

Divergence-based robust inference under proportional hazards model for one-shot device life-test

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Abstract

In this paper, we develop robust estimators and tests for one-shot device testing under proportional hazards assumption based on divergence measures. Through a detailed Monte Carlo simulation study and a numerical example, the developed inferential procedures are shown to be more robust than the classical procedures, based on maximum likelihood estimators.

1 Introduction

A one-shot device is a unit that performs its function only once, and after use the device either gets destroyed or must be rebuilt. For this kind of device, one can only know whether the failure time is either before or after a specific time, and consequently the lifetimes are either left- or right-censored, with the lifetime being less than the inspection time if the test outcome is a failure (resulting in left censoring) and the lifetime being more than the inspection time if the test outcome is a success (resulting in right censoring). Some examples of such one-shot devices include automobile air bags, missiles (Olwell and Sorell [2001]) and fire extinguishers (Newby [2008]).

For devices with long lifetimes, accelerated life-tests (ALTs) are commonly used to induce quick failures. An ALT shortens the life span of the products by increasing the levels of stress factors, such as temperature, humidity, pressure and voltage. Then, a link function relating stress levels and lifetime is applied to extrapolate the lifetimes of units from accelerated conditions to normal operating conditions. The study of one-shot device from ALT data has been discussed considerably recently, mainly motivated by the work of Fan et al. [2009].

Under the classical parametric setup, product lifetimes are assumed to be fully described by a probability distribution involving some model parameters. This has been done with some common lifetime distributions such as exponential (Balakrishnan and Ling [2012]), gamma or Weibull (Balakrishnan and Ling [2013]). However, as data from one-shot devices do not contain actual lifetimes, parametric inferential methods can be very sensitive to violations of the model assumption. Ling et al. [2015] proposed a semi-parametric model, in which, under the proportional hazards assumption, the hazard rate is allowed to change in a non-parametric way. The simulation study carried out by Ling et al. [2015] shows that their proposed method works very well. However, this method suffer

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from lack of robustness, as it is based on the (non-robust) maximum likelihood estimator (MLE) of model parameters. Recently years, some work has been done for developing robust methods for one-shot device testing, most of it based on divergence measures (see, for example, Balakrishnan et al. [2019a,c,b]).

In this paper, we extend the robust approach proposed in the above mentioned papers and develop here robust estimators and tests for one-shot device testing based on divergence measures under proportional hazards model. Section 2 described the model and some basic concepts and results. The estimating equations and asymptotic properties of the proposed estimators are given in Section 3. Wald-type tests are then developed in Section 4 based on the proposed estimators, as a generalization of the classical Wald test. In Section 5, a simulation study is carried out to demonstrate the robustness of the proposed method. A numerical example is finally presented in Section 6, and some concluding comments are finally made in Section 7.

2 Model formulation

Consider S constant-stress accelerated life-tests and I inspection times. For the i -th life-test, K_s devices are placed under stress level combinations with J stress factors, $\mathbf{x}_s = (x_{s1}, \dots, x_{sJ})$, of which K_{is} are tested at the i -th inspection time IT_i , where $K_s = \sum_{i=1}^I K_{is}$ and $0 < IT_1 < \dots < IT_I$. Then, the numbers of devices that have failed by time IT_i at stress \mathbf{x}_s are recorded as n_{is} . One-shot device testing data obtained from such a life-test can then be represented as $(n_{is}, K_{is}, \mathbf{x}_s, IT_i)$, for $i = 1, 2, \dots, I$ and $s = 1, 2, \dots, S$.

Instead of assuming that the true lifetimes of devices follow a specific parametric distribution such as exponential, gamma or Weibull, we assume here that the cumulative hazard function of the lifetimes of devices is of the proportional form

$$H(t, \mathbf{x}; \boldsymbol{\eta}, \boldsymbol{\alpha}) = H_0(t; \boldsymbol{\eta})\lambda(\mathbf{x}; \boldsymbol{\alpha}), \quad (1)$$

where $H_0(t; \boldsymbol{\eta})$ is the baseline cumulative hazard function with $\boldsymbol{\eta} = (\eta_1, \dots, \eta_I)$, and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_J)$ is a vector of coefficients for stress factors. The model in (1) is thus composed of two independent components, with one measuring the changes in the baseline ($H_0(t; \boldsymbol{\eta})$) and the other influencing the stress factors ($\lambda(\mathbf{x}; \boldsymbol{\alpha})$).

The corresponding reliability function is given by

$$R(t, \mathbf{x}; \boldsymbol{\eta}, \boldsymbol{\alpha}) = \exp(-H(t, \mathbf{x}; \boldsymbol{\eta}, \boldsymbol{\alpha})) = R_0(t; \boldsymbol{\eta})^{\lambda(\mathbf{x}; \boldsymbol{\alpha})}, \quad (2)$$

where $R_0(t; \boldsymbol{\eta}) = \exp(-H_0(t; \boldsymbol{\eta}))$ is the baseline reliability function, with $0 < R_0(IT_I; \boldsymbol{\eta}) < R_0(IT_{I-1}; \boldsymbol{\eta}) < \dots < R_0(IT_1; \boldsymbol{\eta}) < 1$. Therefore, we let

$$\gamma(\eta_i) = \begin{cases} 1 - R_0(IT_I; \boldsymbol{\eta}) = 1 - \exp(-\exp(\eta_I)), & i = I, \\ \frac{1 - R_0(IT_i; \boldsymbol{\eta})}{1 - R_0(IT_{i+1}; \boldsymbol{\eta})} = 1 - \exp(-\exp(\eta_i)), & i = 1, \dots, I - 1. \end{cases}$$

We then have

$$R_0(IT_i; \boldsymbol{\eta}) = 1 - \prod_{m=i}^I \{1 - \exp(-\exp(\eta_m))\} = 1 - G_i,$$

where $G_i = \prod_{m=i}^I \{1 - \exp(-\exp(\eta_m))\}$.

We now assume a log-linear link function for relating the stress levels to the failure times of the units in the cumulative hazard function in (1), as

$$\lambda(\mathbf{x}_s; \boldsymbol{\alpha}) = \exp(\boldsymbol{\alpha}^T \mathbf{x}_s) = \exp\left(\sum_{j=1}^J \alpha_j x_{sj}\right).$$

2.1 Maximum likelihood estimator

Consider the proportional hazards model for one-shot devices in (1). The log-likelihood function based on these data is then given by

$$\begin{aligned} \ell(n_{11}, \dots, n_{IS}; \boldsymbol{\eta}, \boldsymbol{\alpha}) &= \sum_{i=1}^I \sum_{s=1}^S n_{is} \log [1 - R(IT_i, \mathbf{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha})] \\ &\quad + (K_{is} - n_{is}) \log [R(IT_i, \mathbf{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha})] + C \\ &= \sum_{i=1}^I \sum_{s=1}^S n_{is} \log \left[1 - (1 - G_i)^{\exp(\sum_{j=1}^J \alpha_j x_{sj})}\right] \\ &\quad + (K_{is} - n_{is}) \log (1 - G_i) \exp\left(\sum_{j=1}^J \alpha_j x_{sj}\right) + C, \end{aligned} \quad (3)$$

where C is a constant not depending on $\boldsymbol{\eta}$ and $\boldsymbol{\alpha}$.

Definition 1 Let $\boldsymbol{\theta} = (\boldsymbol{\eta}, \boldsymbol{\alpha})$. The MLE, $\widehat{\boldsymbol{\theta}}$, of $\boldsymbol{\theta}$, is obtained by maximization of (3), i.e.,

$$\widehat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\theta}} \ell(n_{11}, \dots, n_{IS}; \boldsymbol{\eta}, \boldsymbol{\alpha}). \quad (4)$$

In order to study the relation between the MLE, $\widehat{\boldsymbol{\theta}}$, in Definition 1, with the Kullback-Leibler divergence measure, we introduce the empirical and theoretical probability vectors, as follows:

$$\widehat{\mathbf{p}}_{is} = (\widehat{p}_{is1}, \widehat{p}_{is2})^T = \left(\frac{n_{is}}{K_{is}}, \frac{K_{is} - n_{is}}{K_{is}}\right)^T, \quad i = 1, \dots, I, \quad s = 1, \dots, S, \quad (5)$$

$$\boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha}) = (\pi_{is1}(\boldsymbol{\eta}, \boldsymbol{\alpha}), \pi_{is2}(\boldsymbol{\eta}, \boldsymbol{\alpha}))^T, \quad i = 1, \dots, I, \quad s = 1, \dots, S, \quad (6)$$

where $\pi_{is1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) = 1 - R(IT_i, \mathbf{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha})$ and $\pi_{is2}(\boldsymbol{\eta}, \boldsymbol{\alpha}) = R(IT_i, \mathbf{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha})$.

Definition 2 The Kullback-Leibler divergence measure between $\widehat{\mathbf{p}}_{is}$ and $\boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})$ is given by

$$d_{KL}(\widehat{\mathbf{p}}_{is}, \boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})) = \widehat{p}_{is1} \log\left(\frac{\widehat{p}_{is1}}{\pi_{is1}(\boldsymbol{\eta}, \boldsymbol{\alpha})}\right) + \widehat{p}_{is2} \log\left(\frac{\widehat{p}_{is2}}{\pi_{is2}(\boldsymbol{\eta}, \boldsymbol{\alpha})}\right)$$

and similarly the weighted Kullback-Leibler divergence measure of all the units, where

$K = \sum_{s=1}^S K_s$ is the total number of devices under the life-test, is given by

$$\begin{aligned}
& \sum_{i=1}^I \sum_{s=1}^S \frac{K_{is}}{K} d_{KL}(\hat{\mathbf{p}}_{is}, \boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})) \\
&= \frac{1}{K} \sum_{i=1}^I \sum_{s=1}^S K_{is} \left[\hat{p}_{is1} \log \left(\frac{\hat{p}_{is1}}{\pi_{is1}(\boldsymbol{\eta}, \boldsymbol{\alpha})} \right) + \hat{p}_{is2} \log \left(\frac{\hat{p}_{is2}}{\pi_{is1}(\boldsymbol{\eta}, \boldsymbol{\alpha})} \right) \right] \\
&= \frac{1}{K} \sum_{i=1}^I \sum_{s=1}^S \left[n_{is} \log \left(\frac{\frac{n_{is}}{K_{is}}}{1 - R(IT_i, \mathbf{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha})} \right) + (K_{is} - n_{is}) \log \left(\frac{\frac{K_{is} - n_{is}}{K_{is}}}{R(IT_i, \mathbf{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha})} \right) \right].
\end{aligned} \tag{7}$$

For more details, one may refer to Pardo [2005]. The relation between the MLE and the estimator obtained by minimizing the weighted Kullback-Leibler divergence measure is obtained on the basis on the following theorem.

Theorem 3 *The log-likelihood function $\ell(n_{11}, \dots, n_{IS}; \boldsymbol{\eta}, \boldsymbol{\alpha})$, given in (3), is related to the weighted Kullback-Leibler divergence measure through*

$$\sum_{i=1}^I \sum_{s=1}^S \frac{K_{is}}{K} d_{KL}(\hat{\mathbf{p}}_{is}, \boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})) = c - \frac{1}{K} \ell(n_{11}, \dots, n_{IS}; \boldsymbol{\eta}, \boldsymbol{\alpha}),$$

with c being a constant not dependent on $\boldsymbol{\eta}$ and $\boldsymbol{\alpha}$.

Definition 4 *The MLE, $\hat{\boldsymbol{\theta}}$, of $\boldsymbol{\theta}$, can then be defined as*

$$\hat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\theta}} \sum_{i=1}^I \sum_{s=1}^S \frac{K_{is}}{K} d_{KL}(\hat{\mathbf{p}}_{is}, \boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})). \tag{8}$$

Remark 5 *Suppose the lifetimes of one-shot devices under test follow the Weibull distribution with the same shape parameter $\tau = \exp(b)$ and scale parameters related to the stress levels, $a_s = \exp(\sum_{j=1}^J c_j x_{sj})$, $s = 1, \dots, S$. The cumulative distribution function of the Weibull distribution is then given by*

$$F_T(t; a_s, \tau) = 1 - \exp \left(- \left(\frac{t}{a_s} \right)^\tau \right), \quad t > 0.$$

If the proportional hazards assumption holds, then the baseline reliability and the coefficients of stress factors are given by

$$R_0(t; \beta) = \exp(-t^\tau \exp(-\tau c_0))$$

and $\alpha_s = -\tau c_s$, $s = 1, \dots, S$. Furthermore, we have

$$\begin{aligned}
\eta_i &= \log \left(-\log \left(1 - \frac{1 - R_0(IT_i)}{1 - R_0(IT_{i+1})} \right) \right), \\
\eta_I &= \tau(\log(IT_I) - c_0).
\end{aligned}$$

2.2 Weighted minimum DPD estimator

Given the probability vectors $\widehat{\boldsymbol{p}}_{is}$ and $\boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})$ in (5) and (6), respectively, the density power divergence (DPD) between them, as a function of a single tuning parameter $\beta \geq 0$, is given by

$$d_\beta(\widehat{\boldsymbol{p}}_{is}, \boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})) = \left(\pi_{is1}^{\beta+1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) + \pi_{is2}^{\beta+1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \right) - \frac{\beta+1}{\beta} \left(\widehat{p}_{is1} \pi_{is1}^\beta(\boldsymbol{\eta}, \boldsymbol{\alpha}) + \widehat{p}_{is2} \pi_{is2}^\beta(\boldsymbol{\eta}, \boldsymbol{\alpha}) \right) + \frac{1}{\beta} \left(\widehat{p}_{is1}^{\beta+1} + \widehat{p}_{is2}^{\beta+1} \right), \quad \text{if } \beta > 0, \quad (9)$$

and $d_{\beta=0}(\widehat{\boldsymbol{p}}_{is}, \boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})) = \lim_{\beta \rightarrow 0^+} d_\beta(\widehat{\boldsymbol{p}}_{is}, \boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})) = d_{KL}(\widehat{\boldsymbol{p}}_{is}, \boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha}))$, for $\beta = 0$.

As the term $\frac{1}{\beta} \left(\widehat{p}_{is1}^{\beta+1} + \widehat{p}_{is2}^{\beta+1} \right)$ in (9) has no role in the minimization with respect to $\boldsymbol{\theta}$, we can consider the equivalent measure

$$d_\beta^*(\widehat{\boldsymbol{p}}_{is}, \boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})) = \left(\pi_{is1}^{\beta+1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) + \pi_{is2}^{\beta+1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \right) - \frac{\beta+1}{\beta} \left(\widehat{p}_{is1} \pi_{is1}^\beta(\boldsymbol{\eta}, \boldsymbol{\alpha}) + \widehat{p}_{is2} \pi_{is2}^\beta(\boldsymbol{\eta}, \boldsymbol{\alpha}) \right),$$

and then can redefine the weighted minimum DPD estimator as follows.

Definition 6 *The weighted minimum DPD estimator for $\boldsymbol{\theta}$ is given by*

$$\widehat{\boldsymbol{\theta}}_\beta = \arg \min_{\boldsymbol{\theta}} \sum_{i=1}^I \sum_{s=1}^S \frac{K_{is}}{K} d_\beta^*(\widehat{\boldsymbol{p}}_{is}, \boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})), \quad \text{for } \beta > 0,$$

and for $\beta = 0$, we have the MLE, $\widehat{\boldsymbol{\theta}}$, as defined in (8).

3 Estimation and asymptotic distribution

The estimating equations for the weighted minimum DPD estimator are as given in the following theorem.

Theorem 7 *For $\beta \geq 0$, the estimating equations are given by*

$$\begin{aligned} & \sum_{i=1}^I \sum_{s=1}^S \delta_{is}(\boldsymbol{\eta}) (K_{is}(1 - R(IT_i, \boldsymbol{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha})) - n_{is}) \\ & \quad \times \left[(1 - R(IT_i, \boldsymbol{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha}))^{\beta-1} + R^{\beta-1}(IT_i, \boldsymbol{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha}) \right] = \mathbf{0}_I, \\ & \sum_{i=1}^I \sum_{s=1}^S \delta_{is}(\boldsymbol{\alpha}) (K_{is}(1 - R(IT_i, \boldsymbol{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha})) - n_{is}) \\ & \quad \times \left[(1 - R(IT_i, \boldsymbol{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha}))^{\beta-1} + R^{\beta-1}(IT_i, \boldsymbol{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha}) \right] = \mathbf{0}_J, \end{aligned}$$

where

$$\delta_{is}(\boldsymbol{\eta}) = \frac{\partial R(IT_i, \boldsymbol{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha})}{\partial \boldsymbol{\eta}} = -(1 - G_i)^{\lambda(\boldsymbol{x}_s; \boldsymbol{\alpha})-1} \lambda(\boldsymbol{x}_s; \boldsymbol{\alpha}) \frac{\partial G_i}{\partial \boldsymbol{\eta}}, \quad (10)$$

$$\delta_{is}(\boldsymbol{\alpha}) = \frac{\partial R(IT_i, \boldsymbol{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} = (1 - G_i)^{\lambda(\boldsymbol{x}_s; \boldsymbol{\alpha})} \log(1 - G_i) \lambda(\boldsymbol{x}_s; \boldsymbol{\alpha}) \boldsymbol{x}_s, \quad (11)$$

with

$$\frac{\partial G_i}{\partial \eta_u} = \begin{cases} \exp(\eta_u) \exp(-\exp(\eta_u)) G_i / \gamma(\eta_u) & , i \leq u, \\ 0 & , i > u. \end{cases} \quad (12)$$

Proof. The estimating equations are given by

$$\begin{aligned}\frac{\partial}{\partial \boldsymbol{\eta}} \sum_{i=1}^I \sum_{s=1}^S \frac{K_{is}}{K} d_{\beta}^*(\widehat{\boldsymbol{p}}_{is}, \boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})) &= \sum_{i=1}^I \sum_{s=1}^S \frac{K_{is}}{K} \frac{\partial}{\partial \boldsymbol{\eta}} d_{\beta}^*(\widehat{\boldsymbol{p}}_{is}, \boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})) = \mathbf{0}_I, \\ \frac{\partial}{\partial \boldsymbol{\alpha}} \sum_{i=1}^I \sum_{s=1}^S \frac{K_{is}}{K} d_{\beta}^*(\widehat{\boldsymbol{p}}_{is}, \boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})) &= \sum_{i=1}^I \sum_{s=1}^S \frac{K_{is}}{K} \frac{\partial}{\partial \boldsymbol{\alpha}} d_{\beta}^*(\widehat{\boldsymbol{p}}_{is}, \boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})) = \mathbf{0}_J,\end{aligned}$$

with

$$\begin{aligned}& \frac{\partial}{\partial \boldsymbol{\eta}} d_{\beta}^*(\widehat{\boldsymbol{p}}_{is}, \boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})) \\ &= \left(\frac{\partial}{\partial \boldsymbol{\eta}} \pi_{is1}^{\beta+1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) + \frac{\partial}{\partial \boldsymbol{\eta}} \pi_{is2}^{\beta+1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \right) - \frac{\beta+1}{\beta} \left(\widehat{p}_{is1} \frac{\partial}{\partial \boldsymbol{\eta}} \pi_{i1}^{\beta}(\boldsymbol{\theta}) + \widehat{p}_{is2} \frac{\partial}{\partial \boldsymbol{\eta}} \pi_{is2}^{\beta}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \right) \\ &= (\beta+1) \left(\pi_{is1}^{\beta}(\boldsymbol{\eta}, \boldsymbol{\alpha}) - \pi_{is2}^{\beta}(\boldsymbol{\eta}, \boldsymbol{\alpha}) - \widehat{p}_{is1} \pi_{is1}^{\beta-1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) + \widehat{p}_{is2} \pi_{is2}^{\beta-1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \right) \frac{\partial}{\partial \boldsymbol{\eta}} \pi_{is1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \\ &= (\beta+1) \left((\pi_{i1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) - \widehat{p}_{i1}) \pi_{is1}^{\beta-1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) - (\pi_{is2}(\boldsymbol{\eta}, \boldsymbol{\alpha}) - \widehat{p}_{is2}) \pi_{is2}^{\beta-1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \right) \frac{\partial}{\partial \boldsymbol{\eta}} \pi_{is1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \\ &= (\beta+1) \left((\pi_{is1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) - \widehat{p}_{i1}) \pi_{is1}^{\beta-1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) + (\pi_{i1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) - \widehat{p}_{i1}) \pi_{is2}^{\beta-1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \right) \frac{\partial}{\partial \boldsymbol{\eta}} \pi_{is1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \\ &= (\beta+1) (\pi_{is1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) - \widehat{p}_{is1}) \left(\pi_{is1}^{\beta-1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) + \pi_{is2}^{\beta-1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \right) \frac{\partial}{\partial \boldsymbol{\eta}} \pi_{is1}(\boldsymbol{\eta}, \boldsymbol{\alpha})\end{aligned}\quad (13)$$

and

$$\begin{aligned}& \frac{\partial}{\partial \boldsymbol{\alpha}} d_{\beta}^*(\widehat{\boldsymbol{p}}_{is}, \boldsymbol{\pi}_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha})) \\ &= (\beta+1) (\pi_{is1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) - \widehat{p}_{is1}) \left(\pi_{i1}^{\beta-1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) + \pi_{is2}^{\beta-1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \right) \frac{\partial}{\partial \boldsymbol{\alpha}} \pi_{is1}(\boldsymbol{\eta}, \boldsymbol{\alpha}).\end{aligned}\quad (14)$$

But, $\frac{\partial}{\partial \boldsymbol{\eta}} \pi_{is1}(\boldsymbol{\eta}, \boldsymbol{\alpha})$ and $\frac{\partial}{\partial \boldsymbol{\alpha}} \pi_{is1}(\boldsymbol{\eta}, \boldsymbol{\alpha})$ are as given in (10) and (11), respectively. See equations (25) and (26) of Ling et al. [2015] for details. ■

Theorem 8 *Let $\boldsymbol{\theta}^*$ be the true value of the parameter $\boldsymbol{\theta}$. Then, the asymptotic distribution of the weighted minimum DPD estimator, $\widehat{\boldsymbol{\theta}}_{\beta}$, is given by*

$$\sqrt{K}(\widehat{\boldsymbol{\theta}}_{\beta} - \boldsymbol{\theta}^*) \xrightarrow{K \rightarrow \infty} \mathcal{N} \left(\mathbf{0}_{I+J}, \mathbf{J}_{\beta}^{-1}(\boldsymbol{\theta}^*) \mathbf{K}_{\beta}(\boldsymbol{\theta}^*) \mathbf{J}_{\beta}^{-1}(\boldsymbol{\theta}^*) \right),$$

where $\mathbf{J}_{\beta}(\boldsymbol{\theta})$ and $\mathbf{K}_{\beta}(\boldsymbol{\theta})$ are given by

$$\mathbf{J}_{\beta}(\boldsymbol{\theta}) = \sum_{i=1}^I \sum_{s=1}^S \frac{K_{is}}{K} \Delta_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \left[(1 - R(IT_i, \mathbf{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha}))^{\beta-1} + R^{\beta-1}(IT_i, \mathbf{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha}) \right] \quad (15)$$

$$\begin{aligned}\mathbf{K}_{\beta}(\boldsymbol{\theta}) &= \sum_{i=1}^I \sum_{s=1}^S \frac{K_{is}}{K} \Delta_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha}) (1 - R(IT_i, \mathbf{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha})) R(IT_i, \mathbf{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha}) \\ &\quad \times \left[(1 - R(IT_i, \mathbf{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha}))^{\beta-1} + R^{\beta-1}(IT_i, \mathbf{x}_s; \boldsymbol{\eta}, \boldsymbol{\alpha}) \right]^2,\end{aligned}\quad (16)$$

with

$$\Delta_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha}) = \begin{pmatrix} \delta_{is}(\boldsymbol{\eta}) \delta_{is}^T(\boldsymbol{\eta}) & \delta_{is}(\boldsymbol{\eta}) \delta_{is}^T(\boldsymbol{\alpha}) \\ \delta_{is}(\boldsymbol{\alpha}) \delta_{is}^T(\boldsymbol{\eta}) & \delta_{is}(\boldsymbol{\alpha}) \delta_{is}^T(\boldsymbol{\alpha}) \end{pmatrix},$$

where $\delta_{is}(\boldsymbol{\eta})$ and $\delta_{is}(\boldsymbol{\alpha})$ are as given in (10) and (11), respectively.

Proof. We denote

$$\begin{aligned}\mathbf{u}_{isj}(\boldsymbol{\eta}, \boldsymbol{\alpha}) &= \left(\frac{\partial \log \pi_{isj}(\boldsymbol{\eta}, \boldsymbol{\alpha})}{\partial \boldsymbol{\eta}}, \frac{\partial \log \pi_{isj}(\boldsymbol{\eta}, \boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} \right)^T \\ &= \left(\frac{1}{\pi_{isj}(\boldsymbol{\eta}, \boldsymbol{\alpha})} \frac{\partial \pi_{isj}(\boldsymbol{\eta}, \boldsymbol{\alpha})}{\partial \boldsymbol{\eta}}, \frac{1}{\pi_{isj}(\boldsymbol{\eta}, \boldsymbol{\alpha})} \frac{\partial \pi_{isj}(\boldsymbol{\eta}, \boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} \right)^T \\ &= \left(\frac{(-1)^{j+1}}{\pi_{isj}(\boldsymbol{\eta}, \boldsymbol{\alpha})} \delta_{is}(\boldsymbol{\eta}), \frac{(-1)^{j+1}}{\pi_{isj}(\boldsymbol{\eta}, \boldsymbol{\alpha})} \delta_{is}(\boldsymbol{\alpha}) \right)^T,\end{aligned}$$

with $\delta_{is}(\boldsymbol{\eta})$ and $\delta_{is}(\boldsymbol{\alpha})$ as given in (10) and (11), respectively.

Now, upon using Result 3.1 of Ghosh et al. [2013], we have

$$\sqrt{K} \left(\widehat{\boldsymbol{\theta}}_{\beta} - \boldsymbol{\theta}^* \right) \xrightarrow[K \rightarrow \infty]{\mathcal{L}} \mathcal{N} \left(\mathbf{0}_{I+J}, \mathbf{J}_{\beta}^{-1}(\boldsymbol{\theta}^*) \mathbf{K}_{\beta}(\boldsymbol{\theta}^*) \mathbf{J}_{\beta}^{-1}(\boldsymbol{\theta}^*) \right),$$

where

$$\begin{aligned}\mathbf{J}_{\beta}(\boldsymbol{\theta}) &= \sum_{i=1}^I \sum_{s=1}^S \sum_{j=1}^2 \frac{K_{is}}{K} \mathbf{u}_{isj}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \mathbf{u}_{isj}^T(\boldsymbol{\eta}, \boldsymbol{\alpha}) \pi_{isj}^{\beta+1}(\boldsymbol{\eta}, \boldsymbol{\alpha}), \\ \mathbf{K}_{\beta}(\boldsymbol{\theta}) &= \sum_{i=1}^I \sum_{s=1}^S \sum_{j=1}^2 \frac{K_{is}}{K} \mathbf{u}_{isj}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \mathbf{u}_{isj}^T(\boldsymbol{\eta}, \boldsymbol{\alpha}) \pi_{isj}^{2\beta+1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) - \sum_{i=1}^I \sum_{s=1}^S \frac{K_{is}}{K} \boldsymbol{\xi}_{is,\beta}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \boldsymbol{\xi}_{is,\beta}^T(\boldsymbol{\eta}, \boldsymbol{\alpha}),\end{aligned}$$

with

$$\boldsymbol{\xi}_{i,\beta}(\boldsymbol{\eta}, \boldsymbol{\alpha}) = \sum_{j=1}^2 \mathbf{u}_{isj}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \pi_{isj}^{\beta+1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) = (\delta_{is}(\boldsymbol{\eta}), \delta_{is}(\boldsymbol{\alpha}))^T \sum_{j=1}^2 (-1)^{j+1} \pi_{isj}^{\beta}(\boldsymbol{\eta}, \boldsymbol{\alpha}).$$

Now, for $\mathbf{u}_{isj}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \mathbf{u}_{isj}^T(\boldsymbol{\eta}, \boldsymbol{\alpha})$, we have

$$\mathbf{u}_{isj}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \mathbf{u}_{isj}^T(\boldsymbol{\eta}, \boldsymbol{\alpha}) = \frac{1}{\pi_{isj}^2(\boldsymbol{\eta}, \boldsymbol{\alpha})} \begin{pmatrix} \delta_{is}(\boldsymbol{\eta}) \delta_{is}^T(\boldsymbol{\eta}) & \delta_{is}(\boldsymbol{\eta}) \delta_{is}^T(\boldsymbol{\alpha}) \\ \delta_{is}(\boldsymbol{\alpha}) \delta_{is}^T(\boldsymbol{\eta}) & \delta_{is}(\boldsymbol{\alpha}) \delta_{is}^T(\boldsymbol{\alpha}) \end{pmatrix} = \frac{1}{\pi_{ij}^2(\boldsymbol{\theta})} \Delta_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha}),$$

with

$$\Delta_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha}) = \begin{pmatrix} \delta_{is}(\boldsymbol{\eta}) \delta_{is}^T(\boldsymbol{\eta}) & \delta_{is}(\boldsymbol{\eta}) \delta_{is}^T(\boldsymbol{\alpha}) \\ \delta_{is}(\boldsymbol{\alpha}) \delta_{is}^T(\boldsymbol{\eta}) & \delta_{is}(\boldsymbol{\alpha}) \delta_{is}^T(\boldsymbol{\alpha}) \end{pmatrix}.$$

It then follows that

$$\begin{aligned}\mathbf{J}_{\beta}(\boldsymbol{\theta}) &= \sum_{i=1}^I \sum_{s=1}^S \frac{K_{is}}{K} \Delta_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \sum_{j=1}^2 \pi_{isj}^{\beta-1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \\ &= \sum_{i=1}^I \sum_{s=1}^S \frac{K_{is}}{K} \Delta_{is}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \left(\pi_{is1}^{\beta-1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) + \pi_{is2}^{\beta-1}(\boldsymbol{\eta}, \boldsymbol{\alpha}) \right).\end{aligned}$$

■

From here on, and for simplicity, we will denote $R(IT_i, \mathbf{x}_0; \boldsymbol{\eta}, \boldsymbol{\alpha})$ simply by $R(IT_i, \mathbf{x}_0; \boldsymbol{\theta})$. Based on Theorem 8, the asymptotic variance of the weighted minimum DPD estimator of the reliability at inspection time IT_i under normal operating condition \mathbf{x}_0 is given by

$$\text{Var}(R(IT_i, \mathbf{x}_0; \widehat{\boldsymbol{\theta}}_{\beta})) \equiv \text{Var}(R(\widehat{\boldsymbol{\theta}}_{\beta})) = \mathbf{P}^T \boldsymbol{\Sigma}_{\beta}(\widehat{\boldsymbol{\theta}}_{\beta}) \mathbf{P},$$

where

$$\boldsymbol{\Sigma}_\beta(\widehat{\boldsymbol{\theta}}_\beta) = \mathbf{J}_\beta^{-1}(\widehat{\boldsymbol{\theta}}_\beta) \mathbf{K}_\beta(\widehat{\boldsymbol{\theta}}_\beta) \mathbf{J}_\beta^{-1}(\widehat{\boldsymbol{\theta}}_\beta), \quad (17)$$

$\mathbf{J}_\beta(\boldsymbol{\theta})$, $\mathbf{K}_\beta(\boldsymbol{\theta})$ are as given in (15) and (16), respectively, and \mathbf{P} is a vector of the first-order derivatives of $R(IT_i, \mathbf{x}_0; \boldsymbol{\theta})$ with respect to the model parameters (see (10) and (11)). Consequently, the $100(1 - \alpha)\%$ asymptotic confidence interval for the reliability function $R(\boldsymbol{\theta})$ is given by

$$\left(R(\widehat{\boldsymbol{\theta}}_\beta) - z_{1-\alpha/2} se(R(\widehat{\boldsymbol{\theta}}_\beta)), R(\widehat{\boldsymbol{\theta}}_\beta) + z_{1-\alpha/2} se(R(\widehat{\boldsymbol{\theta}}_\beta)) \right),$$

where $se(R(\widehat{\boldsymbol{\theta}}_\beta)) = \sqrt{Var(R(\widehat{\boldsymbol{\theta}}_\beta))}$ and z_γ is the upper γ percentage point of the standard normal distribution.

However, an asymptotic confidence interval may be satisfactory only for large sample sizes as it is based on the asymptotic properties of the estimators. Balakrishnan and Ling [2013] found that, in the case of small sample sizes, the distribution of the MLE of the reliability is quite skewed, and so proposed a logit-transformation for obtaining a confidence interval for the reliability function, which can be extended to the case of the weighted minimum DPD estimators of the reliabilities as well to obtain a confidence interval of the form:

$$\left(\frac{R(\widehat{\boldsymbol{\theta}}_\beta)}{R(\widehat{\boldsymbol{\theta}}_\beta) + (1 - R(\widehat{\boldsymbol{\theta}}_\beta))T}, \frac{R(\widehat{\boldsymbol{\theta}}_\beta)}{R(\widehat{\boldsymbol{\theta}}_\beta) + (1 - R(\widehat{\boldsymbol{\theta}}_\beta))/T} \right), \quad (18)$$

where $T = \exp\left(z_{1-\alpha/2} \frac{se(R(\widehat{\boldsymbol{\theta}}_\beta))}{R(\widehat{\boldsymbol{\theta}}_\beta)(1-R(\widehat{\boldsymbol{\theta}}_\beta))}\right)$.

4 Wald-type tests

Let us consider the function $\mathbf{m} : \mathbb{R}^{I+J} \rightarrow \mathbb{R}^r$, where $r \leq (I + J)$ and

$$\mathbf{m}(\boldsymbol{\theta}) = \mathbf{0}_r, \quad (19)$$

which corresponds to a composite null hypothesis. We assume that the $(I + J) \times r$ matrix $\mathbf{M}(\boldsymbol{\theta}) = \frac{\partial \mathbf{m}^T(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$ exists and is continuous in $\boldsymbol{\theta}$ and $\text{rank } \mathbf{M}(\boldsymbol{\theta}) = r$. Then, for testing

$$H_0 : \boldsymbol{\theta} \in \boldsymbol{\Theta}_0 \text{ against } H_1 : \boldsymbol{\theta} \notin \boldsymbol{\Theta}_0, \quad (20)$$

where $\boldsymbol{\Theta}_0 = \{\boldsymbol{\theta} \in \mathbb{R}^{(I+J)} : \mathbf{m}(\boldsymbol{\theta}) = \mathbf{0}_r\}$, we can consider the following Wald-type test statistics:

$$W_K(\widehat{\boldsymbol{\theta}}_\beta) = K \mathbf{m}^T(\widehat{\boldsymbol{\theta}}_\beta) \left(\mathbf{M}^T(\widehat{\boldsymbol{\theta}}_\beta) \boldsymbol{\Sigma}(\widehat{\boldsymbol{\theta}}_\beta) \mathbf{M}(\widehat{\boldsymbol{\theta}}_\beta) \right)^{-1} \mathbf{m}(\widehat{\boldsymbol{\theta}}_\beta), \quad (21)$$

where $\boldsymbol{\Sigma}_\beta(\widehat{\boldsymbol{\theta}}_\beta)$ is as given in (17).

Theorem 9 *Under (19), we have*

$$W_K(\widehat{\boldsymbol{\theta}}_\beta) \xrightarrow{K \rightarrow \infty} \chi_r^2,$$

where χ_r^2 denotes a central chi-square distribution with r degrees of freedom.

Proof. Let $\boldsymbol{\theta}^0 \in \Theta_0$ be the true value of the parameter $\boldsymbol{\theta}$. It is clear that

$$\begin{aligned} \mathbf{m}(\widehat{\boldsymbol{\theta}}_\beta) &= \mathbf{m}(\boldsymbol{\theta}^0) + \mathbf{M}^T(\widehat{\boldsymbol{\theta}}_\beta) (\widehat{\boldsymbol{\theta}}_\beta - \boldsymbol{\theta}^0) + o_p(\|\widehat{\boldsymbol{\theta}}_\beta - \boldsymbol{\theta}^0\|) \\ &= \mathbf{M}^T(\widehat{\boldsymbol{\theta}}_\beta) (\widehat{\boldsymbol{\theta}}_\beta - \boldsymbol{\theta}^0) + o_p(K^{-1/2}). \end{aligned}$$

But, under H_0 , $\sqrt{K}(\widehat{\boldsymbol{\theta}}_\beta - \boldsymbol{\theta}^0) \xrightarrow{K \rightarrow \infty} \mathcal{N}(\mathbf{0}_{(I+J)}, \boldsymbol{\Sigma}_\beta(\boldsymbol{\theta}^0))$. Therefore, under H_0 ,

$$\sqrt{K} \mathbf{m}(\widehat{\boldsymbol{\theta}}_\beta) \xrightarrow{K \rightarrow \infty} \mathcal{N}(\mathbf{0}_r, \mathbf{M}^T(\boldsymbol{\theta}^0) \boldsymbol{\Sigma}_\beta(\boldsymbol{\theta}^0) \mathbf{M}(\boldsymbol{\theta}^0))$$

and taking into account that $\text{rank}(\mathbf{M}(\boldsymbol{\theta}^0)) = r$, we obtain

$$K \mathbf{m}^T(\widehat{\boldsymbol{\theta}}_\beta) (\mathbf{M}^T(\boldsymbol{\theta}^0) \boldsymbol{\Sigma}_\beta(\boldsymbol{\theta}^0) \mathbf{M}(\boldsymbol{\theta}^0))^{-1} \mathbf{m}(\widehat{\boldsymbol{\theta}}_\beta) \xrightarrow{K \rightarrow \infty} \chi_r^2.$$

Because $(\mathbf{M}^T(\widehat{\boldsymbol{\theta}}_\beta) \boldsymbol{\Sigma}_\beta(\widehat{\boldsymbol{\theta}}_\beta) \mathbf{M}(\widehat{\boldsymbol{\theta}}_\beta))^{-1}$ is a consistent estimator of $(\mathbf{M}^T(\boldsymbol{\theta}^0) \boldsymbol{\Sigma}_\beta(\boldsymbol{\theta}^0) \mathbf{M}(\boldsymbol{\theta}^0))^{-1}$, we get

$$W_K(\widehat{\boldsymbol{\theta}}_\beta) \xrightarrow{K \rightarrow \infty} \chi_r^2.$$

■

Based on Theorem 9, we shall reject the null hypothesis in (20) if

$$W_K(\widehat{\boldsymbol{\theta}}_\beta) > \chi_{r,\alpha}^2, \tag{22}$$

where $\chi_{r,\alpha}^2$ is the upper α percentage point of χ_r^2 distribution.

Wald-type test statistics based on weighted minimum DPD estimators have been considered previously by a number of authors including Basu et al. [2016], Castilla et al. [2018] and Balakrishnan et al. [2019a,c,b].

5 Monte Carlo Simulation Results

In this section, an extensive simulation study is carried out for evaluating the proposed weighted minimum DPD estimators and Wald-type tests. The simulations results are computed based on 1,000 simulated samples in the R statistical software. Mean square error (MSE) and bias are computed for evaluating the estimators in both balanced and unbalanced data sets, while empirical levels and powers are computed for evaluating the tests.

5.1 Weighted minimum DPD estimators

Suppose the lifetimes of test units follow a Weibull distribution (see Remark 5). All the test units were divided into $S = 4$ groups, subject to different acceleration conditions with $J = 2$ stress factors at two elevated stress levels each, that is, $(x_1, x_2) = \{(55, 70), (55, 100), (85, 70), (85, 100)\}$, and were inspected at $I = 3$ different times, $(IT_1, IT_2, IT_3) = (2, 5, 8)$.

5.1.1 Balanced data

We assume $(c_1, c_2) = (-0.03, -0.03)$, $c_0 \in \{6, 6.5\}$ for different degrees of reliability and $b \in \{0, 0.5\}$. Note that the exponential distribution will be included as a special case when we take $b = 0$. In this framework, we consider outlying cells rather than outlying observations. A cell which does not follow the one-shot device model will be called an outlying cell or outlier. In this cell, the number of devices failed will be different than what is expected. This is in the spirit of principle of inflated models in distribution theory (see Lambert (1992) and Heilbron (1994)). This outlying cell (taken to be $i = 3$, $s = 4$), is generated under the parameters $(\tilde{c}_1, \tilde{c}_2) = (-0.027, -0.027)$ and $\tilde{b} \in \{0.05, 0.45\}$.

Bias of estimates are then computed for different (equal) samples sizes $K_{is} \in \{50, 70, 100\}$ and tuning parameters $\beta \in \{0, 0.2, 0.4, 0.6\}$ for both pure and contaminated data. The obtained results are presented in Tables 4, 5, 6 and 7. As expected, when the sample size increases, errors tend to decrease, while in the contaminated data set, these errors are generally greater than in the case of uncontaminated data. Weighted minimum DPD estimators with $\beta > 0$ present a better behaviour than the MLE in terms of robustness. Note that reliabilities are underestimated and that the estimates are quite precise in all the cases.

5.1.2 Unbalanced data

In this setting, we consider an unbalanced data set, in which at each inspection time i , $(K_{i1}, K_{i2}, K_{i3}, K_{i4}) = (10r, 15r, 20r, 30r)$ for different values of the factor $r \in \{1, 2, \dots, 10\}$. We then assume $(c_0, c_1, c_2) = (6, 5, -0.03, -0.03)$, $b = 0.5$, and $\tilde{c}_2 = -0.027$. MSEs of the parameter θ are then computed and the obtained results are presented in Figure 2.

As expected, when the sample size increases, the MSE decreases, but lack of robustness of the MLE ($\beta = 0$) as compared to the weighted minimum DPD estimators with $\beta > 0$ becomes quite evident.

5.2 Wald-type tests

To evaluate the performance of the proposed Wald-type tests, we consider the scenario of unbalanced data proposed discussed above. We consider the testing problem

$$H_0 : \alpha_1 = 0.04946 \quad \text{against} \quad H_1 : \alpha_1 \neq 0.04946, \quad (23)$$

Under the same simulation scheme as used above in Section 5.1.2, we first evaluate the empirical levels, measured as the proportion of Wald-type test statistics exceeding the corresponding chi-square critical value for a nominal size of 0.05. The empirical powers are computed in a similar manner, with $\alpha_1^0 = 0.05276$ ($c_1 = -0.032$, $c_2 = -0.028$). The obtained results are shown in Figure 3.

In the case of uncontaminated data, the conventional Wald test has level to be close to nominal value and also has good power performance. The robust tests, however, has a slightly inflated level values (as compared to the nominal value), but possesses similar power as the conventional Wald test (which is evident from the Figure 3). But, when the data is contaminated, the level of the conventional Wald test turns out to be quite non-robust and takes on very high values as compared to the nominal level. This, in turn, results in higher power (see Figure 3). However, the proposed robust tests maintain levels close to the nominal value and also possesses good power values (as can be seen in the Figure 3). Thus, taking both level and power into account, the robust tests, though is slightly inferior to the conventional Wald test in the case of uncontaminated data,

turn out to be considerably more efficient than the conventional Walk test in the case of contaminated data

6 Application to Real Data

6.1 Testing on proportional Hazard rates

Based on Balakrishnan and Ling [2012], we suggest a distance-based statistic on the form

$$M_\beta = \max_{i,s} \left| n_{is} - K_{is}(1 - R(IT_i, \mathbf{x}_s; \hat{\boldsymbol{\theta}}_\beta)) \right| \quad (24)$$

as a discrepancy measure for evaluating the fit of the assumed model to the observed data. If the assumed model is not a good fit to the data, we will obtain a large value of M_β . In fact, under the assumed model, we have

$$n_{is} \sim \text{Binomial}(K_{is}, 1 - R(IT_i, \mathbf{x}_s; \boldsymbol{\theta})),$$

and so, by denoting $\Phi_{is} = \lceil K_{is}(1 - R(IT_i, \mathbf{x}_s; \hat{\boldsymbol{\theta}}_\beta)) - M_\beta \rceil$ and $\Psi_{is} = \lfloor K_{is}(1 - R(IT_i, \mathbf{x}_s; \hat{\boldsymbol{\theta}}_\beta)) + M_\beta \rfloor$, the corresponding exact p-value is given by

$$\begin{aligned} p\text{-value} &= Pr \left(\max_{i,s} \left| n_{is} - K_{is}(1 - R(IT_i, \mathbf{x}_s; \hat{\boldsymbol{\theta}}_\beta)) \right| > M_\beta \right) \\ &= 1 - Pr \left(\max_{i,s} \left| n_{is} - K_{is}(1 - R(IT_i, \mathbf{x}_s; \hat{\boldsymbol{\theta}}_\beta)) \right| \leq M_\beta \right) \\ &= 1 - \prod_{i=1}^I \prod_{s=1}^S Pr \left(\left| n_{is} - K_{is}(1 - R(IT_i, \mathbf{x}_s; \hat{\boldsymbol{\theta}}_\beta)) \right| \leq M_\beta \right) \\ &= 1 - \prod_{i=1}^I \prod_{s=1}^S Pr (\Phi_{is} \leq n_{is} \leq \Psi_{is}). \end{aligned} \quad (25)$$

From (25), we can readily validate the proportional hazards assumption if the p -value is sufficiently large.

6.2 Choice of the tuning parameter

In the preceding discussion, we have seen how weighted minimum DPD estimators with $\beta > 0$ tend to be more robust than the classical MLE overall when contamination is present in the data. MLE has been shown to be more efficient when there is no contamination in the data. It is then necessary to provide a data-driven procedure for the determination of the optimal choice of the tuning parameter that would provide a trade-off between efficiency and robustness. One way to do this is as follows: In a grid of possible tuning parameters, apply a measure of discrepancy to the data. Then, the tuning parameter that leads to the minimum discrepancy-statistic can be chosen as the ‘‘optimal’’ one.

A possible choice of the discrepancy measure could be M_β , given in (24). Another idea may be by minimizing the estimated mean square error. This method, originally proposed by Warwick and Jones [2005], was applied in the context of one-shot devices in Balakrishnan et al. [2019a,b]. The estimation of the MSE is as follows:

$$\widehat{MSE}(\beta) = (\boldsymbol{\theta}_\beta - \boldsymbol{\theta}_P)^T (\boldsymbol{\theta}_\beta - \boldsymbol{\theta}_P) + \frac{1}{K} \text{trace} \left\{ \mathbf{J}_\beta^{-1}(\boldsymbol{\theta}_\beta) \mathbf{K}_\beta(\boldsymbol{\theta}_\beta) \mathbf{J}_\beta^{-1}(\boldsymbol{\theta}_\beta) \right\},$$

where $\boldsymbol{\theta}_P$ is a pilot estimator, whose choice will affect the overall procedure. If we take $\boldsymbol{\theta}_P = \hat{\boldsymbol{\theta}}_\beta$, the approach coincides with that of Hong and Kim [2001], but it does not take into account the model misspecification. Note that, the need for a pilot estimator becomes a drawback of this procedure, as will be seen in the next section.

6.3 Electric Current data

We now consider the Electric Current data (Ling et al. [2015]), in which 120 one-shot devices were divided into four accelerated conditions with higher-than-normal temperature and electric current, and inspected at three different times (see Table 1).

In Table 2, estimates of the model parameters by the use of the proportional hazards model and the Weibull distribution (see Balakrishnan et al. [2019b]) are provided, for different values of the tuning parameter. Estimates of reliabilities and confidence intervals under the proportional hazards assumption are given in Table 3.

Table 2 also presents the d -values of the distance-statistic M_β and the corresponding p -values. From these values, it seems that the proportional hazards assumption fits the data at least as well as the Weibull model. The best fit is obtained for $\beta = 0.5$. To complete the study, Warwick and Jones [2005] approach is achieved for different values of the pilot estimator in a grid of width 100. However, as pointed out before, the final choice of the optimal tuning parameter depends too much on the pilot estimator used (see Figure 1). Recently, Basak et al. [2020] proposed an “iterated Warwick and Jones algorithm” trying to solve this problem.

Table 1: Electric Current data

Inspection Time IT_i	2	2	2	2	5	5	5	5	8	8	8	8
Temperature x_{s1}	55	80	55	80	55	80	55	80	55	80	55	80
Electric current x_{s2}	70	70	100	100	70	70	100	100	70	70	100	100
Number of failures n_{is}	4	8	9	8	7	9	9	9	6	10	9	10
Number of tested items K_{is}	10	10	10	10	10	10	10	10	10	10	10	10

Table 2: Electric Current data: one-shot device testing data analysis by using the proportional hazards model and the Weibull distribution

β	Proportional Hazards model							Weibull distribution					
	M_β	p-value	T°	current	η_1	η_2	η_3	M_β	p-value	intercept	T°	current	shape
0	1.80	0.695	0.023	0.018	0.123	0.543	-2.182	1.80	0.695	7.022	-0.053	-0.040	-0.817
0.1	1.72	0.745	0.024	0.018	0.141	0.555	-2.283	1.72	0.745	7.398	-0.055	-0.043	-0.845
0.2	1.65	0.796	0.024	0.019	0.156	0.565	-2.399	1.65	0.796	7.803	-0.057	-0.046	-0.869
0.3	1.58	0.833	0.025	0.020	0.167	0.572	-2.534	1.57	0.833	8.254	-0.060	-0.050	-0.890
0.4	1.49	0.931	0.026	0.022	0.177	0.579	-2.695	1.49	0.931	8.747	-0.064	-0.054	-0.906
0.5	1.40	0.942	0.027	0.023	0.183	0.582	-2.887	1.40	0.942	9.324	-0.068	-0.058	-0.920
0.6	1.51	0.892	0.029	0.025	0.187	0.585	-3.130	1.51	0.892	10.026	-0.073	-0.063	-0.931
0.7	1.64	0.876	0.031	0.027	0.190	0.586	-3.438	1.64	0.876	10.868	-0.079	-0.069	-0.938
0.8	1.76	0.861	0.033	0.030	0.189	0.586	-3.798	1.76	0.861	11.827	-0.086	-0.076	-0.942
0.9	1.84	0.750	0.036	0.032	0.185	0.582	-4.106	1.84	0.750	12.575	-0.091	-0.082	-0.938

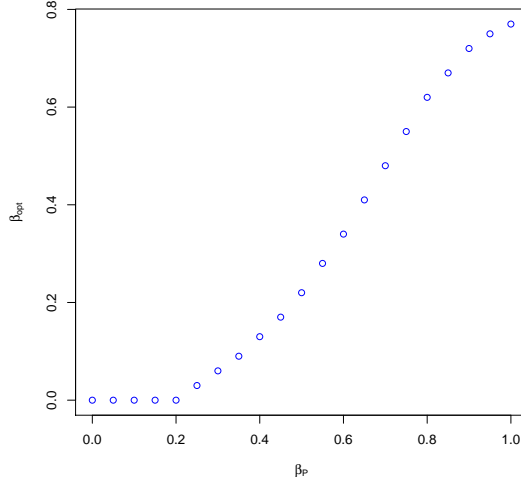


Figure 1: Electric Current data: estimation of the optimal tuning parameter depending on a pilot estimator by Warwick and Jones [2005] procedure

Table 3: Electric Current data: estimates of reliabilities and corresponding confidence intervals

β	$R(2, 25, 35; \hat{\theta}_\beta)$	$R(5, 25, 35; \hat{\theta}_\beta)$	$R(8, 25, 35; \hat{\theta}_\beta)$
0	0.817 (0.516, 0.949)	0.739 (0.397, 0.924)	0.689 (0.336, 0.907)
0.1	0.824 (0.526, 0.952)	0.751 (0.412, 0.928)	0.704 (0.353, 0.912)
0.2	0.833 (0.535, 0.956)	0.765 (0.427, 0.934)	0.721 (0.370, 0.919)
0.3	0.843 (0.545, 0.960)	0.780 (0.442, 0.941)	0.740 (0.387, 0.927)
0.4	0.855 (0.555, 0.965)	0.797 (0.457, 0.948)	0.760 (0.405, 0.936)
0.5	0.868 (0.566, 0.971)	0.816 (0.474, 0.956)	0.782 (0.423, 0.946)
0.6	0.884 (0.581, 0.976)	0.837 (0.493, 0.965)	0.807 (0.445, 0.956)
0.7	0.901 (0.601, 0.982)	0.861 (0.516, 0.973)	0.836 (0.471, 0.967)
0.8	0.918 (0.626, 0.987)	0.885 (0.544, 0.980)	0.863 (0.503, 0.975)
0.9	0.931 (0.649, 0.990)	0.902 (0.570, 0.985)	0.884 (0.533, 0.981)

7 Concluding Remarks

In this paper, we have developed new estimators and tests for one-shot device testing under proportional hazards assumption. This semi-parametric model is presented as an alternative to parametric models, by allowing the hazard rate increasing in a non-parametric way. An extensive simulation study carried out shows the robustness of the proposed methods of inference. Because model selection is an important part of reliability analysis, a test statistic for checking the proportional hazards assumption is presented as well and applied to the numerical example.

As future work, it would be of interest to develop model selection criteria, and also to extend the proposed method to the case of competing risks problem, when there is more than one cause of failure of one-shot devices.

Table 4: Bias for the semi-parametric model with $b = 0$ and $c_0 = 6$.

		Uncontaminated data				Contaminated data				
$K_{is} = 50$	True value	0	0.2	0.4	0.6	0	0.2	0.4	0.6	
	η_1	-0.66688	-0.00494	-0.00276	-0.00053	-0.00372	0.09898	0.06722	0.03547	0.01708
	η_2	-0.01304	-0.00228	-0.00078	0.00109	-0.00131	0.06902	0.04716	0.02531	0.01286
	η_3	-3.92056	-0.02788	-0.01810	-0.01389	-0.01982	0.34916	0.23252	0.12087	0.05402
	α_1	0.03000	0.00010	0.00002	-0.00001	0.00002	-0.00281	-0.00193	-0.00107	-0.00056
	α_2	0.03000	0.00033	0.00027	0.00025	0.00030	-0.00259	-0.00167	-0.00081	-0.00028
	$R(15, \mathbf{x}_0)$	0.79857	-0.00520	-0.00611	-0.00686	-0.00671	-0.03387	-0.02502	-0.01652	-0.01202
		Uncontaminated data				Contaminated data				
$K_{is} = 70$	True value	0	0.2	0.4	0.6	0	0.2	0.4	0.6	
	η_1	-0.66688	-0.00780	-0.00675	-0.00716	-0.00802	0.09876	0.06233	0.03209	0.01278
	η_2	-0.01304	-0.00459	-0.00386	-0.00410	-0.00465	0.06810	0.04315	0.02257	0.00948
	η_3	-3.92056	-0.03954	-0.03763	-0.04073	-0.04458	0.35084	0.21624	0.10254	0.03023
	α_1	0.03000	0.00027	0.00025	0.00026	0.00029	-0.00276	-0.00173	-0.00086	-0.00030
	α_2	0.03000	0.00035	0.00034	0.00038	0.00041	-0.00267	-0.00163	-0.00075	-0.00017
	$R(15, \mathbf{x}_0)$	0.79857	-0.00197	-0.00221	-0.00223	-0.00221	-0.03131	-0.02076	-0.01246	-0.00749
		Uncontaminated data				Contaminated data				
$K_{is} = 100$	True value	0	0.2	0.4	0.6	0	0.2	0.4	0.6	
	η_1	-0.66688	-0.00778	-0.00682	-0.00711	-0.00785	0.09857	0.06207	0.03231	0.01320
	η_2	-0.01304	-0.00477	-0.00412	-0.00429	-0.00477	0.06776	0.04275	0.02248	0.00952
	η_3	-3.92056	-0.02739	-0.02332	-0.02387	-0.02586	0.36315	0.23019	0.12013	0.04993
	α_1	0.03000	0.00031	0.00028	0.00029	0.00031	-0.00271	-0.00169	-0.00084	-0.00029
	α_2	0.03000	0.00016	0.00013	0.00013	0.00015	-0.00287	-0.00185	-0.00099	-0.00045
	$R(15, \mathbf{x}_0)$	0.79857	-0.00144	-0.00176	-0.00185	-0.00187	-0.03085	-0.02034	-0.01214	-0.00720

Table 5: Bias for the semi-parametric model with $b = 0.5$ and $c_0 = 6$.

		Uncontaminated data				Contaminated data				
$K_{is} = 50$	True value	0	0.2	0.4	0.6	0	0.2	0.4	0.6	
	η_1	-1.38827	-0.01224	-0.00988	-0.03700	-0.07648	0.03590	-0.00540	-0.03356	-0.08611
	η_2	-0.48138	-0.00687	-0.00537	-0.02153	-0.04394	0.02362	-0.00239	-0.01962	-0.04962
	η_3	-6.46391	-0.05973	-0.05148	-0.19693	-0.40632	0.14538	-0.03531	-0.17304	-0.47276
	α_1	0.04946	0.00032	0.00023	0.00124	0.00274	-0.00125	0.00010	0.00106	0.00325
	α_2	0.04946	0.00061	0.00056	0.00174	0.00361	-0.00097	0.00043	0.00155	0.00410
	$R(15, \mathbf{x}_0)$	0.91810	-0.00348	-0.00385	-0.00348	-0.00272	-0.01136	-0.00486	-0.00361	-0.00255
		Uncontaminated data				Contaminated data				
$K_{is} = 70$	True value	0	0.2	0.4	0.6	0	0.2	0.4	0.6	
	η_1	-1.38827	-0.02188	-0.01955	-0.05972	-0.10800	0.03391	-0.01254	-0.06567	-0.12770
	η_2	-0.48138	-0.01318	-0.01171	-0.03481	-0.06298	0.02199	-0.00731	-0.03817	-0.07423
	η_3	-6.46391	-0.06923	-0.06287	-0.27869	-0.55900	0.16868	-0.03493	-0.30595	-0.67033
	α_1	0.04946	0.00062	0.00055	0.00202	0.00446	-0.00121	0.00033	0.00217	0.00530
	α_2	0.04946	0.00054	0.00051	0.00235	0.00433	-0.00128	0.00029	0.00260	0.00520
	$R(15, \mathbf{x}_0)$	0.91810	-0.00174	-0.00199	-0.00127	-0.00004	-0.01041	-0.00289	-0.00121	0.00031
		Uncontaminated data				Contaminated data				
$K_{is} = 100$	True value	0	0.2	0.4	0.6	0	0.2	0.4	0.6	
	η_1	-1.38827	-0.01771	-0.01652	-0.06256	-0.08467	0.04334	-0.01518	-0.06228	-0.08025
	η_2	-0.48138	-0.01071	-0.00996	-0.03610	-0.04904	0.02774	-0.00875	-0.03612	-0.04659
	η_3	-6.46391	-0.05209	-0.04718	-0.27304	-0.41072	0.21133	-0.04194	-0.27462	-0.38078
	α_1	0.04946	0.00057	0.00053	0.00204	0.00340	-0.00145	0.00048	0.00226	0.00316
	α_2	0.04946	0.00034	0.00030	0.00217	0.00315	-0.00168	0.00026	0.00204	0.00295
	$R(15, \mathbf{x}_0)$	0.91810	-0.00123	-0.00140	-0.00054	0.00003	-0.01060	-0.00223	-0.00066	-0.00012

Table 6: Bias for the semi-parametric model with $b = 0.5$ and $c_0 = 6.5$.

		Uncontaminated data				Contaminated data				
$K_{is} = 50$	True value	0	0.2	0.4	0.6	0	0.2	0.4	0.6	
	η_1	-0.66879	0.00223	0.00097	0.00020	-0.00900	0.17046	0.14959	0.12180	0.11436
	η_2	-0.01553	0.00320	0.00227	0.00156	-0.00465	0.11845	0.10401	0.08415	0.08029
	η_3	-4.42056	-0.01196	-0.00948	-0.01543	-0.03474	0.56481	0.50586	0.43705	0.36957
	α_1	0.03000	0.00004	0.00001	0.00004	0.00021	-0.00437	-0.00389	-0.00336	-0.00286
	α_2	0.03000	0.00014	0.00013	0.00018	0.00030	-0.00426	-0.00379	-0.00324	-0.00274
	$R(15, \mathbf{x}_0)$	0.87247	-0.00529	-0.00555	-0.00534	-0.00530	-0.03609	-0.03299	-0.02952	-0.02561
		Uncontaminated data				Contaminated data				
$K_{is} = 70$	True value	0	0.2	0.4	0.6	0	0.2	0.4	0.6	
	η_1	-0.66879	0.00050	-0.00537	-0.00598	-0.00468	0.16516	0.14230	0.12082	0.10748
	η_2	-0.01553	0.00166	-0.00294	-0.00333	-0.00217	0.11371	0.09772	0.08279	0.07497
	η_3	-4.42056	-0.03260	-0.03492	-0.03697	-0.04026	0.55639	0.49446	0.42414	0.36377
	α_1	0.03000	0.00016	0.00017	0.00019	0.00021	-0.00434	-0.00386	-0.00329	-0.00322
	α_2	0.03000	0.00029	0.00032	0.00034	0.00036	-0.00418	-0.00367	-0.00314	-0.00318
	$R(15, \mathbf{x}_0)$	0.87247	-0.00224	-0.00231	-0.00226	-0.00211	-0.03358	-0.03044	-0.02652	-0.02275
		Uncontaminated data				Contaminated data				
$K_{is} = 100$	True value	0	0.2	0.4	0.6	0	0.2	0.4	0.6	
	η_1	-0.66879	-0.00302	-0.00428	-0.00423	-0.00453	0.15788	0.14182	0.12319	0.10443
	η_2	-0.01553	-0.00136	-0.00242	-0.00237	-0.00256	0.10788	0.09716	0.08443	0.07154
	η_3	-4.42056	-0.02671	-0.02485	-0.02528	-0.02720	0.55329	0.49534	0.43019	0.36647
	α_1	0.03000	0.00019	0.00018	0.00019	0.00020	-0.00422	-0.00376	-0.00325	-0.00276
	α_2	0.03000	0.00014	0.00014	0.00014	0.00015	-0.00427	-0.00381	-0.00331	-0.00281
	$R(15, \mathbf{x}_0)$	0.87247	-0.00126	-0.00139	-0.00140	-0.00135	-0.03191	-0.02878	-0.02528	-0.02190

Table 7: Bias for the semi-parametric model with $b = 0.5$ and $c_0 = 6.5$.

		Uncontaminated data				Contaminated data				
$K_{is} = 50$	True value	0	0.2	0.4	0.6	0	0.2	0.4	0.6	
	η_1	-1.38845	-0.00292	-0.02634	-0.06961	-0.10441	0.28565	0.19158	0.13289	0.07420
	η_2	-0.48171	-0.00007	-0.01454	-0.03906	-0.08819	0.18184	0.12322	0.12394	0.12467
	η_3	-7.28827	-0.08460	-0.15057	-0.34018	-0.97897	1.21433	0.81382	0.14850	0.33555
	α_1	0.04946	0.00058	0.00102	0.00237	-0.08206	-0.00889	-0.00592	-0.00116	-0.32679
	α_2	0.04946	0.00057	0.00108	0.00246	-0.10152	-0.00891	-0.00594	-0.00112	-0.39906
	$R(15, \mathbf{x}_0)$	0.96322	-0.00163	-0.00176	-0.00145	0.00157	-0.02785	-0.01976	-0.01062	-0.00149
		Uncontaminated data				Contaminated data				
$K_{is} = 70$	True value	0	0.2	0.4	0.6	0	0.2	0.4	0.6	
	η_1	-1.38845	-0.01510	-0.03564	-0.05629	-0.08443	0.28682	0.18935	0.03294	0.02196
	η_2	-0.48171	-0.00878	-0.02069	-0.03247	-0.11914	0.18302	0.12036	0.02563	0.04542
	η_3	-7.28827	-0.06124	-0.15643	-0.22439	-0.99978	1.24650	0.87564	0.14490	-0.01407
	α_1	0.04946	0.00041	0.00110	0.00155	-0.01596	-0.00916	-0.00637	-0.00104	-0.15771
	α_2	0.04946	0.00047	0.00111	0.00168	-0.02075	-0.00910	-0.00631	-0.00113	-0.19334
	$R(15, \mathbf{x}_0)$	0.96322	-0.00119	-0.00105	-0.00084	0.00146	-0.02804	-0.01977	-0.00976	-0.00119
		Uncontaminated data				Contaminated data				
$K_{is} = 100$	True value	0	0.2	0.4	0.6	0	0.2	0.4	0.6	
	η_1	-1.38845	-0.00904	-0.01105	-0.05924	-0.23063	0.28616	0.19928	0.05644	0.03762
	η_2	-0.48171	-0.00531	-0.00635	-0.03368	-0.12308	0.18172	0.12619	0.03911	-0.01015
	η_3	-7.28827	-0.06888	-0.07584	-0.29436	-0.96654	1.22401	0.87425	0.21922	-0.38512
	α_1	0.04946	0.00048	0.00053	0.00202	-0.00879	-0.00897	-0.00635	-0.00168	-0.09403
	α_2	0.04946	0.00041	0.00047	0.00207	-0.01198	-0.00904	-0.00642	-0.00164	-0.11596
	$R(15, \mathbf{x}_0)$	0.96322	-0.00015	-0.00021	0.00022	0.00236	-0.02602	-0.01814	-0.00878	-0.00050

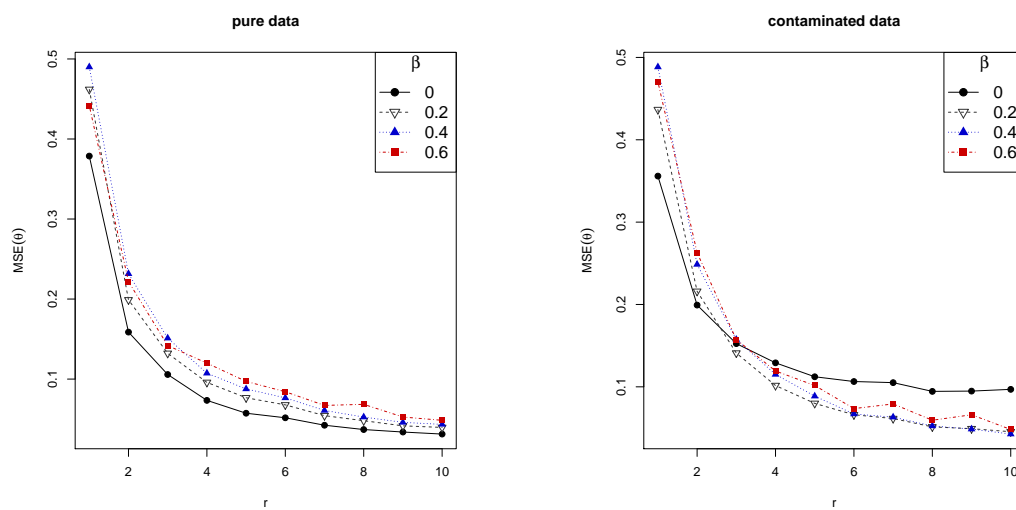


Figure 2: MSEs for unbalanced data

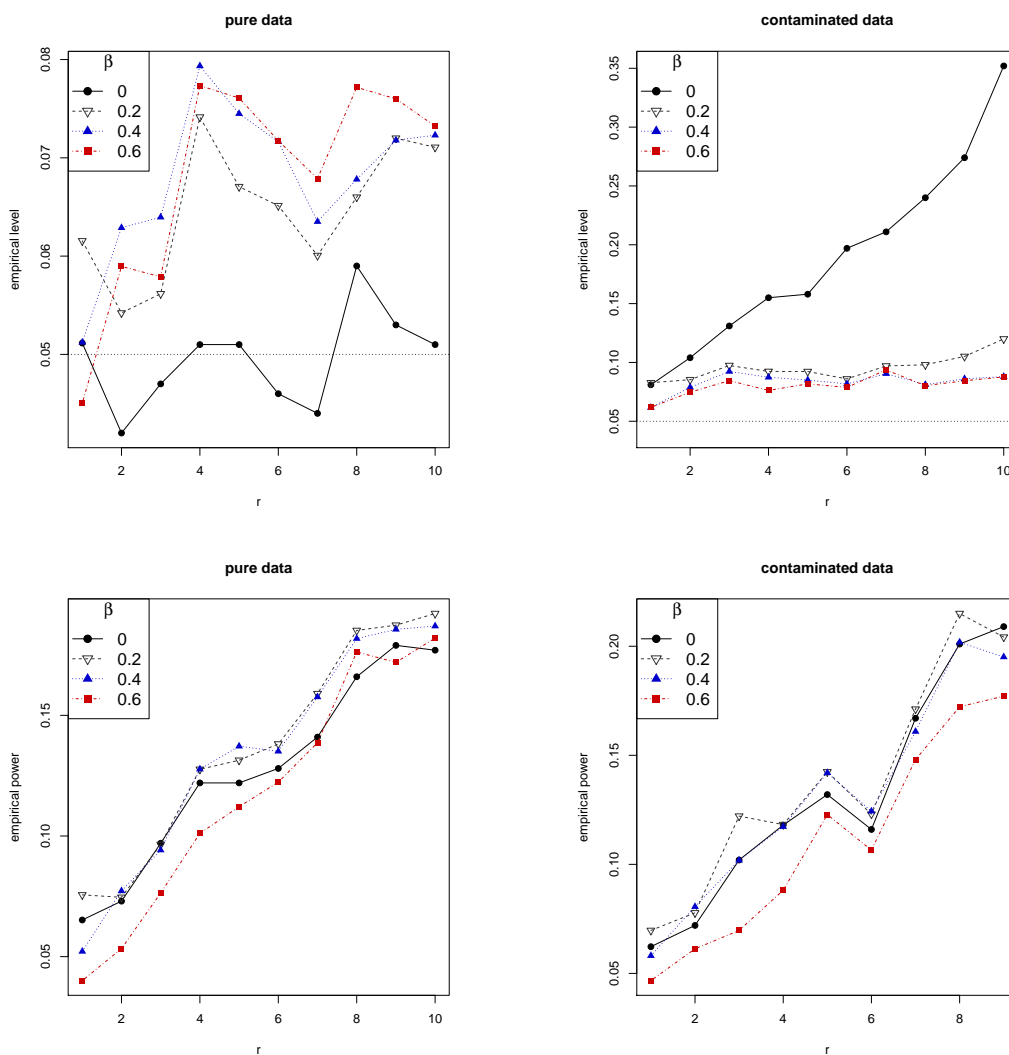


Figure 3: Estimated levels and powers for unbalanced data

A Power function of Wald-type tests

In many cases, the power function of the proposed test procedure cannot be derived explicitly. In the following theorem, we present an useful asymptotic result for approximating the power function of the Wald-type test statistics given in (22). We shall assume that $\boldsymbol{\theta}^* \notin \boldsymbol{\Theta}_0$ is the true value of the parameter such that

$$\widehat{\boldsymbol{\theta}}_\beta \xrightarrow[K \rightarrow \infty]{P} \boldsymbol{\theta}^*,$$

and we denote $\ell_\beta(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \mathbf{m}^T(\boldsymbol{\theta}_1) (\mathbf{M}^T(\boldsymbol{\theta}_2) \boldsymbol{\Sigma}_\beta(\boldsymbol{\theta}_2) \mathbf{M}(\boldsymbol{\theta}_2))^{-1} \mathbf{m}(\boldsymbol{\theta}_1)$. We then have the following result.

Theorem 10 *We have*

$$\sqrt{K} \left(\ell_\beta(\widehat{\boldsymbol{\theta}}_\beta, \boldsymbol{\theta}^*) - \ell_\beta(\boldsymbol{\theta}^*, \boldsymbol{\theta}^*) \right) \xrightarrow[K \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \sigma_{W_{K,\beta}}^2(\boldsymbol{\theta}^*)),$$

where

$$\sigma_{W_{K,\beta}}^2(\boldsymbol{\theta}^*) = \left. \frac{\partial \ell_\beta(\boldsymbol{\theta}, \boldsymbol{\theta}^*)}{\partial \boldsymbol{\theta}^T} \right|_{\boldsymbol{\theta}=\boldsymbol{\theta}^*} \boldsymbol{\Sigma}_\beta(\boldsymbol{\theta}^*) \left. \frac{\partial \ell_\beta(\boldsymbol{\theta}, \boldsymbol{\theta}^*)}{\partial \boldsymbol{\theta}} \right|_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}.$$

Proof. Under the assumption that

$$\widehat{\boldsymbol{\theta}}_\beta \xrightarrow[K \rightarrow \infty]{P} \boldsymbol{\theta}^*,$$

the asymptotic distribution of $\ell_\beta(\widehat{\boldsymbol{\theta}}_1, \widehat{\boldsymbol{\theta}}_2)$ coincides with the asymptotic distribution of $\ell_\beta(\widehat{\boldsymbol{\theta}}_1, \boldsymbol{\theta}^*)$. A first-order Taylor expansion of $\ell_\beta(\widehat{\boldsymbol{\theta}}_\beta, \boldsymbol{\theta}^*)$ at $\widehat{\boldsymbol{\theta}}_\beta$, around $\boldsymbol{\theta}^*$, yields

$$\left(\ell_\beta(\widehat{\boldsymbol{\theta}}_\beta, \boldsymbol{\theta}^*) - \ell_\beta(\boldsymbol{\theta}^*, \boldsymbol{\theta}^*) \right) = \left. \frac{\partial \ell_\beta(\boldsymbol{\theta}, \boldsymbol{\theta}^*)}{\partial \boldsymbol{\theta}^T} \right|_{\boldsymbol{\theta}=\boldsymbol{\theta}^*} (\widehat{\boldsymbol{\theta}}_\beta - \boldsymbol{\theta}^*) + o_p(K^{-1/2}).$$

Now, the result readily follows since

$$\sqrt{K} (\widehat{\boldsymbol{\theta}}_\beta - \boldsymbol{\theta}^*) \xrightarrow[K \rightarrow \infty]{\mathcal{L}} \mathcal{N}(\mathbf{0}_{J+1}, \boldsymbol{\Sigma}_\beta(\boldsymbol{\theta}^*)).$$

■

Remark 11 *Using Theorem 10, we can give an approximation for the power function of the Wald-type test statistic, given in (22), at $\boldsymbol{\theta}^*$, as follows:*

$$\begin{aligned} \pi_{W,K}(\boldsymbol{\theta}^*) &= \Pr \left(W_K(\widehat{\boldsymbol{\theta}}_\beta) > \chi_{r,\alpha}^2 \right) \\ &= \Pr \left(K \left(\ell_\beta(\widehat{\boldsymbol{\theta}}_\beta, \boldsymbol{\theta}^*) - \ell_\beta(\boldsymbol{\theta}^*, \boldsymbol{\theta}^*) \right) > \chi_{r,\alpha}^2 - K \ell_\beta(\boldsymbol{\theta}^*, \boldsymbol{\theta}^*) \right) \\ &= \Pr \left(\frac{\sqrt{K} \left(\ell_\beta(\widehat{\boldsymbol{\theta}}_\beta, \boldsymbol{\theta}^*) - \ell_\beta(\boldsymbol{\theta}^*, \boldsymbol{\theta}^*) \right)}{\sigma_{W_{K,\beta}}(\boldsymbol{\theta}^*)} > \frac{1}{\sigma_{W_{K,\beta}}(\boldsymbol{\theta}^*)} \left(\frac{\chi_{r,\alpha}^2}{\sqrt{K}} - \sqrt{K} \ell_\beta(\boldsymbol{\theta}^*, \boldsymbol{\theta}^*) \right) \right) \\ &= 1 - \Phi_K \left(\frac{1}{\sigma_{W_{K,\beta}}(\boldsymbol{\theta}^*)} \left(\frac{\chi_{r,\alpha}^2}{\sqrt{K}} - \sqrt{K} \ell_\beta(\boldsymbol{\theta}^*, \boldsymbol{\theta}^*) \right) \right) \end{aligned}$$

for a sequence of distributions functions $\Phi_K(x)$ tending uniformly to the standard normal distribution $\Phi(x)$. It is clear that

$$\lim_{K \rightarrow \infty} \pi_{W,K}(\boldsymbol{\theta}^*) = 1,$$

i.e., the Wald-type test statistics are consistent in the sense of Fraser.

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