HODGE POLYNOMIALS OF THE $\mathrm{SL}(2,\mathbb{C})$-CHARACTER VARIETY OF AN ELLIPTIC CURVE WITH TWO MARKED POINTS

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ABSTRACT. We compute the Hodge polynomials for the moduli space of representations of an elliptic curve with two marked points into $\mathrm{SL}(2,\mathbb{C})$. When we fix the conjugacy classes of the representations around the marked points to be diagonal and of modulus one, the character variety is diffeomorphic to the moduli space of strongly parabolic Higgs bundles, whose Betti numbers are known. In that case we can recover some of the Hodge numbers of the character variety. We extend this result to the moduli space of doubly periodic instantons.

1. INTRODUCTION

Let X be a projective algebraic curve of genus $g \geq 1$ and $x_1, \ldots, x_s \in X$ a collection of marked points. Let G be a complex reductive Lie group. The G-character variety of X with marked points x_i is defined as the moduli space of representations of $\pi_1(X - \{x_1, \ldots, x_s\})$ into G. For this, we fix conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_s \subset G$. The G-character variety is the space

$$\mathcal{R}_{\mathcal{C}_1,\ldots,\mathcal{C}_s}(X,G) = \Big\{(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_s) \in G^{2g+s} \mid \prod_{i=1}^{g}[A_i, B_i]\prod_{j=1}^{s} C_j = \mathrm{Id}, C_j \in \mathcal{C}_j, 1 \leq j \leq s\Big\}//G,$$

where G acts by simultaneous conjugation. This is the space of equivalence classes of representations where the holonomy around the punctures x_j has been fixed to be of type $\mathcal{C}_j$, $j = 1, \ldots, s$. If $\mathcal{C}_j = [D_j]$, that is $D_j \in \mathcal{C}_j$, then we may write $\mathcal{R}_{D_1,\ldots,D_s}(X,G)$ instead of $\mathcal{R}_{\mathcal{C}_1,\ldots,\mathcal{C}_s}(X,G)$.

The topology and geometry of G-character varieties has been studied extensively in the last two decades, starting with the foundational work [11]. Recently interest has been given to the algebro-geometric structure of G-character varieties mainly because of the implications to Mirror Symmetry [7, 10]. For this, computations of the Hodge-Deligne polynomials have been done for a number of G-character varieties, mainly using arithmetic and combinatorial techniques [9, 8, 14]. In [12] a geometric method has been

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introduced to compute Hodge-Deligne polynomials of character varieties by analysing the spaces of matrices

$$Z_{\mathcal{C}_1,\ldots,\mathcal{C}_s}(X,G) = \\ = \left\{ (A_1,B_1,\ldots,A_g,B_g,C_1,\ldots,C_s) \in G^{2g+s} \mid \prod_{i=1}^{g}[A_i,B_i]\prod_{j=1}^{s}C_j = \mathrm{Id}, C_j \in \mathcal{C}_j, 1 \leq j \leq s \right\}$$

explicitly (using geometric decompositions of the spaces). The technique is based on the work [15] of the second author. In [12], we dealt with the case $G = \mathrm{SL}(2,\mathbb{C})$, $s = 1$ and $g = 1,2$. In [13] the case $s = 1$, $g = 3$ is done. We address the case $s = 2$, $g = 1$ in this paper. This moduli space is of interest since, when the conjugacy classes are diagonal and the eigenvalues have modulus different from one (see [1, Theorem 0.4]), it appears as the moduli space of doubly periodic instantons. Indeed, the $\mathrm{SL}(2,\mathbb{C})$-character variety with diagonal and different conjugacy classes around the marked points is diffeomorphic to the moduli space of stable parabolic Higgs bundles of parabolic degree 0 and traceless Higgs field. This last moduli space, when the residue of the Higgs field is not nilpotent, is isomorphic, with the same complex structure, to the moduli space of doubly periodic instantons through the Nahm transform [1, 6]. Actually, the moduli space is hyperkähler and both complex structures – the one given as $\mathrm{SL}(2,\mathbb{C})$-character variety, and the one given as moduli space of parabolic Higgs bundles (nilpotent or non-nilpotent) – are two of the complex structures in the family.

When the conjugacy classes are diagonal and the eigenvalues have modulus equal to one, the character variety is diffeomorphic to the moduli space of parabolic Higgs bundles, with nilpotent Higgs field, for which the Betti numbers are known [2]. As the character varieties are diffeomorphic for different values of the eigenvalues (having modulus equal to one is not relevant), the Betti numbers are the same for the moduli space of double periodic instantons. So our results in Section 7 provide some Hodge numbers for the character variety diffeomorphic to the moduli space of doubly periodic instantons.

The case of several marked points has also been studied because of its relation to parabolic bundles [5]. Here we want to address the first case of the computation of Hodge-Deligne polynomials of character varieties for several marked points, namely the case of $G = \mathrm{SL}(2,\mathbb{C})$, $g = 1$ and $s = 2$. For $G = \mathrm{SL}(2,\mathbb{C})$, there are five types of conjugacy classes, determined by the elements Id, $-\mathrm{Id}$, $J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, and the diagonal matrices $\xi_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, for $\lambda \in \mathbb{C} - \{0, \pm 1\}$. The last type is determined by λ up to $\lambda \sim \lambda^{-1}$.

Therefore there are 25 possible character varieties

$$\mathcal{R}_{\mathcal{C}_1,\mathcal{C}_2}(X, \mathrm{SL}(2,\mathbb{C})).$$

The symmetry between $\mathcal{C}_1, \mathcal{C}_2$ reduces the number of cases to 15 (see Lemma 3.1). Moreover, the cases where $\mathcal{C}_1 = [\mathrm{Id}]$ or $\mathcal{C}_2 = [-\mathrm{Id}]$, correspond basically to the case of a one puncture elliptic curve, computed in [12]. This means that there are 6 cases left. Our main result is the computation of the Hodge-Deligne polynomials of these character varieties (see Section 2 for the definition of the Hodge-Deligne polynomial of an algebraic variety). These are as follows.

Theorem 1.1. *We have the following:*

(1) $e(\mathcal{R}_{J_+,J_+}(X, \mathrm{SL}(2,\mathbb{C}))) = e(\mathcal{R}_{J_-,J_-}(X, \mathrm{SL}(2,\mathbb{C}))) = q^4 + q^3 - q + 7$.
(2) $e(\mathcal{R}_{J_+,J_-}(X, \mathrm{SL}(2,\mathbb{C}))) = q^4 - 3q^2 - 6q$.
(3) $e(\mathcal{R}_{J_+,\xi_\lambda}(X, \mathrm{SL}(2,\mathbb{C}))) = e(\mathcal{R}_{J_-,\xi_\lambda}(X, \mathrm{SL}(2,\mathbb{C}))) = q^4 + q^3 + 2q^2 + q + 1$.
(4) $e(\mathcal{R}_{\xi_\lambda,\xi_\mu}(X, \mathrm{SL}(2,\mathbb{C}))) = q^4 + 2q^3 + 6q^2 + 2q + 1$, *for* $\mu \neq \lambda^{\pm 1}$.
(5) $e(\mathcal{R}_{\xi_\lambda,\xi_\lambda}(X, \mathrm{SL}(2,\mathbb{C}))) = q^4 + q^3 + 8q^2 + q + 1$.

The moduli spaces $\mathcal{R}_{\mathcal{C}_1,\mathcal{C}_2}(X, \mathrm{SL}(2,\mathbb{C}))$ *contain reducibles in cases (1) and (5).*

We recall that in a GIT quotient, the reducibles are the points with non-trivial stabilizers. These produce lower-dimensional orbirts. All orbits which contain such lower-dimensional orbits in their closure must be identified in the GIT quotient (these are the semistable points, and the identification is commonly known as S-equivalence).

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2. Hodge-Deligne polynomials

Our main goal is to compute the Hodge-Deligne polynomial of the $\mathrm{SL}(2,\mathbb{C})$-character variety of an elliptic curve with two marked points. We will follow the methods in [12], so we collect some basic results from [12] in this section.

We start by reviewing the definition of the Hodge-Deligne polynomial. A pure Hodge structure of weight k consists of a finite dimensional complex vector space H with a real structure, and a decomposition $H = \bigoplus_{k=p+q} H^{p,q}$ such that $H^{q,p} = \overline{H^{p,q}}$, the bar meaning complex conjugation on H. A Hodge structure of weight k gives rise to the so-called Hodge filtration, which is a descending filtration $F^p = \bigoplus_{s\geq p} H^{s,k-s}$. We define $\mathrm{Gr}^p_F(H) := F^p/F^{p+1} = H^{p,k-p}$.

A mixed Hodge structure consists of a finite dimensional complex vector space H with a real structure, an ascending (weight) filtration $\ldots \subset W_{k-1} \subset W_k \subset \ldots \subset H$ (defined over $\mathbb{R}$) and a descending (Hodge) filtration F such that F induces a pure Hodge structure of weight k on each $\mathrm{Gr}^W_k(H) = W_k/W_{k-1}$. We define

$$H^{p,q} := \mathrm{Gr}^p_F \mathrm{Gr}^W_{p+q}(H)$$

and write $h^{p,q}$ for the *Hodge number* $h^{p,q} := \dim H^{p,q}$.

Let Z be any quasi-projective algebraic variety (maybe non-smooth or non-compact). The cohomology groups $H^k(Z)$ and the cohomology groups with compact support $H^k_c(Z)$

are endowed with mixed Hodge structures [4]. We define the *Hodge numbers* of Z by

$$\begin{aligned} h^{k,p,q}(Z) &= h^{p,q}(H^k(Z)) = \dim \mathrm{Gr}^p_F \mathrm{Gr}^W_{p+q} H^k(Z), \\ h^{k,p,q}_c(Z) &= h^{p,q}(H^k_c(Z)) = \dim \mathrm{Gr}^p_F \mathrm{Gr}^W_{p+q} H^k_c(Z). \end{aligned}$$

The Hodge-Deligne polynomial, or E-polynomial is defined as

$$e(Z) = e(Z)(u,v) := \sum_{p,q,k} (-1)^k h^{k,p,q}_c(Z) u^p v^q.$$

When $h^{k,p,q}_c = 0$ for $p \neq q$, the polynomial $e(Z)$ depends only on the product uv. This will happen in all the cases that we shall investigate here. In this situation, it is conventional to use the variable $q = uv$. If this happens, we say that the variety is *of balanced type*. For instance, $e(\mathbb{C}^n) = q^n$.

The key property of Hodge-Deligne polynomials that permits their calculation is that they are additive for stratifications of Z. If Z is a complex algebraic variety and $Z = \bigsqcup_{i=1}^n Z_i$, where all Z_i are locally closed in Z, then $e(Z) = \sum_{i=1}^n e(Z_i)$,

Proposition 2.1 (Proposition 2.4 in [12])**.** *Suppose that B is connected and $\pi : Z \to B$ is an algebraic fibre bundle with fibre F (not necessarily locally trivial in the Zariski topology) and that the action of $\pi_1(B)$ on $H^*_c(F)$ is trivial. Suppose that Z, F, B are smooth. Then $e(Z) = e(F)e(B)$.*

The hypotheses of Proposition 2.1 hold in particular in the following cases:

- B is irreducible and π is locally trivial in the Zariski topology.
- π is a principal G-bundle with G a connected algebraic group.

We shall use the above in the following form. Suppose that Z is a space with a free action of an algebraic group G, $H \subset G$ is a connected subgroup and $\overline{Z} \subset Z$ is a subset such that $G\overline{Z} = Z$ and

$$Hz_0 = Gz_0 \cap \overline{Z}, \text{ for any } z_0 \in \overline{Z}. \tag{1}$$

Then, in particular $Z/G = \overline{Z}/H$. In this case we have an H-bundle $G \times \overline{Z} \to Z$. Applying the above, $e(Z) = e(G \times \overline{Z})/e(H)$. Hence we can write

$$e(Z) = e(\overline{Z})e(G/H). \tag{2}$$

We need to recall some Hodge-Deligne polynomials from [12]. First, we have that $e(\mathrm{SL}(2,\mathbb{C})) = q^3 - q$ and $e(\mathrm{PGL}(2,\mathbb{C})) = q^3 - q$. Consider the following subsets of $\mathrm{SL}(2,\mathbb{C})$:

- $W_0 :=$ conjugacy class of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It has $e(W_0) = 1$.
- $W_1 :=$ conjugacy class of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. It has $e(W_1) = 1$.

- W_2 := conjugacy class of $J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It is $W_2 \cong \mathrm{PGL}(2,\mathbb{C})/U$, with $U = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \mid y \in \mathbb{C} \right\}$. It has $e(W_2) = q^2 - 1$.
- W_3 := conjugacy class of $J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. It is $W_3 \cong \mathrm{PGL}(2,\mathbb{C})/U$ and $e(W_3) = q^2 - 1$.
- $W_{4,\lambda}$:= conjugacy class of $\xi_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $\lambda \in \mathbb{C} - \{0, \pm 1\}$. Note that $W_{4,\lambda} = W_{4,\lambda^{-1}}$, since the matrices ξ_λ and $\xi_{\lambda^{-1}}$ are conjugated. We have $W_{4,\lambda} \cong \mathrm{PGL}(2,\mathbb{C})/D$, where $D = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in \mathbb{C}^* \right\}$. So $e(W_{4,\lambda}) = q^2 + q$.
- We also need the set $W_4 := \{A \in \mathrm{SL}(2,\mathbb{C}) \mid \mathrm{Tr}\,(A) \neq \pm 2\}$, which is the union of the conjugacy classes $W_{4,\lambda}$, $\lambda \in \mathbb{C} - \{0, \pm 1\}$. This has $e(W_4) = e(\mathrm{SL}(2,\mathbb{C})) - e(W_0) - e(W_1) - e(W_2) - e(W_3) = q^3 - 2q^2 - q$.

Now consider the map

$$\begin{array}{rcl} f : \mathrm{SL}(2,\mathbb{C})^2 & \longrightarrow & \mathrm{SL}(2,\mathbb{C}) \\ (A,B) & \mapsto & [A,B] = ABA^{-1}B^{-1} \end{array}$$

Note that f is equivariant under the action of $\mathrm{SL}(2,\mathbb{C})$ by conjugation on both spaces. We stratify $X = \mathrm{SL}(2,\mathbb{C})^2$ as follows

- $X_0 := f^{-1}(W_0)$,
- $X_1 := f^{-1}(W_1)$,
- $X_2 := f^{-1}(W_2)$,
- $X_3 := f^{-1}(W_3)$,
- $X_4 := f^{-1}(W_4)$.

We also introduce the varieties $f^{-1}(C)$ for fixed $C \in \mathrm{SL}(2,\mathbb{C})$ and define accordingly

- $\overline{X}_0 := f^{-1}(\mathrm{Id}) = X_0$,
- $\overline{X}_1 := f^{-1}(-\mathrm{Id}) = X_1$,
- $\overline{X}_2 := f^{-1}(J_+)$. Then there is a fibration $U \to \mathrm{PGL}(2,\mathbb{C}) \times \overline{X}_2 \to X_2$, and by (2), $e(X_2) = (q^2 - 1)e(\overline{X}_2)$. Note that the action of $\mathrm{PGL}(2,\mathbb{C})$ on X_2 is free because there are no reducibles (see [12]).
- $\overline{X}_3 := f^{-1}(J_-)$. Again there is a fibration $U \to \mathrm{PGL}(2,\mathbb{C}) \times \overline{X}_3 \to X_3$, and $e(X_3) = (q^2 - 1)e(\overline{X}_3)$.
- $\overline{X}_{4,\lambda} := f^{-1}(\xi_\lambda)$, for $\lambda \neq 0, \pm 1$. We define also $X_{4,\lambda} = f^{-1}(W_{4,\lambda})$. There is a fibration $D \to \mathrm{PGL}(2,\mathbb{C}) \times \overline{X}_{4,\lambda} \to X_{4,\lambda}$, and $e(X_{4,\lambda}) = (q^2 + q)e(\overline{X}_{4,\lambda})$.

It will also be convenient to define

- $\overline{X}_4 := \left\{ (A, B, \lambda) \mid [A, B] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \neq 0, \pm 1,\ A,\ B \in \mathrm{SL}(2,\mathbb{C}) \right\}$.

There is an action of $\mathbb{Z}_2$ on $\overline{X}_4$ given by interchanging $(A,B,\lambda)\mapsto(P^{-1}AP,P^{-1}BP,\lambda^{-1})$, with $P=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. The map $\overline{X}_4\to\overline{X}_4/\mathbb{Z}_2$ is equivalent to the map $(A,B,\lambda)\to(A,B)$, so

$$\overline{X}_4/\mathbb{Z}_2=\left\{(A,B)\,|\,[A,B]=\begin{pmatrix}\lambda&0\\0&\lambda^{-1}\end{pmatrix},\lambda\neq 0,\pm1,\ A,\ B\in \mathrm{SL}(2,\mathbb{C})\right\}$$

coincides with the union of all $\overline{X}_{4,\lambda}$.

The Hodge-Deligne polynomials computed in [12] are as follows:

$$\begin{aligned}
e(X_0)&=q^4+4q^3-q^2-4q\\
e(X_1)&=e(\mathrm{PGL}(2,\mathbb{C}))=q^3-q\\
e(\overline{X}_2)&=q\big((q-1)^2-4\big)=q^3-2q^2-3q\\
e(X_2)&=(q^2-1)(q^3-2q^2-3q)=q^5-2q^4-4q^3+2q^2+3q\\
e(\overline{X}_3)&=q(q^2+3q)=q^3+3q^2\\
e(X_3)&=(q^2-1)(q^3+3q^2)=q^5+3q^4-q^3-3q^2\\
e(\overline{X}_{4,\lambda})&=(q-1)(q^2+4q+1)=q^3+3q^2-3q-1\\
e(X_{4,\lambda})&=(q^2+q)(q^3+3q^2-3q-1)=q^5+4q^4-4q^2-q\\
e(\overline{X}_4)&=q^4-3q^3-6q^2+5q+3\\
e(\overline{X}_4/\mathbb{Z}_2)&=q^4-2q^3-3q^2+3q+1\\
e(X_4)&=q^6-2q^5-4q^4+3q^2+2q
\end{aligned}$$

We will call holonomies of Jordan type those which belong to one of the conjugacy classes J_+ and J_-, and of diagonalisable type those which belong to Id, $-\mathrm{Id}$ or ξ_λ.

3. Holonomies of Jordan type

Let $\mathcal{C}_1,\mathcal{C}_2$ be conjugacy classes in $\mathrm{SL}(2,\mathbb{C})$. We want to study the set

$$Z(\mathcal{C}_1,\mathcal{C}_2)=\{(A,B,C_1,C_2)\in \mathrm{SL}(2,\mathbb{C})\,|\,[A,B]C_1C_2=\mathrm{Id},C_1\in\mathcal{C}_1,C_2\in\mathcal{C}_2\}$$

and

$$\mathcal{R}_{\mathcal{C}_1,\mathcal{C}_2}=Z(\mathcal{C}_1,\mathcal{C}_2)//\,\mathrm{PGL}(2,\mathbb{C}).$$

Recall that there are five types of conjugacy classes in $\mathrm{SL}(2,\mathbb{C})$, namely Id, $-\mathrm{Id}$, $J_+=\begin{pmatrix}1&1\\0&1\end{pmatrix}$, $J_-=\begin{pmatrix}-1&1\\0&-1\end{pmatrix}$, and $\xi_\lambda=\begin{pmatrix}\lambda&0\\0&\lambda^{-1}\end{pmatrix}$, for $\lambda\neq0,\pm1$, and defined up to $\lambda\sim\lambda^{-1}$. Therefore there is a total of 25 possible combinations for $\mathcal{C}_1,\mathcal{C}_2$. The following symmetry reduces the number of cases.

Lemma 3.1. *$Z(\mathcal{C}_1,\mathcal{C}_2)\cong Z(\mathcal{C}_2,\mathcal{C}_1)$.*

Proof. The equation $[A,B]C_1C_2=\mathrm{Id}$ is equivalent to $[A,B]=C_2^{-1}C_1^{-1}$. This can be inverted to give $[B^{-1},A^{-1}]=C_1C_2$, i.e. $[B^{-1},A^{-1}]C_2^{-1}C_1^{-1}=\mathrm{Id}$. So the map $(A,B,C_1,C_2)\mapsto(B^{-1},A^{-1},C_2^{-1},C_1^{-1})$ gives the required isomorphism. Note that when $C\in\mathcal{C}$, C^{-1} runs over $\mathcal{C}$ again. □

If $\mathcal{C}_1 = W_0 = \{\mathrm{Id}\}$, then

- $Z_{0j} = Z(W_0, W_j) = X_j$, $j = 0, 1, 2, 3$
- $Z_{04}^\lambda = Z(W_0, W_{4,\lambda}) = X_{4,\lambda}$
- $Z_{04} = \bigcup Z_{04}^\lambda = X_4$

If $\mathcal{C}_1 = W_1 = \{-\mathrm{Id}\}$, then

- $Z_{11} = Z(W_1, W_1) = X_0$
- $Z_{12} = Z(W_1, W_2) = X_3$
- $Z_{13} = Z(W_1, W_3) = X_2$
- $Z_{14}^\lambda = Z(W_0, W_{4,\lambda}) = X_{4,-\lambda}$
- $Z_{14} = \bigcup Z_{14}^\lambda = X_4$

The Hodge-Deligne polynomials have been computed in [12]. They are

- $e(Z_{00}) = e(Z_{11}) = q^4 + 4q^3 - q^2 - 4q$
- $e(Z_{01}) = q^3 - q$
- $e(Z_{02}) = e(Z_{13}) = q^5 - 2q^4 - 4q^3 + 2q^2 + 3q$
- $e(Z_{03}) = e(Z_{12}) = q^5 + 3q^4 - q^3 - 3q^2$
- $e(Z_{04}^\lambda) = e(Z_{14}^\lambda) = q^5 + 4q^4 - 4q^2 - q$

and the Hodge-Deligne polynomials of the character varieties are [12, Theorem 1.1]

- $e(\mathcal{R}_{\mathrm{Id},\mathrm{Id}}) = e(\mathcal{R}_{-\mathrm{Id},-\mathrm{Id}}) = q^2 + 1$
- $e(\mathcal{R}_{\mathrm{Id},-\mathrm{Id}}) = 1$
- $e(\mathcal{R}_{\mathrm{Id},J_+}) = e(\mathcal{R}_{-\mathrm{Id},J_-}) = q^2 - 2q + 3$
- $e(\mathcal{R}_{\mathrm{Id},J_-}) = e(\mathcal{R}_{-\mathrm{Id},J_+}) = q^2 + 3q$
- $e(\mathcal{R}_{\mathrm{Id},\xi_\lambda}) = e(\mathcal{R}_{-\mathrm{Id},\xi_\lambda}) = q^2 + 4q + 1$

Now we move to the cases when the holonomy around the punctures is given by the Jordan forms.

3.1. The case $\mathcal{C}_1 = W_2 = [J_+]$, $\mathcal{C}_2 = W_2 = [J_+]$.

Let

$$Z(W_2, W_2) = Z_{22} = \{(A, B, C_1, C_2) \,|\, C_1, C_2 \in W_2, [A, B]C_1 = C_2^{-1}\},$$
$$\overline{Z}(W_2, W_2) = \overline{Z}_{22} = \{(A, B, C) \,|\, C \in W_2, [A, B]C = J_+\}.$$

The action of PGL$(2,\mathbb{C})$ on Z_{22} is free except when a non-trivial element P fixes simultaneously A, B, C_1, C_2. Write $C_2 = QJ_+Q^{-1}$, for some $Q \in \mathrm{PGL}(2,\mathbb{C})/U$. Then $P \in QUQ^{-1}$, and hence $A, B \in Q(U \cup (-U))Q^{-1}$. So $[A, B] = \mathrm{Id}$ and $C_1 = C_2^{-1}$. Analogously, the action of U on points of $\overline{Z}_{22}$ is free except when $A, B \in U \cup (-U)$, $C = J_+$. The set of reducibles is thus

$$\mathcal{D} = \{(A, B, C) | A, B \in Q(U \cup (-U))Q^{-1}, C_1^{-1} = C_2 = QJ_+Q^{-1} \in W_2\} \subset Z_{22},$$
$$\overline{\mathcal{D}} = \{(A, B, C) | A, B \in U \cup (-U), C = J_+\} \subset \overline{Z}_{22}.$$

Denote by $Z_{22}^* = Z_{22} - \mathcal{D}$ and $\overline{Z}_{22}^* = \overline{Z}_{22} - \overline{\mathcal{D}}$ the set of irreducible representations. Then PGL$(2,\mathbb{C})$ acts freely on Z_{22}^*, clearly PGL$(2,\mathbb{C})\,\overline{Z}_{22} = Z_{22}$, and also Condition (1) holds:

if $(A,B,C) \in \overline{Z}_{22}$ and $P \in \mathrm{PGL}(2,\mathbb{C})$ satisfies that $(PAP^{-1}, PBP^{-1}, PCP^{-1}) \in \overline{Z}_{22}$, then P fixes J_+, so $P \in U$. Therefore there is a fibration

$$U \to \mathrm{PGL}(2,\mathbb{C}) \times \overline{Z}_{22}^* \to Z_{22}^*,$$

so $e(Z_{22}^*) = (q^2-1)e(\overline{Z}_{22}^*)$.

Let $(A,B,C) \in \overline{Z}_{22}$. As $C \sim J_+$, there is some $P \in \mathrm{SL}(2,\mathbb{C})$, well-defined up to sign and up to multiplication by U on the right, such that $C = PJ_+P^{-1}$. The equation $[A,B]PJ_+P^{-1} = J_+$ is rewritten as $[A,B][P,J_+] = \mathrm{Id}$. So

$$\overline{Z}_{22} = \{(A,B,P) \in \mathrm{SL}(2,\mathbb{C})^3 |\, [A,B][P,J_+] = \mathrm{Id}\}/\mathbb{Z}_2 \times U,$$

where $\mathbb{Z}_2$ acts by $P \mapsto -P$, and U acts by multiplication on the right on P. This is a free action.

Write $P = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, then we have

$$[P,J_+] = \begin{pmatrix} 1-xz & -1+x(x+z) \\ -z^2 & 1+(x+z)z \end{pmatrix}. \tag{3}$$

Note that the trace is

$$t = \mathrm{Tr}\,[P,J_+] = 2 + z^2 \,.$$

The action of U on P moves y and w, so the class of P modulo $\mathbb{Z}_2 \times U$ is determined by $(x,z) \in \mathbb{C}^2 - \{0\}$, up to sign. Hence

$$\overline{Z}_{22} \cong \{(A,B,(x,z)) \in \mathrm{SL}(2,\mathbb{C})^2 \times ((\mathbb{C}^2 - \{0\})/\pm) \,|\, [A,B]^{-1} = [P,J_+]\}.$$

The space $\overline{Z}_{22}$ is stratified as follows:

(1) If $z=0$ then $x \neq 0$. So $P = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}$. For $x \neq \pm 1$, we have $[P,J_+] \sim J_+$ so $(A,B) \in \overline{X}_2$. Hence the contribution to the Hodge-Deligne polynomial is $e((\mathbb{C}^* - \{\pm 1\})/\pm)e(\overline{X}_2) = (q-2)e(\overline{X}_2)$.
(2) $z=0$ and $x = \pm 1$. As it is defined up to sign, we can arrange $x=1$. So $P = \mathrm{Id}$, and $(A,B) \in X_0$. The contribution is $e(X_0)$.
(3) $z = \pm 2i$. As it is defined up to sign, can choose $z = 2i$. Then $[P,J_+] \sim J_-$, and hence the contribution is $q\, e(\overline{X}_3)$.
(4) $z \neq 0, \pm 2i$. Now we have a fibration $(A,B) \mapsto t = 2+z^2$, where z is defined up to sign, so $z^2 \in \mathbb{C} - \{0,-4\}$. Take $v = xz$, $v \in \mathbb{C}$. So $[P,J_+] = \begin{pmatrix} 1-v & * \\ -z^2 & 1+v+z^2 \end{pmatrix}$. Hence the Hodge-Deligne polynomial is $e(\mathbb{C})e(\overline{X}_4/\mathbb{Z}_2)$.

Putting all together, and using $e(\overline{\mathcal{D}}) = 4q^2$, we have

$$\begin{aligned} e(\overline{Z}_{22}) &= (q-2)e(\overline{X}_2) + qe(\overline{X}_3) + e(X_0) + qe(\overline{X}_4/\mathbb{Z}_2) \\ &= q^5 + q^4 + 3q^2 + 3q, \\ e(\overline{Z}_{22}^*) &= e(\overline{Z}_{22}) - e(\overline{\mathcal{D}}) = q^5 + q^4 - q^2 + 3q \\ e(Z_{22}^*) &= (q^2-1)e(\overline{Z}_{22}^*) = q^7 + q^6 - q^5 - 2q^4 + 3q^3 + q^2 - 3q. \end{aligned}$$

Finally, we compute the Hodge-Deligne polynomial of $\mathcal{R}_{W_2,W_2} = Z_{22}// \mathrm{PGL}(2,\mathbb{C})$. It is clear that $Z_{22}^*/ \mathrm{PGL}(2,\mathbb{C}) = \overline{Z}_{22}^*/U$. The contribution of the reducibles is as follows. First note that $\mathcal{D}/\mathrm{PGL}(2,\mathbb{C}) = \overline{\mathcal{D}}/U$. Now let $(A,B,C) \in \overline{\mathcal{D}}$. Then $A, B \in U \cup (-U), C = J_+$. If $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, we consider $A' = \begin{pmatrix} x & a \\ 0 & x^{-1} \end{pmatrix}$, $B' = \begin{pmatrix} y & b \\ 0 & y^{-1} \end{pmatrix}$, with $(x-x^{-1})b = (y-y^{-1})a =: \eta ab$, so $[A', B'] = [A, B]$. When $x, y \to 1$, we have $A' \to A, B' \to B$. The action of $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ on A', B' takes $(a, b) \mapsto (a + \alpha(x - x^{-1}), b + \alpha(y - y^{-1})) = (a + \alpha\eta a, b + \alpha\eta b)$. So going to the limit $x, y \to 1$, $(a, b) \sim (a+\alpha\eta a, b+\alpha\eta b)$. For $\alpha = -\eta^{-1}$, they converge to $(\mathrm{Id}, \mathrm{Id})$. Taking into account the possible signs, this means that the contribution of $\overline{\mathcal{D}}$ in the quotient consists of 4 points.

So the contribution is

$$e(\mathcal{R}_{W_2,W_2}) = e(\overline{Z}_{22}^*)/e(U) + 4 = q^4 + q^3 - q + 7.$$

3.2. **The case $\mathcal{C}_1 = W_3 = [J_-]$, $\mathcal{C}_2 = W_3 = [J_-]$.** , There is an isomorphism

$$\begin{aligned} Z(\mathcal{C}_1, \mathcal{C}_2) &= \{(A, B, C_1, C_2) \,|\, C_1 \in \mathcal{C}_1, C_2 \in \mathcal{C}_2, [A, B]C_1C_2 = \mathrm{Id}\} \\ &= \{(A, B, C_1, C_2) \,|\, C_1 \in \mathcal{C}_1, C_2 \in \mathcal{C}_2, [A, B](-C_1)(-C_2) = \mathrm{Id}\} \\ &= Z(-\mathcal{C}_1, -\mathcal{C}_2). \end{aligned}$$

In particular

$$Z(W_3, W_3) = Z(W_2, W_2),$$

using that $W_3 = [J_-] = [-J_+] = -W_2$.

Therefore

$$e(\mathcal{R}_{W_3,W_3}) = q^4 + q^3 - q + 7.$$

3.3. **The case $\mathcal{C}_1 = W_2 = [J_+]$, $\mathcal{C}_2 = W_3 = [J_-]$.** Now we choose $W_2 = [J_+]$ but $W_3 = [-J_+]$, using that $-J_+ \sim J_-$. So let

$$\begin{aligned} Z(W_2, W_3) = Z_{23} &= \{(A, B, C_1, C_2) \,|\, C_1 \in [J_+], C_2 \in [J_+], [A, B]C_1 = -C_2^{-1}\}, \\ \overline{Z}(W_2, W_3) = \overline{Z}_{23} &= \{(A, B, C) \,|C \in W_2,\ [A, B]C = -J_+\}. \end{aligned}$$

So

$$\overline{Z}(W_2, W_3) = \overline{Z}_{23} = \{(A, B, P) \,|\, [A, B][P, J_+] = -\mathrm{Id}\}/\mathbb{Z}_2 \times U.$$

Note that $\mathrm{PGL}(2,\mathbb{C})$ acts freely on Z_{23}, since if a non-trivial P fixes (A, B, C_1, C_2) then $[A, B] = \mathrm{Id}$, and this cannot be possible. Also Condition (1) holds here, which is proved as in Subsection 3.1. This means that there is a fibration $U \to \mathrm{PGL}(2,\mathbb{C}) \times \overline{Z}_{23} \to Z_{23}$, so $e(Z_{23}) = (q^2 - 1)e(\overline{Z}_{23})$.

Writing $P = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, we have

$$[P, J_+] = \begin{pmatrix} 1 - xz & -1 + x(x + z) \\ -z^2 & 1 + (x + z)z \end{pmatrix},$$

and the trace is

$$t = \operatorname{Tr}[P, J_+] = 2 + z^2.$$

The space $\overline{Z}_{23}$ is stratified as follows:

(1) If $z = 0$ and $x \neq \pm 1$, then $[P, J_+] \sim J_+$ so $[A, B] \sim J_-$ and $(A, B) \in \overline{X}_3$. As x is defined up to sign, we get the contribution $e((\mathbb{C}^* - \{\pm 1\})/\pm) e(\overline{X}_3) = (q-2)e(\overline{X}_3)$.
(2) $z = 0$ and $x = \pm 1$. As it is defined up to sign, we can arrange $x = 1$. So $[P, J_+] = \mathrm{Id}$ and $(A, B) \in X_1$. So we get the contribution $e(X_1)$.
(3) If $z = \pm 2i$, as it is defined up to sign, can choose $z = 2i$. Then $[P, J_+] \sim J_-$. As $x \in \mathbb{C}$, we have the contribution $q\, e(\overline{X}_2)$.
(4) $z \neq 0, \pm 2i$. Now we have a fibration $(A, B) \mapsto t = 2 + z^2$, where z is defined up to sign, so $z^2 \in \mathbb{C} - \{0, -4\}$. Take $v = xz$, $v \in \mathbb{C}$. So $[P, J_+] = \begin{pmatrix} 1 - v & * \\ -z^2 & 1 + v + z^2 \end{pmatrix}$. Hence the Hodge-Deligne polynomial is $e(\mathbb{C}) e(\overline{X}_4/\mathbb{Z}_2)$.

Putting all together,

$$\begin{aligned} e(\overline{Z}_{23}) &= (q-2)e(\overline{X}_3) + e(X_1) + qe(\overline{X}_2) + qe(\overline{X}_4/\mathbb{Z}_2) \\ &= q^5 - 3q^3 - 6q^2, \\ e(Z_{23}) &= (q^2 - 1)e(\overline{Z}_{23}) = q^7 - 4q^5 - 6q^4 + 3q^3 + 6q^2. \end{aligned}$$

Finally, we want to compute the Hodge-Deligne polynomial of $\mathcal{R}_{W_2,W_3} = Z_{23}/\!/ \operatorname{PGL}(2, \mathbb{C})$. In this case the action is free, and there are no reducibles. So

$$e(\mathcal{R}_{W_2,W_3}) = q^4 - 3q^2 - 6q.$$

4. One holonomy of Jordan type and the other of diagonalisable type

4.1. **The case $\mathcal{C}_1 = W_2$, $\mathcal{C}_2 = W_{4,\lambda}$.** Now we have

$$\begin{aligned} Z(W_2, W_{4,\lambda}) &= Z_{24}^\lambda = \{(A, B, C_1, C_2) \,|\, C_1 \in W_2, C_2 \in W_{4,\lambda}, [A, B]C_1 = C_2^{-1}\}, \\ \overline{Z}(W_2, W_{4,\lambda}) &= \overline{Z}_{24}^\lambda = \{(A, B, C) \,|\, C \in W_2, [A, B]C = D\}, \end{aligned}$$

where $D = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$. The action of $\operatorname{PGL}(2, \mathbb{C})$ is free as in Subsection 3.3, and Condition (1) holds again, which is proved as in Subsection 3.1. Therefore there is a fibration $\mathbb{C}^* \to \operatorname{PGL}(2, \mathbb{C}) \times \overline{Z}_{24}^\lambda \to Z_{24}^\lambda$, and $e(Z_{24}^\lambda) = (q^2 + q)e(\overline{Z}_{24}^\lambda)$.

Writing $C = PJ_+P^{-1}$, we get

$$\overline{Z}_{24}^\lambda \cong \{(A, B, P) \,|\, [A, B][P, J_+] = DJ_+^{-1}\}/\mathbb{Z}_2 \times U,$$

where $DJ_+^{-1} = \begin{pmatrix} \lambda^{-1} & -\lambda^{-1} \\ 0 & \lambda \end{pmatrix}$.

Let $[A,B]=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$, so

(4)
$$[P,J_+]=[A,B]^{-1}DJ_+^{-1}=\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}\begin{pmatrix} \lambda^{-1} & -\lambda^{-1} \\ 0 & \lambda\end{pmatrix}=\begin{pmatrix} \lambda^{-1}d & -\lambda^{-1}d-\lambda b \\ -\lambda^{-1}c & \lambda^{-1}c+\lambda a\end{pmatrix}.$$

Writing $P=\begin{pmatrix} x & y \\ z & w\end{pmatrix}$, we have

$$[P,J_+]=\begin{pmatrix} 1-xz & -1+x(x+z) \\ -z^2 & 1+(x+z)z\end{pmatrix}. \tag{5}$$

Quotienting by the action of U is equivalent to forgetting y and w. Hence the quotient is determined by $(x,z)\in\mathbb{C}^2-\{(0,0)\}$, modulo sign. Equating (4) and (5), we get

$$\begin{aligned} a&=\lambda^{-1}(1+xz)\\ b&=-\lambda^{-1}x^2\\ c&=\lambda z^2\\ d&=\lambda(1-xz). \end{aligned}$$

We have

$$t=\mathrm{Tr}\,([A,B])=a+d=\lambda(1-xz)+\lambda^{-1}(1+xz)=\lambda+\lambda^{-1}-xz(\lambda-\lambda^{-1}).$$

So we get the following strata for $\overline{Z}_{24}^{\lambda}$:

- For $xz=0$, $t=\lambda+\lambda^{-1}$. The contribution from the values (x,z) is $2q-1$, so we get the contribution $(2q-1)e(\overline{X}_{4,\lambda})$.
- For $t=2$, we have $xz=c_0$ for some $c_0\neq 0$. Quotienting by the change of sign, (x,z) move in a $\mathbb{C}^*$. Also $[A,B]\neq \mathrm{Id}$, so $[A,B]\sim J_+$. This gives the contribution $(q-1)e(\overline{X}_2)$.
- For $t=-2$, the computation is similar and the contribution is $(q-1)e(\overline{X}_3)$.
- For $t\neq \pm 2, \lambda+\lambda^{-1}$. We have a fibration with fiber $\mathbb{C}^*$ parametrizing the values of (x,z), for fixed $xz=(\lambda+\lambda^{-1}-t)/(\lambda-\lambda^{-1})$. For each value of $t\in\mathbb{C}-\{\pm 2,\lambda+\lambda^{-1}\}$, we have $(A,B)\in\overline{X}_{4,\lambda}$. The union of all of them is $\overline{X}_4/\mathbb{Z}_2$. The total contribution is thus $(q-1)\left(e(\overline{X}_4/\mathbb{Z}_2)-e(\overline{X}_{4,\lambda})\right)$.

Putting all together,

$$\begin{aligned} e(\overline{Z}_{24}^{\lambda})&=(2q-1)e(\overline{X}_{4,\lambda})+(q-1)e(\overline{X}_2)+(q-1)e(\overline{X}_3)+(q-1)\left(e(\overline{X}_4/\mathbb{Z}_2)-e(\overline{X}_{4,\lambda})\right)\\ &=q^5+q^3-q^2-1,\\ e(Z_{24}^{\lambda})&=(q^2+q)e(\overline{Z}_{24}^{\lambda})=q^7+q^6+q^5-q^3-q^2-q. \end{aligned}$$

To get $\mathcal{R}_{W_2,W_{4,\lambda}}$, we need to quotient by the action of $\mathbb{C}^*$, corresponding to the diagonal matrices acting by conjugation on (A,B,P). There are no fixed points, since in such case $[A,B]=\mathrm{Id}$, which does not happen. Hence

$$e(\mathcal{R}_{W_2,W_{4,\lambda}})=q^4+q^3+2q^2+q+1.$$

4.2. **The case $\mathcal{C}_1 = W_3$ and $\mathcal{C}_2 = W_{4,\lambda}$.** Note that $W_3 = -W_2$ and $W_{4,\lambda} = -W_{4,-\lambda}$. Hence the isomorphism $Z(\mathcal{C}_1, \mathcal{C}_2) = Z(-\mathcal{C}_1, -\mathcal{C}_2)$, described at the beginning of Subsection 3.2, gives here

$$Z(W_3, W_{4,\lambda}) = Z(W_2, W_{4,-\lambda}) = Z_{24}^{-\lambda}.$$

Therefore using the results of Subsection 4.1, we have

$$\begin{aligned} e(\overline{Z}_{34}^{\lambda}) &= q^5 + q^3 - q^2 - 1, \\ e(Z_{34}^{\lambda}) &= q^7 + q^6 + q^5 - q^3 - q^2 - q, \\ e(\mathcal{R}_{W_3, W_{4,\lambda}}) &= q^4 + q^3 + 2q^2 + q + 1. \end{aligned}$$

5. Holonomies of diagonalizable type

Let $D_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$ and $D_2 = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$, with $\lambda_1, \lambda_2 \neq 0, \pm 1$. We want to understand the set

$$Z_{44}^{\lambda_1 \lambda_2} = \{(A, B, C_1, C_2) | C_1 \in [D_1], C_2 \in [D_2], [A, B]C_1 = C_2^{-1}\}$$

and the quotient $\mathcal{R}_{\xi_{\lambda_1}, \xi_{\lambda_2}} = Z_{44}^{\lambda_1 \lambda_2} // \mathrm{PGL}(2, \mathbb{C})$. All orbits have trivial stabilizers, except in one case: when A, B, C_1, C_2 are diagonal with respect to the same basis, and hence $\lambda_1 \lambda_2 = 1$, that is $\lambda_2 = \lambda_1^{-1}$. As λ is defined up to $\lambda \sim \lambda^{-1}$, we also have reducibles in the case $\lambda_2 = \lambda_1$.

Suppose that $\lambda_1 \neq \lambda_2, \lambda_2^{-1}$ from now on in this section. Let

$$\overline{Z}_{44}^{\lambda_1 \lambda_2} = \{(A, B, C) | C \in [D_1], [A, B]C = D_2^{-1}\}.$$

As we said above, the action of $\mathrm{PGL}(2, \mathbb{C})$ is free on $Z_{44}^{\lambda_1 \lambda_2}$. We check this as follows: write $C_1 = Q D_1 Q^{-1}$. Then if a non-trivial P fixes A, B, C_1, C_2, it must be that $P \in Q \mathbb{C}^* Q^{-1}$, where $\mathbb{C}^* \subset \mathrm{PGL}(2, \mathbb{C})$ denotes the diagonal matrices. This forces that $A, B, C_2 \in Q \mathbb{C}^* Q^{-1}$, hence $[A, B] = \mathrm{Id}$ and $C_2 = C_1^{-1}$, which contradicts our assumption. In the second place, it is clear that $\mathrm{PGL}(2, \mathbb{C}) \, \overline{Z}_{44}^{\lambda_1 \lambda_2} = Z_{44}^{\lambda_1 \lambda_2}$. Finally, Condition (1) holds as follows: let $P \in \mathrm{PGL}(2, \mathbb{C})$ such that $(A, B, C), (PAP^{-1}, PBP^{-1}, PCP^{-1}) \in \overline{Z}_{44}^{\lambda_1 \lambda_2}$. Then P fixes D_2^{-1}, so $P \in \mathbb{C}^*$. All together, we have a fibration

$$\mathbb{C}^* \to \mathrm{PGL}(2, \mathbb{C}) \times \overline{Z}_{44}^{\lambda_1 \lambda_2} \to Z_{44}^{\lambda_1 \lambda_2}.$$

So $\mathcal{R}_{\xi_{\lambda_1}, \xi_{\lambda_2}} = \overline{Z}_{44}^{\lambda_1 \lambda_2} // \mathbb{C}^*$.

To describe $\overline{Z}_{44}^{\lambda_1 \lambda_2}$, note that as $C \sim D_1$, there exists $P \in \mathrm{SL}(2, \mathbb{C})$ with $C = P D_1 P^{-1}$. Such P is defined up to $\mathbb{C}^*$, the diagonal matrices acting by multiplication on the right on P (whcih is a free action). So we can rewrite

$$\overline{Z}_{44}^{\lambda_1 \lambda_2} = \{(A, B, P) | [A, B][P, D_1] = D_2^{-1} D_1^{-1}\} / \mathbb{C}^*.$$

Set $D := D_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $\lambda = \lambda_1$, and $\xi_\mu := D_2^{-1} D_1^{-1} = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$, where $\mu = \lambda_1^{-1} \lambda_2^{-1} \neq 0, 1$ and $\lambda^2 \mu = \lambda_1^{-1} \lambda_2 \neq 0, 1$.

The action of $Q \in \mathbb{C}^*$ on the set of (A, B, P) is by conjugation $A \mapsto Q^{-1} A Q$, $B \mapsto Q^{-1} B Q$, $PDP^{-1} \mapsto Q^{-1} PDP^{-1} Q$, or equivalently, $P \mapsto Q^{-1} P$.

We have $\lambda_1 = \lambda$, $\lambda_2 = \lambda^{-1}\mu^{-1}$. The case $\mu = -1$ corresponds to $\lambda_2 = -\lambda^{-1}$. This case is equivalent to $\lambda_2 = -\lambda_1$, i.e. to $\lambda^2\mu = -1$. So we will assume $\mu \neq -1$ henceforth without loss of generality.

We focus on the following equation

$$[A, B][P, D] = \xi_\mu. \tag{6}$$

Our purpose is to find the solutions of (6).

Consider the following invariants:

$$t_1 = \mathrm{Tr}\,([A, B]),$$
$$t_2 = \mathrm{Tr}\,([P, D]).$$

Denote by $\eta = [A, B] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\delta = [P, D]$ so that

$$\delta = \eta^{-1}\,\xi_\mu = \begin{pmatrix} d\mu & -b\mu^{-1} \\ -c\mu & a\mu^{-1} \end{pmatrix}.$$

Now $t_1 = a + d$ and $t_2 = a\mu^{-1} + d\mu$, so

$$\begin{cases} a = \dfrac{\mu t_1 - t_2}{\mu - \mu^{-1}} \\ d = \dfrac{t_2 - \mu^{-1} t_1}{\mu - \mu^{-1}} \\ ad - bc = 1 \end{cases} \tag{7}$$

Consider the following set of matrices $P \in \mathrm{SL}(2, \mathbb{C})/\mathbb{C}^*$,

$$\mathcal{P} = \left\{ P \,|\, [P, D] = \delta = \begin{pmatrix} d\mu & -b\mu^{-1} \\ -c\mu & a\mu^{-1} \end{pmatrix}, a, b, c, d \text{ satisfy (7)} \right\} \subset \mathrm{SL}(2, \mathbb{C})/\mathbb{C}^*.$$

Denote $P = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, where $xw - yz = 1$. Also

$$[P, D] = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} w & -y \\ -z & x \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} xw - \lambda^{-2} yz & xy(1 - \lambda^2) \\ zw(1 - \lambda^{-2}) & xw - \lambda^2 yz \end{pmatrix}.$$

Hence, equality $[P, D] = \delta$ is now

$$\begin{pmatrix} xw - \lambda^{-2} yz & xy(1 - \lambda^2) \\ zw(1 - \lambda^{-2}) & xw - \lambda^2 yz \end{pmatrix} = \begin{pmatrix} d\mu & -b\mu^{-1} \\ -c\mu & a\mu^{-1} \end{pmatrix}. \tag{8}$$

It gives us the following equations:

$$xw - \lambda^{-2} yz = d\mu \tag{9}$$
$$xy(1 - \lambda^2) = -b\mu^{-1} \tag{10}$$
$$zw(1 - \lambda^{-2}) = -c\mu \tag{11}$$
$$xw - \lambda^2 yz = a\mu^{-1} \tag{12}$$
$$xw - yz = 1. \tag{13}$$

Lemma 5.1. *If (9)–(13) have solutions then*

$$\mu(\lambda^2-1)t_1+(1-\mu^2\lambda^2)t_2=(1-\mu^2)(1+\lambda^2). \tag{14}$$

This is actually equivalent to $(a\mu^{-1}-1)+\lambda^2(d\mu-1)=0$.

Proof. From (9) and (13), we have $yz=\frac{d\mu-1}{1-\lambda^{-2}}$. From (12) and (13), we have $yz=\frac{a\mu^{-1}-1}{1-\lambda^2}$. Equating both, we get

$$\frac{d\mu-1}{1-\lambda^{-2}}=\frac{a\mu^{-1}-1}{1-\lambda^2}$$

that is rewritten as

$$(a\mu^{-1}-1)+\lambda^2(d\mu-1)=0. \tag{15}$$

Now using (7), we get the result. □

Lemma 5.1 says that we can write t_1 in terms of t_2 as

$$t_1=\frac{\mu^2\lambda^2-1}{\mu(\lambda^2-1)}t_2+\frac{(1-\mu^2)(1+\lambda^2)}{\mu(\lambda^2-1)}.$$

Therefore we have a projection

$$\begin{array}{rcl} \pi:\mathcal{P} & \longrightarrow & \mathbb{C} \\ P & \mapsto & t_2=\mathrm{Tr}\,([P,D]). \end{array} \tag{16}$$

Lemma 5.2. *Assuming (14) holds, the condition* $bc=0$ *is equivalent to one of the following:*

- $a=d^{-1}=\mu$, $\delta=\begin{pmatrix}1 & 0\\ -c\mu & 1\end{pmatrix}$ *or* $\begin{pmatrix}1 & -b\mu^{-1}\\ 0 & 1\end{pmatrix}$. *Here* $t_1=\mu+\mu^{-1}$ *and* $t_2=2$,
- $a=d^{-1}=\mu\lambda^2$, $\delta=\begin{pmatrix}\lambda^2 & 0\\ -c\mu & \lambda^{-2}\end{pmatrix}$ *or* $\begin{pmatrix}\lambda^2 & -b\mu^{-1}\\ 0 & \lambda^{-2}\end{pmatrix}$. *Here* $t_1=\mu\lambda^2+\mu^{-1}\lambda^{-2}$ *and* $t_2=\lambda^2+\lambda^{-2}$.

Proof. The equality $bc=0$ means that the matrix δ in (8) is equal to $\begin{pmatrix}d\mu & 0\\ -c\mu & a\mu^{-1}\end{pmatrix}$ or $\begin{pmatrix}d\mu & -b\mu^{-1}\\ 0 & a\mu^{-1}\end{pmatrix}$, depending on whether $b=0$ or $c=0$ (or both). Note that in this case $ad=1$, so $a=d^{-1}$. By Lemma 5.1 (see equation (15)), we have

$$\begin{aligned} & a\mu^{-1}-1=(1-d\mu)\lambda^2 \\ & \Longleftrightarrow \mu^{-1}a+\lambda^2 d\mu-\lambda^2=1 \\ & \Longleftrightarrow (\mu d-1)(\mu^{-1}a-\lambda^2)=ad-1=bc=0. \end{aligned} \tag{17}$$

Hence it must be $d=\mu^{-1}$ or $a=\mu\lambda^2$. In the first case, $a=\mu$, $t_1=\mu+\mu^{-1}$ and $t_2=a\mu^{-1}+d\mu=2$. In the second case, $a=\mu\lambda^2$, $t_1=\mu\lambda^2+\mu^{-1}\lambda^{-2}$ and $t_2=\lambda^2+\lambda^{-2}$. □

Remark 5.3. Note that $\lambda^2+\lambda^{-2}\neq 2$, as $\lambda\neq\pm 1$.

Theorem 5.4. *Consider* $\mathcal{P}_{t_2}=\pi^{-1}(t_2)$, *where* π *is given in (16), and* (t_1,t_2) *satisfy (14). Then*

(1) *If* $t_2 \in \mathbb{C} - \{2,\, \lambda^2+\lambda^{-2}\}$ *then*

$$\mathcal{P}_{t_2} \cong \left\{ (x,y,z,w,b,c) \,|\, x\neq 0,\ y = \frac{-\mu^{-1}b}{x(1-\lambda^2)},\ z = \frac{(\mu-a)x}{b},\ w = \frac{\mu^{-1}a-\lambda^2}{(1-\lambda^2)x}, bc = k \right\} /\mathbb{C}^*$$

for some $k = k(t_2) \neq 0$.

(2) *If* $t_2 = 2$ *then*

$$\mathcal{P}_{t_2} \cong \left\{ (x,y,z,w,b,c) \,|\, x\neq 0,\ y = \frac{-\mu\, b}{x(1-\lambda^2)},\ z = \frac{-\mu^{-1}c}{(1-\lambda^{-2})w}\,, w = \frac{1}{x}\,, bc = 0 \right\} /\mathbb{C}^*$$

(3) *If* $t_2 = \lambda^2+\lambda^{-2}$ *then*

$$\mathcal{P}_{t_2} \cong \left\{ (x,y,z,w,b,c) \,|\, x = \frac{-\mu\, b}{y(1-\lambda^2)},\ y\neq 0,\ z = -\frac{1}{y},\ w = \frac{-\mu^{-1}c}{(1-\lambda^{-2})z}\,, bc = 0 \right\} /\mathbb{C}^*$$

where $\mathbb{C}^*$ *acts as* $(x,y,z,w)\mapsto(\alpha x,\alpha^{-1}y,\alpha z,\alpha^{-1}w)$. *So* $\mathcal{P}_{t_2}\cong\mathbb{C}^*$ *in (1), and* $\mathcal{P}_{t_2}\cong\{bc=0\}$ *in (2) and (3).*

Proof. To prove (1), note that Lemma 5.2 implies that $bc\neq 0$. Now a,d are fixed by (7) and $bc = ad-1 = k\neq 0$.

We show that $\mathcal{P}_{t_2}\subset\left\{(x,y,z,w)\in\mathbb{C}^4 \,|\, x\neq 0,\ y=\frac{-\mu b}{x(1-\lambda^2)},\ z=\frac{(1-\mu^{-1}a)x}{\mu^2 b},\ w=\frac{a-\mu\lambda^2}{(\lambda^2-1)x}\right\}$. First (10) implies $xy = \frac{-\mu b}{1-\lambda^2}$, and as $x\neq 0$ (since $b\neq 0$) then we get $y=\frac{-\mu b}{x(1-\lambda^2)}$. Now equations (12) and (13) imply that $yz = \frac{\mu^{-1}a-1}{1-\lambda^2}$. We can divide by y since $y\neq 0$, hence $z = \frac{\mu^{-1}a-1}{y(1-\lambda^2)} = \frac{(1-\mu^{-1}a)x}{\mu b}$. Finally equation (11) implies $zw = \frac{-\mu^{-1}c}{1-\lambda^{-2}}$ hence, dividing by z (since $c\neq 0$), $w=\frac{-\mu^{-1}c}{z(1-\lambda^{-2})} = \frac{-bc}{(1-\lambda^{-2})(1-\mu^{-1}a)x}$. Using that $bc = (\mu d-1)(\mu^{-1}a-\lambda^2)$ (equation (17)) and $d\mu - 1 = -\lambda^{-2}(a\mu^{-1}-1)$ (equation (15)), we get $w=\frac{\mu^{-1}a-\lambda^2}{(1-\lambda^2)x}$.

For the reverse inclusion, take $x\neq 0$, $y=\frac{-\mu b}{x(1-\lambda^2)}$, $z=\frac{(1-\mu^{-1}a)x}{\mu b}$, $w=\frac{\mu^{-1}a-\lambda^2}{(1-\lambda^2)x}$. Then clearly (10) holds. Now

$$\begin{aligned} xw &= \frac{\mu^{-1}a-\lambda^2}{1-\lambda^2}, \\ yz &= \frac{\mu^{-1}a-1}{1-\lambda^2}. \end{aligned}$$

So (9), (12) and (13) are

$$\begin{aligned} xw - yz &= 1, \\ xw - \lambda^2 yz &= \mu^{-1}a, \\ xw - \lambda^{-2}yz &= 1-\lambda^{-2}(a\mu^{-1}-1) = d\mu. \end{aligned}$$

Finally,

$$zw = \frac{(1-\mu^{-1}a)(\mu^{-1}a-\lambda^2)}{\mu b(1-\lambda^2)} = \frac{\lambda^2(d\mu-1)(\mu^{-1}a-\lambda^2)}{\mu b(1-\lambda^2)} = \frac{\lambda^2 bc}{\mu b(1-\lambda^2)} = -\frac{c\mu^{-1}}{1-\lambda^{-2}},$$

using (17) and (15) again.

To prove (2), note that by Lemma 5.2, $a=\mu$. Then $yz=\frac{1-a\mu^{-1}}{1-\lambda^2}=0$. So $xw=1$. The formulas for y and z follow straight away. The reverse inclusion is equally easy.

To prove (3), note that Lemma 5.2 says that $a=\mu\lambda^2$. From (12) and (13), $xw=\frac{a\mu^{-1}-\lambda^2}{1-\lambda^2}=0$. So $yz=-1$. The formulas for x and w follow clearly, and the reverse inclusion is easy. □

Now we compute the Hodge-Deligne polynomials.

5.1. **Case $\lambda^2\mu\neq\pm1$.** There are several contributions depending on the values of the parameter t_2.

- When $t_2=2$ then $t_1=\mu+\mu^{-1}$. Hence $[P,D]=\begin{pmatrix}\mu^{-1}d & *\\ * & \mu a\end{pmatrix}$ and $[A,B]=\begin{pmatrix}\mu & b\\ c & \mu^{-1}\end{pmatrix}\sim\begin{pmatrix}\mu & 0\\ 0 & \mu^{-1}\end{pmatrix}$, as $bc=0$. The contribution is $\overline{X}_{4,\mu}$ from $[A,B]$, and from $[P,D]$ we get $\{bc=0\}$ by Theorem 5.4(2).
  The contribution of this fiber is thus
$$\begin{aligned}e(F_1)&=(2q-1)e(\overline{X}_{4,\mu})\\&=(2q-1)(q^3+3q^2-3q-1)\\&=2q^4+5q^3-9q^2+q+1.\end{aligned}$$
- When $t_2=\lambda^2+\lambda^{-2}$, Lemma 5.1 says that $t_1=\mu\lambda^2+\mu^{-1}\lambda^{-2}\neq\pm2$ (here we use $\lambda^2\mu\neq\pm1$). The contribution is then, by Theorem 5.4(3),
$$\begin{aligned}e(F_2)&=(2q-1)e(\overline{X}_{4,\mu\lambda^2})\\&=2q^4+5q^3-9q^2+q+1.\end{aligned}$$
- When $t_1=2$ then $bc\neq0$ hence the matrix $\eta\neq\mathrm{Id}$. So $\eta\sim J_+$. It cannot be $t_2=2$ or $t_2=\lambda^2+\lambda^{-2}$, as in these cases $t_1\neq2$. So Theorem 5.4(1) says that the contribution of $\mathcal{P}$ is $\mathbb{C}^*$. Hence
$$\begin{aligned}e(F_3)&=(q-1)e(\overline{X}_2)\\&=(q-1)(q^3-2q^2-3q)\\&=q^4-3q^3-q^2+3q.\end{aligned}$$
- The case $t_1=-2$ is analogous, the only difference being that the matrix η is of Jordan type J_-. The contribution is then
$$\begin{aligned}e(F_4)&=(q-1)e(\overline{X}_3)\\&=(q-1)(q^3+3q^2)\\&=q^4+2q^3-3q^2.\end{aligned}$$
- The generic case is a fibration with base $t_1\in L=\mathbb{C}-\{\pm2,\mu+\mu^{-1},\lambda^2\mu+\lambda^{-2}\mu^{-1}\}$. The fibration $\mathcal{P}\to L$ is trivial with fibers $\mathbb{C}^*$. The fibration $\{(A,B)\}\to L$ has

total space $\overline{X}_4/\mathbb{Z}_2$, and we remove two fibers. Therefore

$$\begin{aligned} e(F_5) &= (q-1)\left(e(\overline{X}_4/\mathbb{Z}_2) - e(\overline{X}_{4,\mu}) - e(\overline{X}_{4,\lambda^2\mu})\right) = \\ &= (q-1)(q^4-2q^3-3q^2+3q+1) - 2(q^3+3q^2-3q-1)) \\ &= q^5-5q^4-5q^3+18q^2-6q-3. \end{aligned}$$

The total sum of all contributions is

$$e(\overline{Z}_{44}^{\lambda_1\lambda_2}) = e(F_1)+e(F_2)+e(F_3)+e(F_4)+e(F_5) = q^5+q^4+4q^3-4q^2-q-1.$$

We quotient by $\mathbb{C}^*$ to get the sought moduli space. So

$$e(\mathcal{R}_{\xi_{\lambda_1},\xi_{\lambda_2}}) = e(\overline{Z}_{44}^{\lambda_1\lambda_2}/\mathbb{C}^*) = q^4+2q^3+6q^2+2q+1.$$

5.2. **Case $\lambda^2\mu = -1$.**

- The set F_1 is defined by $t_2 = 2$ and $t_1 = \mu + \mu^{-1}$. As in the previous case, we have $e(F_1) = 2q^4+5q^3-9q^2+q+1$.
- The set F_2 is defined by $t_2 = \lambda^2 + \lambda^{-2}$ and $t_1 = -2$. We have $bc = 0$ now, so $\mathcal{P}$ is computed in Theorem 5.4(3) as $\{bc = 0\}$. There are three cases:
  - $b = 0$, $c \neq 0$. This gives $(q-1)e(\overline{X}_3)$, since $[A, B] \sim J_-$,
  - $b \neq 0$, $c = 0$. This gives $(q-1)e(\overline{X}_3)$, since $[A, B] \sim J_-$,
  - $b = c = 0$. This gives $e(X_1)$, since $[A, B] = -\mathrm{Id}$.

  So the sum is

$$\begin{aligned} e(F_2) &= 2(q-1)e(\overline{X}_3) + e(X_1) \\ &= (q^3-q) + 2(q-1)(q^3+3q^2) \\ &= 2q^4+5q^3-6q^2-q. \end{aligned}$$

- The set F_3 is defined by $t_1 = 2$. We note that $bc \neq 0$, so $\eta \sim J_+$. The contribution is

$$\begin{aligned} e(F_3) &= (q-1)e(\overline{X}_2) \\ &= q^4-3q^3-q^2+3q. \end{aligned}$$

- Finally, the generic case is a fibration with base $t_1 \in L = \mathbb{C} - \{\pm 2, \mu+\mu^{-1}\}$. The fibration $\mathcal{P} \to L$ is trivial with fibers $\mathbb{C}^*$. The fibration $\{(A, B)\} \to L$ has total space $\overline{X}_4/\mathbb{Z}_2$, and we remove one fiber. Therefore

$$\begin{aligned} e(F_4) &= (q-1)\left(e(\overline{X}_4/\mathbb{Z}_2) - e(\overline{X}_{4,\mu})\right) \\ &= q^5-4q^4-3q^3+12q^2-4q-2. \end{aligned}$$

The total sum is

$$e(\overline{Z}_{44}^{\lambda_1\lambda_2}) = q^5+q^4+4q^3-4q^2-q-1$$

and

$$e(\mathcal{R}_{\xi_{\lambda_1},\xi_{\lambda_2}}) = e(\overline{Z}_{44}^{\lambda_1\lambda_2}/\mathbb{C}^*) = q^4+2q^3+6q^2+2q+1.$$

6. The case $\lambda_1 = \lambda_2$

We end up with the moduli space $\mathcal{R}_{\xi_{\lambda_1},\xi_{\lambda_1}}$, in which case the moduli space has reducibles. For convenience, we are going to take $\lambda_2 = \lambda_1^{-1}$, since $\xi_{\lambda_2} \sim \xi_{\lambda_1}$. Then $\mu = \lambda_1^{-1}\lambda_2^{-1} = 1$. Now we have

$$Z_{44}^{\lambda_1\lambda_2} = \{(A, B, C_1, C_2) | C_1, C_2 \in W_{4,\lambda}, [A, B]C_1C_2 = \mathrm{Id}\},$$
$$\overline{Z}_{44}^{\lambda_1\lambda_2} = \{(A, B, P) | [A, B][P, D] = \mathrm{Id}\}/\mathbb{C}^*$$

where $D = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. Here $\lambda = \lambda_1$ and $\mathbb{C}^*$ acts on P by multiplication on the right. Also $t_1 = \mathrm{Tr}\,([A, B])$, $t_2 = \mathrm{Tr}\,([P, D])$ satisfy $t_2 = t_1$.

Denote by $[A, B] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and

$$\mathcal{P} = \left\{ P \,|\, [P, D] = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right\} \subset \mathrm{SL}(2, \mathbb{C})/\mathbb{C}^*.$$

The computations in Section 5 remain valid with $\mu = 1$ except that we cannot use equation (7). This is only used to reduce equation (15). But note that this equation can be reduced directly by using $t_1 = a + d$, $t_2 = a\mu^{-1} + d\mu$. In particular, Theorem 5.4 also holds for $\mu = 1$. The computation of the Hodge-Deligne polynomials of the strata now runs as follows.

- When $t_2 = 2$, we have the subset F_1 is as in the first case of Subsection 5.1. So $e(F_1) = 2q^4 + 5q^3 - 9q^2 + q + 1$.
- When $t_2 = \lambda^2 + \lambda^{-2} = t_1$, we have $bc = 0$. There are three cases:
  - $b = 0$, $c \neq 0$. This gives $(q - 1)e(\overline{X}_2)$, since $[A, B] \sim J_+$,
  - $b \neq 0$, $c = 0$. This gives $(q - 1)e(\overline{X}_2)$, since $[A, B] \sim J_+$,
  - $b = c = 0$. This gives $e(X_0)$, since $[A, B] = \mathrm{Id}$.

  So the sum is
  $$e(F_2) = 2(q - 1)e(\overline{X}_2) + e(X_0) = 3q^4 - 2q^3 - 3q^2 + 2q.$$
- The stratum defined by $t_1 = -2$ contributes
  $$e(F_3) = (q - 1)e(\overline{X}_3) \\ = q^4 + 2q^3 - 3q^2.$$
- The remaining contribution is a fibration with base $t_1 \in L = \mathbb{C} - \{\pm 2, \mu + \mu^{-1}\}$. So, as in the fourth case of Subsection 5.2, we have
  $$e(F_4) = (q - 1)\left(e(\overline{X}_4/\mathbb{Z}_2) - e(\overline{X}_{4,\mu})\right) \\ = q^5 - 4q^4 - 3q^3 + 12q^2 - 4q - 2.$$

The total sum is

$$e(\overline{Z}_{44}^{\lambda_1\lambda_2}) = q^5 + 2q^4 + 2q^3 - 3q^2 - q - 1\,.$$

To finish, we compute the Hodge-Deligne polynomial of

$$\mathcal{R}_{\xi_\lambda,\xi_\lambda} = Z_{44}^{\lambda_1\lambda_2} // \,\mathrm{PGL}(2, \mathbb{C}) = \overline{Z}_{44}^{\lambda_1\lambda_2} // \mathbb{C}^*.$$

The reducibles $(A, B, C_1, C_2) \in Z_{44}^{\lambda_1\lambda_2}$ satisfy that $[A, B] = \mathrm{Id}$, $C_1 = C_2^{-1} \in W_{4,\lambda}$, and the stabilizer of C_1 also stabilize A, B. Correspondingly, the reducibles $(A, B, P) \in \overline{Z}_{44}^{\lambda_1\lambda_2}$ satisfy that A, B, P are diagonal matrices and $C = [P, D] = [A, B]^{-1} = \mathrm{Id}$. We also have to see which orbits have reducibles in their closure under the action of the group $\mathbb{C}^*$ of diagonal matrices. For this, A, B should be either upper triangular or lower triangular. The matrices of the form $A = \begin{pmatrix} x & a \\ 0 & x^{-1} \end{pmatrix}$, $B = \begin{pmatrix} y & b \\ 0 & y^{-1} \end{pmatrix}$, $C = [A, B]^{-1}$, converge to $A' = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$, $B' = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$, $C' = \mathrm{Id}$. Hence they go to a single orbit in the GIT quotient. Let $\overline{\mathcal{D}}$ be the space of A, B both either upper-triangular or lower-triangular. Then $e(\overline{\mathcal{D}}) = (q-1)^2(2(q-1)^2 + 4(q-1) + 1) = (q-1)^2(2q^2 - 1)$. So the irreducibles $\overline{Z}_{44}^{\lambda_1\lambda_2*} = \overline{Z}_{44}^{\lambda_1\lambda_2} - \overline{\mathcal{D}}$ have

$$e(\overline{Z}_{44}^{\lambda_1\lambda_2*}) = q^5 + 6q^3 - 4q^2 - 3q.$$

In the quotient, the contribution of the reducibles is $(q-1)^2$. Hence

$$\begin{aligned} e(\mathcal{R}_{\xi_\lambda,\xi_\lambda}) &= e(\overline{Z}_{44}^{\lambda_1\lambda_2*}/\mathbb{C}^*) + (q-1)^2 \\ &= q^4 + q^3 + 8q^2 + q + 1. \end{aligned}$$

7. Betti numbers and Hodge numbers

With the simultaneous information of the Hodge-Deligne polynomial and the Poincaré polynomial we are able to recover some of the Hodge numbers of the spaces. Consider the variety $Z = \mathcal{R}_{\xi_\lambda,\xi_\mu}(X, \mathrm{SL}(2,\mathbb{C}))$, $\lambda \neq \mu^{\pm 1}$, whose Hodge-Deligne polynomial was computed in Section 5,

$$e(\mathcal{R}_{\xi_\lambda,\xi_\mu}(X, \mathrm{SL}(2,\mathbb{C}))) = q^4 + 2q^3 + 6q^2 + 2q + 1.$$

Recall that the Hodge-Deligne polynomial is $e(Z) = \sum_p \sum_k (-1)^k h_c^{k,p,p}(Z) q^p = \sum_p e_p\, q^p$.

For $Z = \mathcal{R}_{\xi_\lambda,\xi_\mu}(X, \mathrm{SL}(2,\mathbb{C}))$, and $|\lambda| = |\mu| = 1$, the Poincaré polynomial was computed in [2],

$$P_t(\mathcal{R}_{\xi_\lambda,\xi_\mu}(X, \mathrm{SL}(2,\mathbb{C}))) = 10t^4 + 2t^3 + 3t^2 + 1.$$

Now note that the family $\mathcal{R}_{\xi_\lambda,\xi_\mu}(X, \mathrm{SL}(2,\mathbb{C}))$ over $\{(\lambda,\mu) \in (\mathbb{C}^* - \{\pm 1\})^2 | \lambda \neq \mu, \mu^{-1}\}$ is analytically locally trivial. In particular, the spaces are diffeomorphic, and hence all have the same Poincaré polynomial, for any $\lambda \neq \mu, \mu^{-1}$, $\lambda, \mu \neq 0, \pm 1$.

This character variety is smooth and of complex dimension 4 so its compactly supported Poincaré polynomial is

$$P_t^c(\mathcal{R}_{\xi_\lambda,\xi_\mu}(X, \mathrm{SL}(2,\mathbb{C}))) = t^8 + 3t^6 + 2t^5 + 10t^4.$$

We write $h_c^{k,p,p} = h_c^{k,p,p}(\mathcal{R}_{\xi_\lambda,\xi_\mu}(X, \mathrm{SL}(2,\mathbb{C})))$ and $b_c^k = b_c^k(\mathcal{R}_{\xi_\lambda,\xi_\mu}(X, \mathrm{SL}(2,\mathbb{C})))$. Note that the elements in the columns of Table 1 add as $b_c^k = \sum_p h_c^{k,p,p}$, and the elements of the rows satisfy that their alternate sums are $e_p = \sum_k (-1)^k h_c^{k,p,p}$. So we have

	$b_c^4=10$	$b_c^5=2$	$b_c^6=3$	$b_c^7=0$	$b_c^8=1$	$\sum_k(-1)^k h_c^{k,p,p}$
$h_c^{k,0,0}$	$h_c^{4,0,0}=1+a-\epsilon_1$	$h_c^{5,0,0}=a$	$h_c^{6,0,0}=\epsilon_1$	0	0	1
$h_c^{k,1,1}$	$h_c^{4,1,1}=2+b-\epsilon_2$	$h_c^{5,1,1}=b$	$h_c^{6,1,1}=\epsilon_2$	0	0	2
$h_c^{k,2,2}$	$h_c^{4,2,2}=6+c-\epsilon_3$	$h_c^{5,2,2}=c$	$h_c^{6,2,2}=\epsilon_3$	0	0	6
$h_c^{k,3,3}$	0	0	2	0	0	2
$h_c^{k,4,4}$	0	0	0	0	1	1

Table 1.

where $\epsilon_1, \epsilon_2, \epsilon_3$ are $1,0,0$ or $0,1,0$ or $0,0,1$; and a,b,c are positive integers such that $a+b+c=2$. This means that there are 18 possible solutions. In any case, at least we get that $h_c^{7,p,p}(Z)=0$ for all $p=0,1,2,3,4$, $h_c^{8,p,p}(Z)=0$ for $p=0,1,2,3$, $h_c^{8,4,4}=1$, $h_c^{k,3,3}(Z)=0$ for $k=4,5,7,8$ and $h_c^{6,3,3}(Z)=2$.

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