

L^∞ a-priori estimates for subcritical semilinear elliptic equations with a Carathéodory nonlinearity

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Abstract

We present new L^∞ *a priori* estimates for weak solutions of a wide class of subcritical elliptic equations in bounded domains. No hypotheses on the sign of the solutions, neither of the non-linearities are required. This method is based in Gagliardo-Nirenberg and Caffarelli-Kohn-Nirenberg interpolation inequalities.

Let us consider a semilinear boundary value problem $-\Delta u = f(x, u)$, in Ω , with Dirichlet boundary conditions, where $\Omega \subset \mathbb{R}^N$, with $N > 2$, is a bounded smooth domain, and f is a subcritical Carathéodory non-linearity. We provide L^∞ a priori estimates for weak solutions, in terms of their L^{2^*} -norm, where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent.

By a subcritical non-linearity we mean, for instance, $|f(x, s)| \leq |x|^{-\mu} \tilde{f}(s)$, where $\mu \in (0, 2)$, and $\tilde{f}(s)/|s|^{2_\mu^*-1} \rightarrow 0$ as $|s| \rightarrow \infty$, here $2_\mu^* := \frac{2(N-\mu)}{N-2}$ is the critical Sobolev-Hardy exponent. Our non-linearities includes non-power non-linearities.

In particular we prove that when $f(x, s) = |x|^{-\mu} \frac{|s|^{2_\mu^*-2} s}{[\log(e+|s|)]^\beta}$, with $\mu \in [1, 2)$, then, for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that for any solution $u \in H_0^1(\Omega)$, the following holds

$$\left[\log(e + \|u\|_\infty) \right]^\beta \leq C_\varepsilon \left(1 + \|u\|_{2^*} \right)^{(2_\mu^*-2)(1+\varepsilon)}.$$

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1 Introduction

Let us consider the following semilinear boundary value problem

$$-\Delta u = f(x, u), \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega, \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$, $N > 2$, is a bounded, connected, open subset with C^2 boundary $\partial\Omega$, and the non-linearity $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is Carathéodory function (that is, the mapping $f(\cdot, s)$ is measurable for all $s \in \mathbb{R}$, and the mapping $f(x, \cdot)$ is continuous for almost all $x \in \Omega$), and *subcritical* (see definition 1.1).

We analyze the effect of the smoothness of the non-linearity $f = f(x, u)$ on the $L^\infty(\Omega)$ a priori estimates of *weak solutions* to (1.1). Degree theory combined with a priori bounds in the sup-norm of solutions of parametrized versions of (1.1), is a very classical topic in elliptic equations, posed by Leray and Schauder in [18]. It provides a great deal of information about existence of solutions and the structure of the solution set. This study is usually focused on positive classical solutions, see the classical references of de Figueiredo-Lions-Nussbaum, and of Gidas-Spruck [11, 14], see also [7, 8].

A natural question concerning the class of solutions is the following one,

(Q1) *those $L^\infty(\Omega)$ estimates can be extended to a bigger class of solutions, in particular to weak solutions (with possibly changing sign solutions)?*

Another natural question with respect to the class of non-linearities, can be stated as follows,

(Q2) *those estimates can be extended to a bigger class of non-linearities, in particular to non-smooth non-linearities (with possibly changing sign weights)?*

In this paper we provide sufficient conditions guarantying uniform $L^\infty(\Omega)$ a priori estimates for any $u \in H_0^1(\Omega)$ weak solution to (1.1), in terms of their

$L^{2^*}(\Omega)$ bounds, in the class of Caratheodory subcritical generalized problems. In this class, we state that any set of weak solutions uniformly $L^{2^*}(\Omega)$ a priori bounded is universally $L^\infty(\Omega)$ a priori bounded. Our theorems allow changing sign weights, and singular weights, and also apply to changing sign solutions.

Problem (1.1) with $f(x, s) = |x|^{-\mu}|s|^{p-1}s$, $\mu > 0$, is known as Hardy's problem, due to its relation with the Hardy-Sobolev inequality. The Caffarelli-Kohn-Nirenberg interpolation inequality for radial singular weights [4], states that whenever $0 \leq \mu \leq 2$,

$$2_\mu^* := \frac{2(N - \mu)}{N - 2}, \quad (1.2)$$

is the *critical* exponent of the Hardy-Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2_\mu^*}(\Omega, |x|^{-\mu})$. Using variational methods, one obtains the existence of a nontrivial solution to (1.1) in $H_0^1(\Omega)$ whenever $1 < p < 2_\mu^* - 1$. For the case $0 < \mu < 2$, using a Pohozaev type identity, we have that for $p \geq 2_\mu^* - 1$ there is no solution to Hardy's problem in star-shaped domains with respect to the origin. But, there exist positive solutions for the problem with $p = 2_\mu^* - 1$ depending on the geometry of the domain Ω , see [16] and [5].

If $\mu \geq 2$, it is known that Hardy's problem has no positive solution in any domain Ω containing the origin, see [13], [1, Lemma 6.2], [12].

Usually the term subcritical non-linearity is reserved for power like non-linearities. Next, we expand this concept:

Definition 1.1. *By a subcritical non-linearity we mean that f satisfies one of the following two growth conditions:*

(H0)

$$|f(x, s)| \leq |x|^{-\mu} \tilde{f}(s), \quad (1.3)$$

where $\mu \in (0, 2)$, and $\tilde{f} : \mathbb{R} \rightarrow [0, +\infty)$ is continuous and satisfy

$$\tilde{f}(s) > 0 \text{ for } |s| > s_0, \quad \text{and} \quad \lim_{|s| \rightarrow \infty} \frac{\tilde{f}(s)}{|s|^{2_\mu^* - 1}} = 0;^1 \quad (1.4)$$

or

¹Observe that $2_\mu^* > 2$ for $\mu \in (0, 2)$. Let $a(x) = |x|^{-\mu}$, $a \in L^p(\Omega)$ for any $p < N/\mu$. Moreover, the sharp Caffarelli-Kohn-Nirenberg inequality implies that if $u \in H_0^1(\Omega)$, then $f(\cdot, u) \in L^{\frac{2N}{N+2}}(\Omega)$

(H0)'

$$|f(x, s)| \leq |a(x)| \tilde{f}(s) \quad (1.5)$$

where $a \in L^r(\Omega)$ with $r > N/2$, and $\tilde{f} : \mathbb{R} \rightarrow [0, +\infty)$ is continuous and satisfy

$$\tilde{f}(s) > 0 \text{ for } |s| > s_0, \quad \text{and} \quad \lim_{s \rightarrow \pm\infty} \frac{\tilde{f}(s)}{|s|^{2_{N/r}^* - 1}} = 0, \quad (1.6)$$

where

$$2_{N/r}^* := \frac{2^*}{r'} = 2^* \left(1 - \frac{1}{r}\right), \quad (1.7)$$

and where r' is the conjugate exponent of r , $1/r + 1/r' = 1$.²

Our analysis shows that non-linearities satisfying either (H0): (1.3)-(1.4) (either (H0)': (1.5)-(1.6)), widen the class of subcritical non-linearities to non-power non-linearities, sharing with power like non-linearities properties such as L^∞ a priori estimates. Our definition of a subcritical non-linearity includes non-linearities such as

$$f^{(1)}(x, s) := \frac{|x|^{-\mu} |s|^{2_\mu^* - 2} s}{[\log(e + |s|)]^\alpha}, \quad \text{or} \quad f^{(2)}(x, s) := \frac{a(x) |s|^{2_{N/r}^* - 2} s}{[\log[e + \log(1 + |s|)]]^\alpha},$$

for any $\alpha > 0$, and $\mu \in (0, 2)$, or any $a \in L^r(\Omega)$, with $r > N/2$.

In particular, if $f(x, s) = f^{(1)}(x, s)$ with $\mu \in [1, 2)$, then for any $\varepsilon > 0$ there exists a constant $C > 0$ depending on ε , μ , N , and Ω , such that for any $u \in H_0^1(\Omega)$ solution to (1.1), the following holds:

$$\left[\log(e + \|u\|_\infty) \right]^\alpha \leq C \left(1 + \|u\|_{2^*}\right)^{(2_\mu^* - 2)(1 + \varepsilon)},$$

and where C is independent of the solution u , see Theorem 3.2. Related results concerning those non-power non-linearities can be found in [10] for the

²Since $r > N/2$, then $2_{N/r}^* > 2$. Moreover, thanks to Sobolev embeddings, for any $u \in H_0^1(\Omega)$,

$$\tilde{f}(u) \in L^{\frac{2^*}{2_{N/r}^* - 1}}(\Omega) \quad \text{with} \quad \frac{2_{N/r}^* - 1}{2^*} = \frac{1}{2} + \frac{1}{N} - \frac{1}{r},$$

and $f(\cdot, u) \in L^{\frac{2N}{N+2}}(\Omega)$.

p -laplacian case, [9] analyzing what happen when $\alpha \rightarrow 0$, [19] for systems, [25] for the radial case, and [23, 24] for a summary.

Moreover, if $f(x, s) = f^{(2)}(x, s)$ with $a \in L^r(\Omega)$ for $r \in (N/2, N]$, then for any $\varepsilon > 0$ there exists a constant $C > 0$ depending only on ε , Ω , r and N such that for any $u \in H_0^1(\Omega)$ solution to (1.1), the following holds:

$$\left[\log [e + \log (1 + \|u\|_\infty)] \right]^\alpha \leq C \|a\|_r^{1+\varepsilon} \left(1 + \|u\|_{2^*}\right)^{(2_{N/r}^* - 2)(1+\varepsilon)},$$

where C is independent of the solution u , see Theorem 2.2.

Definition 1.2. *By a solution we mean a weak solution $u \in H_0^1(\Omega)$ such that $f(\cdot, u) \in L^{\frac{2N}{N+2}}(\Omega)$, and*

$$\int_{\Omega} \nabla u \nabla \varphi = \int_{\Omega} f(x, u) \varphi, \quad \forall \varphi \in H_0^1(\Omega). \quad (1.8)$$

By an estimate of Brezis-Kato [3], based on Moser's iteration technique [21], and according to elliptic regularity, any weak solution to (1.1) with a Caratheodory subcritical non-linearity is a continuous function, and in fact is an strong solution, see Lemma 2.1 and Lemma 3.1.

Joseph and Lundgren in [17] show us that those L^∞ a priori estimates are not applicable for L^1 - weak solutions, neither to super-critical non-linearities.

Definition 1.3. *We will say that a function u is an L^1 - weak solution to (1.1) if*

$$u \in L^1(\Omega), \quad f(\cdot, u) \delta_\Omega \in L^1(\Omega)$$

where $\delta_\Omega(x) := \text{dist}(x, \partial\Omega)$ is the distance function with respect to the boundary, and

$$\int_{\Omega} \left(u \Delta \varphi + f(x, u) \varphi \right) dx = 0, \quad \text{for all } \varphi \in C^2(\overline{\Omega}), \quad \varphi|_{\partial\Omega} = 0.$$

They posed the study of singular solutions. Working on non-linearities such as $f(s) := e^s$ or $f(s) := (1 + s)^p$, they consider the following BVP depending on a multiplicative parameter $\lambda \in \mathbb{R}$,

$$- \Delta u = \lambda f(u), \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega, \quad (1.9)$$

and look for classical radial positive solutions in the unit ball B_1 . They obtain singular solutions as limit of classical solutions.

In particular, they obtain the explicit weak solution

$$u_1^*(x) := \log \frac{1}{|x|^2}, \quad u_1^* \in H_0^1(B_1),$$

to (1.10), when $N > 2$, $\lambda = 2(N - 2)$, and $f(s) := e^s$, see [17, p. 262].

They also found the explicit L^1 -weak solution

$$u_2^*(x) := \left(\frac{1}{|x|} \right)^{\frac{2}{p-1}} - 1, \quad \text{with } p > \frac{N}{N-2}, \quad N > 2, \quad u_2^* \in W_0^{1, \frac{N}{N-1}}(B_1),$$

to (1.10), where $f(s) := (1 + s)^p$, and $\lambda = \frac{2}{p-1}(N - \frac{2p}{p-1}) > 0$, see [17, (III.a)]. It holds that $u_2^* \in H_0^1(B_1)$ only when $p > 2^* - 1$. So, in the subcritical range u_2^* is a singular L^1 -weak solution, not in H^1 .

Let us focus on BVP with radial singular weights,

$$-\Delta u = \lambda |x|^{-\mu} (1 + u)^p, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega, \quad (1.10)$$

with $N > 2$, $\mu < 2$ and $p > 1$. It can be checked that

$$u_3^*(x) := \left(\frac{1}{|x|} \right)^{\frac{2-\mu}{p-1}} - 1, \quad \text{with } p > \frac{N - \mu}{N - 2}, \quad u_3^* \in W_0^{1, \frac{N}{N-1}}(B_1),$$

and u_3^* is an L^1 -weak solution to (1.10), for $\lambda = \frac{2-\mu}{p-1}(N - 2 - \frac{2-\mu}{p-1}) > 0$. It also holds that $u_3^* \in H_0^1(B_1)$ only when $p > 2_\mu^* - 1$. So, in the subcritical range u_3^* is a singular L^1 -weak solution to (1.10), not in H^1 .

Those examples of radially symmetric singular solutions to BVP's on spherical domains, solve either super-critical problems (u_1^*) or are L^1 -weak solutions not in $H_0^1(\Omega)$ (u_2^* and u_3^*). Consequently, we restrict our study for $u \in H_0^1(\Omega)$ *weak solutions* to (1.1), in the class of subcritical generalized problems. It is natural to ask for uniform L^∞ a priori estimates over non power non-linearities in non-spherical domains.

To state our main results, for a non-linearity f satisfying (1.3)-(1.4), let us define

$$h(s) = h_\mu(s) := \frac{|s|^{2_\mu^* - 1}}{\max_{\{-s, s\}} \tilde{f}}, \quad \text{for } |s| > s_0. \quad (1.11)$$

And for a non-linearity f satisfying (1.5)-(1.6), let us now define

$$h(s) = h_{N/r}(s) := \frac{|s|^{2_{N/r}^* - 1}}{\max\{\tilde{f}(-s), \tilde{f}(s)\}} \quad \text{for } |s| > s_0. \quad (1.12)$$

By sub-criticality, (see (1.4) or (1.6) respectively),

$$h(s) \rightarrow \infty \quad \text{as } s \rightarrow \infty. \quad (1.13)$$

Let u be a solution to (1.1). We estimate $h(\|u\|_\infty)$, in terms of its L^{2^*} -norm. This result is robust, and holds for positive, negative and changing sign non-linearities, and also for positive, negative and changing sign solutions.

As an immediate consequence, as soon as we have a universal *a priori* L^{2^*} - norm for weak solutions in $H_0^1(\Omega)$, then solutions are a priori universally bounded in the L^∞ - norm, see Corollary 2.3.

This paper is organized in the following way. In Section 2, using Gagliardo–Nirenberg inequality, we analyze the case when $a \in L^r(\Omega)$ with $r > N/2$, see Theorem 2.2. In Section 3, we analyze the more involved case of a radial singular weight, see Theorem 3.2. It needs the Caffarelli-Kohn-Nirenberg inequality.

2 Carathéodory non-linearities

In this section, assuming that f satisfy the subcritical growth condition, we state our first main result concerning Carathéodory non-linearities, see Theorem 2.2.

We first collect a regularity Lemma for any weak solution to (1.1) with a non-linearity of polynomial critical growth.

Lemma 2.1 (Improved regularity). *Assume that $u \in H_0^1(\Omega)$ weakly solves (1.1) for a Carathéodory non-linearity $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ with polynomial critical growth*

$$|f(x, s)| \leq |a(x)|(1 + |s|^{2_{N/r}^* - 1}), \quad \text{with } a \in L^r(\Omega), \quad N/2 < r \leq \infty. \quad (2.1)$$

Then, the following hold:

- (i) If $r < N$, then $u \in C^\nu(\overline{\Omega}) \cap W^{2,r}(\Omega)$ for $\nu = 2 - \frac{N}{r} \in (0, 1)$.
- (ii) If $r = N$, then $u \in C^\nu(\overline{\Omega}) \cap W^{2,r}(\Omega)$ for any $\nu < 1$.
- (iii) If $N < r < \infty$, then $u \in C^{1,\nu}(\overline{\Omega}) \cap W^{2,r}(\Omega)$ for $\nu = 1 - \frac{N}{r} \in (0, 1)$.
- (iv) If $r = +\infty$, then $u \in C^{1,\nu}(\overline{\Omega}) \cap W^{2,p}(\Omega)$ for any $\nu < 1$ and any $p < \infty$.

Proof. Let $u \in H_0^1(\Omega)$ be a solution to (1.1). Since an estimate of Brezis-Kato [3], if

$$|f(x, u)| \leq b(x)(1 + |u|), \quad \text{with } 0 \leq b \in L^{N/2}(\Omega), \quad (2.2)$$

then, $u \in L^q(\Omega)$ for any $q < \infty$ (see [26, Lemma B.3]).

Assume that f satisfies (2.1), then assumption (2.2) is satisfied with

$$b(x) = \frac{|a(x)|(1 + |u|^{2_{N/r}^* - 1})}{1 + |u|} \leq C |a(x)|(1 + |u|^{2_{N/r}^* - 2}) \in L^{N/2}(\Omega).$$

Consequently, $u \in L^q(\Omega)$ for any $q < \infty$. The growth condition for f (see (2.1)), implies that $-\Delta u = f(x, u) \in L^p(\Omega)$ for any $p < r$. Thus, by the Calderon-Zygmund inequality (see [15, Theorem 9.14]), $u \in W^{2,p}(\Omega)$, for any $p \in (1, r)$.

- (i) Assume $r < N$. Choosing any $p \in (N/2, r)$, by Sobolev embeddings, $u \in W^{1,p^*}(\Omega)$, where $\frac{1}{p^*} := \frac{1}{p} - \frac{1}{N} < \frac{1}{N}$. Since $p^* > N$, $u \in C^\nu(\overline{\Omega})$ for any $\nu < 2 - \frac{N}{p}$. Now, from elliptic regularity $u \in C^{\nu_0}(\overline{\Omega}) \cap W^{2,r}(\Omega)$ for $\nu_0 = 2 - \frac{N}{p}$.
- (ii) Assume $r = N$. Choosing any $p \in (N/2, N)$, and reasoning as in (i), $u \in W^{1,p^*}(\Omega)$, where $\frac{1}{p^*} := \frac{1}{p} - \frac{1}{N} < \frac{1}{N}$. Also $u \in C^\nu(\overline{\Omega})$ for any $\nu < 1$. Now, from elliptic regularity $u \in C^\nu(\overline{\Omega}) \cap W^{2,r}(\Omega)$ for any $\nu < 1$.
- (iii) Assume $r > N$. Choosing any $p \in (N, r)$, and reasoning as above, $u \in C^{1,\nu_0}(\overline{\Omega}) \cap W^{2,r}(\Omega)$ for $\nu_0 = 1 - \frac{N}{r}$.
- (iv) Assume $r = +\infty$. Since elliptic regularity and Sobolev embeddings, $u \in C^{1,\nu}(\overline{\Omega}) \cap W^{2,p}(\Omega)$ for any $\nu < 1$ and any $p < \infty$.

□

2.1 Estimates of the L^∞ -norm of the solutions

We assume that the non-linearity f satisfies the growth condition (H0)', and that $\tilde{f} : \mathbb{R} \rightarrow (0, +\infty)$ satisfies the following hypothesis:

(H1) there exists a uniform constant $c_0 > 0$ such that

$$\limsup_{s \rightarrow +\infty} \frac{\max_{[-s,s]} \tilde{f}}{\max \{ \tilde{f}(-s), \tilde{f}(s) \}} \leq c_0.^3 \quad (2.3)$$

Under hypothesis (H0)'-(H1), we establish an estimate for the function h applied to the $L^\infty(\Omega)$ -norm of any $u \in H_0^1(\Omega)$ solution to (1.1), in terms of their $L^{2^*}(\Omega)$ -norm.

From now on, C denotes several constants that may change from line to line, and are independent of u .

Our first main results is the following theorem.

Theorem 2.2. *Assume that $f : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying (H0)'-(H1).*

Then, for any $u \in H_0^1(\Omega)$ weak solution to (1.1), the following holds:

- (i) *either there exists a constant $C > 0$ such that $\|u\|_\infty \leq C$, where C is independent of the solution u ,*
- (ii) *either for any $\varepsilon > 0$ there exists a constant $C > 0$ such that*

$$h(\|u\|_\infty) \leq C \|a\|_r^{A+\varepsilon} \left(1 + \|u\|_{2^*}\right)^{(2_{N/r}^* - 2)(A+\varepsilon)},$$

where h is defined by (1.12),

$$A := \begin{cases} 1, & \text{if } r \leq N, \\ 1 + \frac{2}{N} - \frac{2}{r}, & \text{if } r > N, \end{cases} \quad (2.4)$$

and C depends only on ε , c_0 (defined in (2.3)), r , N , and Ω , and it is independent of the solution u .

³Observe that in particular, if $\tilde{f}(s)$ is monotone, then (H1) is obviously satisfied with $c_0 = 1$.

As an immediate corollary, we prove that any sequence of solutions in $H_0^1(\Omega)$, uniformly bounded in the $L^{2^*}(\Omega)$ -norm, is also uniformly bounded in the $L^\infty(\Omega)$ -norm.

Corollary 2.3. *Let $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying (H0)'–(H1).*

Let $\{u_k\} \subset H_0^1(\Omega)$ be any sequence of solutions to (1.1) such that there exists a constant $C_0 > 0$ satisfying

$$\|u_k\|_{2^*} \leq C_0.$$

Then, there exists a constant $C > 0$ such that

$$\|u_k\|_\infty \leq C. \quad (2.5)$$

Proof. We reason by contradiction, assuming that (2.5) does not hold. So, at least for a subsequence again denoted as u_k , $\|u_k\|_\infty \rightarrow \infty$ as $k \rightarrow \infty$. Now part (ii) of the Theorem 2.2 implies that

$$h(\|u_k\|_\infty) \leq C. \quad (2.6)$$

From hypothesis (H0)' (see in particular (1.13)), for any $\varepsilon > 0$ there exists $s_1 > 0$ such that $h(s) \geq 1/\varepsilon$ for any $s \geq s_1$, and so $h(\|u_k\|_\infty) \geq 1/\varepsilon$ for any k big enough. This contradicts (2.6), ending the proof. \square

We next state a straightforward corollary, assuming that the non-linearity $\tilde{f} : \mathbb{R} \rightarrow (0, +\infty)$ satisfies also the following hypothesis:

(H1)' there exists a uniform constant $c_0 > 0$ such that

$$\sup_{s>0} \frac{\max_{[-s,s]} \tilde{f}}{\max\{\tilde{f}(-s), \tilde{f}(s)\}} \leq c_0. \quad (2.7)$$

Corollary 2.4. *Assume that $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying (H0)'–(H1)'.*

Then, for any $u \in H_0^1(\Omega)$ weak solution to (1.1), the following holds: for any $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$h(\|u\|_\infty) \leq C \|a\|_r^{A+\varepsilon} \left(1 + \|u\|_{2^*}\right)^{(2_{N/r}^* - 2)(A+\varepsilon)},$$

where h is defined by (1.12), A is defined by (2.4), $C = C(c_0, r, N, \varepsilon, |\Omega|)$, and C is independent of the solution u .

⁴In particular, if $\tilde{f}(s)$ is monotone, then (H1)' is satisfied with $c_0 = 1$.

2.2 Proof of Theorem 2.2

The arguments of the proof use Gagliardo-Nirenberg interpolation inequality (see [22]), and are inspired in the equivalence between uniform $L^{2^*}(\Omega)$ *a priori* bounds and uniform $L^\infty(\Omega)$ *a priori* bounds for solutions to subcritical elliptic equations, see [6, Theorem 1.2] for the semilinear case and $f = f(u)$, and [20, Theorem 1.3] for the p -laplacian and $f = f(x, u)$.

We first use elliptic regularity and Sobolev embeddings, and next, we invoke the Gagliardo-Nirenberg interpolation inequality (see [22]).

Proof of Theorem 2.2. Let $\{u_k\} \subset H_0^1(\Omega)$ be any sequence of weak solutions to (1.1). Since Lemma 2.1, in fact $\{u_k\} \subset H_0^1(\Omega) \cap L^\infty(\Omega)$.

If $\|u_k\|_\infty \leq C$, then (i) holds.

Now, we argue on the contrary, assuming that there exists a sequence $\|u_k\|_\infty \rightarrow +\infty$ as $k \rightarrow \infty$.

We split the proof in two steps. Firstly, we write an $W^{2,q}$ estimate for $q \in (N/2, \min\{r, N\})$, then through Sobolev embeddings we get a W^{1,q^*} estimate with $1/q^* = 1/q - 1/N < 1/N$. Secondly, we invoke the Gagliardo-Nirenberg interpolation inequality for the L^∞ -norm in terms of its W^{1,q^*} -norm and its L^{2^*} -norm.

Step 1. $W^{2,q}$ estimates for $q \in (N/2, \min\{r, N\})$.

Let us denote by

$$M_k := \max \left\{ \tilde{f}(-\|u_k\|_\infty), \tilde{f}(\|u_k\|_\infty) \right\} \geq (c_0/2)^{-1} \max_{[-\|u_k\|_\infty, \|u_k\|_\infty]} \tilde{f}, \quad (2.8)$$

where the inequality holds by hypothesis (H1), see (2.3).

Let us take q in the interval $(N/2, N) \cap (N/2, r)$. Growth hypothesis (H0)' (see (1.5)), hypothesis (H1) (see (2.3)), and Hölder inequality, yield

the following

$$\begin{aligned}
\int_{\Omega} |f(x, u_k(x))|^q dx &\leq \int_{\Omega} |a(x)|^q \left(\tilde{f}(u_k(x)) \right)^q dx \\
&= \int_{\Omega} |a(x)|^q \left(\tilde{f}(u_k(x)) \right)^t \left(\tilde{f}(u_k(x)) \right)^{q-t} dx \\
&\leq C \left[\int_{\Omega} |a(x)|^q \left(\tilde{f}(u_k(x)) \right)^t dx \right] M_k^{q-t} \\
&\leq C \left(\int_{\Omega} |a(x)|^{qs} dx \right)^{\frac{1}{s}} \left(\int_{\Omega} \left(\tilde{f}(u_k(x)) \right)^{ts'} dx \right)^{\frac{1}{s'}} M_k^{q-t} \\
&\leq C \|a\|_r^q \left(\|\tilde{f}(u_k)\|_{\frac{2^*}{2^{N/r}-1}} \right)^t M_k^{q-t},
\end{aligned}$$

where $\frac{1}{s} + \frac{1}{s'} = 1$, $qs = r$, $C = c_0^{q-t}$ (for c_0 defined in (2.3)), and $ts' = \frac{2^*}{2^{N/r}-1}$, so

$$\begin{aligned}
t &:= \frac{2^*}{2^{N/r}-1} \left(1 - \frac{q}{r} \right) < q & (2.9) \\
\iff \frac{1}{q} - \frac{1}{r} &< \frac{2^{N/r}-1}{2^*} = 1 - \frac{1}{r} - \frac{1}{2} + \frac{1}{N} \\
\iff \frac{1}{q} &< \frac{1}{2} + \frac{1}{N} \iff q > \frac{2N}{N+2} \checkmark
\end{aligned}$$

since $q > N/2 > \frac{2N}{N+2}$.

Now, elliptic regularity and Sobolev embedding imply that

$$\|u_k\|_{W^{1,q^*}(\Omega)} \leq C \|a\|_r \left(\|\tilde{f}(u_k)\|_{\frac{2^*}{2^{N/r}-1}} \right)^{\frac{t}{q}} M_k^{1-\frac{t}{q}},$$

where $1/q^* = 1/q - 1/N$, and $C = C(c_0, r, N, q, |\Omega|)$ and it is independent of u . Observe that since $q > N/2$, then $q^* > N$.

Step 2. Gagliardo-Nirenberg interpolation inequality.

Thanks to the Gagliardo-Nirenberg interpolation inequality, there exists a constant $C = C(N, q, |\Omega|)$ such that

$$\|u_k\|_{\infty} \leq C \|\nabla u_k\|_{q^*}^{\sigma} \|u_k\|_{2^*}^{1-\sigma}$$

where

$$\frac{1-\sigma}{2^*} = \sigma \left(\frac{2}{N} - \frac{1}{q} \right). \quad (2.10)$$

Hence

$$\|u_k\|_\infty \leq C \left[\|a\|_r \left(\|\tilde{f}(u_k)\|_{\frac{2^*}{2^{N/r}-1}} \right)^{\frac{t}{q}} M_k^{1-\frac{t}{q}} \right]^\sigma \|u_k\|_{2^*}^{1-\sigma}, \quad (2.11)$$

where $C = C(c_0, r, N, q, |\Omega|)$.

From definition of M_k (see (2.8)), and definition of h (see (1.12)), we deduce that

$$M_k = \frac{\|u_k\|_\infty^{2^{N/r}-1}}{h(\|u_k\|_\infty)}.$$

From (2.10)

$$\frac{1}{\sigma} = 1 + 2^* \left(\frac{2}{N} - \frac{1}{q} \right) = 2^* - 1 - \frac{2^*}{q} = 2_{N/q}^* - 1. \quad (2.12)$$

Moreover, since definition of t (see (2.9)), and definition of $2_{N/r}^*$ (see (1.7))

$$1 - \frac{t}{q} = \frac{2^* \left(1 - \frac{1}{r}\right) - 1 - 2^* \left(\frac{1}{q} - \frac{1}{r}\right)}{2_{N/r}^* - 1} = \frac{2_{N/q}^* - 1}{2_{N/r}^* - 1}, \quad (2.13)$$

which, joint with (2.12), yield

$$\sigma \left[1 - \frac{t}{q} \right] (2_{N/r}^* - 1) = 1.$$

Now (2.11) can be rewritten as

$$h(\|u_k\|_\infty)^{(1-\frac{t}{q})\sigma} \leq C \left[\|a\|_r \left(\|\tilde{f}(u_k)\|_{\frac{2^*}{2^{N/r}-1}} \right)^{\frac{t}{q}} \right]^\sigma \|u_k\|_{2^*}^{1-\sigma},$$

or equivalently

$$h(\|u_k\|_\infty) \leq C \|a\|_r^\theta \left(\|\tilde{f}(u_k)\|_{\frac{2^*}{2^{N/r}-1}} \right)^{\theta-1} \|u_k\|_{2^*}^\vartheta,$$

where

$$\theta := (1 - t/q)^{-1} = \frac{2_{N/r}^* - 1}{2_{N/q}^* - 1}, \quad (2.14)$$

$$\vartheta := \frac{1 - \sigma}{\sigma} (1 - t/q)^{-1} = \theta (2_{N/q}^* - 2), \quad (2.15)$$

see (2.13) and (2.10). Observe that since $q < r$, then $\theta > 1$. Moreover, since (2.15), and (2.14)

$$\theta + \vartheta = \theta(2_{N/q}^* - 1) = 2_{N/r}^* - 1. \quad (2.16)$$

Furthermore, from sub-criticality, see (1.6)

$$\int_{\Omega} |\tilde{f}(u_k)|^{\frac{2^*}{2_{N/r}^* - 1}} \leq C \left(1 + \int_{\Omega} |u_k|^{2^*} dx \right),$$

so

$$\|\tilde{f}(u_k)\|_{\frac{2^*}{2_{N/r}^* - 1}} \leq C \left(1 + \|u_k\|_{2^*}^{2_{N/r}^* - 1} \right).$$

Consequently

$$h(\|u_k\|_{\infty}) \leq C \|a\|_r^{\theta} \left(1 + \|u_k\|_{2^*}^{\Theta} \right),$$

with

$$\Theta := (2_{N/r}^* - 1)(\theta - 1) + \vartheta = (2_{N/r}^* - 2)\theta,$$

where we have used (2.16).

Fixed $N > 2$ and $r > N/2$, the function $q \rightarrow \theta = \theta(q)$ for $q \in (N/2, \min\{r, N\})$, is decreasing, so

$$\inf_{q \in (N/2, \min\{r, N\})} \theta(q) = \theta(\min\{r, N\}) = A := \begin{cases} 1, & \text{if } r \leq N, \\ 1 + \frac{2}{N} - \frac{2}{r}, & \text{if } r > N. \end{cases}$$

Finally, and since the infimum is not attained in $(N/2, \min\{r, N\})$, for any $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$h(\|u_k\|_{\infty}) \leq C \|a\|_r^{A+\varepsilon} \left(1 + \|u_k\|_{2^*}^{(2_{N/r}^* - 2)(A+\varepsilon)} \right),$$

where A is defined by (2.4), and $C = C(\varepsilon, c_0, r, N, |\Omega|)$, ending the proof. \square

3 Radial singular weights

In this section, assuming that $0 \in \Omega$ and that $|f(x, s)| \leq |x|^{-\mu} \tilde{f}(s)$ for some $\mu \in (0, 2)$, we state our second main result concerning weak solutions for singular subcritical non-linearities, see Theorem 3.2.

First, we also collect a regularity Lemma for any weak solution to (1.1) with $\tilde{f}(s)$ of polynomial critical growth, according to Caffarelli-Kohn-Nirenberg inequality.

Lemma 3.1 (Improved regularity). *Assume that $u \in H_0^1(\Omega)$ weakly solves (1.1) for a Carathéodory non-linearity $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ with polynomial critical growth*

$$|f(x, s)| \leq |x|^{-\mu} (1 + |s|^{2^*_\mu - 1}), \quad \text{with } \mu \in (0, 2).$$

Then, the following hold:

- (i) If $\mu < 1$, then $u \in C^{1,\nu}(\bar{\Omega}) \cap W^{2,p}(\Omega)$ for any $p < N/\mu$, and any $\nu < 1 - \mu$.
- (ii) If $\mu = 1$, then $u \in C^\nu(\bar{\Omega}) \cap W^{2,p}(\Omega)$ for any $p < N$, and $\nu < 1$.
- (iii) If $1 < \mu < 2$, then $u \in C^\nu(\bar{\Omega}) \cap W^{2,p}(\Omega)$ for any $p < N/\mu$, and $\nu < 1 - \mu$.

Proof. Let $u \in H_0^1(\Omega)$ be a solution to (1.1). We reason as in Lemma 2.1.

If f satisfies (1.4), then Caffarelli-Kohn-Nirenberg interpolation inequality (see [4]) implies that assumption (2.2) is satisfied with

$$b(x) = \frac{|x|^{-\mu} (1 + |u|^{2^*_\mu - 1})}{1 + |u|} \leq C |x|^{-\mu} (1 + |u|^{2^*_\mu - 2}) \in L^{N/2}(\Omega).$$

Indeed, since Caffarelli-Kohn-Nirenberg, there exists a constant $C > 0$ depending on the parameters N , and μ , such that

$$\| |x|^\gamma u \|_t \leq C \| \nabla u \|_2^\theta \| u \|_{2^*}^{1-\theta},$$

where $\gamma = -\frac{\mu}{2^*_\mu - 2} = -\frac{\mu(N-2)}{2(2-\mu)}$, $t = (2^*_\mu - 2)\frac{N}{2} = \frac{N(2-\mu)}{N-2}$, and $\frac{1}{t} + \frac{\gamma}{N} = \frac{1}{2^*} = \theta \left(\frac{1}{2} - \frac{1}{N} \right) + (1 - \theta) \frac{1}{2^*}$, with $\theta \in (0, 1)$.

Consequently, $u \in L^q(\Omega)$ for any $q < \infty$. The growth condition for f (see (1.3)-(1.4)), implies that $-\Delta u = f(x, u) \in L^p(\Omega)$ for any $p < N/\mu$. Thus, by the Calderon-Zygmund inequality (see [15, Theorem 9.14]), $u \in W^{2,p}(\Omega)$, for any $p \in (1, N/\mu)$.

- (i) Assume $\mu < 1$. Choosing any $p \in (N, N/\mu)$, by elliptic regularity, $u \in W^{2,p}(\Omega)$, with $p > N$. Then $u \in C^{1,\nu}(\overline{\Omega})$ for any $\nu < 1 - \frac{N}{p}$, and finally, $u \in C^{1,\nu}(\overline{\Omega}) \cap W^{2,p}(\Omega)$ for any $p < N/\mu$, and any $\nu < 1 - \mu$.
- (ii) Assume $\mu = 1$. Choosing any $p \in (N/2, N)$, by elliptic regularity and Sobolev embeddings, $u \in W^{1,p^*}(\Omega)$, where $\frac{1}{p^*} := \frac{1}{p} - \frac{1}{N} < \frac{1}{N}$. Also $u \in C^\nu(\overline{\Omega})$ for any $\nu < 1$. Finally $u \in C^\nu(\overline{\Omega}) \cap W^{2,p}(\Omega)$ for any $p < N$, and $\nu < 1$.
- (iii) Assume $1 < \mu < 2$. Choosing any $p \in (N/2, N/\mu)$, and reasoning as above, $u \in W^{1,p^*}(\Omega)$, where $\frac{1}{p^*} := \frac{1}{p} - \frac{1}{N} < \frac{1}{N}$. Finally $u \in C^\nu(\overline{\Omega}) \cap W^{2,p}(\Omega)$ for any $p < N/\mu$, and $\nu < 1 - \mu$.

□

3.1 Estimates of the L^∞ -norm of the solutions

Our second main results is the following theorem.

Theorem 3.2. *Assume that $f : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying (H0) and (H1).*

Then, for any $u \in H_0^1(\Omega)$ solution to (1.1), the following holds:

- (i) *either there exists a constant $C > 0$ such that $\|u\|_\infty \leq C$, where C is independent of the solution u ,*
- (ii) *either for any $\varepsilon > 0$ there exists a constant $C > 0$ such that*

$$h(\|u\|_\infty) \leq C_\varepsilon \left(1 + \|u\|_{2^*}\right)^{(2_\mu^* - 2)(B + \varepsilon)},$$

where h is defined by (1.11),

$$B := \begin{cases} 1 + \frac{2}{N} - \frac{2\mu}{N}, & \text{if } \mu \in (0, 1), \\ 1, & \text{if } \mu \in [1, 2), \end{cases} \quad (3.1)$$

and C depends only on ε , c_0 (defined in (2.3)), μ , N , and Ω , and it is independent of the solution u .

3.2 Proof of Theorem 3.2

Proof of Theorem 3.2. Let $\{u_k\} \subset H_0^1(\Omega)$ be any sequence of solutions to (1.1). Since Lemma 3.1, $\{u_k\} \subset H_0^1(\Omega) \cap L^\infty(\Omega)$. If $\|u_k\|_\infty \leq C$, then (i) holds.

Now, we argue on the contrary, assuming that there exists a sequence $\{u_k\} \subset H_0^1(\Omega)$ of solutions to (1.1), such that $\|u_k\|_\infty \rightarrow +\infty$ as $k \rightarrow \infty$. By Morrey's Theorem (see [2, Theorem 9.12]), observe that also

$$\|\nabla u_k\|_p \rightarrow +\infty \quad \text{as } k \rightarrow \infty, \quad (3.2)$$

for any $p > N$.

Step 1. $W^{2,q}$ estimates for $q \in (N/2, \min\{N, N/\mu\})$.

As in the proof of Theorem (2.2), let us denote by

$$M_k := \max \left\{ \tilde{f}(-\|u_k\|_\infty), \tilde{f}(\|u_k\|_\infty) \right\} \geq (c_0/2)^{-1} \max_{[-\|u_k\|_\infty, \|u_k\|_\infty]} \tilde{f}, \quad (3.3)$$

where the inequality is due to hypothesis (H1), see (2.3).

Let us take q in the interval $(N/2, N) \cap (N/2, N/\mu)$. Using growth hypothesis (H0) (see (1.3)), hypothesis (H1) (see (2.3)), and Hölder inequality, we deduce

$$\begin{aligned} \int_{\Omega} |f(x, u_k(x))|^q dx &\leq \int_{\Omega} |x|^{-\mu q} \left(\tilde{f}(u_k(x)) \right)^q dx \\ &= \int_{\Omega} |x|^{-\mu q} \left(\tilde{f}(u_k(x)) \right)^{\frac{t}{2_\mu^* - 1}} \left(\tilde{f}(u_k(x)) \right)^{q - \frac{t}{2_\mu^* - 1}} dx \\ &\leq C \left[\int_{\Omega} |x|^{-\mu q} (1 + u_k(x)^t) dx \right] M_k^{q - \frac{t}{2_\mu^* - 1}} \\ &\leq C \left(1 + \int_{\Omega} |x|^{-\gamma} u_k^t dx \right) M_k^{q - \frac{t}{2_\mu^* - 1}}, \end{aligned}$$

where $\gamma = \frac{\mu q}{t}$, $t \in (0, q(2_\mu^* - 1))$, $C = c_0^{q - \frac{t}{2_\mu^* - 1}}$ (for c_0 defined in (2.3)), and where M_k is defined by (3.3).

Combining now elliptic regularity with Sobolev embedding, we have that

$$\|\nabla u_k\|_{q^*} \leq C \left(1 + \int_{\Omega} |x|^{-\gamma} u_k^t dx \right)^{\frac{1}{q}} M_k^{1 - \frac{t}{q(2_\mu^* - 1)}}, \quad (3.4)$$

where $1/q^* = 1/q - 1/N$ (since $q > N/2$, then $q^* > N$), and $C = C(N, q, |\Omega|)$.

Step 2. Caffarelli-Kohn-Nirenberg interpolation inequality.

Since the Caffarelli-Kohn-Nirenberg interpolation inequality for singular weights (see [4]), there exists a constant $C > 0$ depending on the parameters N , q , μ , and t , such that

$$\| |x|^{-\gamma} u_k \|_t \leq C \|\nabla u_k\|_{q^*}^\theta \|u_k\|_{2^*}^{1-\theta}, \quad (3.5)$$

where

$$\frac{1}{t} - \frac{\mu q}{Nt} = -\theta \left(\frac{2}{N} - \frac{1}{q} \right) + (1-\theta) \frac{1}{2^*}, \quad \text{with } \theta \in (0, 1). \quad (3.6)$$

Substituting now (3.5) into (3.4) we can write

$$\|\nabla u_k\|_{q^*} \leq C \left(1 + \|\nabla u_k\|_{q^*}^{\theta t} \|u_k\|_{2^*}^{(1-\theta)t} \right)^{\frac{1}{q}} M_k^{1 - \frac{t}{q(2_\mu^* - 1)}},$$

now, dividing by $\|\nabla u_k\|_{q^*}^{\theta t/q}$ and using (3.2) we obtain

$$\|\nabla u_k\|_{q^*}^{1-\theta t/q} \leq C \left(1 + \|u_k\|_{2^*}^{\frac{(1-\theta)t}{q}} \right) M_k^{1 - \frac{t}{q(2_\mu^* - 1)}}. \quad (3.7)$$

Let us check that

$$1 - \theta \frac{t}{q} > 0 \quad \text{for any } t < q(2_\mu^* - 1). \quad (3.8)$$

Indeed, observe first that (3.6) is equivalent to

$$\theta = \frac{\frac{1}{2^*} - \frac{1}{t} + \frac{\mu q}{Nt}}{\frac{1}{2} + \frac{1}{N} - \frac{1}{q}}, \quad (3.9)$$

moreover, from (3.9)

$$\theta \frac{t}{q} = \frac{\frac{1}{q} \left(\frac{t}{2^*} - 1 \right) + \frac{\mu}{N}}{\frac{1}{2} + \frac{1}{N} - \frac{1}{q}}, \quad (3.10)$$

consequently

$$\begin{aligned} \theta \frac{t}{q} < 1 &\iff \frac{1}{q} \left(\frac{t}{2^*} - 1 \right) + \frac{\mu}{N} < \frac{1}{2} + \frac{1}{N} - \frac{1}{q} \\ &\iff \frac{1}{q} \frac{t}{2^*} < \frac{1}{2} + \frac{1}{N} - \frac{\mu}{N} \pm 1 \\ &\iff \frac{t}{q} < 2^* \left(1 - \frac{\mu}{N} \right) - 2^* \left(\frac{1}{2} - \frac{1}{N} \right) = 2_\mu^* - 1 \\ &\iff t < q(2_\mu^* - 1), \end{aligned}$$

so, (3.8) holds.

Consequently,

$$\|\nabla u_k\|_{q^*} \leq C \left(1 + \|u_k\|_{2^*}^{\frac{(1-\theta)t}{q-\theta t}}\right) M_k \left(1 - \frac{t}{q(2^*_\mu-1)}\right)^{(1-\theta t/q)^{-1}}. \quad (3.11)$$

Step 3. Gagliardo-Nirenberg interpolation inequality.

Thanks to the Gagliardo-Nirenberg interpolation inequality (see [22]), there exists a constant $C = C(N, q, |\Omega|)$ such that

$$\|u_k\|_\infty \leq C \|\nabla u_k\|_{q^*}^\sigma \|u_k\|_{2^*}^{1-\sigma}, \quad (3.12)$$

where

$$\frac{1-\sigma}{2^*} = \sigma \left(\frac{2}{N} - \frac{1}{q} \right). \quad (3.13)$$

Hence, substituting (3.11) into (3.12) we deduce

$$\|u_k\|_\infty \leq C \left(1 + \|u_k\|_{2^*}^{\sigma \frac{(1-\theta)t}{q-\theta t} + 1 - \sigma}\right) M_k^{\sigma \left(1 - \frac{t}{q(2^*_\mu-1)}\right)^{(1-\theta t/q)^{-1}}}. \quad (3.14)$$

From definition of M_k (see (2.8)), and of h (see (1.11)), we obtain

$$M_k = \frac{\|u_k\|_\infty^{2^*_\mu-1}}{h(\|u_k\|_\infty)}. \quad (3.15)$$

From (3.13)

$$\frac{1}{\sigma} = 1 + 2^* \left(\frac{2}{N} - \frac{1}{q} \right) = 2^*_{N/q} - 1. \quad (3.16)$$

From (3.10), we deduce

$$\begin{aligned} 1 - \theta \frac{t}{q} &= \frac{\frac{1}{2} + \frac{1}{N} - \frac{t}{2^*q} - \frac{\mu}{N} \pm 1}{\frac{1}{2} + \frac{1}{N} - \frac{1}{q}} \\ &= \frac{\left(1 - \frac{\mu}{N}\right) - \frac{1}{2^*} - \frac{t}{2^*q}}{\frac{1}{2} + \frac{1}{N} - \frac{1}{q}} = \frac{2^*_\mu - 1 - \frac{t}{q}}{2^*_{N/q} - 1}, \end{aligned} \quad (3.17)$$

where we have used that, by definition of 2^*_μ (see (1.2)), $\frac{2^*_\mu}{2^*} = 1 - \frac{\mu}{N}$.

Moreover, since (3.17),

$$\begin{aligned} & \left(1 - \frac{t}{q(2_\mu^* - 1)}\right) (2_\mu^* - 1) \frac{1}{(1 - \theta t/q)} \\ &= \left(2_\mu^* - 1 - \frac{t}{q}\right) \frac{1}{(1 - \theta t/q)} = 2_{N/q}^* - 1. \end{aligned} \quad (3.18)$$

Taking into account (3.16) and (3.18) we obtain

$$\sigma \left(1 - \frac{t}{q(2_\mu^* - 1)}\right) (2_\mu^* - 1) (1 - \theta t/q)^{-1} = 1. \quad (3.19)$$

Consequently, since (3.15), and (3.19), we can rewrite (3.14) in the following way

$$h(\|u_k\|_\infty)^{\frac{1}{2_\mu^* - 1}} \leq C \left(1 + \|u_k\|_{2^*}^{\sigma \frac{(1-\theta)t/q}{1-\theta t/q} + 1 - \sigma}\right), \quad (3.20)$$

or equivalently

$$h(\|u_k\|_\infty) \leq C \left(1 + \|u_k\|_{2^*}^\Theta\right), \quad (3.21)$$

where

$$\Theta := (2_\mu^* - 1) \left[1 + \sigma \frac{t/q - 1}{1 - \theta t/q}\right].$$

Since (3.19), $\sigma(1 - \theta t/q)^{-1} = (2_\mu^* - 1 - \frac{t}{q})^{-1}$, and substituting it into the above equation we obtain

$$\Theta = (2_\mu^* - 1) \left(\frac{2_\mu^* - 2}{2_\mu^* - 1 - \frac{t}{q}}\right).$$

Fixed $N > 2$ and $\mu \in (0, 2)$, the function $(t, q) \rightarrow \Theta = \Theta(t, q)$ for $(t, q) \in (0, q(2_\mu^* - 1)) \times (N/2, \min\{N, N/\mu\})$, is increasing in t and decreasing in q .

For $\mu \in [1, 2)$, $\min\{N, N/\mu\} = N/\mu$. If $q_k \rightarrow N/\mu$, equation (3.6) with $q = q_k$, $\theta = \theta_k < 1$ and an arbitrary $t \in (0, (2_\mu^* - 1)N/\mu)$ fixed, yields $\theta_k \rightarrow \frac{1}{2_\mu^* - 1} < 1$ (since $\mu < 2$). Hence, when $\mu \in [1, 2)$,

$$\inf_{t \in (0, (2_\mu^* - 1)\frac{N}{\mu}), q \in (\frac{N}{2}, \frac{N}{\mu})} \Theta(t, q) = \Theta\left(0, \frac{N}{\mu}\right) = 2_\mu^* - 2.$$

On the other hand, for $\mu \in (0, 1)$, $\min\{N, N/\mu\} = N$. If $q_k \rightarrow N$, equation (3.6) with $q = q_k$, $\theta = \theta_k > 0$ and t fixed, yields $\theta_k \rightarrow \frac{2}{2^*} - \frac{2(1-\mu)}{t} \geq 0$, so $t \geq 2^*(1 - \mu)$. Hence, when $\mu \in (0, 1)$,

$$\inf_{t \in [2^*(1-\mu), (2_\mu^* - 1)N], q \in (\frac{N}{2}, N)} \Theta(t, q) = \Theta(2^*(1 - \mu), N) = (2_\mu^* - 2)B,$$

where B is defined by (3.1).

Since the infimum is not attained, for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, c_0, \mu, N, \Omega)$ such that

$$h(\|u_k\|_\infty) \leq C \left(1 + \|u_k\|_{2_\mu^*}^{(2_\mu^* - 2)(B + \varepsilon)} \right), \quad (3.22)$$

which ends the proof. \square

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References

- [1] M.-F. Bidaut-Veron. Local behaviour of the solutions of a class of nonlinear elliptic systems. *Adv. Differential Equations*, 5(1-3):147–192, 2000.
- [2] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.
- [3] H. Brézis and T. Kato. Remarks on the Schrödinger operator with singular complex potentials. *J. Math. Pures Appl. (9)*, 58(2):137–151, 1979.
- [4] L. Caffarelli, R. Kohn, and L. Nirenberg. First order interpolation inequalities with weights. *Compositio Math.*, 53(3):259–275, 1984.
- [5] P. Caldiroli and A. Malchiodi. Singular elliptic problems with critical growth. *Comm. Partial Differential Equations*, 27(5-6):847–876, 2002.

- [6] A. Castro, N. Mavinga, and R. Pardo. Equivalence between uniform $L^{2^*}(\Omega)$ a-priori bounds and uniform $L^\infty(\Omega)$ a-priori bounds for subcritical elliptic equations. *Topol. Methods Nonlinear Anal.*, 53(1):43–56, 2019.
- [7] A. Castro and R. Pardo. A priori bounds for positive solutions of subcritical elliptic equations. *Rev. Mat. Complut.*, 28(3):715–731, 2015.
- [8] A. Castro and R. Pardo. A priori estimates for positive solutions to subcritical elliptic problems in a class of non-convex regions. *Discrete Contin. Dyn. Syst. Ser. B*, 22(3):783–790, 2017.
- [9] M. Clapp, R. Pardo, A. Pistoia, and A. Saldaña. A solution to a slightly subcritical elliptic problem with non-power nonlinearity. *J. Differential Equations*, 275:418–446, 2021.
- [10] L. Damascelli and R. Pardo. A priori estimates for some elliptic equations involving the p -Laplacian. *Nonlinear Anal. Real World Appl.*, 41:475–496, 2018.
- [11] D. G. de Figueiredo, P.-L. Lions, and R. D. Nussbaum. A priori estimates and existence of positive solutions of semilinear elliptic equations. *J. Math. Pures Appl. (9)*, 61(1):41–63, 1982.
- [12] L. Dupaigne and A. C. Ponce. Singularities of positive supersolutions in elliptic PDEs. *Selecta Math. (N.S.)*, 10(3):341–358, 2004.
- [13] B. Gidas and J. Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.*, 34(4):525–598, 1981.
- [14] B. Gidas and J. Spruck. A priori bounds for positive solutions of nonlinear elliptic equations. *Comm. Partial Differential Equations*, 6(8):883–901, 1981.
- [15] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

- [16] E. Jannelli and S. Solimini. Critical behaviour of some elliptic equations with singular potentials. *Rapporto no. 41/96, Dipartimento Di Matematica, Universita degli Studi di Bari, 70125 Bari, Italia.*
- [17] D. D. Joseph and T. S. Lundgren. Quasilinear Dirichlet problems driven by positive sources. *Arch. Rational Mech. Anal.*, 49:241–269, 1972/73.
- [18] J. Leray and J. Schauder. Topologie et équations fonctionnelles. *Ann. Sci. École Norm. Sup. (3)*, 51:45–78, 1934.
- [19] N. Mavinga and R. Pardo. A priori bounds and existence of positive solutions for semilinear elliptic systems. *J. Math. Anal. Appl.*, 449(2):1172–1188, 2017.
- [20] N. Mavinga and R. Pardo. Equivalence between uniform L^{p^*} a priori bounds and uniform L^∞ a priori bounds for subcritical p -Laplacian equations. *Mediterr. J. Math.*, 18(1):Paper No. 13, 24, 2021.
- [21] J. Moser. A new proof of De Giorgi’s theorem concerning the regularity problem for elliptic differential equations. *Comm. Pure Appl. Math.*, 13:457–468, 1960.
- [22] L. Nirenberg. On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 13:115–162, 1959.
- [23] R. Pardo. On the existence of a priori bounds for positive solutions of elliptic problems, I. *Rev. Integr. Temas Mat.*, 37(1):77–111, 2019.
- [24] R. Pardo. On the existence of a priori bounds for positive solutions of elliptic problems, II. *Rev. Integr. Temas Mat.*, 37(1):113–148, 2019.
- [25] R. Pardo and A. Sanjuán. Asymptotic behavior of positive radial solutions to elliptic equations approaching critical growth. *Electron. J. Differential Equations*, pages Paper No. 114, 17, 2020.
- [26] M. Struwe. *Variational methods*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, fourth edition, 2008. Applications to nonlinear partial differential equations and Hamiltonian systems.