

# Polarization versus photon spin

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**Abstract:** We examine whether the Stokes parameters of a two-mode electromagnetic field results from the superposition of the spins of the photons it contains. To this end we express any  $n$ -photon state as the result of the action on the vacuum of  $n$  creation operators generating photons which can have many different polarization states in general. We find that the macroscopic polarization holds as sum of the single-photon Stokes parameters only for the SU(2) orbits of photon-number states. The states that lack this property are entangled in every basis of independent field modes, so this is a class of entanglement beyond the reach of SU(2) transformations.

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## 1. Introduction

From a quantum perspective there is the widespread idea that polarization is the spin of the photon. Accordingly, the polarization of a light beam should result from the combination of the spins of the photons it contains [1, 2]. This polarization-spin connection is reinforced when expressing polarization by the Stokes operators, since they are formally equivalent to an angular momentum [3].

In this work we examine in more detail the representation of two-mode polarization as the superposition of the spin of individual photons. To this end in Sec. 2 we express any  $n$ -photon state as the result of the action on the vacuum of  $n$  creation operators, generating photons with many different polarization states in general. Then, in Sec. 3 we investigate whether the Stokes parameters of any  $n$ -photon state is the sum of the Stokes parameters of the  $n$  individual photons that appear in the expression derived in Sec. 2. We find that this is true only for the SU(2) orbits of photon-number states. We provide also a simple criterion to determine whether a given state is in the SU(2) orbit of a photon-number state via the Stokes-operators covariance matrix. We illustrate these results with some relevant examples.

## 2. Quantum polarization of a two-mode field

In a typical mode decomposition of the transverse electromagnetic field in terms of plane waves of wave-vector  $\mathbf{k}$  we have

$$\mathbf{E}(\mathbf{r}, t) \propto \sum_{\mathbf{k}, \pm} a_{\mathbf{k}, \pm} \boldsymbol{\varepsilon}_{\mathbf{k}, \pm} e^{i(\mathbf{k}\mathbf{r} - \omega_{\mathbf{k}}t)}, \quad (1)$$

where the complex two-dimensional vectors  $\boldsymbol{\varepsilon}_{\mathbf{k}, \pm}$  with  $\boldsymbol{\varepsilon}_{\mathbf{k}, +}^* \cdot \boldsymbol{\varepsilon}_{\mathbf{k}, -} = 0$  express the vibration state of the field mode with complex amplitude operator  $a_{\mathbf{k}, \pm}$ , the subscript  $\pm$  representing circular polarization for example. Throughout we will consider a two-mode approach with a single  $\mathbf{k}$ , so no subscript  $\mathbf{k}$  will be necessary from now on.

In the most general terms quantum polarization is addressed in terms of the Stokes operators [3]

$$\begin{aligned} S_0 &= a_+^\dagger a_+ + a_-^\dagger a_-, & S_z &= a_+^\dagger a_+ - a_-^\dagger a_-, \\ S_x &= a_-^\dagger a_+ + a_-^\dagger a_+^\dagger, & S_y &= i(a_-^\dagger a_+ - a_- a_+^\dagger), \end{aligned} \quad (2)$$

so that the Stokes parameters are their mean values  $s_j = \langle S_j \rangle$ . The Stokes operators satisfy the commutation relations of an angular momentum

$$[S_x, S_y] = 2iS_z, \quad (3)$$

and its cyclic permutations, with

$$\mathbf{S}^2 = S_0(S_0 + 2), \quad [\mathbf{S}, S_0] = \mathbf{0}, \quad (4)$$

where  $\mathbf{S} = (S_x, S_y, S_z)$ . Thus there is a complete formal equivalence between the subspace  $\mathcal{H}_n$  of fixed total photon number  $n$  with an spin  $j = n/2$ . In particular, a single photon  $n = 1$  is equivalent to an spin  $1/2$ . The Stokes operators are also the infinitesimal generators of SU(2) transformations [4, 5]

$$U = \exp(i\theta \mathbf{u} \cdot \mathbf{S}/2), \quad (5)$$

with  $\theta$  a real parameter, and  $\mathbf{u}$  a unit three-dimensional real vector. These are all linear, energy preserving transformations of the field amplitudes, embracing very fundamental optical operations such as lossless beam splitters, phase plates, and all linear interferometers [6]. It can be seen that the action of  $U$  on  $\mathbf{S}$  is a rotation  $R$  of angle  $\theta$  and axis  $\mathbf{u}$  [5]

$$U^\dagger \mathbf{S} U = R\mathbf{S}, \quad (6)$$

where  $R^t R = R R^t = I$ , the superscript  $t$  denotes matrix transposition, and  $I$  is the  $3 \times 3$  identity matrix. Throughout, by SU(2) invariance we mean that two field states connected by a SU(2) transformation are fully equivalent concerning polarization statistics, leaving aside their mean polarization state. Equivalently, SU(2) invariance means that the conclusions which one could draw are independent of which polarization basis one chooses.

In order to link quantum field states with individual-photon properties we demonstrate in the Appendix A that any state with  $n$  photons  $|\psi_n\rangle \in \mathcal{H}_n$  can be expressed as the result of the action on the vacuum of  $n$  creation operators generating photons that in general will have many with different polarization states. This is

$$|\psi_n\rangle = \mathcal{N} \prod_{m=1}^n a_m^\dagger |0, 0\rangle, \quad (7)$$

where  $\mathcal{N}$  is a normalization constant,  $|0, 0\rangle$  is the two-mode vacuum state, and the complex amplitude operators  $a_m$  are

$$a_m = \cos \theta_m a_+ + e^{-i\phi_m} \sin \theta_m a_-, \quad (8)$$

where  $\theta_m$  and  $\phi_m$  are independent parameters. In appendix A we show the close relation of expressions (7) and (8) with the Majorana representation of spins [7].

The action of each creation operator  $a_m^\dagger$  on the vacuum generates the single-photon pure state

$$|\varepsilon_m\rangle = a_m^\dagger |0, 0\rangle = \cos \theta_m |1, 0\rangle + e^{i\phi_m} \sin \theta_m |0, 1\rangle, \quad (9)$$

where  $|n_+, n_-\rangle$  are the photon-number states of modes  $a_\pm$ . These states  $|\varepsilon_m\rangle$  are actually SU(2) coherent states [5, 8]. Their Stokes parameters are

$$\begin{aligned} s_z &= \cos(2\theta_m), & s_x &= \sin(2\theta_m) \cos \phi_m, \\ s_y &= \sin(2\theta_m) \sin \phi_m, & s_0 &= 1, \end{aligned} \quad (10)$$

so they reach the maximum of the standard definition of degree of polarization,  $\mathcal{P} = |s|/s_0 = 1$ . Thus, we may say that  $a_m^\dagger$  creates photons with the polarization state expressed by the two-dimensional complex vector  $\varepsilon_m$

$$\varepsilon_m = \cos \theta_m \varepsilon_+ + e^{i\phi_m} \sin \theta_m \varepsilon_-. \quad (11)$$

Note that in general the polarization states of the photons appearing in Eqs. (7) and (9) are not orthogonal,  $\varepsilon_\ell^\dagger \cdot \varepsilon_m \neq 0$  for  $\ell \neq m$ .

### 3. Polarization versus single-photon spins

The question to be addressed is whether the Stokes parameters of a  $n$ -photon state are the sum of the Stokes parameters of the individual photons:

$$\langle \mathbf{S} \rangle_n = \sum_{m=1}^n \langle \mathbf{S} \rangle_{1,m}, \quad (12)$$

or, equivalently,

$$\langle S_z \rangle_n = \sum_{m=1}^n \langle S_z \rangle_{1,m}, \quad \langle a_+^\dagger a_- \rangle_n = \sum_{m=1}^n \langle a_+^\dagger a_- \rangle_{1,m}, \quad (13)$$

where  $\langle \mathbf{S} \rangle_n$  are the Stokes parameters in the  $n$ -photon state  $|\psi_n\rangle \in \mathcal{H}_n$  in Eq. (7), while  $\langle \mathbf{S} \rangle_{1,m}$  are the Stokes parameters (10) of the corresponding one-photon states  $|\varepsilon_m\rangle$  in Eq. (9). In the transit from Eqs. (12) to (13) we have replaced a pair of real equations for  $S_x$  and  $S_y$  by a single complex equation for  $S_x + iS_y = 2a_+^\dagger a_-$ .

We are going to demonstrate the following proposition: The property (12) holds exclusively for the SU(2) orbits of the photon-number states  $|n_+, n_-\rangle$ , i. e., for every state of the form  $U|n_+, n_-\rangle$ , where  $U$  is any SU(2) unitary transformation (5).

### 3.1. Proof of the proposition

The proposition can be demonstrated via induction, starting with the simplest nontrivial case with two photons  $n = 2$ .

#### 3.1.1. Two photons

Let us take full advantage of the SU(2) invariance stated above considering without loss of generality the properly normalized two-photon state

$$|\psi_2\rangle = \mathcal{N} a_\alpha^\dagger \left( \cos \theta a_\alpha^\dagger + e^{i\phi} \sin \theta a_\beta^\dagger \right) |0, 0\rangle, \quad (14)$$

this is

$$|\psi_2\rangle = \frac{1}{\sqrt{1 + \cos^2 \theta}} \left( \sqrt{2} \cos \theta |2, 0\rangle + e^{i\phi} \sin \theta |1, 1\rangle \right), \quad (15)$$

where  $a_{\alpha,\beta}$  are two arbitrary field modes with orthogonal polarizations  $\varepsilon_\alpha^* \cdot \varepsilon_\beta = 0$ , and  $|n, m\rangle$  the associated photon-number basis. On the other hand, the corresponding single-photon states are, in the same photon-number basis,

$$|\varepsilon_1\rangle = |1, 0\rangle, \quad |\varepsilon_2\rangle = \cos \theta |1, 0\rangle + e^{i\phi} \sin \theta |0, 1\rangle. \quad (16)$$

After replacing  $a_{+,-}$  by  $a_{\alpha,\beta}$  the equalities (13) read,

$$\frac{4 \cos^2 \theta}{1 + \cos^2 \theta} = 2 \cos^2 \theta, \quad \frac{2 e^{i\phi} \cos \theta \sin \theta}{1 + \cos^2 \theta} = e^{i\phi} \cos \theta \sin \theta. \quad (17)$$

It can be easily seen that these equalities are satisfied only when  $\theta = 0, \pi/2$  modulus  $\pi$ , for any  $\phi$ . This means that the photons must have either the same polarization state,  $\varepsilon_1 = \varepsilon_2 = \varepsilon_\alpha$  for  $\theta = 0$ , or orthogonal polarization states for  $\theta = \pi/2$ ,  $\varepsilon_1 = \varepsilon_\alpha$ ,  $\varepsilon_2 = \varepsilon_\beta$ . In other words, Eq. (12) holds just for the SU(2) orbits of the number states  $|2, 0\rangle$  and  $|1, 1\rangle$ .

#### 3.1.2. $n+1$ photons

Now we assume that Eqs.(13) hold for an state  $|\psi_n\rangle$  with  $n$  photons, this is  $|\psi_n\rangle = |n_+, n_-\rangle$ , modulus SU(2) transformations. Then we add another photon in an arbitrary polarization state

$$|\psi_{n+1}\rangle = \mathcal{N} \left( \cos \theta a_+^\dagger + e^{i\phi} \sin \theta a_-^\dagger \right) |n_+, n_-\rangle, \quad (18)$$

this is

$$|\psi_{n+1}\rangle = \mathcal{N} \left( \sqrt{n_+ + 1} \cos \theta |n_+ + 1, n_-\rangle + \sqrt{n_- + 1} e^{i\phi} \sin \theta |n_+, n_- + 1\rangle \right), \quad (19)$$

where  $\mathcal{N} = 1/\sqrt{1 + n_+ \cos^2 \theta + n_- \sin^2 \theta}$ . Next we examine whether Eqs. (13) hold as

$$\langle S_z \rangle_{n+1} = \langle S_z \rangle_n + \langle S_z \rangle_\varepsilon, \quad \langle a_+^\dagger a_- \rangle_{n+1} = \langle a_+^\dagger a_- \rangle_n + \langle a_+^\dagger a_- \rangle_\varepsilon, \quad (20)$$

where the subscripts  $n+1$ ,  $n$ , and  $\varepsilon$  refer to the states  $|\psi_{n+1}\rangle$ ,  $|\psi_n\rangle$ , and  $|\varepsilon\rangle = \cos \theta |1, 0\rangle + e^{i\phi} \sin \theta |0, 1\rangle$ , respectively. An straightforward calculation implies that Eqs. (20) are equivalent to

$$\frac{(n_+ + 1)(n_+ - n_- + 1) \cos^2 \theta + (n_+ + 1)(n_+ - n_- - 1) \sin^2 \theta}{1 + n_+ \cos^2 \theta + n_- \sin^2 \theta} = n_+ - n_- + \cos^2 \theta - \sin^2 \theta, \quad (21)$$

$$\frac{(n_+ + 1)(n_+ - 1) e^{i\phi} \sin \theta \cos \theta}{1 + n_+ \cos^2 \theta + n_- \sin^2 \theta} = e^{i\phi} \sin \theta \cos \theta.$$

It can be easily seen that these equalities are satisfied only when  $\theta = 0, \pi/2$  modulus  $\pi$ , for any  $\phi$ , so that either  $|\psi_{n+1}\rangle = |n_+ + 1, n_-\rangle$ , or  $|\psi_{n+1}\rangle = |n_+, n_- + 1\rangle$ .

This completes the proof of the proposition. This is that the polarization-sum property (12) holds only for the SU(2) orbits of all number states. These are the states that result from the addition to the vacuum of photons either with the same or orthogonal polarization states.

### 3.2. Sum property and covariance matrix

Let us provide a simple criterion to determine whether a given state satisfies the sum property (12) or not. We demonstrate that property (12) holds if and only if  $\det M = 0$ , where  $M$  is the covariance matrix of Stokes-operators [4]

$$M_{\ell,m} = \frac{1}{2} (\langle S_\ell S_m \rangle + \langle S_m S_\ell \rangle) - \langle S_\ell \rangle \langle S_m \rangle. \quad (22)$$

The diagonal elements of  $M$  are the variances of the Stokes operators  $\mathbf{S}$ , while the variance of any other Stokes component  $S_u = \mathbf{u} \cdot \mathbf{S}$  is computed as  $(\Delta S_u)^2 = \mathbf{u}^t M \mathbf{u}$ , where  $\mathbf{u}$  is any unit three-dimensional real vector.

The states satisfying Eq. (12) are SU(2) transforms of the eigenstates of  $S_z$ , which are  $|n_+, n_-\rangle$ . The number states  $|n_+, n_-\rangle$  have the covariance matrix

$$M = (n_+ + n_- + 2n_+n_-) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (23)$$

with  $\det M = 0$ . Under SU(2) transformations (5) we have  $M \rightarrow R^t M R$ , so that the determinant is preserved  $\det(R^t M R) = 0$ . Thus, if the state satisfies Eq. (12) then  $\det M = 0$ .

The reverse is also true. If the state has  $\det M = 0$  then  $M$  has a vanishing eigenvalue, say  $M \mathbf{u} = 0$ , and the variance of the corresponding Stokes component  $S_u = \mathbf{u} \cdot \mathbf{S}$  vanish  $\Delta S_u = 0$ . Since  $S_u$  and  $S_z$  can be always related by an SU(2) transformation we get that the states with  $\det M = 0$  are SU(2) transforms of the eigenstates of  $S_z$ , so that the sum property (12) is fulfilled.

### 3.3. Sum property and entanglement

Let us note that for two-mode field states with exactly  $n$  photons the only states that factorize as product of single-mode states are the number states  $|n, m\rangle$  for any polarization-orthogonal mode basis. This is to say that all the states that satisfy the sum property (12) can be rendered factorized by an SU(2) transformation.

The other way round, the states that lack property (12) are entangled states of  $n$  photons that cannot be rendered factorized by any choice of polarization-orthogonal mode basis. This means that condition (12) reveals a definite class of entanglement beyond the reach of devices performing SU(2) transformations.

### 3.4. Examples

Let us consider three relevant examples.

#### 3.4.1. SU(2) coherent states

All the SU(2) coherent states satisfy property (12) since they can be actually defined as the SU(2) orbit of the number states  $|n, 0\rangle$  [5]. This is to say that all the photons are in the same polarization state. They can be regarded as the output of an ideal polarizer since we can always find a mode which is in the vacuum state. Moreover, the SU(2) coherent states are considered as the most classical states regarding spin properties [8].

### 3.4.2. Twin-number states

On the other hand, we can consider a typical example of nonclassical states satisfying property (12) as the orbits of the twin-photon number states  $U|n, n\rangle$  [9]. In this case half of the photons are in one polarization state while the other half are in the orthogonal polarization state. These states have found a lot of attention by their good properties in quantum metrology and they can be regarded as the limiting case of large SU(2) squeezing [10].

### 3.4.3. N00N states

Finally let us consider a relevant family of nonclassical states that do not satisfy property (12). These are the SU(2) orbits of the so-called N00N states, which are proper examples of Schrödinger-cat states [11, 12, 13]

$$|\psi_n\rangle = \frac{1}{\sqrt{2}}(|n, 0\rangle - |0, n\rangle), \quad n > 2, \quad (24)$$

with covariance matrix [4]

$$M = \begin{pmatrix} n & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n^2 \end{pmatrix}, \quad (25)$$

so that  $\det M \neq 0$  as expected. Following the program outlined in the Appendix the factorized form (7) of these states is

$$|\psi_n\rangle \propto \prod_{m=1}^n \left( a_+^\dagger - e^{i2\pi m/n} a_-^\dagger \right) |0, 0\rangle. \quad (26)$$

This is to say that there are no two photons in the same polarization state. All them have the same  $\theta_m = \pi/4$  but different  $\phi_m = 2\pi m/n$ .

## 4. Conclusions

We have shown that every state can be regarded as the result of the action on the vacuum of creation operators generating photons with different spin states. Then we have found the states whose polarization Stokes vector results from the sum of the spins of the individual photons it contains. These are the SU(2) orbits of number states and correspond to the addition of photons either in the same or in orthogonal polarization modes. This is that the Stokes vectors of all the photons are either parallel or antiparallel. Moreover, we have shown that the states that lack such sum property have a distinguished entanglement behavior since they are entangled for every choice of field modes. This is entanglement that cannot be reached from factorized states via SU(2) transformations.

### A. Multi-photon states as photon-added states

Let us demonstrate that every  $n$ -photon state  $|\psi_n\rangle \in \mathcal{H}_n$  can be expressed in the form (up to a normalization constant)

$$|\psi_n\rangle \propto \prod_{m=1}^n a_m^\dagger |0, 0\rangle, \quad (27)$$

where  $a_m$  are in Eq. (8). This is equivalent to say that there are  $k$  complex number  $\xi_m$  such that

$$|\psi_n\rangle \propto a_-^{\dagger n-k} \prod_{m=1}^k \left( a_+^\dagger - \xi_m a_-^\dagger \right) |0, 0\rangle, \quad (28)$$

where  $\xi_m = -e^{i\phi_m} \tan \theta_m$ , and we have singled out the potential  $n - k$  photons with  $\cos \theta_m = 0$ , so that all the  $k$  parameters  $\xi_m$  are finite.

The existence and uniqueness of factorization (28) can be demonstrated by projecting  $|\psi_n\rangle$  on the two-mode Glauber coherent states  $|\alpha_+, \alpha_-\rangle$ , with  $a_\pm|\alpha_+, \alpha_-\rangle = \alpha_\pm|\alpha_+, \alpha_-\rangle$ ,

$$\langle\alpha_+, \alpha_-|\psi_n\rangle \propto \alpha_-^{*n} \prod_{m=1}^k (x - \xi_j) e^{-\frac{|\alpha_+|^2 + |\alpha_-|^2}{2}}, \quad x = \frac{\alpha_+^*}{\alpha_-^*}. \quad (29)$$

On the other hand, every  $|\psi_n\rangle$  can be expressed in the photon number basis as

$$|\psi_n\rangle = \sum_{m=0}^k c_m |n_+ = m, n_- = n - m\rangle, \quad (30)$$

for suitable  $c_m$  and  $k$ . Using that each photon-number state  $|n\rangle$  can be expressed as the  $n$ -times action on the vacuum state  $|0\rangle$  of the corresponding creation operator

$$|n\rangle = \frac{1}{\sqrt{n!}} a^\dagger{}^n |0\rangle, \quad (31)$$

we get

$$|\psi_n\rangle = \sum_{m=0}^k \frac{c_m}{\sqrt{m!(n-m)!}} (a_+^\dagger)^m (a_-^\dagger)^{n-m} |0, 0\rangle. \quad (32)$$

Projecting from the left on Glauber coherent states  $|\alpha_+, \alpha_-\rangle$  we have

$$\langle\alpha_+, \alpha_-|\psi_n\rangle = \sum_{m=0}^k \frac{c_m}{\sqrt{m!(n-m)!}} \alpha_+^{*m} \alpha_-^{*n-m} e^{-\frac{|\alpha_+|^2 + |\alpha_-|^2}{2}}, \quad (33)$$

so that after extracting a common factor  $\alpha_-^{*n}$  we get

$$\langle\alpha_+, \alpha_-|\psi_n\rangle = \alpha_-^{*n} P(x) e^{-\frac{|\alpha_+|^2 + |\alpha_-|^2}{2}}, \quad (34)$$

where

$$P(x) = \sum_{m=0}^k \frac{c_m}{\sqrt{m!(n-m)!}} x^m, \quad x = \frac{\alpha_+^*}{\alpha_-^*}. \quad (35)$$

The key point is that for any  $n$ -photon state  $\langle\alpha_+, \alpha_-|\psi_n\rangle$  is a complex polynomial of the complex variable  $\alpha_+^*/\alpha_-^*$  of degree  $k \leq n$ . Thus, comparing Eqs. (29) and (34), the equality in Eq. (27) is the standard factorization of a complex polynomial  $P(x)$  in terms of its  $k$  roots  $\xi_m$ , maybe degenerate. Thus, the factorization in Eq. (27) always exists and is unique.

It is worth noting that this way of expressing quantum states in Eqs. (27) and (28) is actually equivalent to the Majorana representation of angular momentum states [7], where we can take advantage of the formal similarity between the subspace  $\mathcal{H}_n$  of  $n$  photons and an angular momentum  $j = n/2$ , and also to the fully symmetric states of  $n$  qubits. In the Majorana approach angular-momentum states are represented by the zeros of the wave-function in the coherent-state basis. These are the zeros  $\xi_m$  of  $\langle\alpha_+, \alpha_-|\psi_n\rangle$  in Eqs. (29) or (34), sometimes referred to as vortices, or constellation of Majorana stars. This representation is currently being used in quantum information science [14, 15, 16], and other areas [17, 18].

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