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Diffraction by cylinders illuminated in oblique, off-axis incidence

Luis Miguel Sanchez-Brea, Eusebio Bernabeu

Departamento de Óptica, Universidad Complutense de Madrid, Facultad de Ciencias Físicas, Ciudad Universitaria s/n. 28040 Madrid, Spain

Abstract: In this paper we present a model to determine the light scattered by a metallic cylinder when it is illuminated with a light beam in oblique incidence. This model is based on an approximate solution to the *Helmholtz-Kirchhoff* integral by means of the *Stationary Phase Method.* The polarization of the beam, its width, and the misalignment between the beam and the cylinder are considered, as well as the reflection coefficient of the surface.

Key words: Cylinder diffraction – metallic cylinder

1. Introduction

The diffraction of light by cylinders is a well-known canonical problem. Its rigorous solution, when the cylinder is illuminated by a plane wave in normal incidence, was obtained many years ago by Rayleigh [1]. This problem has also been deeply studied by several diffraction models, both for dielectric and conductor materials: Boundary Diffraction Wave (BDW) [3], Geometrical Theory of Diffraction (GTD) [1], Uniform Theory of Diffraction (UTD) [4], Combined method of ray tracing and diffraction [5], etc. Also several studies have been developed in which the incident light is a Gaussian beam [6–8].

A rigorous solution to the diffraction of plane waves in oblique incidence was also studied by Wait [2, 9]. The solution obtained is given in terms of a summatory whose calculation is quite timeconsuming. There have also been some works studying the scattering of a diagonally incident focused Gaussian beam by dielectric cylinder [10, 11].

Recently diffraction by metallic cylinders in oblique incidence has grown in interest since there have appeared some devices to obtain information of the cylinder surface which works in oblique incidence [12]. Some theoretical work, based in geometrical optics, has been done [13, 14]. However, to the knowledge of the authors, there has been no deep analysis of this configuration, in particular considering intensity distributions different to a plane wave with misalignments between the beam and the cylinder, or considering the reflection coefficient of the cylinder surface. Experimentally, it is observed that the diffracted light by a metallic cylinder generates a cone. At the direction of incidence, there is a maximum of intensity accompanied of several diffraction maxima and minima. By means of these minima the cylinder diameter can be determined [15]. These maxima and minima gradually disappears as we separate from the direction of incidence and, in the rest of the cone, there is a smooth intensity distribution (fig. 1).

In this paper we present an approximate solution to the problem of diffraction by cylinders when an off-axis Gaussian beam illuminates a metallic cylinder. We have used the *Kirchhoff* approximation with the boundary conditions given by *Beckmann* and *Spizzichino* [16]. The integrals involved have been solved by means of the Stationary Phase method [17].

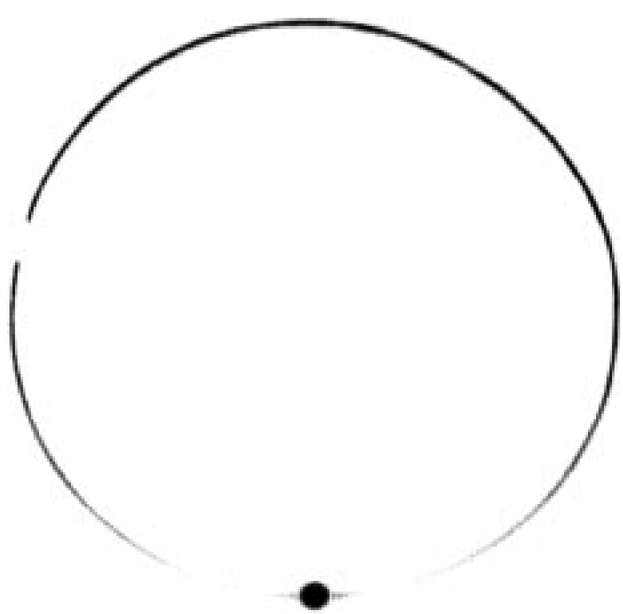

Fig. 1. Experimental diffracted intensity when a laser beam with a Gaussian profile impinges upon a silver cylinder with a diameter of 200 μm. The incidence angle is *a* = 30°. The maximum of intensity at the bottom of the figure corresponds to the direction of incidence. The intensity increases as we separate from this incidence direction. The lack of light at the left of the circle corresponds to the wire holder.

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Correspondence to: E. Bernabeu
Fax: ++34-913-944674
E-mail: ebernabeu@fis.ucm.es

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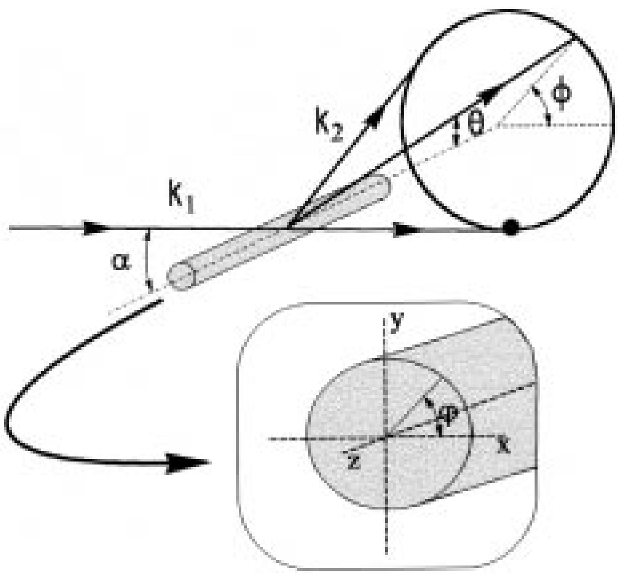

Fig. 2. Diagram showing the parameters involved in the calculations.

2. Diffraction by cylinders

A well-known approximation for the diffraction of light is the *Helmholtz-Kirchhoff* integral theorem [18]. To solve the equations involved it is necessary to know the field over the surface and/or the value of its derivative in the direction normal to the surface. As these values are rarely known several models that predict these values have been developed, as the *Kirchhoff* approximation (also known as the geometrical optics approximation) and the *Rayleigh-Sommerfeld* approximations [19]. When the light is reflected by the surface, the usual approximation for the field and its derivative at the surface is the one proposed by *Beckmann* and *Spizzichino* [16]

$$E|_S = (1+R)\,E_1|_S\,,$$
$$\left.\frac{\partial E}{\partial n}\right|_S = (1-R)\,E_1|_S\;\boldsymbol{k}_1\cdot\boldsymbol{n}\,, \tag{1}$$

where E_1 is the incident field over the surface, R is the reflection coefficient (which depends on the polarization and on the angle of incidence according to *Fresnel equations* [18], $\boldsymbol{n}$ is the normal to the surface, and $\boldsymbol{k}_1 = k\,(0, -\sin\alpha, \cos\alpha)$, where k is the wavenumber, describes the direction of the incident field, that we have supposed to be contained in the plane YZ (fig. 2). We consider the scattered field at infinity and, thus, the diffraction pattern is represented in terms of the wavevector direction $\boldsymbol{k}_2 = k(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Under the boundary conditions of eq. (1) the *Helmholtz-Kirchhoff* integral results

$$E_2(\phi,\theta) \propto \iint_S E_1(R\boldsymbol{v}-\boldsymbol{p})\cdot\boldsymbol{n}\,\exp(\mathrm{i}\,\boldsymbol{v}\cdot\boldsymbol{r})\,dS\,, \tag{2}$$

where a_0 is the radius of the cylinder, $\boldsymbol{r} = (a_0\cos\varphi, a_0\sin\varphi, z)$ defines the surface, $\boldsymbol{n} = (\cos\varphi, \sin\varphi, 0)$ is the normal to the surface, the surface element is given by $dS = a_0\,d\varphi dz$ since we use cylindrical coordinates to describe the surface, and

$$\boldsymbol{v} = \boldsymbol{k}_1 - \boldsymbol{k}_2 = k(v_x, v_y, v_z) \tag{3}$$
$$= k(-\sin\theta\cos\phi, -\sin\theta\sin\phi - \sin\alpha, \cos\alpha - \cos\theta)\,,$$

$$\boldsymbol{p} = \boldsymbol{k}_1 + \boldsymbol{k}_2 = k(-v_x, -v_y - 2\sin\alpha, \cos\alpha + \cos\theta)\,. \tag{4}$$

We have supposed that the incident field is a gaussian beam, whose beam waist is on the wire surface. Therefore, the incidence amplitude can be expressed by

$$E_1(\varphi, z) = E_0 \exp\left[-\left(\frac{a_0\cos\varphi - x_0}{w_x}\right)^2 - \left(\frac{\sin\alpha\, z}{w_z}\right)^2\right],$$

where w_x, w_z, are the principal widths of the beam and x_0 is the misalignment between the incident beam and the cylinder. In order to simplify the calculations we have supposed that the principal width w_z is parallel to the cylinder axis.

Introducing eqs. (3), (4) and (5) in eq. (2) it results that the double integral $E_2(\phi, \theta)$ can be divided in the product of two separated integrals: one with a dependence in z and another with a dependence in φ:

$$E_2(\phi, \theta) = C\,k\,E_0\,U_1 U_2\,. \tag{6}$$

The first integrals is

$$U_1(\phi,\theta) = \int_{-\infty}^{\infty} \exp\left[-\left(\frac{\sin\alpha\, z}{w_z^2}\right)^2\right] \cdot \exp(\mathrm{i}k\, v_z\, z)\,dz\,, \tag{7}$$

that it is easily solved by

$$U_1(\phi,\theta) = \sqrt{\pi}\,\frac{w_z}{\sin\alpha}\exp\left[-\left(\frac{k\,v_z\,w_z}{2\sin\alpha}\right)^2\right]. \tag{8}$$

The second integral can be written in the following way

$$U_2(\phi, \theta) = \int g(\varphi)\exp[i\,k\,f(\varphi)]\,d\varphi, \tag{9}$$

where

$$g(\varphi) = a_0\exp[-(a_0\cos\varphi - x_0)^2/w_x^2]$$
$$\cdot\,[(R+1)(v_x\cos\varphi + v_y\sin\varphi) + 2\sin\alpha\sin\varphi],$$
$$f(\varphi) = a_0\,(v_x\cos\varphi + v_y\sin\varphi)\,. \tag{10}$$

An analytical solution to this integral is only possible when the surface behaves as a perfect reflector, the beam is aligned with the cylinder axis ($x_0 = 0$) and its width w_x is large enough with respect to the cylinder radius a_0 so as the first term of the exponential in eq. (5) be zero. On the other hand, when the cylinder radius is much larger than the wavelength λ, an approximate solution to this integral can be obtained by means of the so called *Stationary Phase Method* (SPM) [17]. As the exponential term of the equation (9) is strongly variable, the positive and negative parts cancel except at the points φ_S were $df(\varphi)/d\varphi|_{\varphi_S} = 0$. The approximate solution to this integral by SPM is

$$U_2(\phi,\theta) \approx \left(\frac{\pi}{k\,|f_2|}\right)^{1/2} g_0 \cdot \exp\left[\mathrm{i}\left(k\,f_0 + \frac{\pi}{4} - \frac{\arg f_2}{2}\right)\right], \tag{11}$$

where $h_n = d^n h(x)/dx^n|_{xs}$, $(h = f, g)$, and

$$\cos\varphi_S = \frac{v_x}{(v_x^2+v_y^2)^{1/2}} = \frac{-\sin\theta\sin\phi - \sin\alpha}{\sqrt{\sin^2\theta+\sin^2\alpha+2\sin\theta\sin\phi\sin\alpha}},$$
$$\sin\varphi_S = \frac{v_y}{(v_x^2+v_y^2)^{1/2}} = \frac{-\sin\theta\cos\phi}{\sqrt{\sin^2\theta+\sin^2\alpha+2\sin\theta\sin\phi\sin\alpha}}. \quad (12)$$

In order to determine φ_S we have neglected the imaginary part of R, as it is much smaller than $f(\varphi)$. With this stationary value we obtain

$$f_0 = -f_2 = -a_0(v_x^2+v_y^2)^{1/2},$$
$$g_0 = a_0\exp\left[-\left(\frac{a_0 v_x}{(v_x^2+v_y^2)^{1/2}} - x_0\right)^2 / w_x^2\right] \cdot \left\{(R+1)(v_x^2+v_y^2)^{1/2} + 2\sin\alpha\, v_y/(v_x^2+v_y^2)^{1/2}\right\}. \quad (13)$$

Finally, introducing these results in eq. (11) the diffracted light by the cylinder is

$$E_2(\phi,\theta) \propto E_0\frac{a_0^{1/2}}{\sin\alpha} \cdot \exp\left[-\left(\frac{a_0 v_x}{(v_x^2+v_y^2)^{1/2}} - x_0\right)^2 / w_x^2\right] w_z \cdot \exp\left[-\left(\frac{k v_z w_z}{2\sin\alpha} - x_0\right)^2\right] \cdot \left\{(R(\gamma)+1)(v_x^2+v_y^2)^{1/4} + 2\sin\alpha\frac{v_y}{(v_x^2+v_y^2)^{3/4}}\right\} \cdot \exp\left[-\mathrm{i}\left(k a_0 (v_x^2+v_y^2)^{1/2} - \frac{\pi}{4}\right)\right], \quad (14)$$

where γ is the angle between the direction of the incident beam and the normal to the surface

$$\cos\gamma = -\frac{\boldsymbol{k}_1\cdot\boldsymbol{n}}{|\boldsymbol{k}_1||\boldsymbol{n}|} = \sin\alpha\sin\varphi_S. \quad (15)$$

The intensity is, of course,

$$I_2(\phi,\theta) = E_2(\phi,\theta)\,E_2^*(\phi,\theta). \quad (16)$$

In eq. (14) we can see that, when using the SPM approximation, the diffraction minima and maxima around the direction of incidence are lost.

2.1. Large beam with w_z

And important case occurs when the width w_z of the incident beam is much larger than the wavelength λ. Then, an easier solution can be obtained. For this, we will use the following limit

$$\lim_{n\to\infty}\frac{1}{\sqrt{\pi}}\,n\exp(-n^2x^2) \to \delta(x). \quad (17)$$

where δ is the Dirac delta. If $n = w_z$, $x = k\,v_z/2\sin\alpha$ in eq. (8), then U_1 results

$$U_1(\theta) = \frac{\pi}{\sin\alpha}\,\delta\left(\frac{k(\cos\alpha-\cos\theta)}{2\sin\alpha}\right), \quad (18)$$

and since

$$\delta(h(\theta)) = \sum_j \frac{\delta(\theta-\theta_j)}{|h'(\theta_j)|}, \quad (19)$$

where θ_j are the zeros of $h(\theta)$ and $h'(\theta_j)$ is the derivative of h evaluated in θ_j; U_1 becomes

$$U_1(\theta) = \frac{2\pi}{k\sin\alpha}\,\delta(\theta-\alpha). \quad (20)$$

When $\theta = \alpha$, the vector $\boldsymbol{v}$ results

$$\boldsymbol{v} = -k\sin\alpha\,(\cos\phi,\ 1+\sin\phi,\ 0), \quad (21)$$

and, as a consequence, eq. (14) results

$$E_2(\phi,\theta) \propto \left(\frac{a_0}{\sin\alpha}\right)^{1/2} E_0\,R(\varphi_S) \cdot \exp\left[-\left(\frac{a_0\sin\left(\frac{\pi/2-\phi}{2}\right)+x_0}{w_x}\right)^2\right] \cdot \cos^{1/2}\left(\frac{\pi/2-\phi}{2}\right)\delta(\theta-\alpha) \cdot \exp\left[-\mathrm{i}\left(k a_0(v_x^2+v_y^2)^{1/2} - \frac{\pi}{4}\right)\right],$$

being the intensity

$$I_2(\phi,\theta) \propto I_0\frac{a_0}{\sin\alpha}\,\mathcal{R}(\phi,\theta) \cdot \exp\left[-2\left(\frac{a_0\sin\left(\frac{\pi/2-\phi}{2}\right)+x_0}{w_x}\right)^2\right] \cdot \cos\left(\frac{\pi/2-\phi}{2}\right)\delta(\theta-\alpha), \quad (22)$$

where $I_0 = |E_0|^2$ and $\mathcal{R} = |R|^2$. The intensity is proportional to the reflectivity and to the cylinder radius a_0, and inversely proportional to $\sin\alpha$.

3. Analysis of the results

A. Width of the incident beam. In the first place, we will study the behaviour of the diffraction pattern in terms of the widths of the Gaussian beam. In fig. 3, the diffraction pattern, obtained according to eq. (16), for several values for the width w_z is shown. For large values of w_z (figs. 3a–b) the cone of light becomes very narrow, as we have seen in the previous section. In the other hand, when w_z decreases the cone widens, as it is shown in figs. 3c–f.

We have also studied how the intensity in the diffraction pattern changes when the parameter w_z varies. In fig. 4 we

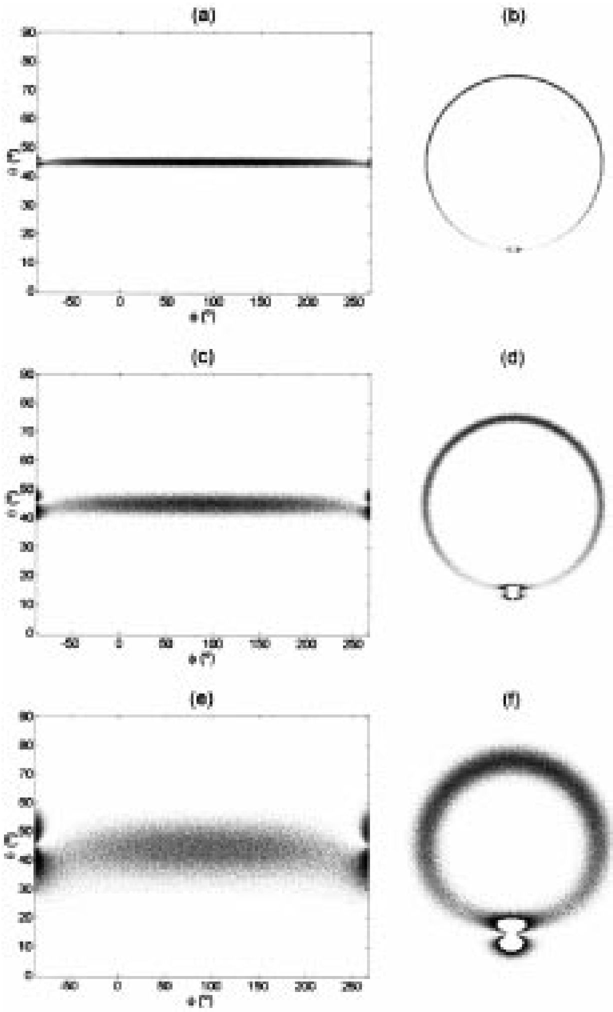

Fig. 3. Intensity distribution at the diffracted cone for different values of the width w_z a) and b) w_z = 1 μm, c) and d) w_z = 3 μm, e) and f) w_z = 10 μm. As the width w_x decreases the cone of light becomes wider. The wire radius is 50 μm with a perfect reflector behaviour, $\alpha = \pi/4$ and the wavelength is λ = 0.63 μm.

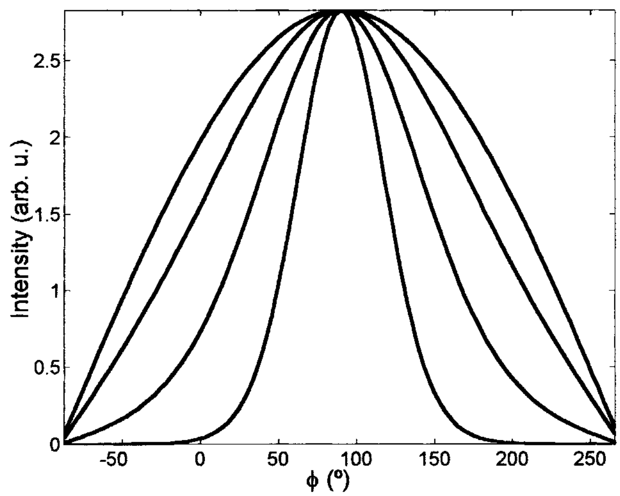

Fig. 4. Intensity profile at the diffracted cone for perfect reflector cylinder for several values of the width w_x = 1000, 200, 100, 50 μm. The cylinder radius is 100 μm with a perfect reflector behaviour, $\alpha = \pi/4$ and the wavelength is λ = 0.63 μm.

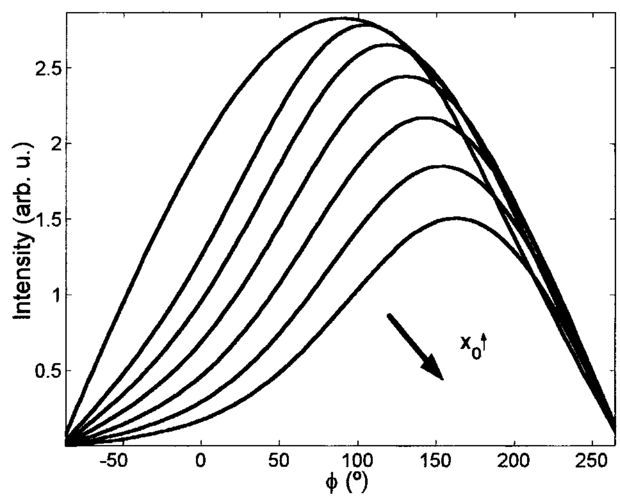

Fig. 5. Intensity profile for a cylinder, which surface is a perfect reflector, of radius a_0 = 100 μm, w_x = 200 μm and different misalignment parameter x_0 = 0, 25, 50, 75, 100, 125 and 150 μm. We have considered $w_z \gg \lambda$ and the angle of incidence is $\alpha = \pi/4$.

can see the intensity profile, when $w_z \gg \lambda$, at the scattered cone of light for several values of w_x. As the parameter w_z/a_0 decreases the edges of the cylinder have less incident light and the intensity distribution becomes narrower.

B. Misaligment. The misaligment x_0 between the cylinder and the Gaussian beam has also been considered in eqs. (14) and (22). In fig. 5 we can see that when the parameter x_0 increases the light that reaches the cylinder decreases, and the diffraction pattern becomes asymmetrical.

C. Angle of incidence. Another parameter that we have studied is the angle of incidence α. When $w_z \gg \lambda$ the intensity distribution only depends on the angle α indirectly through the reflection coefficient R (eq. 15), as it is shown in fig. 6a–b. However, when the beam width w_z is only a few microns, then the intensity distribution depends strongly on the angle of incidence, (fig. 6c).

D. Polarization and reflectivity. We have also analyzed the effect of the beam polarization and the cylinder's material on the diffracted intensity distribution. In eq. (14) it is shown that the effect of these parameters is included in the equations by the reflection coefficient R. In fig. 7 we show the diffracted intensity at the cone for several materials, for parallel and perpendicular polarizations, and for two angles of incidence. We can see that for normal incidence

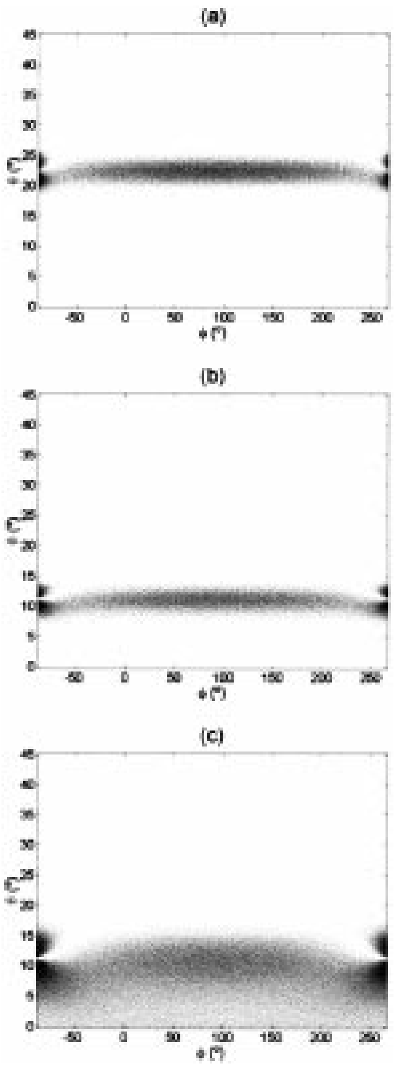

Fig. 6. Intensity distribution for a cylinder of radius $a_0 = 100$ μm for several incidence directions α. When the parameter w_z is higher than a few microns the intensity distribution does not depends appreciably on α as it can be seen in a) $\alpha = \pi/8$, $w_z = 5$ μm and b) $\alpha = \pi/16$, $w_z = 5$ μm. However, when w_z and α are small the intensity distribution of light changes, as it can be seen in c) $\alpha = \pi/16$, $w_z = 2$ μm. In this case the wavelength is $\lambda = 0.63$ μm.

$(\alpha = \pi/2)$, the intensity is the same for both polarizations for $\phi = \pi/2$. However, when the beam incides obliquely on the cylinder the intensity increases for the perpendicular polarization and it decreases for the parallel polarization. The difference between both polarizations is notorious for the case of tungsten (fig. 7e–f).

4. Conclusions

In this paper an analytical solution for the diffraction of a diagonally and off-axis Gaussian beam over a metallic cylinder is given. We see that the diffracted ligth generates a cone which intensity depends on the cylinder radius, the polarization of the beam and the reflection coefficient of the surface. We have also studied the effect of the beam widths and the misalignment between the cylinder and the Gaussian beam.

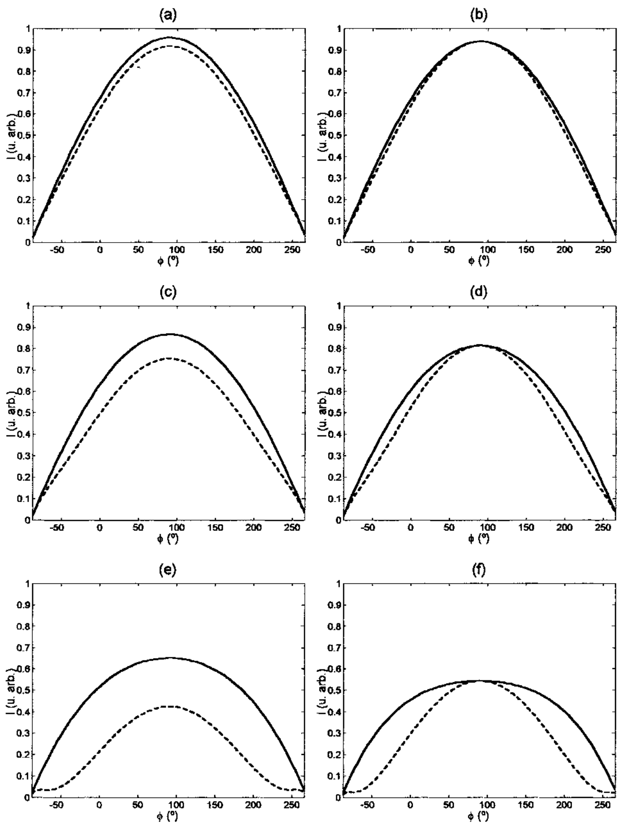

Fig. 7. Intensity profile in the specular direction ($\theta = \alpha$) for cylinders of different materials, for parallel (——) and perpendicular (– – –) polarizations: a) and b) silver $n = 0.2 + 3.44i$, c) and d) gold $n = 0.47 + 2.83i$, and e) and f) tungsten $n = 3.46 + 3.25i$. For a given material the first figure is for $\alpha = \pi/4$, and the second figure for $\alpha = \pi/2$. The radius of the cylinder is $a_0 = 100$ μm.

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