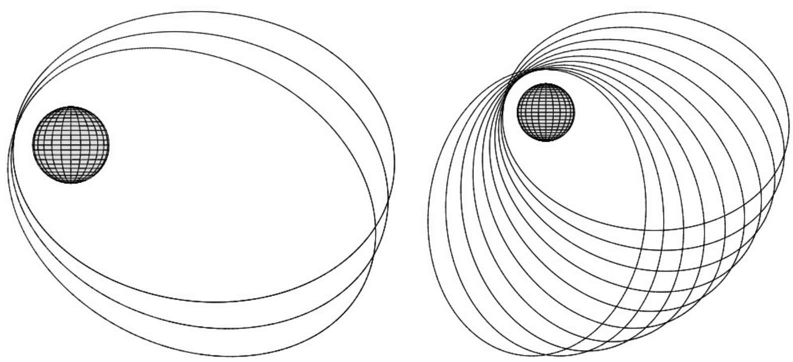
by taking into account that

$$\cos\left[\phi\left(1-\alpha\right)\right] = \cos\phi\cos\alpha\phi + \sin\phi\sin\alpha\phi \approx \cos\phi + \alpha\phi\sin\phi \tag{8.88}$$

for

$$\alpha \equiv \frac{3\left(GM\right)^2}{h^2} \ll 1\,. \tag{8.89}$$

The solution (8.87) shows that the orbit is still periodic, but with a period that is not longer 2π, but rather $2\pi(1-\alpha)$. The values of r repeats on cycles larger than 2π and the orbit precesses.



The advance of the perihelion in one revolution is

$$\Delta\phi = \frac{2\pi}{1-\alpha} - 2\pi \approx 2\phi\alpha = \frac{6\pi G^2 M^2}{h^2}\,, \tag{8.90}$$

which taking into account (8.77) can be written as[16] (note that we restore the c factors)

$$\Delta\phi = \frac{6G^2M^2}{h^2c^2} = \frac{6\pi GM}{a(1-e^2)c^2} \,. \tag{8.91}$$

Because it is a small effect, let's accumulate this over 100 years to get the observable quantity

$$\Delta\phi_{100} \equiv \frac{\Delta\phi}{T} \times \frac{100\,\text{years}}{\text{century}} \,, \tag{8.92}$$

with T the period of the orbit in years. In terms of observable orbits within the solar system, Mercury is the closest planet to the Sun, and so it should have the largest precession.

Object	Mass (10^{24} kg)	Mean Equatorial Radius (10^3 km)	Period (days)	Semimajor axis (10^8 km)	Eccentricity
Mercury	0.33010	2.4397	87.869	0.57909227	0.20563593
Venus	4.8673	6.0518	224.701	1.0820948	0.00677672
Earth	5.9722	6.3710	365.256	1.4959826	0.01671123
Mars	0.64169	3.3895	686.98	2.2794382	0.0933941
Jupiter	1898.1	69.911	4332.71	7.7834082	0.04838624
Saturn	568.32	58.232	10759.50	14.266664	0.05386179
Uranus	86.810	25.362	30685.00	28.706582	0.04725744
Neptune	102.41	24.622	60190.00	44.983964	0.00859048

Taking into account the values for Mercury's orbit, we obtain

$$\Delta\phi_{100} \approx 43.03'' \tag{8.93}$$

The major axis of Mercury precesses at a rate of 43 arcsecs per century. The observational results are in excellent agreement with General Relativity

Planet	Observed residual	GR prediction
Mercury	$(43.11 \pm 0.45)''$	$43.03''$
Venus	$(8.4 \pm 4.8)''$	$8.6''$
Earth	$(5.0 \pm 1.2)''$	3.8"

[16]The use of the expressions for the unperturbed solution is justified by the fact that we are looking to a very small quantity.

43 arcseconds and the end of the Newtonian empire

Newton's theory had been a very successful theory, extensively used by astronomers for centuries. It had predicted the return of comet Halley (1758) with an error of 33 days, the elliptical character of the recently discovered Uranus (1781) and even more surprisingly the location, mass and orbit parameters of Neptune, even before it was directly observed (1846). Leverrier discovered it just *with the point of his pen!*[a]; clearly an amazing proof of the universality of the gravitational interaction. Nevertheless, at the end of the 19th century there were still some caveats related to Mercury's orbit. As you should know the $1/r^2$ dependence of the Newton's force gives rise to elliptical trajectories on a plane, and the corresponding perihelion is *a priori* a fixed point[b]. However, different perturbations (due for instance to the presence of other massive objects in the Solar system, such as Jupiter, or to the quadrupole moment of the Sun), give rise to a perihelion advance, and therefore to an ellipse turning on the plane. Even when all those effects were taken into account there was a residual contribution to the shift. As pointed out by Leverrier and Newcomb at the end of the 19th, Mercury's perihelion precesses at a rate of $575''$ per century, but only $532''$ can be explained by the perturbations associated to the other planets. The remaining $43''$ per century could not be accounted for by the Newtonian theory even when errors were taken into account. The observational problem was basically closed for everyone (apart from Leverrier[c]), but the theoretical problem would remain open till the introduction of General Relativity in 1915.

[a] Francois Arago 1786-1853.

[b] The Laplace-Runge-Lenz vector is conserved.

[c] He died believing that the history of the discovery of Neptune would repeat and a new planet with a mass enough to account for the $43''$ per century would be encountered between the Sun and Mercury.

8.9 The massless case: Gravitational deflection of light

Let us consider now the massless case where $\epsilon = 0$ and (8.61) becomes

$$\frac{d^2u}{d\phi^2} + u = 3GMu^2\,. \tag{8.94}$$

In the absence of the term $3GMu^2$, the previous solution reduces to the simple harmonic oscillator equation

$$\frac{d^2u_0}{d\phi^2} + u_0 = 0\,, \tag{8.95}$$

whose solutions

$$u_0 = \frac{\sin\phi}{b}\,, \tag{8.96}$$

can be interpreted as straight lines passing at a distance b from the origin. Following a similar procedure to the one used in the previous section, we look for perturbative solutions of the form

$$u = u_0 + \Delta u = \frac{\sin\phi}{b} + \Delta u \tag{8.97}$$

with u_0 given by (8.96). Substituting (8.97) into (8.94) we get a linear equation in Δu

$$\frac{d^2\Delta u}{d\phi^2} + \Delta u = \frac{3GM}{b^2}\sin^2\phi\,, \tag{8.98}$$

whose solution is given by

$$\Delta u = \frac{3GM}{2b^2}\left(1 + \frac{1}{3}\cos(2\phi)\right)\,. \tag{8.99}$$

Adding this to the unperturbed solution we get

$$u = \frac{\sin\phi}{b} + \frac{3GM}{2b^2}\left(1 + \frac{1}{3}\cos(2\phi)\right)\,, \tag{8.100}$$

which in the limit $r \to \infty$, $u \to 0$ and for small ϕ gives rise to

$$\phi \approx -\frac{2GM}{bc^2}\,. \tag{8.101}$$

The total deflection angle is twice this value

$$\Delta\phi = \frac{2R_s}{b} = \frac{4GM}{bc^2}\,. \tag{8.102}$$

For rays coming from a distant stars and grazing the surface of the Sun[17]

$$b \approx R_\odot = 6.96 \times 10^5\,\mathrm{km} \qquad M_\odot = 2 \times 10^{30}\,\mathrm{Kg} \tag{8.103}$$

we get

$$\Delta\phi = \frac{4GM_\odot}{c^2 R_\odot} = 1.75''\,. \tag{8.104}$$

Light paths so close to the Sun are of course not visible by day, but they become visible at the time of a total eclipse. Their position relative to the other background stars during the total eclipse appears shifted relative to the position in the usual night sky. This prediction of General Relativity was verified in 1919 just a few years later the formulation of the theory. Two separate groups led by Arthur Eddington and Andrew Crommelin moved to Guinea and Brazil to observe the total eclipse of May 29, 1919. They reported deflections of $(1.61 \pm 0.40)''$ and $(1.98 \pm 0.16)''$, in reasonable agreement with Einstein's prediction (8.104).

8.10 The post-Newtonian formalism

Nowdays, the agreement between theory and observation is at the level of a few parts in a thousand. The deviations from the General Relativity are usually parametrized in terms of the so-called post-Newtonian parameters β and γ measuring respectively the non-linearity in the superposition law for gravity and the spatial curvature produced by unit rest mass

$$ds^2 = -\left(1 - \frac{2GM}{r} + 2(\beta - \gamma)\frac{G^2M^2}{r^2}\right)dt^2 + \left(1 - \frac{2\gamma GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2\,. \tag{8.105}$$

[17] In this case, the effect is maximized and easier to observe.

When this parameters are taken into account Eqs. (8.91) and (8.102) become respectively

$$\Delta\phi = \left(\frac{2-\beta+2\gamma}{3}\right)\frac{6\pi GM}{a(1-e^2)c^2}\,, \qquad \Delta\phi = \left(\frac{1+\gamma}{2}\right)\frac{4GM}{bc^2} \tag{8.106}$$

The General Relativity limit corresponds to $\gamma = \beta = 1$. Recent measurements provide values $\gamma = 0.9998 \pm 0.0003$ and $|2\gamma - \beta - 2| < 3 \times 10^{-3}$, in excellent agreement with GR.

CHAPTER 9

GENERAL RELATIVITY: THE FIELD THEORY APPROACH

We move now to the modern approach to General Relativity: field theory. The chief advantage of this formulation is that it is simple and easy; the only thing to specify is the so-called Lagrangian density. We start by presenting a simple introduction to classical field theory in flat spacetime which we later generalize to curved spacetime. The last part of the Chapter is devoted to the action for the gravitational field and the recovery of Einstein equations from it.

9.1 Classical mechanics

The fundamental problem of classical mechanics is to determine the way particles move given a potential. Dynamical systems can be described by equations of motion or by action functionals

$$S = \int dt\, L\left(q_j(t), \dot{q}_j(t)\right) \tag{9.1}$$

depending on generalized coordinates and velocities $\{q_j(t), \dot{q}_j(t)\}$. The classical trajectory is defined as the unique path that extremizes the action functional ($\delta S = 0$) for all variations $q_j \to q_j + \delta q_j$ with fixed initial $q_j(t_0)$ and final values $q_j(t_f)$. An explicit variation of the action gives

$$\delta S = \int_{t_0}^{t_f} dt \left(\frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right) . \tag{9.2}$$

Integrating the last term by parts to flip the temporal derivative onto $\partial L / \partial \dot{q}_j$ we get

$$\delta S = \int_{t_0}^{t_f} dt \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_j \,, \tag{9.3}$$

where we have omitted a total derivative that vanishes because of the boundary conditions $\delta q_j(t_0) = \delta q_j(t_f) = 0$. Since δq_j is arbitrary, the extremization of the action translates into the so-called Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_j} = 0\,. \tag{9.4}$$

This variational formulation has several advantages:

i) The properties of the system are compactly summarized in one function, the Lagrangian.

ii) There is a direct connection between invariances of the Lagrangian and constants of motion[1].

iii) There is a close relation between the Lagrangian formulation of classical mechanics and quantum mechanics.

9.2 From Classical Mechanics to Field theory

The Lagrangian formalism presented above can be extended to continuous systems involving an infinite number of degrees of freedom. This is achieved by taking the appropriate limit of a system with a finite number of degrees of freedom. Consider a one dimensional chain of length l made of N equal masses m. The masses are separated by a distance a and connected by identical massless springs with force constant k. The total length of the system is $l = (N+1)a$. The displacement of the particles with respect to its equilibrium position $\bar{x}_j = ja$ is described by a generalized coordinate $\phi(x_j, t) \equiv x_j(t) - \bar{x}_j$ with $j = 1, \ldots, N$ and $\phi_0 = \phi_{N+1} = 0$. The Lagrangian of the full system includes the kinetic energy of the particles and the energy stored into the springs, i.e.

$$L = T - U = \frac{1}{2} m \sum_{j=1}^{N} \dot{\phi}_j^2(t) - \frac{1}{2} k \sum_{j=0}^{N} \left(\phi_{j+1}(t) - \phi_j(t)\right)^2 . \tag{9.5}$$

The continuous limit of the previous expression can be taken by sending $N \to \infty$ and $a \to 0$ in such a way that the total length of the chain, $l = (N+1)a$, remains fixed. To keep the total mass of the system and the force between particles finite we require m/a and ka to go to some finite values μ and Y playing the role of the mass density and the Young modulus in the continuous theory. We have

$$L = \frac{1}{2}\sum_{j=1}^{N} a\left(\frac{m}{a}\right)\dot{\phi}_j^2(t) - \frac{1}{2}\sum_{j=0}^{N} a\,(ka)\left(\frac{\phi_{j+1}(t) - \phi_j(t)}{a}\right)^2 \quad \longrightarrow \quad L = \frac{1}{2}\int_0^l dx \left[\mu\dot{\phi}^2 - Y\left(\partial_x \phi\right)^2\right] ,$$

with the finite number of generalized coordinates ϕ_j replaced by a continuous function $\phi(x, t)$. The antisymmetric dependence of Eq.(9.6) on the derivatives suggests the introduction of a set of coordinates $x^\mu = (c_s t, x)^T$ with $c_s = \sqrt{Y/\mu}$ and a Lorentzian metric $\eta_{\mu\nu} = \text{diag}(-1, 1)$. This allows us to write

$$S = \int d(c_s t) dx\, \mathcal{L} \tag{9.6}$$

[1] For instance, if the Lagrangian is invariant under rotations, angular momentum is conserved.

with

$$\mathcal{L} = -\frac{\mu c_s}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\,, \tag{9.7}$$

the so-called *Lagrangian density.* The jump from fields existing within a physical medium to fields in vacuum is now straightforward: we must simply replace c_s by the speed of light c. Generalizing the metric $\eta_{\mu\nu}$ to arbitrary dimensions , we can write generically write the action for relativistic fields as

$$S = \int d^n x\,\mathcal{L}\left(\phi, \partial_\mu\phi\right)\,, \tag{9.8}$$

where we have allowed for a dependence of the Lagrangian density on the fields.

Exercise

Consider again the chain of masses connected by springs. Modify the system to give rise to an explicit dependence of the Lagrangian on ϕ. Hint: Eq. (9.7) is shift-invariant.

Gauge freedom in the Lagrangian

The Lagrangian of a physical system is not unique. To see this, consider a transformation of the form

$$\mathcal{L} \longrightarrow \mathcal{L}' = \mathcal{L} + \partial_\sigma g^\sigma\,, \tag{9.9}$$

and its effect on the action (9.8)

$$S' = \int d^n x\mathcal{L}' = S + \int_{\mathcal{R}} d^n x\,\partial_\sigma g^\sigma = S + \int_{\partial\mathcal{R}} g^\sigma dS_\sigma\,. \tag{9.10}$$

The term associated to $\partial_\sigma g^\sigma$ turns out to be a boundary term, which does not contribute to the equations of motion. Lagrangians differing by a contribution $\partial_\sigma g^\sigma$ give rise to the same equations of motion.

The equations of motion for the field $\phi(x,t)$ can be obtained by considering the change of the action under an infinitesimal change $\phi(x,t) \longrightarrow \phi(x,t) + \delta\phi(x,t)$. The only requirement to be satisfied by the variations $\delta\phi$ is to be differentiable and to vanish outside some bounded region of spacetime (to allow an integration by parts). Performing this variation we get

$$\delta S = \int d^n x\left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\delta\phi_{,\mu}\right]\delta\phi = \int d^n x\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\delta\phi\,. \tag{9.11}$$

Requiring the action to be stationary ($\delta S = 0$) and taking into account that $\delta\phi$ is completely arbitrary, we obtain the continuous version of the Euler-Lagrange equations

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0\,. \tag{9.12}$$

A worked-out example

As a direct application of Eq. (9.12), let me consider the action (9.7)

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \quad \longrightarrow \quad \partial_\mu \left(\eta^{\mu\nu} \partial_\nu \phi \right) = 0 \quad \longrightarrow \quad -\frac{1}{c_s^2} \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} = 0 \,. \tag{9.13}$$

As expected, we get a wave equation.

9.3 Principles of Lagrangian construction

What kind of Lagrangian density should we choose? To be in the safe side, the Lagrangian density of a *relativistic theory* is required to satisfy the following requirements:

1. $\mathcal{L}$ must be a real-valued function, since it enters into expressions of physical significance, like the Hamiltonian.
2. $\mathcal{L}$ must have dimension 4 in units of energy, since in natural units the action is dimensionless and $[d^4x] = -4$.
3. $\mathcal{L}$ must be a linear combination of Lorentz invariant quantities constructed from the fields, their first partial derivatives and the universally available objects $\eta_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$.
4. The coefficients of this linear combination can be restricted by the symmetries of the problem (internal symmetries/ gauge symmetries).
5. $\mathcal{L}$ should be bounded from below.

The power of the previous program is made most vividly evident by considering some examples.

9.3.1 A complex scalar field with $U(1)$ symmetry

Consider a complex scalar field

$$\Phi = \phi_1 + i\phi_2 \,, \qquad \Phi^* = \phi_1 - i\phi_2 \,. \tag{9.14}$$

The quadratic Lorentz invariants which can be constructed from $\Phi, \Phi^*, \partial_\mu\Phi$ and $\partial_\mu\Phi^*$ lead to a Lagrangian density of the form

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu} \left[a\partial_\mu\Phi\partial_\nu\Phi + a^*\partial_\mu\Phi^*\partial_\nu\Phi^* + 2a_0\partial_\mu\Phi^*\partial_\nu\Phi \right] + \frac{1}{2} \left[b\Phi\Phi + b^*\Phi\Phi^* + 2b_0\Phi^*\Phi \right] \tag{9.15}$$

where the reality condition $\mathcal{L} = \mathcal{L}^*$ imposes the appearance of the pairs a, a^* and b, b^* and requires the coefficients a_0 and b_0 to be real. The previous Lagrangian density can be written in a more compact way by introducing the arrays

$$\tilde{\Phi} \equiv \begin{pmatrix} \Phi \\ \Phi^* \end{pmatrix} \qquad \text{and} \qquad \tilde{\Phi}^\dagger \equiv \begin{pmatrix} \Phi^* \\ \Phi \end{pmatrix}^T = (\Phi^* \;\, \Phi) \,. \tag{9.16}$$

to get[2]

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\tilde{\Phi}^{\dagger}_{,\mu}\begin{pmatrix} a_0 & a^* \\ a & a_0 \end{pmatrix}\tilde{\Phi}_{,\nu} + \frac{1}{2}\tilde{\Phi}^*\begin{pmatrix} b_0 & b^* \\ b & b_0 \end{pmatrix}\tilde{\Phi}\,. \tag{9.17}$$

The number of terms appearing in this Lagrangian can be reduced in cases in which we have symmetries on top of Lorentz invariance. As an illustration of this, imagine the field Φ to possess an internal symmetry

$$\Phi \to e^{i\omega}\Phi\,, \qquad \Phi^* \to e^{-i\omega}\Phi^*\,. \tag{9.18}$$

In this case, we necessarily have $a = b = 0$ and the matrices in (9.17) become diagonal. This leaves us with a simpler Lagrangian, that with some notational adjustments, can be written as

$$\mathcal{L} = \frac{1}{2}K\left[-\eta^{\mu\nu}\partial_\mu\Phi^*\partial_\nu\Phi - \kappa^2\Phi^*\Phi\right]\,. \tag{9.19}$$

A direct application of the Euler-Lagrange equations (9.12) provides two uncoupled equations for Φ and Φ^*, namely

$$\left(\Box - \kappa^2\right)\Phi = 0\,, \qquad \left(\Box - \kappa^2\right)\Phi^* = 0\,, \tag{9.20}$$

that we can use to provide a physical interpretation for the parameter κ^2. Indeed, setting $\Phi = e^{ik_\mu x^\mu}$ with $p^\mu = (E, \mathbf{p})$ in any of these two equations, we get a dispersion relation $E^2 - \mathbf{p}^2 = \kappa^2$, which makes it natural to identify the parameter κ^2 with the *mass* m^2 of the field.

9.3.2 A worked-out example: Vector and tensor fields

The previous analysis can be easily extended to other kinds of fields. Consider for instance a vector field A_μ and a tensor field $B_{\mu\nu}$. Which is the most general Lagrangian $\mathcal{L}(A, \partial A, B, \partial B)$ that can be constructed out of these two fields? To clarify the procedure, let me split the problem into several pieces

$$\mathcal{L}(A, B, \partial A, \partial B) = \mathcal{L}_A(A, \partial A) + \mathcal{L}_B(B, \partial B) + \mathcal{L}_{\text{int}}(A, B, \partial A, \partial B)\,. \tag{9.21}$$

The lagrangian $\mathcal{L}_A$ is a *free lagrangian* for A_μ, i.e. a linear combination of Lorentz invariant terms quadratic in A_μ[3],

$$\mathcal{L}_A = a_1\partial_\mu A^\mu\partial_\nu A^\nu + a_2\partial_\mu A_\nu\partial^\mu A^\nu + a_3\partial_\mu A_\nu\partial^\nu A^\mu + a_4 A_\mu A^\mu + a_5\,\epsilon_{\mu\nu\rho\sigma}(\partial^\nu A^\mu)(\partial^\sigma A^\rho)\,. \tag{9.22}$$

The lagrangian $\mathcal{L}_B$ is a free lagrangian for $B_{\mu\nu}$

$$\begin{aligned}\mathcal{L}_B = {} & b_1\partial_\mu B^\nu{}_\nu\partial^\mu B^\kappa{}_\kappa + b_2\partial_\mu\left(B^{\kappa\nu} + B^{\nu\kappa}\right)\partial^\mu\left(B_{\kappa\nu} + B_{\nu\kappa}\right) + b_3\partial_\mu\left(B^{\mu\nu} + B^{\nu\mu}\right)\partial_\nu B^\kappa{}_\kappa + \\ & + b_4\partial_\mu\left(B^{\kappa\mu} + B^{\mu\kappa}\right)\partial^\nu\left(B_{\kappa\nu} + B_{\nu\kappa}\right) + b_5\epsilon_{\mu\nu\rho\sigma}B^{\mu\nu}B^{\rho\sigma} + b_6 B_{\mu\nu}B^{\mu\nu} + b_7 B_{\mu\nu}B^{\nu\mu} + b_8\left(B^\mu{}_\mu\right)^2\,.\end{aligned} \tag{9.23}$$

The interaction Lagrangian $\mathcal{L}_{\text{int}}$ follows the same logic, but now involving the interactions between the two fields,

$$\begin{aligned}\mathcal{L}_{\text{int}} = {} & i_1\partial_\mu A_\nu\left(B^{\mu\nu} + B^{\nu\mu}\right) + i_2 A_\mu\partial_\nu\left(B^{\mu\nu} + B^{\nu\mu}\right) + i_3\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\nu)B^{\rho\sigma} + \\ & + i_4\epsilon_{\mu\nu\rho\sigma}A^\mu(\partial^\nu B^{\rho\sigma}) + i_5 A_\mu A_\nu B^{\mu\nu}\,.\end{aligned} \tag{9.24}$$

[2]In this notation, the reality condition $\mathcal{L} = \mathcal{L}^*$ results from the hermiticity of the 2×2 matrices.

[3]Quadratic actions give rise to linear equations of motion, where the superposition principle can be applied.

I maybe missed some terms, but I think you get the idea. As in the scalar case, the list of operators and independent coefficients can be shortened cases in which there are extra symmetries on top on Lorentz invariance. Let me illustrate this with a simpler example.

The action for the electromagnetic field

Consider the action (9.22) alone. The first thing that can simplify our life is the gauge freedom in the choice of the Lagrangian density. In particular notice that choosing $g_\sigma = \epsilon_{\mu\nu\rho\sigma}(\partial^\nu A^\mu)A^\rho$ in (9.9) allows us to eliminate the term a_5 in (9.22), since

$$\partial^\sigma g_\sigma = \epsilon_{\mu\nu\rho\sigma}\left(\partial^\nu A^\mu\right)\left(\partial^\sigma A^\rho\right) + \underbrace{\epsilon_{\mu\nu\rho\sigma}\left(\partial^\nu\partial^\sigma A^\mu\right)A^\rho}_{0\,\text{by symmetry}} . \tag{9.25}$$

Taking this into account we are left with an action containing 4 pieces

$$S = \int d^4x \left[a_1\partial_\mu A^\mu\partial_\nu A^\nu + a_2\partial_\mu A_\nu\partial^\mu A^\nu + a_3\partial_\mu A_\nu\partial^\nu A^\mu + a_4 A_\mu A^\mu\right] . \tag{9.26}$$

Imagine now that A_μ is the field of a *gauge theory*. In that case the field configurations A_μ and

$$A'_\mu = A_\mu + \partial_\mu\chi\,, \tag{9.27}$$

with arbitrary scalar function χ give rise to the same physical observables[4]. This automatically forbids the a_4 term in (9.26)[5] and puts some restrictions on the other coefficients. To see this, let me split $\partial_\mu A_\nu$ into its symmetric $S_{\mu\nu} \equiv \partial_{(\mu}A_{\nu)}$ and antisymmetric $F_{\mu\nu} \equiv \partial_{[\mu}A_{\nu]}$ parts

$$\partial_\mu A_\nu = S_{\mu\nu} + F_{\mu\nu}\,, \tag{9.28}$$

and rewrite the action (9.26) as

$$S = \int d^4x \left[a_1 S^\mu{}_\mu S^\nu{}_\nu + (a_2 + a_3)S_{\mu\nu}S^{\mu\nu} + (a_2 - a_3)F_{\mu\nu}F^{\mu\nu}\right] . \tag{9.29}$$

The invariance of the action under the gauge transformation $A_\mu \to A_\mu + \partial_\mu\chi$ requires $a_1 = 0$ and $a_3 = -a_2$. This restriction leaves us with an action

$$S = \int d^4x F_{\mu\nu}F^{\mu\nu}\,, \tag{9.30}$$

where we have omitted an overall normalization factor that can be determined by choosing the coupling of the gauge field A_μ to matter and the units of that coupling. The equations of motion associated with this action can be computed via the Euler-Lagrange equations (9.12)

[4] In the same way that *physicality* cannot be attributed to $\mathcal{L}$, we cannot make any claim about the physicality of A_μ. Physicality might be attributed to the set $\{A_\mu\}$ of gauge-equivalent 4-potentials or to any gauge invariant attribute of that set, but not to its individual elements.

[5] It cannot be compensated by the transformation of the other (derivative) terms.

or by varying the action with respect to A_μ. We follow the second procedure to get

$$\begin{aligned}\delta S &= \int d^4x \left[F^{\mu\nu}\delta F_{\mu\nu} + F_{\mu\nu}\delta F^{\mu\nu}\right] = 2\int d^4x F^{\mu\nu}\delta F_{\mu\nu} \\ &= 2\int d^4x F^{\mu\nu}(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) = 4\int d^4x F^{\mu\nu}\partial_\mu \delta A_\nu \\ &= -4\int d^4x \partial_\mu F^{\mu\nu}\delta A_\nu + \text{boundary terms} ,\end{aligned} \tag{9.31}$$

where we used the symmetry properties of $F_{\mu\nu}$ and performed an integration by parts. Imposing finally the condition $\delta S = 0$ for arbitrary δA_ν, we arrive to the very familiar result

$$\partial_\mu F^{\mu\nu} = 0. \tag{9.32}$$

The Maxwell equations in vacuum are recovered from an action (9.30) constructed with very limited principles, namely, quadraticity in the fields, Lorentz invariance and gauge invariance.

9.4 The action for the graviton

The procedure outlined in the previous section is quite powerful. As an interesting application for General Relativity, let me consider the action for a second rank symmetric and massless tensor field $h_{\mu\nu}$. As in the vector field case, the kinetic term is constructed out of scalars that are quadratic in the derivatives $\partial_\rho h_{\mu\nu}$. The most general expression will be the sum of the different scalars obtained by contracting pairs of indices in all possible ways. The resulting action takes the form

$$S = \int d^4x \left[c_1 \partial_\mu h^\nu{}_\nu \partial^\mu h^\kappa{}_\kappa + c_2 \partial_\mu h^{\kappa\nu}\partial^\mu h_{\kappa\nu} + c_3 \partial_\mu h^{\mu\nu}\partial_\nu h^\kappa{}_\kappa + c_4 \partial_\mu h^{\kappa\mu}\partial^\nu h_{\kappa\nu}\right] , \tag{9.33}$$

with c_1, c_2, c_3, c_4 some undetermined constants. These constants can be determined up to an overall factor by requiring the action to be invariant under the *gauge transformations*

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \, . \tag{9.34}$$

Plugging (9.34) into (9.33) and performing some simple manipulations we get

$$\begin{aligned}S \to S + \int d^4x [&-2(2c_1+c_3)\partial_\mu h^\kappa{}_\kappa \partial^\mu \partial_\lambda \xi^\lambda - 2(2c_2+c_4)\partial^\mu h_{\mu\nu}\Box\xi^\nu - 2(c_3+c_4)\partial^\mu h_{\mu\nu}\partial^\nu\partial_\kappa \xi^\kappa \\ &+ (4c_1+2c_2+4c_3+3c_4)\partial_\mu\partial_\kappa \xi^\kappa \partial^\mu \partial_\lambda \xi^\lambda + (2c_2+c_4)\Box\xi_\mu \Box\xi^\mu] \, ,\end{aligned} \tag{9.35}$$

which imposing $\delta S = 0$, provides the constraints

$$c_2 = -c_1, \quad c_4 = -c_3 = 2c_1 \, . \tag{9.36}$$

Taking this into account, the action (9.33) takes the form

$$S = \int d^4x \left[\partial_\mu h^{\kappa\nu}\partial^\mu h_{\kappa\nu} + 2\partial_\mu h^{\mu\nu}\partial_\nu h^\kappa{}_\kappa - 2\partial_\mu h^{\kappa\mu}\partial^\nu h_{\kappa\nu} - \partial_\mu h^\nu{}_\nu \partial^\mu h^\kappa{}_\kappa\right] , \tag{9.37}$$

where we have omitted an overall normalization factor that can be determined by specifying the coupling to matter and setting the units of the coupling. The associated equations of motion can be obtained by varying the action with respect to the field. This leads to

$$\delta S = \int d^4x \Big[2\partial^\mu h_{\kappa\nu}\partial_\mu \delta h^{\kappa\nu} + 2\partial_\nu h^\kappa{}_\kappa \partial_\mu \delta h^{\mu\nu} + 2\eta_{\kappa\lambda}\partial_\mu h^{\mu\nu}\partial_\nu \delta h^{\kappa\lambda} \\ -2\partial^\nu h_{\kappa\nu}\partial_\mu \delta h^{\kappa\mu} - 2\partial^\mu h_{\kappa\mu}\partial_\nu \delta h^{\kappa\nu} - 2\eta_{\kappa\lambda}\partial^\mu h^\nu{}_\nu \partial_\mu \delta h^{\kappa\lambda} \Big] \ , \qquad (9.38)$$

which, integrating by parts, dropping boundary terms and renaming indices can be written as

$$\delta S = 2\int d^4x \Big[-\Box h_{\kappa\nu}\delta h^{\kappa\nu} - \partial_\mu\partial_\nu h^\kappa{}_\kappa \delta h^{\mu\nu} - \eta_{\kappa\lambda}\partial_\mu\partial_\nu h^{\mu\nu}\delta h^{\kappa\lambda} + \partial_\mu\partial^\nu h_{\kappa\nu}\delta h^{\kappa\mu} + \partial^\mu\partial_\nu h_{\kappa\mu}\delta h^{\kappa\nu} + \eta_{\kappa\lambda}\Box h^\nu{}_\nu \delta h^{\kappa} \\ = 2\int d^4x \left[-\Box h_{\mu\nu} - \partial_\mu\partial_\nu h^\kappa{}_\kappa - \eta_{\mu\nu}\partial_\kappa\partial_\lambda h\kappa\lambda + \partial_\mu\partial^\kappa h_{\kappa\nu} + \partial_\nu\partial^\kappa h_{\kappa\mu} + \eta_{\mu\nu}\Box h^\kappa{}_\kappa \right] \delta h^{\mu\nu} \ . \qquad (9.3$$

A simple inspection reveals that the quantity inside the square brackets is nothing else than the linearized version of the Einstein tensor $G_{\mu\nu}$

$$\delta S \propto \int d^4x\, G_{\mu\nu}\delta h^{\mu\nu} \ . \qquad \longrightarrow \qquad G_{\mu\nu} = 0 \ . \qquad (9.40)$$

The linearized version of Einstein equations in vacuum are recovered from an action (9.39) constructed with very limited principles, namely, quadraticity in the fields, Lorentz invariance and gauge invariance.

9.5 Field theory in curved spacetime

The variational approach for fields presented in the previous section can be generalized to include the interaction with gravity. We guess the form of the action in this case with the help of the Equivalence Principle:

- Replace the Minkowski metric $\eta_{\mu\nu}$ by $g_{\mu\nu}$.
- Replace partial derivatives by covariant derivatives (*colon-goes-to semicolon rule*).
- Replace the Lorentz invariant volume element d^nx by the covariant volume element $d^nx\sqrt{|g|}$.

Since $\tilde{\mathcal{L}}$ and $d^nx\sqrt{|g|}$ are scalars under general coordinate transformations, the resulting action

$$S = \int d^nx \underbrace{\sqrt{|g|}\,\tilde{\mathcal{L}}(\phi, \nabla_\mu\phi, g_{\mu\nu})}_{\mathcal{L}} \qquad (9.41)$$

is guaranteed to provide covariant equations of motion. Note that the untilded quantity $\mathcal{L} \equiv \sqrt{|g|}\tilde{\mathcal{L}}$ is a scalar density of weight 1.

9.6 The energy-momentum tensor

Consider now an arbitrary *infinitesimal* coordinate transformation

$$x^\mu \to \bar{x}^\mu = x^\mu + \xi^\mu(x)\,. \tag{9.42}$$

This transformation generates a perturbation to both the fields and the metric in such a way that the Lagrangian density $\mathcal{L}$ (no tilde) becomes

$$\mathcal{L}(\phi+\delta\phi, \phi_{,\mu}+\delta\phi_{,\mu}, g_{\mu\nu}+\delta g_{\mu\nu}) \approx \mathcal{L}(\phi, \phi_{,\mu}, g_{\mu\nu}) + \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}}\delta\phi_{,\mu} + \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}\delta g_{\mu\nu}\,. \tag{9.43}$$

Integrating by parts, we get two pieces

$$\delta S = \int d^n x \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right]}_{=0}\delta\phi + \int d^n x \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}\delta g_{\mu\nu}\,. \tag{9.44}$$

The first one is associated to a *particular variation* $\delta\phi$ and vanishes when taking into account the Euler-Lagrange equation for ϕ. The second term must be then equal to zero for S to remain unchanged. The integrand $\partial\mathcal{L}/\partial g_{\mu\nu}$ is a scalar density. Let's define a symmetric second-rank tensor out of such a density

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{|g|}}\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}\,, \tag{9.45}$$

and write

$$\delta S = \frac{1}{2}\int d^n x \sqrt{|g|}\, T^{\mu\nu}\delta g_{\mu\nu}\,. \tag{9.46}$$

Although is tempting to simply set $\sqrt{|g|}T_{\mu\nu} = 0$, this condition is overly restrictive, since $\delta g_{\mu\nu}$ refers here to a *specific* type of variation, not to an arbitrary one. The variation $\delta g_{\mu\nu}$ can be however expressed in terms of the arbitrary perturbation ξ_μ by taking into account that $\delta g_{\mu\nu} = -(\xi_{\mu;\nu} + \xi_{\nu;\mu})$ (cf. Eq. (6.60) and notice (9.50)). This gives

$$\begin{aligned}\delta S &= \frac{1}{2}\int d^n x\sqrt{|g|}\,T^{\mu\nu}\delta g_{\mu\nu} = -\int d^n x\sqrt{|g|}\,T^{\mu\nu}\xi_{\mu;\nu} \\ &= \int d^n x \sqrt{|g|}\,T^{\mu\nu}{}_{;\nu}\xi_\mu - \underbrace{\int d^n x\left(\sqrt{|g|}\,T^{\mu\nu}\xi_\mu\right)_{;\nu}}_{=0}\,,\end{aligned} \tag{9.47}$$

where we have made use of the symmetry property of $T^{\mu\nu}$ and integrated by parts to get a total derivative that vanishes by assumption on the boundary of integration. Since ξ_μ is arbitrary we must have

$$\nabla_\nu T^{\mu\nu} = 0\,, \tag{9.48}$$

which is a continuity equation suggesting that we can identify the tensor (9.45) with the energy-momentum tensor of any physical system.

A common sign mistake

You will find some books giving an alternative definition of the energy momentum tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \,, \tag{9.49}$$

in terms of $\delta g^{\mu\nu}$ rather than $\delta g_{\mu\nu}$. The difference in sign between these two equivalent expressions comes from

$$g_{\mu\nu} g^{\nu\lambda} = \delta^{\mu}{}_{\lambda} \quad \rightarrow \quad \delta\left(g_{\mu\nu} g^{\nu\lambda}\right) = 0 \quad \rightarrow \quad \delta g^{\mu\nu} = -g^{\mu\lambda} g^{\nu\rho} \delta g_{\lambda\rho} \,. \tag{9.50}$$

9.6.1 A particular case

When the Lagrangian $\tilde{\mathcal{L}}$ on $\mathcal{L} = \sqrt{|g|}\tilde{\mathcal{L}}$ depends only on the metric and not on the first derivatives of the metric[6], i.e. $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}(\phi, \partial_\mu \phi, g_{\mu\nu})$, it is possible to derive an alternative expression for the energy-momentum tensor. In particular, taking into account that $\frac{\partial\sqrt{|g|}}{\partial g_{\mu\nu}} = \frac{1}{2}\sqrt{|g|} g^{\mu\nu}$, we can write

$$T^{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = \frac{2}{\sqrt{|g|}} \left(\frac{\partial \sqrt{|g|}}{\partial g_{\mu\nu}} \tilde{\mathcal{L}} + \sqrt{|g|} \frac{\partial \tilde{\mathcal{L}}}{\partial g_{\mu\nu}} \right) = \frac{2}{\sqrt{|g|}} \left(\frac{1}{2} \sqrt{|g|} g^{\mu\nu} \tilde{\mathcal{L}} + \sqrt{|g|} \frac{\partial \tilde{\mathcal{L}}}{\partial g_{\mu\nu}} \right) \tag{9.51}$$

or equivalently

$$T^{\mu\nu} = g^{\mu\nu} \tilde{\mathcal{L}} + 2 \frac{\partial \tilde{\mathcal{L}}}{\partial g_{\mu\nu}} \,. \tag{9.52}$$

Exercise

Compute the energy-momentum tensor for (9.7).

9.7 The Einstein-Hilbert action

In order to construct an action for the gravitational field we must define a Lagrangian. This Lagrangian must transform as a scalar under general coordinate transformations and depend on the metric tensor and its derivatives. Note however that gravity is completely different from all other fundamental interactions, since non-trivial quantity can be constructed from the metric and its *first derivatives* alone. This can be easily seen by considering an arbitrary scalar combination $f(g_{\mu\nu}, \partial_\rho g_{\mu\nu})$ of this quantities around a small region in spacetime. According to the local flatness theorem, in such a region, it is always possible to find coordinates such

[6] Particular cases are the lagrangian for scalar fields or that for the electromagnetic field, where the covariant derivative reduces to the standard derivative.

that $g_{\mu\nu} = \eta_{\mu\nu}$ and $\partial_\rho g_{\mu\nu} = 0$ and therefore $f =$ constant. But ,since we are dealing with a scalar quantity, this will be the case in any other coordinate system. In other words, any covariant scalar function constructed just from the metric tensor and its derivatives will be a trivial constant.

The simplest non-trivial quantity that can be constructed from the metric and its derivatives is the Ricci scalar, which depends on the metric and its first and second order derivatives. The resulting action reads

$$S_{EH} = \int d^4x \mathcal{L}_{EH} = \frac{1}{2\kappa^2}\int d^4x\sqrt{|g|}R = \frac{1}{2\kappa^2}\int d^4x\sqrt{|g|}g^{\mu\nu}R_{\mu\nu} \tag{9.53}$$

and is known as the *Einstein-Hilbert action*. The constant of proportionality κ^2 is included on dimensional grounds and will be determined of the end of the computation.

Exercise

Which is the dimension of κ^2?

Consider the variation of (9.53) resulting from the variation of the metric tensor

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}\,, \tag{9.54}$$

where $\delta g_{\mu\nu}$ and its first derivative are assumed to vanish at the boundary of the integration region. We obtain

$$\delta\mathcal{L}_{EH} \propto \underbrace{\sqrt{|g|}\delta g^{\mu\nu}R_{\mu\nu} + \delta\sqrt{|g|}R}_{\delta\mathcal{L}_1} + \underbrace{\sqrt{|g|}g^{\mu\nu}\delta R_{\mu\nu}}_{\delta\mathcal{L}_2}\,, \tag{9.55}$$

where we have defined two pieces, $\mathcal{L}_1$ and $\mathcal{L}_2$. The first one can be easily evaluated by taking into account the variation and the variation of the metric determinant

$$\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|}g_{\mu\nu}\delta g^{\mu\nu}\,. \tag{9.56}$$

Exercise

Prove Eq. (9.56).

Collecting the terms, we get

$$\delta\mathcal{L}_1 = \sqrt{|g|}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}\,. \tag{9.57}$$

The evaluation of $\delta\mathcal{L}_2$ is slightly more involved since it requires to perform the variation of the Ricci tensor. The easiest way of doing this is to consider the variation of the Riemann tensor and perform the required contractions at the end of the computation. Schematically, the variation of the Riemann tensor has the structure

$$\text{Riemann} \sim \partial\Gamma + \Gamma\Gamma \quad \longrightarrow \quad \delta\,(\text{Riemann}) \sim \delta\,(\partial\Gamma) + (\delta\Gamma)\,\Gamma\,, \tag{9.58}$$

so the first thing that we have to compute is the variation of Christoffel symbols $\delta\Gamma^{\rho}_{\mu\nu}$ defined by

$$\Gamma^{\rho}{}_{\mu\nu} \xrightarrow[g_{\mu\nu}+\delta g_{\mu\nu}]{} \tilde{\Gamma}^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} + \delta\Gamma\rho_{\mu\nu} \quad \Longrightarrow \quad \delta\Gamma^{\rho}{}_{\mu\nu} = \tilde{\Gamma}^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\mu\nu}\,. \tag{9.59}$$

Be careful! We are just performing a variation of the metric, not transforming it.

It is easy to see that, even though a connection is *not* a tensor, the difference of two connections $\delta\Gamma^{\rho}_{\mu\nu}$ transforms as a tensor, i.e.

$$\delta\bar{\Gamma}^{\rho}{}_{\mu\nu} = \frac{\partial\bar{x}^{\rho}}{\partial x^{\sigma}}\frac{\partial x^{\lambda}}{\partial\bar{x}^{\mu}}\frac{\partial x^{\kappa}}{\partial\bar{x}^{\nu}}\delta\Gamma^{\rho}{}_{\lambda\kappa}\,. \tag{9.60}$$

Exercise
Check it.

The property (9.60) extremely simplifies the computation of the variation of the Riemann tensor. Indeed, we can always go to a local free fall reference frame in which $\Gamma^{\rho}_{\mu\nu} = 0$ at some arbitrary point P. In such a point the expression (9.58) becomes

$$\delta R_{\mu\nu} = \partial_{\rho}(\delta\Gamma^{\rho}{}_{\mu\nu}) - \partial_{\nu}(\delta\Gamma^{\rho}{}_{\rho\mu}) = \nabla_{\rho}(\delta\Gamma^{\rho}{}_{\mu\nu}) - \nabla_{\nu}(\delta\Gamma^{\rho}{}_{\rho\mu})\,, \tag{9.61}$$

where in the last step used of the fact that the partial and covariant derivatives coincide when $\Gamma^{\rho}{}_{\mu\nu} = 0$. The resulting *Palatini equation*

$$\delta R_{\mu\nu} = \nabla_{\rho}(\delta\Gamma^{\rho}{}_{\mu\nu}) - \nabla_{\nu}(\delta\Gamma^{\rho}{}_{\rho\mu}) \tag{9.62}$$

is *a tensorial equation* (remember the property (9.60)) valid in *any arbitrary coordinate system* (and *not only* in the free fall reference frame at P).

Exercise
Prove that the second term in the previous expression is symmetric, as it should be.

A similar trick can be applied to get the explicit expression of $\delta\Gamma$, that takes the same form that the definition of the Christoffel symbols, with the metric replaced by the metric variation and the partial derivatives replaces by covariant derivatives, i.e.

$$\delta\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(\nabla_{\nu}\delta g_{\sigma\rho} + \nabla_{\rho}\delta g_{\sigma\nu} - \nabla_{\sigma}\delta g_{\nu\rho}\right)\,. \tag{9.63}$$

Exercise
Check the previous expression by explicit computation.

Taking (9.61) into account, $\delta\mathcal{L}_2$ becomes

$$\delta\mathcal{L}_2 = \sqrt{|g|}\; g^{\mu\nu}\left[\nabla_\rho(\delta\Gamma^\rho{}_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\rho{}_{\rho\mu})\right] = \sqrt{|g|}\;\nabla_\sigma\left[g^{\mu\nu}\delta\Gamma^\sigma{}_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^\rho{}_{\rho\mu}\right] \;, \quad (9.64)$$

where we have used the metric compatibility condition (4.68). We are left therefore with the covariant divergence of a vector. Using the property

$$V^\mu{}_{;\mu} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}V^\mu\right)_{,\mu}\;, \quad (9.65)$$

we get the boundary term

$$\delta\mathcal{L}_2 = \sqrt{|g|}\;\nabla_\sigma\left[g^{\mu\nu}\delta\Gamma^\sigma{}_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^\rho{}_{\rho\mu}\right] = \;\partial_\sigma\left[\sqrt{|g|}g^{\mu\nu}(\delta\Gamma^\sigma{}_{\mu\nu}) - \sqrt{|g|}g^{\mu\sigma}(\delta\Gamma^\rho{}_{\rho\mu})\right] . \quad (9.66)$$

As I said before, gravity is a quite particular field theory. The existence of *second derivatives* in the Einstein-Hilbert action gives rise to a contribution depending on the value of the first derivatives on the boundary. To deal with these, we have two options:

- Extend the variational principle and require the fields *and* their derivatives to be fixed at the boundary. This would give rise to reasonable field equations. A clear example from classical mechanics illustrating this would be

$$S = \int_{t_0}^{t_f} dt\left(\ddot{q} + \frac{1}{2}\dot{q}^2\right) = \dot{q}(t_f) - \dot{q}(t_0) + \frac{1}{2}\int_{t_0}^{t_f}\dot{q}^2\,, \quad (9.67)$$

  with the assumption that *both* $\dot{q}$ and q are fixed at the boundary. This approach has however some caveats. On the one hand, it does not obey a composition rule of the kind

$$S(0 \to 1 \to 2) = S(0 \to 1) + S(1 \to 2)\,, \quad (9.68)$$

  where the paths connecting (q_0, t_0) and (q_2, t_2) are decomposed at an intermediate time t_1 with $t_0 < t_1 < t_2$. Although the paths are expected to be continuous at $t = t_1$, they do not need to be smooth at that point which requires leaving $\dot{q}_1$ free at $t = t_1$. On the other hand, the action principle has its roots in quantum mechanics, where the simultaneous fixing of q and $\dot{q}$ is inappropriate.

- Add the so-called Gibbons-Hawking-York counterterm to the action

$$S = S_{EH} + S_{HGY} = \frac{1}{2\kappa^2}\int_{\mathcal{R}} d^4x\sqrt{|g|}R + \frac{1}{\kappa^2}\int_{\partial\mathcal{R}} d^3x\sqrt{|h|}K\,, \quad (9.69)$$

with h the determinant of the induced metric on the boundary and K the trace of the extrinsic curvature. The Gibbons-Hawking-York is constructed in such a way that its variation cancels the unwanted term associated to the second derivatives of the metric, keeping only the part associated to the quadratic part of the action. Proving this statement is beyond the scope of this course. The interested reader is referred to the excellent discussion in Padmanabhan's book.

Forgetting about the boundary term, the Einstein-Hilbert action (9.53) becomes

$$\delta S_{EH} = \frac{1}{2\kappa^2} \int d^4x\sqrt{|g|} \left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right] \delta g^{\mu\nu} \,, \tag{9.70}$$

which, demanding it to vanish for arbitrary variations $\delta g^{\mu\nu}$, gives us the Einstein's equations in the absence of matter

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \,. \tag{9.71}$$

9.8 Einstein equations in the presence of matter

Having obtained the Einstein equations in the vacuum, let us now derive its full form in the presence of matter. Consider the action

$$S = S_{EH} + S_M \,, \tag{9.72}$$

with S_M containing all the non-gravitational fields. The variation of S_M with respect to $\delta g^{\mu\nu}$ (upper indices) gives

$$\delta S_M = -\frac{1}{2}\int d^4x\sqrt{|g|}T_{\mu\nu}\delta g^{\mu\nu} \,, \tag{9.73}$$

where we have made use of the covariant definition (9.49). Putting everything together we get

$$\delta S_{EH} + \delta S_M = \frac{1}{2\kappa^2} \int d^4x\sqrt{|g|} \left[R_{\mu\nu} - \frac{1}{2}R - \kappa^2 T_{\mu\nu}\right] \delta g^{\mu\nu} \,. \tag{9.74}$$

Since $\delta g^{\mu\nu}$ is arbitrary, we must have

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu} \,, \tag{9.75}$$

which confirms the identification of (9.45), or (9.49), with the energy-momentum tensor and allows us to identify the proportionality constant κ^2 with $8\pi G$.

Exercise

Modify the Einstein-Hilbert action to obtain the Einstein equations with cosmological constant.