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All the quantum tetrahedra: representations and their unitary equivalences

Abstract

In the quantum gravity literature, background-independent theories are based on the canonical quantization of the phase space of General Relativity (GR) in terms of connection and flux variables. There is also a dual description in terms of spin-network states, which were found to be related to simplicial dual geometries in the classical limit.

We want to describe the mathematical structure of a single tetrahedron in 3 dimensions, as the dual simplicial geometry of a 4-valent spin-network vertex, and develop in detail the mathematics applied to the quantization of this system, obtaining its Hilbert spaces with its corresponding unitary equivalent representations.

Resúmen

Las teorías de gravedad cuántica independientes de un fondo se basan en la cuantización canónica del espacio de fases de la Relatividad General (RG) en términos de variables de conexión y flujo, las cuales pueden ofrecer una descripción dual en términos de estados de redes de espín, que demostraron estar relacionadas con geometrías duales simpliciales en el límite clásico.

Queremos describir la estructura matemática de un solo tetraedro en 3 dimensiones, como la geometría dual de un vértice de red de espín de valencia 4, y desarrollar en detalle las matemáticas aplicadas a la cuantización de este sistema, obteniendo sus espacios de Hilbert asociados con sus correspondientes representaciones equivalentes.

Contents

Objectives

This project aims to introduce the quantization of the phase space of a tetrahedron and to highlight the different equivalent representations or polarizations that can be employed to describe its Hilbert space, once a quantization map between the phase space and an abstract algebra of quantum operators has been established. Additionally, it is demonstrated how unitary maps can be defined between the different representations, thus indicating their equivalence. Finally, we also aim to show how the closure condition for the variables of the algebra implies the possibility of interpreting the graphs of spin network states, commonly used in quantum gravity, in geometric terms as polyhedra connected together by their faces.

Methodology

For the completion of this bachelor thesis project, a prior self-study of the fundamental aspects of canonical quantization theories of gravity has been conducted through various introductory online courses. This preliminary work took place during the first semester of the course as an initial conceptual approach to the commonly used mathematics. Subsequently, in the second semester, a more specific and focused research effort was carried out based on the bibliography provided by the tutor, with the aim of achieving the stated objectives. I would like to thank Daniele Oriti for his guidance in formulating the main goals to address in this report in a reasoned and clear way, as well as for his advice and corrections or clarifications regarding some mathematical results. The report was written in English to simplify the reading and correction process for the tutor.

I Introduction

In quantum gravity literature it is very common to use spin networks to describe the quantum states on a space manifold, but it is also possible to use other representations and interpret the spin network basis in terms of a dual simplicial geometry of polyhedrons glued together by their faces.

This review is divided into two main parts. In section II, the general method used to quantize a classical system is presented, as in the usual QM. First, the case of a single free point particle moving in flat space is analyzed starting by describing the quantization map from the algebra of its classical observables, given by a Poisson structure, to an abstract operator algebra, with the Dirac commutators instead of Poisson brackets. Then, it is shown that the usual Fourier transform in $\mathbb{R}^d$ can be applied to obtain the unitary equivalence between the position and momenta representations of the Hilbert space of states, which would not be the case for a more general algebra structure where the algebra elements do not have to commute. Finally, it is shown how in a more complex case we would need to generalize the notion of a Fourier transform to a non-commutative one (NCFT), as introduced in [1–3]. This is exactly the case when describing the quantization of the same particle restricted to an $T^*SO(3)$ phase space, studied in part B of this section.

Then in section III the quantization procedure is reproduced to quantize a single tetrahedron, following the steps developed in [4], [5] and [6], showing that it is possible to interpret the spin-network states leaving on graphs in geometrical terms as a space of polyhedra glued together. This is due to the dual relation between the $SU(2)$ gauge invariance imposed on each vertex of the graph with a closure condition for three-dimensional vectors. Finally, in section IV the useful representations used in the quantum version are presented. We can show that they are unitary equivalent, since it is possible to find a unitary map between them, so that their Hilbert spaces carry the same physical information.

II The point particle and quantization procedure

The structure of a quantum theory consists of a pair of elements $(\mathcal{H}, \mathcal{A})$, where $\mathcal{H}$ is a Hilbert space and $\mathcal{A}$ an algebra of observables. The steps to follow to quantize the classical phase space of a system are; first to introduce a quantization map between the classical phase space algebra and that of the quantum operators, then to describe the representations of the Hilbert space $\mathcal{H}$, and finally we can construct equivalence maps between these representations.

A The free point particle

We will follow the above steps to quantize the phase space of a point particle in $\mathbb{R}^d$.

i) The classical phase space of a free point particle is given by $T^*\mathbb{R}^d = \mathbb{R}^d \times (Lie\mathbb{R}^d)^*$ with $(Lie\mathbb{R}^d)^* \cong (\mathbb{R}^d)^* \cong \mathbb{R}^d$ the dual of the Lie algebra of $\mathbb{R}^d$ ($d \in \mathbb{N}$). The phase space of a point particle can be parameterized using canonical coordinates x^i and its conjugate momenta p_i, on some basis of $\mathbb{R}^d$, with Poisson brackets

$$\{x^i, x^j\} = 0,\ \{x^i, p_j\} = \delta^i_j,\ \{p_i, p_j\} = 0 \tag{1}$$

This Poisson structure defined on $C^\infty(T^*\mathbb{R}^d)$ by the canonical symplectic structure, together with ordinary pointwise multiplication $(\cdot)$, gives rise to the full Poisson algebra $\mathcal{P}_{\mathbb{R}^d} = (C^\infty(\mathbb{R}^{2d}), \{\cdot,\cdot\}, \cdot)$ of the classical observables of the point particle in Euclidean space.

ii) To quantize this algebra, we take a quantization map between a maximal subalgebra $\mathcal{A}$, of $\mathcal{P}_{\mathbb{R}^d}$, to an abstract operator $^* - algebra\ \mathfrak{H}$[1], like $\mathcal{Q} : \mathcal{A} \to \mathfrak{H}$, with $\mathcal{Q}(x^i) = \widehat{X}^i$, $\mathcal{Q}(p_i) = \widehat{P}_i$ and $\mathcal{Q}(\{\cdot,\cdot\}) = i[\cdot,\cdot]$, such that the Poisson brackets are mapped to the commutators:

$$[\widehat{X}^i, \widehat{X}^j] = 0,\ [\widehat{X}^i, \widehat{P}_j] = i\delta^i_j \mathbb{1},\ [\widehat{P}_i, \widehat{P}_j] = 0. \tag{2}$$

Due to the principles of quantum mechanics and the fact that the position and conjugate momenta operators do not commute with each other, it is not possible to construct a complete set

[1]An *-algebra is a complex *involutive* algebra, i.e. there is an anti-linear map called hermitian conjugation $a \mapsto a^*$, with $(a^*)^* = a$, $(\lambda a)^* = \overline{\lambda} a^*$, and $(ab)^* = b^* a^*$, $\forall\ a, b \in \mathfrak{H}$ and $\lambda \in \mathbb{C}$.

of commuting operators (CCO) and a basis of states with all the eigenvalues defined simultaneously. The total degrees of freedom of the original classical phase space of the system are reduced, and we end up with a Hilbert space with different representations such that the observables can only be functions of some set of the total operators, being forced to limit our description to a particular representation or a mixture of them. Also, and due to this, many equivalent classical functions are mapped to different quantum operators, and there is not a one-to-one correspondence between classical and quantum systems.

iii) Now we consider representations [2] Π of $\mathfrak{H}$ as a concrete algebra of unbounded operators on some dense subspace of a Hilbert space $\mathcal{H}$, like the $^*-$homomorphism $\Pi : \mathfrak{H} \to Aut(\mathcal{H})$. Since the commutativity of the $\hat{X}^i$ and the $\hat{P}_i$ operators, separately, allows us to diagonalize all of them simultaneously, we have two main different representations of such an algebra, the position Π_x (on $L^2(\mathbb{R}^d, d^dx)$), and the momenta Π_p (on $L^2(\mathbb{R}^d, d^dp/(2\pi)^d)$) representations.

The action of the position and momenta operators, in both representations, on the set of states ψ in the L^2 spaces, is given in the table below

	$\widehat{X}^i$	$\widehat{P}_i$
Π_x	$(\Pi_x(\widehat{X}^i)\psi)(\vec{x}) = x^i\psi(\vec{x})$	$(\Pi_x(\widehat{P}_i)\psi)(\vec{x}) = -i\frac{\partial}{\partial x^i}\psi(\vec{x})$
Π_p	$(\Pi_p(\widehat{X}^i)\tilde{\psi})(\vec{p}) = i\frac{\partial}{\partial p_i}\tilde{\psi}(\vec{p})$	$(\Pi_p(\widehat{P}_i)\tilde{\psi}(\vec{p}) = p_i\tilde{\psi}(\vec{p})$

Table 1: Operator action in different representations of $\mathfrak{H}$

iv) Finally, in Euclidean space $\mathbb{R}^d$, we can define the Fourier transform $\mathcal{F}$ as the unique unitary intertwiner between these two representations, establishing a unitary equivalence[3].

The Fourier transform, with $\Pi_p(\widehat{O}) \circ \mathcal{F} = \mathcal{F} \circ \Pi_x(\widehat{O})$ for all $\widehat{O} \in \mathfrak{H}$, can be defined as

$$\tilde{\psi}(\vec{k}) = \mathcal{F}(\psi)(\vec{k}) = \int_{\mathbb{R}^d} d^dx E(\vec{x}, \vec{k})\psi(\vec{x}) = c\int_{\mathbb{R}^d} d^dx e^{-i\vec{k}\cdot\vec{x}}\psi(\vec{x}), \ \psi \in L^2(\mathbb{R}^d), \tag{3}$$

with $E(\vec{x}, \vec{k}) = ce^{-i\vec{k}\cdot\vec{x}}$ the kernel of the transformation. Setting $c = 1$ the transformation is unitary i.e. $\mathcal{F} \circ \mathcal{F}^* = id_{L^2(\mathbb{R}^d)} = \mathcal{F}^* \circ \mathcal{F}$ and invertible, with inverse map given by

$$\mathcal{F}^{-1}(\tilde{\psi})(\vec{x}) = \int_{\mathbb{R}^d} \frac{d^dk}{(2\pi)^d}\overline{E(\vec{x}, \vec{k})}\tilde{\psi}(\vec{k}) = \psi(\vec{x}), \ \psi \in L^2(\mathbb{R}^d). \tag{4}$$

Here, the plane waves $e^{i\vec{k}\cdot\vec{x}}$ constitute a dual representation of the translation group, which follows from the form of its representations, being the triviality of the Euclidean space an important clue due to the integrability of the action of partial derivatives. When considering a more general Lie group, with phase space the cotangent bundle $T^*G \simeq G \times \mathfrak{g}^*$, other complications arise.

As we shall see later, for a generic curved manifold a representation in terms of $L^2(\mathfrak{g}^*)$ commutative functions on its cotangent space cannot be defined. Despite this, for a Lie group, which is a symmetric space, the notion of a Fourier transform can be generalized as an expansion in terms of unitary irreducible representations of the group, being possible to define equivalence maps between representations and, following the tools introduced in [2], it is also possible to define representations in terms of non-commutative functions with a deformation quantization product.

[2] A representation is an homomorphism (linear map) between the *-algebra $\mathfrak{H}$ and a vector space $\mathcal{H}$, $\Pi : \mathfrak{H} \mapsto \mathcal{H}$, such that $\Pi(a \cdot b) = \Pi(a) \cdot \Pi(b), \ \forall a, b \in \mathfrak{H}$

[3] Two representations Π_1 and Π_2 of the same *-algebra $\mathcal{A}$, in Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, are called unitarily equivalent iff there exists a unitary map $U : \mathcal{H}_1 \mapsto \mathcal{H}_2$ such that $\Pi_2(a) = U\Pi_1(a)U^{-1} \ \forall a \in \mathcal{A}$, and $\Pi_1(\mathcal{A})$ and $\Pi_2(\mathcal{A})$ are isomorphic.

B The point particle on a sphere

If the point particle is restricted to move on a space which has the topology of $SO(3)$ or the half of a 3-sphere, we can formulate quantum mechanics on $SO(3)$ using non-commutative dual representations for the quantum states, and the non-commutative quantum variables have a clear connection to the corresponding classical ones [1].

i) The system evolves in a global time t and we choose $SO(3)$ to be its the configuration space.[4] Starting from the canonical symplectic structure of the cotangent bundle of $SO(3)$ and due to the existence of left-invariant 1-forms P_g, induced by the group action as $P_{gh} = L_g^* P_h \in T_{gh}^* SO(3)$ (with L^* the pull-back of the left multiplication $L_g h = gh$ on 1-forms), at the unit element in $T_e^* SO(3) \simeq \mathfrak{so}(3) \simeq \mathbb{R}^3$ the 1-forms $P_e = P$ correspond to the momentum space of the system. Left-invariant vector fields are given by $l_o^{-1} T_{i,g} = l_o^{-1} L_{*g} T_{i,e}$, where $t_i = T_{i,e} = T_i(e) = \sigma_i$ are the Pauli matrices in the fundamental representation[5]. They form an orthonormal set of vector fields in $TSO(3)$, and left-invariant 1-forms are given by $l_o T_g^i = l_o L_g^* T_e^i$, where again $t^i = T_e^i = \sigma^i$ forms an orthonormal set of 1-forms.

The vector fields respect the structure of the $\mathfrak{so}(3)$ Lie algebra

$$\left[\mathcal{L}_{l_o^{-1}T_i}, \mathcal{L}_{l_o^{-1}T_j}\right] = c_{ij}^k \mathcal{L}_{l_o^{-1}T_k} = -2 l_o^{-1} \epsilon_{ij}^k \mathcal{L}_{l_o^{-1}T_k} \tag{5}$$

where $\mathcal{L}_{T_i} f(g) = \frac{d}{ds} f(g e^{i s t_i})\big|_{s=0}$ is the Lie derivative of $f \in C^1(SO(3))$ with respect to T_i.

The canonical symplectic 1-form[6] $\theta \in T^*(T^*SO(3)) \approx T^*SO(3) \times TSO(3)$ is given by $\theta_{P_g} = (P_g, 0) \in T_{P_g}^*(T^*SO(3))\ \forall\ P_g \in T^*SO(3)$ and the symplectic 2-form can be defined as

$$\omega = -d\theta = dg^i \wedge dP_i - 2 l_o^{-1} \epsilon_{ij}^k P_k dg^i \wedge dg^j \tag{6}$$

with $dg^i = (l_o T^i, 0)$, $dP_i = (0, l_o^{-1} T_i)$ and $P_i = P_g \cdot T_{i,g} \in \mathfrak{so}(3)$, Euclidean coordinates in the momentum space. Finally, given Hamiltonian vector fields $X_f \in T(T^*SO(3))$, corresponding to a function $f \in C^1(T^*SO(3))$, defined through the equation $\iota_{X_f}\omega = \omega(X_f, \cdot) = df$, and being the symplectic structure conserved under the flow generated by X_f (i.e. $\mathcal{L}_{X_f}\omega = 0$), the Poisson bracket of two functions f and g is given by[7]

$$\begin{aligned} \{f, g\} = \iota_{X_g}\iota_{X_f}\omega = \omega(X_f, X_g) = \frac{\partial f}{\partial P_i}\mathcal{L}_{(l_o^{-1}T_i,0)} g \ - \ \mathcal{L}_{(l_o^{-1}T_i,0)} f \frac{\partial g}{\partial P_i} \ + \ c_{ij}^k P_k \frac{\partial f}{\partial P_i}\frac{\partial g}{\partial P_j} = \\ = \frac{\partial f}{\partial P_i}\frac{\partial g}{\partial g^i} \ - \ \frac{\partial f}{\partial g^i}\frac{\partial g}{\partial P_i} - 2 l_o^{-1} \epsilon_{ij}^k P_k \frac{\partial f}{\partial P_i}\frac{\partial g}{\partial P_j} \end{aligned} \tag{7}$$

for $f, g \in C^1(T^*SO(3))$, $\mathcal{L}_{(0, l_o T^i)} = \frac{\partial}{\partial P_i}$, and $\mathcal{L}_{(l_o^{-1}T_i, 0)} = \frac{\partial}{\partial g^i}$. Then, the Poisson brackets for the coordinate functions ($X_h^i : SO(3) \to \mathbb{R}^3$) and momentum variables read

[4] Here we are choosing an specific polarization of the phase space $T^*SO(3) \simeq SO(3) \times \mathfrak{so}(3)$ ($T^*G \simeq G \times \mathfrak{g}^*$, with $\mathfrak{g}^* \simeq \mathfrak{g} \simeq \mathbb{R}^d$ and $d = dim(G)$ for a general Lie group G) where $SO(3)$ is the configuration space and $\mathfrak{so}(3)$ the momentum space.

[5] $l_o \in R_+$ is a constant with dimensions of length which determines the length scale of the group manifold (i.e. the radius of the 3-sphere $S^3 \simeq SU(2) \simeq SO(3) \times \mathbb{Z}_2$, when antipodes $g, -g \in SU(2)$ are identified). We can absorb this constant on the basis T_i.

[6] On the contangent bundle of a group T^*G the canonical symplectic 1-form θ is obtained via the pull-back $\pi^* : T^*G \to T^*(T^*G)$ of the canonical bundle projection map $\pi : T^*G \to G,\ \pi(a) = p \in G\ \forall \alpha \in T_p^*G$.

[7] The Poisson brackets are modified from the usual ones, due to the the non-commutativity of the left-invariant derivations. The classical Poisson algebra is $\mathcal{P}_G = (C^\infty(SO(3) \times \mathfrak{so}(3)), \{\cdot,\cdot\}, \cdot)$

$$\{X_h^i, X_h^j\} = 0 \ , \ \{X_h^i, P_j\} = -\delta_j^i \ , \ \{P_i, P_j\} = -2l_o^{-1}\epsilon_{ij}^k P_k \tag{8}$$

for $X_h^i(g)$ a coordinate system of any element $h \in SO(3)$, such that $g = h\, exp\left[iX_h^i(g)t_i/l_o\right]$, with $|X_h^i(g)| < \pi l_o/2$. With this definition, we can appreciate that the Poisson algebra of the momentum space coordinate functions reflects the non-commutativity structure of the $\mathfrak{so}(3)$ algebra.

ii) We have chosen as canonical variables the group elements $g \in SO(3)$ and the Lie algebra elements $P \in \mathfrak{so}(3) \approx \mathbb{R}^3$ of the phase space $SO(3) \times \mathfrak{so}(3)$, where the momenta correspond to the translation generators in the configuration space $SO(3)$. Now we take the quantization map $\mathcal{Q} : g \to \widehat{g}$, such that $\mathcal{Q}(P_i) = \widehat{P}_i$ and $\mathcal{Q}(X_h^i) = \widehat{X}_h^i = X_h^i(\widehat{g})$, and on the "group basis" $\{|g\rangle\, ; g \in SO(3)\}$ we have the properties

$$\langle g|g'\rangle = \frac{1}{\pi^2 l_o^3}\delta(g^{-1}g'),\ \int_{SO(3)} \pi^2 l_o^3 dg\, |g\rangle\langle g| = \mathbb{1},\ f(\widehat{g})\,|g\rangle = f(g)\,|g\rangle\,. \tag{9}$$

$f(\widehat{g})\,|g\rangle = f(g)\,|g\rangle$ guarantees that for any coordinate system in $SO(3)$, $|g\rangle$ are eigenstates of the coordinates and we can define the states $|\psi\rangle$ of the Hilbert space $\mathcal{H}$ in this basis.

$$|\psi\rangle = \int_{SO(3)} \pi^2 l_o^3 dg\ \psi(g)\,|g\rangle \tag{10}$$

where $\psi \in L^2(SO(3), \pi^2 l_o^3 dg)$, with the states normalized[8] to $\langle\psi|\psi\rangle = 1$. In this "group basis, the momentum operators take the form

$$\widehat{P}_i = -i\frac{\hbar}{l_o}\mathcal{L}_{(l_o^{-1}T_i,0)} \tag{11}$$

For a general Lie group G and the quantization map $\mathcal{Q} : \mathcal{A} \to \mathfrak{U}$, for all $f \in \mathcal{A}_G \subset C^\infty(G)$, with $\mathcal{A}_G$ a subalgebra of $\mathcal{A} \subset C^\infty(G \times \mathfrak{g}^*)$, such that $\mathfrak{U}_G = \mathcal{Q}(\mathcal{A}_G)$ is the commutative subalgebra of $\mathfrak{U}$ (our quantum algebra for T^*G), the Poisson brackets (following [2]) are mapped to

$$\left[\widehat{f}, \widehat{g}\right] = 0 \ , \ \left[\widehat{P}_i, \widehat{f}\right] = i\widehat{\mathcal{L}_i f} \in \mathfrak{U}_G \ , \ \left[\widehat{P}_i, \widehat{P}_j\right] = ic_{ij}{}^k\widehat{P}_k \tag{12}$$

and if we define the set of coordinates $\zeta^i : G \to \mathbb{R}$, by imposing $\widehat{f} = f_\zeta(\widehat{\zeta^i})$, where $f_\zeta \circ \vec{\zeta} = f$ for all $f \in C^\infty(G)$, we find the commutators[9]

$$\left[\widehat{\zeta^i}, \widehat{\zeta^j}\right] = 0 \ , \ \left[\widehat{P}_i, \widehat{\zeta^j}\right] = i\widehat{\mathcal{L}_i\zeta^j} \ , \ \left[\widehat{P}_i, \widehat{P}_j\right] = ic_{ij}{}^k\widehat{P}_k \tag{13}$$

where we have used the notation $\mathcal{L}_{(l_o^{-1}T_i,0)} = \mathcal{L}_i$.

For $SO(3)$, the commutators, obtained from the quantization of (8), in the quantum algebra are

$$\left[\widehat{X}_h^i, \widehat{X}_h^j\right] = 0 \ , \ \left[\widehat{X}_h^i, \widehat{P}_j\right] = i\hbar\delta_j^i \ , \ \left[\widehat{P}_i, \widehat{P}_j\right] = 2i\frac{\hbar}{l_o}\epsilon_{ij}{}^k\widehat{P}_k \tag{14}$$

[8] In general for a Lie group G, in the group representation, the inner product for $\psi, \psi' \in L^2(G)$ is given by $\langle\psi|\psi'\rangle_G = \int_G dg\ \overline{\psi(g)}\psi'(g)$, with dg the left-invariant Haar measure on G.

[9] Assuming $\zeta^i(e) = 0$ and $\mathcal{L}_i\zeta^j(e) = \delta_i^j$, the operator $\widehat{\mathcal{L}_i\zeta^j}$ may be obtained from the Taylor series in a neighborhood of the identity, and $\mathfrak{U}$ will differ in general from the Heisenberg algebra $\mathfrak{H}$ as $\left[\widehat{P}_i, \widehat{\zeta^j}\right]$ is not a multiple of $\mathbb{1}$, for any choice of coordinates ζ^j. For more details, see [2].

and the canonical dynamics of the quantum system, corresponding to the classical one, is obtained by applying the quantization map to the Hamiltonian function.

iii) Now, given a Lie group G (configuration space) of weakly exponential type, such that $exp(\mathfrak{g}) \subset G$ is dense in G and we are able to determine plane waves of the exponential type, a group representation π_G on $L^2(G)$ is the one diagonalizing all operators $\widehat{f} \in \mathfrak{U}_G$:

$$(\pi_G(\widehat{f})\psi)(g) = f(g)\psi(g) \in C_c^\infty(G) \subset L^2(G) \tag{15}$$

for all $f \in \mathcal{A}_G$, with $\widehat{f} = \mathcal{Q}(f) \in \mathfrak{U}_G$ acting on the space of compactly supported smooth functions $\psi \in C_c^\infty(G) \subset L^2(G)$. In this representation, the Lie algebra operators act as

$$(\pi_G(\widehat{P}_i)\psi)(g) = i\mathcal{L}_i\psi(g) \tag{16}$$

However, in the algebra representation $\pi_{\mathfrak{g}^*}$ the route taken to obtain the group representation, based on the simultaneous diagonalization of the operators, can no longer be used due to the non-commutativity of $\widehat{P}_i \in \mathfrak{U}_{\mathfrak{g}^*}$. We need to introduce an operation $(\star)$, for all $i = 1, ..., d$, that deforms the action of operators over functions $\varphi(P) \in L^2(\mathfrak{g}^*)$ as

$$(\pi_{\mathfrak{g}^*}(\widehat{P}_i)\varphi)(P) = P_i \star \varphi(P) \tag{17}$$

turning the commutation relations into

$$(\widehat{P}_i \star \widehat{P}_j - \widehat{P}_j \star \widehat{P}_i) \star \varphi(P) = ic_{ij}{}^k \widehat{P}_k \star \varphi(P) \tag{18}$$

Imposing the stronger condition

$$(\pi_{\mathfrak{g}^*}(f(\widehat{P}_i))\varphi)(P) = f_\star(P) \star \varphi(P) \tag{19}$$

for all $f_\star \in \mathcal{A}_{\mathfrak{g}^*} \subset C^\infty(\mathfrak{g}^*)$, such that $f(\widehat{P}_i) = \mathcal{Q}(f_\star) \in \mathfrak{U}_{\mathfrak{g}^*}$, ensures that $f_\star$ is the function which upon quantification gives $f(\widehat{P}_i)$. We conclude that the choice of a quantization map determines uniquely the $\star$-product[10] in the $\pi_{\mathfrak{g}^*}$ representation of $\mathfrak{U}$.

With $\star$ a deformation quantization product on $\mathfrak{U}_{\mathfrak{g}^*}$ and $\widehat{\zeta}^i$ coordinate operators corresponding to a specific parameterization[11] of G, the operators $\widehat{\zeta}^i$ and $\widehat{P}_i$ in the algebra representation $\pi_{\mathfrak{g}^*}$ acts on $\varphi \in C_c^\infty(\mathfrak{g}^*)$ by

$$\begin{aligned} (\pi_{\mathfrak{g}^*}(\widehat{P}_i)\varphi)(P) &= P_i \star \varphi(P) \\ (\pi_{\mathfrak{g}^*}(\widehat{\zeta}^i)\varphi)(P) &= -i\partial^i\varphi(P) \\ (\pi_{\mathfrak{g}^*}(\widehat{f})\varphi)(P) &= -if_k(\vec{\partial})\varphi(P) \end{aligned} \tag{20}$$

where $\partial^i = \dfrac{\partial}{\partial P_i}$ and $f_k(k) = f(e^{ik}) \in C^\infty(\mathfrak{g})$ for all $f \in C^\infty(G)$. For more details see [2].

Finally, being the $\star$-product of functions in $C_c^\infty(\mathfrak{g}^*)$ also defined on $C_c^\infty(\mathfrak{g}^*)$, we can compute an integral sesquilinear form

$$\langle \varphi | \varphi' \rangle_{\mathfrak{g}^*} = \int_{\mathfrak{g}^*} \frac{dP}{(2\pi)^d} (\overline{\varphi} \star \varphi')(P) \tag{21}$$

[10] Given a quantization map $\mathcal{Q}$, for all $f_\star, f'_\star \in \mathcal{A}_{\mathfrak{g}^*}$ its $\star$-product can be obtained as $f_\star \star f'_\star = \mathcal{Q}^{-1}(\mathcal{Q}(f_\star)\mathcal{Q}(f'_\star))$.

[11] Coordinates $\zeta : G \to \mathfrak{g} \approx \mathbb{R}^d$ on G satisfy $\zeta^i_k(e) = 0$ and $\dfrac{\partial}{\partial k^i}\zeta^j_k(e) = \delta^j_i$, where $\zeta_k(k) = \zeta(e^{ik})$.

for $\varphi, \varphi' \in C_c^\infty(\mathfrak{g}^*)$. Now, it can be proven that there exist a unitary intertwiner between the two representation spaces $L^2(G)$ and $L^2_\star(\mathfrak{g}^*)$[12].

iv) Assuming there exists a unitary intertwiner $\mathcal{F} : L^2(G) \to L^2_\star(\mathfrak{g}^*)$ as an integral transform of the form:

$$\tilde{\psi}(P) = \mathcal{F}(\psi)(P) = \int_G dg\ E(g,P)\psi(g) \in L^2_\star(\mathfrak{g}^*) \tag{22}$$

where $\psi \in L^2(G)$ and $E(g,P)$ is the integral kernel of the transformation, the goal then is to identify the differential equations for it.

From the property $\mathcal{F} \circ \pi_G(\widehat{T}) = \pi_{\mathfrak{g}^*}(\widehat{T}) \circ \mathcal{F}$, where $\widehat{T} \in \mathfrak{U}$, for the operators $\widehat{P}_i$ we have[13]

$$\mathcal{F}(\pi_G(\widehat{P}_i)\psi)(P) = \int_G dg\ E(g,P)(i\mathcal{L}_i\psi)(g) = \int_G dg\ (-i\mathcal{L}_i E)(g,P)\psi(g)$$

and, on the other hand

$$(\pi_{\mathfrak{g}^*}(\widehat{P}_i)\mathcal{F}(\psi))(P) = \int_G dg\ (P_i \star E(g,P))\psi(g).$$

Thus, the previous equality requires $E(g,P)$ to satisfy

$$-i\mathcal{L}_i E(g,P) = P_i \star E(g,P). \tag{23}$$

Integrating the equation by right-invariant Lie derivatives we obtain

$$E(hg,P) = e^{k(h)\cdot\vec{\mathcal{L}}}E(g,P) = e_\star^{ik(h)\cdot P} \star E(g,P), \tag{24}$$

where we used the $\star$-exponential notation

$$e_\star^{\lambda f(P)} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} f(P) \star \cdots \star f(P). \tag{25}$$

Weak exponentiality of G is a sufficient condition for integrating any group element h in this way.[14] This makes it possible to define non-commutative plane waves in terms of $k(g)$ under the $\star$-product as

$$\begin{aligned} E_g(P) = e_\star^{ik(g)\cdot P/\hbar} &= \sum_{n=0}^{\infty} \frac{i^n}{n!\hbar^n} k^{i_1}(g) \cdots k^{i_n}(g)(P_{i_1} \star \cdots \star P_{i_n}) = \\ = \sum_{n=0}^{\infty} \frac{1}{n! l_o^n} k^{i_1}(g) \cdots k^{i_n}(g)(\mathcal{L}_{T_{i_1}} \cdots \mathcal{L}_{T_{i_n}} E_e(P)) &= e^{k^i(g)\mathcal{L}_{T_i}/l_o} E_e(P) \end{aligned} \tag{26}$$

We can then express the $\star$-product of $P's$, and in the lowest-order term in $\hbar l_o^{-1}$ we find:

[12] In general there exist a degenerate non-empty subspace $\mathcal{N} = \{\varphi \in C_c^\infty(\mathfrak{g}^*);\ \langle\varphi|\varphi'\rangle_{\mathfrak{g}^*} = 0\}$ and to define a proper inner product we need to quotient $C_c^\infty(\mathfrak{g}^*)$ by $\mathcal{N}$, such that the norm $\|\varphi\| = \sqrt{\langle\varphi|\varphi'\rangle_{\mathfrak{g}^*}}$ lies on $L^2_\star(\mathfrak{g}^*)$, making it possible to define the intertwiner.

[13] Here we integrate by parts using the fact that $\mathcal{L}_i(f \cdot f') = (\mathcal{L}_i f) \cdot f' + f \cdot (\mathcal{L}_i f')$ as it is shown in [2].

[14] The Kernel $E(g,P)$ can only be considered under integration, since the definition of the $\star$-product of non-commutative plane waves does only make sense under it. For more details see [1].

$$P_{i_1} \star \cdots \star P_{i_n} = P_{i_1} \cdots P_{i_n} + \mathcal{O}(\hbar/l_o) \tag{27}$$

showing that the non-commutative $\star$-product coincides with the commutative one when both the classical $\hbar \to 0$ and the Euclidean limit $l_o \to \infty$ are taken. Explicit calculation shows

$$P_i \star P_j = P_i P_j + i\frac{\hbar}{l_o}\epsilon_{ij}^{k} P_k. \tag{28}$$

when working on $SO(3)$. This justifies the commutation relations we have defined previously in (18), directly from the properties of the Lie derivatives.

For the operators $\widehat{\zeta^i}$, the Fourier transform of them in the group representation is

$$\mathcal{F}(\pi_G(\widehat{\zeta^i})\psi)(P) = \int_G dg\ E(g,P)\zeta^i(g)\psi(g)$$

and in the algebra representation

$$(\pi_{\mathfrak{g}^*}(\widehat{\zeta^i})\mathcal{F}(\psi))(P) = \int_G dg\ (-i\partial^i E)(g,P)\psi(g)$$

We must therefore require

$$(-i\partial^i E)(g,P) = \zeta^i(g)E(g,P) \tag{29}$$

which through integration yields

$$E(g, P+Q) = e^{Q\cdot\vec{\partial}}E(g,P) = e^{i\zeta(g)\cdot Q}E(g,P) \tag{30}$$

Thus, we have found two equivalent forms for defining the Kernel $E(g,P)$. From (24), we get

$$E(g,P) = e_\star^{ik(g)\cdot P} \tag{31}$$

and from (30)

$$E(g,P) = \eta(g)e^{i\zeta(g)\cdot P} \tag{32}$$

with the factor $\eta(g) = E(g,0)$. These expressions are equivalent for a specific choice of coordinates ζ^i for which the equality is satisfied. The existence of an algebra representation and of the corresponding non-commutative Fourier transform is guaranteed only under the assumption that those coordinates can be found. Non-commutative plane waves $E(g,P)$ as $\star$-exponentials are obtained by applying $\mathcal{Q}^{-1}$ to $e^{ik(g)\cdot\widehat{P}} \in \mathfrak{U}_{\mathfrak{g}^*}$ and, in the coordinates $\zeta^i(g)$, the non-commutative plane waves take the form of classical exponentials times a multiplicative factor $\eta(g)$, uniquely determined from the choice of a quantization map together with its $\star$-product.

Finally, we define non-commutative plane waves as equivalence classes of elements $E_g(P) = \{e_\star^{ik(g)\cdot P} \in C^\infty(\mathfrak{g}\times\mathfrak{g}^*)\}$, with properties

$$\begin{aligned}
E_g(P) = e_\star^{ik(g)\cdot P} &= \eta(g)e^{i\zeta(g)\cdot P} \\
E_e(P) = E_e(0) &= \eta(e) = 1 \\
\mathcal{Q}(E_g(P)) &= e^{ik(g)\cdot\widehat{P}} \in \mathfrak{U}_{\mathfrak{g}^*} \\
E_{g^{-1}}(P) = \overline{E_g(P)} &= E_g(-P) \\
E_{gh}(P) &= E_g(P) \star E_h(P).
\end{aligned} \tag{33}$$

We have finally found an integral transform $\mathcal{F} : L^2(G) \to L^2_\star(\mathfrak{g}^*)$, for the intertwiner between the representations π_G and $\pi_{\mathfrak{g}^*}$, in the form

$$\tilde{\psi}(P) = \mathcal{F}(\psi)(P) = \int_G dg\ e_\star^{ik(g)\cdot P}\psi(g) \tag{34}$$

and its inverse $\mathcal{F}^{-1} \equiv \mathcal{F}^* : L^2_\star(\mathfrak{g}^*) \to L^2(G)$, between $\pi_{\mathfrak{g}^*}$ and π_G, is defined as

$$\psi(g) = \mathcal{F}^{-1}(\tilde{\psi})(g) = \int_{\mathfrak{g}^*} \frac{d^dP}{(2\pi)^d}\ \overline{E_g(P)} \star \tilde{\psi}(P) = \sigma(g)\int_{\mathfrak{g}^*} \frac{d^dP}{(2\pi)^d}\ E_{g^{-1}}(P)\tilde{\psi}(P) \tag{35}$$

where $\sigma(g) = (\omega(\zeta(g))|\eta(g)|^2)^{-1}$, with $\eta(g) = \eta(\zeta(g)) = E(g,0)$ and $dg \equiv \omega(\zeta(g))d\zeta(g)$ the left-invariant Haar measure.[15]

Thus, particularizing again for $G = SO(3)$ and $\mathfrak{g}^* \approx \mathbb{R}^3$, with $\mathcal{F} : L^2(SO(3)) \to L^2_\star(\mathbb{R}^3)$ we have

$$\tilde{\psi}(P) = \mathcal{F}(\psi)(P) = \int_{SO(3)} \pi^2 l_o^3 dg\ E_g(P)\psi(g) \tag{36}$$

and its inverse is

$$\psi(g) = \mathcal{F}^{-1}(\tilde{\psi})(g) = \int_{\mathbb{R}^3} \frac{d^3P}{(2\pi\hbar)^3}\ \overline{E_g(P)} \star_p \tilde{\psi}(P) = \sigma(g)\int_{\mathbb{R}^3} \frac{d^3P}{(2\pi\hbar)^3}\ \overline{E_g(P)}\tilde{\psi}(P) \tag{37}$$

It is also possible, from the study of harmonic analysis on locally compact groups G, to define a Fourier transform as a unitary isometry between spaces $L^2(G)$ and $L^2(\widehat{G})$, where $\widehat{G}$ is the set of equivalence classes of unitary irreducible representations of G. Therefore, in the cases where $\widehat{G}$ differs from $\mathfrak{g}^*$ it is not possible to obtain a dual representation in terms of functions of momenta from usual harmonic analysis. However, for locally compact groups the Peter-Weyl theorem gives us an expansion of functions on G in terms of unitary irreducible representations, and in this context we have both Fourier transforms at our disposal, which would be crucial in Section IV to describe the equivalence between the spinor representation and the one in terms of the non-commutative flux variables, or its equivalence with the spin network representation.

The Peter-weyl theorem allows us to make a decomposition of functions on the configuration space in terms of spin representations using the Wigner D-matrices $D^j_{mn}(g)$, which are a kind of generalization of the spherical harmonics, with m and n the magnetic numbers. Any state $|\psi\rangle \in \mathcal{H}$ can be expanded in the spin basis as

$$|\psi\rangle = \frac{1}{\pi^2 l_o^3}\sum_{j,m,n} |j;m,n\rangle\,\langle j;m,n|\psi\rangle \tag{38}$$

with $\langle j;m,n|\psi\rangle = \int_{SO(3)} \pi^2 l_o^3 dg\sqrt{2j+1}\ \overline{D^j_{mn}(g)}\psi(g)$ and a set of coherent states $|j;m,n\rangle$ defined by

$$\langle g|j;\vec{m},\vec{n}\rangle = \sqrt{\frac{2j+1}{\pi^2 l_o^3}}D^j_{jj}(g_{\vec{n}}^{-1}gg_{\vec{m}}) \tag{39}$$

We can also identify the non-commutative momentum variables $P \in \mathbb{R}^3_\star$ in this spin basis by $P = 2(\hbar/l_o)j\vec{n}$, and conclude that there is an isometry between the non-commutative momentum basis and the spin basis. This identification has its importance too in the description of a tetrahedron.

[15] A proof of this result can also be found in the appendix of [2].

III The quantum tetrahedron

In the process of canonically quantizing classical gravitation it is also possible and more convenient to work on with the so-called "frame fields" and with connections, instead of describing classical GR as is usually done with the metric. This new formulation leads us directly to the Ashtekar variables. We will briefly discuss the usage of these new variables and the derivation of the Poisson structure of the classical phase space of GR in terms of them, following the steps developed from chapters 3 to 5 of part III of the book [7], and then how it is this description related to a quantum tetrahedron and the later related to twisted geometries.

In this new formalism of GR we start making a $3+1$ splitting of the space-time manifold $\mathcal{M}$, making it diffeomorphic to $\mathbb{R}\times S$, where S is the space manifold and $t\in\mathbb{R}$ the standard time coordinate. Then, we take space-like slices Σ of $\mathbb{R}\times S$ in coordinates such that ∂_0 is normal to Σ and ∂_i tangent to it. The Hamiltonian form of GR can be obtained by calculating the Poisson brackets of the 3-metric and its conjugate momentum with an Hamiltonian that is a linear combination of some constraints. If one starts from the tetrad formalism and attempts to canonically quantize gravity, one can write the constraints as polynomials of fields satisfying some commutation relations. The constraints are not closed under the Poisson structure and to avoid this problem one is forced to make a modification on it, in terms of the new Ashtekar variables.

A frame field is defined as the bundle isomorphism $e:\mathcal{M}\times\mathbb{R}^n\to T\mathcal{M}$, sending each fiber $\{p\}\in\mathcal{M}$ of the trivial bundle $\mathcal{M}\times\mathbb{R}^n$ to the corresponding tangent space $T_p\mathcal{M}$. If $\mathcal{M}$ is 3-dimensional we call this frame field triad and if it is 4-dimensional, tetrad. This allows us to rewrite the metric as $g_{\alpha\beta}=\eta_{IJ}e^I_\alpha e^J_\beta$, with η the internal metric of the internal space[16] $\mathbb{R}^n$.

Besides, we can talk about a Lorentz connection only in the sense of a vector potential A, which is an End($\mathbb{R}^n$)-valued 1-form on $\mathcal{M}$ with $A^{IJ}_\alpha=-A^{JI}_\alpha$, or equivalently, an $\bigwedge^2\mathbb{R}^n$-valued 1-form. Finally, with the choice of coordinates made above, we can restrict ourselves to a self-dual connection A^{IJ}_i on Σ, and define $A_i=-\frac{i}{2}A^a_i\sigma_a$ with σ_a the Pauli matrices. The momentum conjugate to A^a_i is the "densitized" triad $E^i_a=\frac{1}{\gamma}\sqrt{q}e^i_a$, where q is the determinant of the 3-metric and γ the Immirzi parameter, and the Poisson bracket they satisfy is

$$\{E^i_a(x),A^b_j(y)\}=\gamma\delta^b_a\delta^i_j\delta^{(3)}(x-y) \tag{40}$$

This corresponds to the Poisson structure of the classical phase space of GR. We will discuss more about this phase space then, in section IV, when dealing with the different representations we can use to describe the quantum space of states.

There is a close relationship between the spin representation of the quantum space of a graph in Σ and vectors in $\mathbb{R}^3$ (up to a $U(1)$ phase). This provides an interpretation of these vectors as the vectorial areas of the faces of some elementary polyhedra that glued together form a piecewise flat manifold. For simplicity, we can choose these polyhedrons to be tetrahedrons. Then, a tetrahedron can be understood as the convex envelope of four points in three-dimensional Euclidean space E^3. A triad $\vec{e}_1,\vec{e}_2,\vec{e}_3$ of independent edge vectors, pointing out from a common vertex, completely defines the geometry of the tetrahedron and we have 9 parameters, where the relevant independent ones are only $\{\vec{e}_1\cdot\vec{e}_1,\ \vec{e}_2\cdot\vec{e}_2,\ \vec{e}_3\cdot\vec{e}_3,\ \vec{e}_1\cdot\vec{e}_2,\ \vec{e}_2\cdot\vec{e}_3,\ \vec{e}_1\cdot\vec{e}_3\}$. The normals to the areas are obtained from the edge vectors as

[16] Here $I,J,K,L,\dots$ are internal indices associated with $\mathbb{R}^n$, and $\mu,\nu,\alpha,\beta,\dots$ space-time indices associated with coordinate vector fields ∂_μ on a chart.

$$\vec{n}_1 = -\vec{e}_2 \times \vec{e}_3 \tag{41}$$
$$\vec{n}_2 = -\vec{e}_3 \times \vec{e}_1 \tag{42}$$
$$\vec{n}_3 = -\vec{e}_1 \times \vec{e}_2 \tag{43}$$

with closure condition[17] $\vec{n}_4 = -\vec{n}_1 - \vec{n}_2 - \vec{n}_3$. Only three of these normals are independent, and described this way we have 6 independent parameters. With these quantities, the invariants are their squares, mutual scalar products, and $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3)$, which gives us the classical volume of the tetrahedron[18].

The three triangular faces of the tetrahedron meeting at the vertex common to the three edge vectors can also be defined using bivectors[19], or elements of $\bigwedge^2 \mathbb{R}^3$ with the wedge-product $\wedge$, in the form:

$$E_1 = e_2 \wedge e_3,\ E_2 = e_3 \wedge e_1,\ E_3 = e_1 \wedge e_2 \tag{44}$$

and with the same closure condition $E_4 = -E_1 - E_2 - E_3$. We can compute the volume in the same way using the positivity constraint, and the fact that in three dimensions $\bigwedge^2 \mathbb{R}^3 \cong \mathbb{R}^3$ so that the $\wedge$ operation can be replaced by the cross product $\times$, obtaining $E_1 \cdot (E_2 \times E_3) = V^2 > 0$.

The bivectors $E_e,\ e = 1, 2, 3, 4$, are elements of $\mathfrak{so}(3)^*$ (or $\mathfrak{su}(2)^*$) and $\{E_e^i\}$ form a basis of this algebra, since there is an isomorphism $\beta : \bigwedge^2 \mathbb{R}^n \to \mathfrak{so}(n)^*$ provided by the internal metric η on $\mathbb{R}^n$. We can then identify bivectors with elements of $\mathfrak{so}(3)^*$ and they must obey its Poisson brackets[20]

$$\{E^i, E^j\} = \epsilon^{ij}{}_k E^k \tag{45}$$

on an edge. In general, for different edges of a graph, the Poisson structure is

$$\{E_e^i, E_{e'}^j\} = \epsilon^{ij}{}_k E^k \delta_{e,e'} \tag{46}$$

Now, when promoting these bivectors to quantum self-adjoint operators $\widehat{E}_e$ on a particular Hilbert space $\mathcal{H}$, with geometric quantization, they satisfy the usual angular momentum commutation relations:

$$[\widehat{E}_e^i, \widehat{E}_{e'}^j] = i\epsilon^{ij}{}_k \widehat{E}^k \delta_{e,e'} \tag{47}$$

Since they do not commute, the components of a quantum bivector cannot in general be measured simultaneously with complete precision, and as in the case of the particle in an $T^*SO(3)$ phase space described in part B of section II, it is not possible to have a usual representation of functions in this space without introducing a $\star$-product.

This is the algebra of the $SO(3)$ group, and we can associate to the four faces of the tetrahedron an irreducible representation of it acting on $\mathcal{H}_{j_e}$, where j_e is the spin of the representation and e

[17] This equality is obtained directly from the definition of the fourth normal expressed in terms of the edges left, i.e. $\vec{n}_4 = \vec{e}_4 \times \vec{e}_5 = (\vec{e}_2 - \vec{e}_1) \times (\vec{e}_3 - \vec{e}_2) = \vec{e}_2 \times \vec{e}_3 - \vec{e}_1 \times \vec{e}_3 + \vec{e}_1 \times \vec{e}_2 = -\vec{n}_1 - \vec{n}_2 - \vec{n}_3$.

[18] Classically $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = -(\vec{e}_1 \cdot \vec{e}_2 \times \vec{e}_3)^2 = -36V^2$.

[19] Every bivector determines an oriented 2-dimensional plane in $\mathbb{R}^n$, with its norm been twice the area of the triangle.

[20] The commutators can be obtained applying geometric quantization. For more details on the Kirillov-Kostant Poisson structure and geometric quantization, see [8].

labels the faces (edge on the dual graph) of the tetrahedron. The quantum version of the vectorial areas[21] $\vec{n}_e$ are the generators $\widehat{E}_e$ acting on

$$\mathcal{H}_v = \mathcal{H}_{j_1,j_2,j_3,j_4} = \bigotimes_{e=1}^{4} \mathcal{H}_{j_e} = \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4}, \tag{48}$$

the Hilbert space of a single vertex.

If a wave function ψ represents an invariant state under the group

$$\sum_e \widehat{\vec{E}}_e \psi(h_e) = 0 \tag{49}$$

and to obey this closure condition, $\mathcal{H}_{j_1,j_2,j_3,j_4}$ must contain a subspace $\mathcal{H}^0_{j_1,j_2,j_3,j_4}$ which is the invariant part of $\mathcal{H}_v$[22]. This closure constraint will be considered in different forms, later in section IV, when describing other quantum representations or parameterizations of the phase space. The total Hilbert space on which we will be working is

$$\mathcal{H}_\tau = \bigoplus_{\{J\}} \mathcal{H}^0_{\{J\}} \tag{50}$$

where $\{J\}$ runs over the set of 4-ordered tuples of integers or half-integers. In this space[23], the well-defined operators (invariant objects) that define the quantum tetrahedron are $\widehat{E}_e^2 = \widehat{E}_e \cdot \widehat{E}_e$ and $\widehat{E}_{ee'} = \widehat{E}_e \cdot \widehat{E}_{e'}$. With these operators, we can construct volume and area operators for the tetrahedron.

As we showed earlier, the continuum phase space of quantum gravity is given by the Ashtekar connection A^i_a and the triad field E^a_i, with the Poisson algebra in (40). On a graph γ, embedded in the 3-d hypersurface slice Σ of the space manifold S, we can replace (A^i_a, E^a_i) by the pair $(h_e, X_e) \in SU(2) \times \mathfrak{su}(2)$ on each edge. The group element $h_e = \mathcal{P}\exp\left(\int_e A\right)$ is the holonomy along a path e on Σ, with $\mathcal{P}$ the path ordering operator, and the algebra element $X_e = \int_{e^*} (gE)^a N_a d^2S = \int_{S_e} *(e \wedge e)$ its conjugate $\mathfrak{su}(2)$-valued flux of the triad field through a surface S_e (the face of the polyhedron, dual to an edge of the graph, with normal N_a). The abstract graph γ with n vertices, labeled with a quantum number of volume m_n, and l links between these vertices labeled by the spins j_e, forms a triple $s = (\gamma, m_n, j_e)$ called "spin network" [9]. The form of the spin network (its valence[24]) will depend on the fundamental dual simplicial geometry into which we triangulate space. Then, by imposing the closure condition, as a gauge invariance condition, we can interpret the dual quantum spin network model in terms of these geometrical objects. In this case, we have chosen tetrahedrons, so that we have a 4-valent graph spin network. In this picture, a tetrahedron is dual to a vertex on the graph, and triangles (the common face of two adjacent tetrahedrons) are dual to the edges.

A different parametrization of the phase space leads to the so-called "twisted geometry", in the sense that they are locally well defined, but the local patches lack a consistent gluing among each other. To each triangle of its oriented area there is assigned two unit normals from the two

[21]The area opposite tho the vertex defined by $\vec{e}_k$ is $\frac{|\vec{n}_k|}{2} = A_k = A_{ij}$.

[22]Even though the faces by themselves transform non trivially, since they carry a non-zero spin, this closure condition induces an invariance under rotations onto the full tetrahedron, such that the total spin created inside must sum up to zero.

[23]We work on this space since there the spins aren't fixed, but we have a linear combination of all possible spins.

[24]The valence is the number of links attached to each vertex. For example, a 4-valent graph is one that has 4 links leaving or ending at the same vertex.

polyhedra sharing it, and an additional angle related to the extrinsic curvature. The Hilbert space is $L^2(G_\Gamma) = \oplus_{j_e}(\otimes_v \mathcal{H}_{j_v}) = \mathcal{H}_\mathcal{T}$ and each n-valent vertex is labeled by n unit vectors in $\mathbb{R}^3$. Adding the closure condition, there are $2(n-3)$ labels, allowing this geometric dual interpretation. If we consider two adjacent vertices and the edge e connecting them, calling s the source of it and t its target, denoting the normals $N_e = N_e(s)$ and $\tilde{N}_e = N_e(t)$, on each edge of the graph we have a triple $(N_e, \tilde{N}_e, j_e)$ and two types of conditions for them. The closure constraint is given by

$$C_{j_v} = \sum_{e \supset v} j_e N_e(v) = 0, \tag{51}$$

with P_v the space of a vertex on the graph. This constraint leads us to the constrained space

$$\mathrm{Tet}_{j_v} = \{N_e(v) \in P_v | C_{j_v} = 0\} \tag{52}$$

that defines the geometry of a flat tetrahedron in $\mathbb{R}^3$.

The closure constraint guarantees that the geometry is unique up to rotations, and the space of shapes of a tetrahedron is then given by

$$S_{j_v} \equiv \mathrm{Tet}_{j_v}/SU(2) \tag{53}$$

The gluing constraint requires the compatibility condition

$$N_e = R(h_e)\tilde{N}_e \tag{54}$$

for $h_e \in SU(2)$ and $R(h_e)$ a rotation matrix. The h_e defines a notion of parallel transport by which the normal N_e of the triangle in the frame of its source is mapped into the frame of its target tetrahedron. However, if $\bar{h}_e$ is a solution, so is $h_e(N_e, \tilde{N}_e, \xi_e) = \bar{h}_e e^{\xi_e \tilde{N}_e^i \tau_i}$ with $\tau_i = -i\sigma_i/2$ and $\xi \in [-\pi, \pi]$ an arbitrary angle. We can construct group elements n_e and $\tilde{n}_e$, by rotating τ_3 into $N_e^i \tau_i$ or into $\tilde{N}_e^i \tau_i$ as $R(n_e)\tau_3 = N_e^i \tau_i$ and $R(\tilde{n}_e)\tau_3 = \tilde{N}_e^i \tau_i$, respectively. Once n_e and $\tilde{n}_e$ are found, the general solution is

$$h_e(N_e, \tilde{N}_e, \xi_e) = n_e e^{\xi_e \tau_3} \tilde{n}_e^{-1}. \tag{55}$$

Including the angle ξ, the space of variables of an edge of the graph $(N_e, \tilde{N}_e, j_e, \xi_e)$ is 6 dimensional and the space of twisted geometries is

$$P_\Gamma = \bigtimes_e P_e, \quad P_e = S_e^2 \times S_e^2 \times (S_e^1 \times \mathbb{R}_e). \tag{56}$$

This is a presymplectic manifold, and to construct a phase space we take its reduction[25] $\bar{P}_\Gamma$. On a given edge, $\bar{P}_e$ has the same dimensionality of $T^*SU(2)$, (j_e, ξ_e) are conjugate variables, and the space is globally symplectomorphic to $T^*SU(2)_e$.

Considering now the closure constraint, we have the space of closed twisted geometries

$$S_\Gamma = \bigtimes_e T^* S_e^1 \bigtimes_v S_{j_v} \tag{57}$$

and its reduction is symplectomorphic to $\bar{S}_\Gamma \cong \bigtimes_e T^*SU(2)_e // SU(2)^{V_\Gamma}$, where V_Γ is the total number of vertices in the graph[26].

[25] A presymplectic manifold is equipped with a closed but possibly degenerate 2-form Ω, and the reduction is then the quotient by the kernel of Ω.

[26] The double quotient means imposing the Gauss law constraints at each vertex and dividing out the action of the $SU(2)$ gauge transformation it generates as described in detail in [6].

Considering now two sections $n(N)$ and $\tilde{n}(\tilde{N})$, such that $N = n\tau_3 n^{-1}$ and $\tilde{N} = \tilde{n}\tau_3\tilde{n}^{-1}$, that defines the map

$$(N_e, \tilde{N}_e, j_e, \xi_e) \to (X_e, h_e): \begin{aligned} X_e &= j_e n \tau_3 n^{-1} \\ h_e &= n e^{\xi_e \tau_3} \tilde{n}^{-1} \end{aligned} \tag{58}$$

which implies $\tilde{X}_e = -h_e^{-1} X_e h_e = -j_e \tilde{N}_e$. Since the map is two-to-one due to the $\mathbb{Z}_2$ identification $\sigma : (N_e, \tilde{N}_e, j_e, \xi_e) \to (-N_e, -\tilde{N}_e, -j_e, -\xi_e)$, we get the same pair (X_e, h_e). We then conclude that we have the symplectomorphism

$$\bar{P}_e/\mathbb{Z}_2 \cong T^*SU(2)_e \tag{59}$$

that preserves the Poisson structure

$$\{X_e^i, X_e^j\} = \epsilon^{ij}{}_k X_e^k, \quad \{X_e^i, h_e\} = -\tau^i h_e, \quad \{\tilde{X}_e^i, h_e\} = h_e \tau^i, \tag{60}$$

with the identification of $\mathfrak{su}(2)$ with $\mathbb{R}^3$ via $X_e^i = tr\{\tau^i X_e\}$. This is in fact the Poisson structure that we will use later in part A of section IV, when dealing with the non-commutative flux representation of quantum gravity.

IV Unitary equivalent representations

Finally, we study the two main quantum representations or polarizations, and present the unitary equivalence maps between them.

A The holonomy and non-commutative flux representations

We start working with an underlying classical phase space based on the cotangent bundle over $SU(2)$, $T^*SU(2) \simeq SU(2) \times \mathfrak{su}(2)$. Here, the group elements encode the degrees of freedom of the gravitational connection A through its holonomies $h_e[A]$ (configuration space $SU(2)$), and the elements of the Lie algebra are related directly to the densitized triad fields (momentum space $\mathfrak{su}(2)$) by its fluxes across 2-surfaces [3]. We need a representation of the functions of $\mathfrak{su}(2)$ elements to bring the geometric aspects of theory to the forefront. Even at the classical level, flux variables do not commute, since dually to a single edge they act as invariant vector fields and have the structure of $\mathfrak{su}(2)$. Then, the Poisson structure corresponds to that of (60) for a single edge

$$\begin{aligned} \{h[A], h[A]\} &= 0 \\ \{E^i, h[A]\} &= h[A]\,\tau^i \\ \{E^i, E^j\} &= \epsilon^{ij}{}_k E^k. \end{aligned} \tag{61}$$

As developed above for a particle in an $T^*SO(3)$ phase space, this non-commutativity of the flux variables naturally encodes a definition of a non-commutative Fourier transform with an $\star$-product. We can use this fact to construct a well-defined non-commutative flux representation of states. This representation would help us clarify the quantum geometry and its relation to simplicial geometries.

We first take the Hilbert space in the connection representation $\mathcal{H}_\gamma = L^2(A_\gamma, d\mu_\gamma)$, where γ is the 4-valent graph embedded in the spatial manifold Σ, A_γ the space of connections on γ, identified with the product $SU(2)^4$ of 4 copies of the gauge group, and $d\mu_\gamma$ the product Haar measure on

$SU(2)^4$. The fundamental operators arising from the quantization of the classical phase space given by $T^*SU(2)^4$ act, respectively, by multiplication by a smooth function φ_γ of $SU(2)^4$ in the form

$$(\widehat{\varphi}_\gamma f_\gamma)(h_1,\dots,h_4) = \varphi_\gamma(h_1,\dots,h_4) f_\gamma(h_1,\dots,h_4)$$
$$(\widehat{L}^i_e f_\gamma)(h_1,\dots,h_4) = \frac{d}{dt} f_\gamma(h_1,\dots,h_e e^{t\tau_i},\dots,h_4)$$

where $\widehat{L}^i_e$ is the left-invariant vector field on the copy of $SU(2)$ associated to the edge e. They provide the momenta operators of $\mathfrak{su}(2)$, or flux operators $E^i_{S_e} = E(S_e,\tau_i)$ acting on functions as

$$\widehat{E}^i_S f_\gamma = \sum_{v\in\gamma'\cap S}\sum_{e\supset v} \epsilon(S,e)\widehat{L}^i_e f_\gamma \tag{62}$$

being $\epsilon(S,e) = \pm$, depending on the relative orientation between the edge and the surface.

The non-commutative Fourier transform will provide a decomposition in terms of the flux Lie algebra variables. The $SO(3)$ Fourier transform $\mathcal{F}$ isometrically maps $L^2(SO(3), d\mu_H)$ onto $L^2_\star(\mathbb{R}^3, d\mu)$ of functions on $\mathfrak{su}(2) \sim \mathbb{R}^3$, equipped with a $\star$-product and the standard Lebesgue measure $d\mu$.

From now on, we identify functions of $SO(3) \simeq SU(2)/\mathbb{Z}_2$ with the ones of $SU(2)$ which are invariant under the transformation $g \to -g$, and set $\tau_i = i\sigma_i$. Given the plane waves

$$E_h : \mathfrak{su}(2) \to U(1); \quad E_h(X) = e^{\vec{p}_h\cdot\vec{X}} \tag{63}$$

depending on a specific choice of coordinates $\vec{p}_h$ on the group manifold, and using the general expressions for the Fourier transform and its inverse in (36) and (37) respectively, in natural units where $\hbar = 1$ and with $l_o = 1$ we get

$$(\mathcal{F}(f))(X) = \int dh E_h(X) f(h), \quad f(h) = \frac{1}{8\pi}\int d^3X (E_{h^{-1}} \star \mathcal{F}(f))(X). \tag{64}$$

Choosing a generic element $k \in \mathfrak{su}(2)$, that can be written as $k = k^i\sigma_j$, $k^i \in \mathbb{R}$, the group element is $h = e^{ik^j\sigma_j} \in SU(2)$. Or in the other parametrization with $\vec{p}_h$; $h = p^0\mathbb{1} + ip^j\sigma_j$, $p^j \in \mathbb{R}$. This last parametrization naturally identifies $SU(2)$ with $\mathcal{S}^3$, where $p^0 \geq 0$ and $p^0 \leq 0$ corresponds to the upper and lower hemispheres of $\mathcal{S}^3$, respectively. The relation between the two parameterizations is established by the following change of coordinates.

$$\vec{p} = \frac{\sin|\vec{k}|}{|\vec{k}|}\vec{k}, \quad p^0 = \cos|\vec{k}|, \quad \text{where} \quad |\vec{k}| = [0,\pi/2] \quad \text{or} \quad |\vec{k}| = [\pi/2,\pi] \tag{65}$$

With this choice, the group element assumes the form $h = \cos|\vec{k}|\mathbb{1} + i\frac{\sin|\vec{k}|}{|\vec{k}|}\vec{k}\cdot\vec{\sigma} = e^{i\vec{k}\cdot\vec{\sigma}}$, and then the Haar measure takes the form

$$\begin{gathered} dh = d^3\vec{k}\left(\frac{\sin|\vec{k}|}{|\vec{k}|}\right)^2, |\vec{k}| \in [0,\pi] \\ dh = \frac{d^3\vec{p}}{\sqrt{1-|\vec{p}|^2}}, |\vec{p}|^2 < 1. \end{gathered} \tag{66}$$

So, in this coordinates, the non-commutative Fourier transform is expressed as

$$\mathcal{F}(f)(X) = \frac{1}{\pi}\int_{|\vec{p}|\leq 1} \frac{d^3\vec{p}}{\sqrt{1-|\vec{p}|^2}} f(h(\vec{p}))e^{i\vec{p}\cdot\vec{X}}. \tag{67}$$

We can compute an extension to functions of several copies of the group to get the Fourier transform $\mathcal{F}_\gamma$ in the space $\mathcal{H}_\gamma \simeq L^2(SU(2)^4, dh)$ of any graph with 4 edges. For $h = (h_1, \ldots, h_4) \in SU(2)^4$, the plane waves $E_h^{(4)} : \mathfrak{su}(2) \to U(1)$ are defined as a product of 4 copies of them, and the image of the Fourier transform, endowed with an inner product, forms a Hilbert space $L^2_\star(\mathbb{R}^3)^{\otimes 4} = \mathcal{H}_{\star,\gamma}$. This Fourier transform provides a unitary equivalence between $\mathcal{H}_\gamma$ and $\mathcal{H}_{\star,\gamma}$.

For a given fixed graph γ, and considering an elementary surface S_e intersecting γ at a single point of an edge e, the action of the flux operator $\widehat{E}^i_{S_e}$ on $\mathcal{H}_\gamma$ coincides with that of the invariant vector fields $\widehat{L}^i, \widehat{R}^i$ on $SO(3)$, depending on the orientation of e with respect to S_e. They act dually on $L^2_\star(\mathbb{R}^3)$ as $\widehat{L}^i u = \mathcal{F}(\widehat{L}^i f)$, and $\widehat{R}^i u = \mathcal{F}(\widehat{R}^i f)$, where $u = \mathcal{F}(f)$. The action is given by:

$$\begin{aligned}\mathcal{F}(\widehat{R}^i f)(X) &= -i\widehat{X}^i \star \mathcal{F}(f)\\ \mathcal{F}(\widehat{L}^i f)(X) &= i\mathcal{F}(f) \star \widehat{X}^i\end{aligned} \tag{68}$$

with the choice $\widehat{X}^i(X) = -\frac{1}{2}tr\{X\tau^i\}$. In addition, the dual action of holonomy operators $\widehat{h}(a)$ (with $a \in \mathfrak{su}(2)$), generated by the plane waves $h \to E_h(a)$, on $u = \mathcal{F}(f) \in L^2_\star(\mathbb{R}^3)$ is given by:

$$(\widehat{h}(a)u)(X) = \mathcal{F}(\widehat{h}(a)f)(X) = \int dh E_h(a) f(h) E_h(X) = \mathcal{F}(f)(X+a) = u(X+a) \tag{69}$$

so that holonomy operators act by translation on the states. Note also that, as $\mathcal{F} : \mathcal{H}_\gamma \to \mathcal{H}_{\star,\gamma}$ is a unitary transformation, it preserves the spectra of all operators.

Finally, for a given graph γ, a gauge transformation at a vertex v generated by a group element h_v corresponds to the action of $\widehat{h}_v$ in $\mathcal{H}_\gamma$, which acting in a dual state $u_\gamma = \mathcal{F}_\gamma(f_\gamma)$ as $\widehat{h}_v u_\gamma = \mathcal{F}_\gamma(\widehat{h}_v f_\gamma)$ reads [3]

$$(\widehat{h}_v u_\gamma)(X_1, \ldots, X_4) = u_\gamma(h_{t_1}^{-1} X_1 h_{s_1}, \ldots, h_{t_4}^{-1} X_4 h_{s_4}) \tag{70}$$

In addition, gauge invariance can be imposed by acting with the gauge averaging operator $\mathcal{P}_\gamma = \bigotimes_v \int dh_v \widehat{h}_v$, and assuming only outgoing edges we get

$$\left(\int dh_v \widehat{h}_v u_\gamma\right)(X) = (\widehat{C}_v \star u)(X_1, \ldots, X_4) \tag{71}$$

with $\widehat{C}_v$ a closure constraint, defined at a vertex v as

$$\widehat{C}_v(X_i) = \int dh \prod_{e_i \supset v} E_h(X_i) = 8\pi\delta_0\left(\sum_{e_i\supset v} X_i\right). \tag{72}$$

Thus, we can conclude that gauge invariance is a strong closure constraint for the $\mathfrak{su}(2)$ variables X_i of the edges incident at v. This shows spin network functions are gauge-invariant functions on graphs that directly defines three-dimensional space and can be interpreted as discrete chunks of space. The Lie algebra variables are fluxes associated with surfaces dual to the edges of γ, closing around vertices to form these 3-cells of space.

It is also important to mention, for the next part, that the Fourier transform has a discrete basis, given by applying it to the decomposition of functions on $SU(2)$ in terms of spin representations. The Peter-Weyl theorem ensures the existence of a spin basis of states through $\langle h|j;m,n\rangle = \sqrt{2j+1}D^j_{mn}(h)$ in $\mathcal{H} \simeq L^2(SU(2))$, where $D^j_{mn}(h)$ are again the Wigner D-functions.

Applying the Fourier transform on the Peter-weyl formula $\widetilde{f}(X) = \sum_{j_e,m_e,n_e} f^{j_e}_{m_e n_e} \widetilde{D}^{j_e}_{m_e n_e}(X)$ we get this discrete basis in terms of the matrix elements of the dual Wigner matrices $\widetilde{D}^{j_e}_{m_e n_e}(X) = \int dh E_h(X) D^{j_e}_{m_e n_e}(h)$, in the spin j_e representation of $SU(2)$ on an edge. This representation matrix elements $D^{j_e}_{m_e n_e}(h) = \langle j_e, m_e| D^{j_e}(h_e) |j_e, n_e\rangle$ can also be used to define the set of coherent sates, with $\langle h|j_e; n_e, m_e\rangle = \sqrt{2j_e+1} D^{j_e}_{j_e j_e}(h^{-1}_{n_e} h h_{m_e})$ a sort of plane waves, giving rise to an alternative Fourier transform. This Wigner matrices can also be expressed in terms of spinors, instead of group elements, by implementing a unitary map $\mathcal{T}$ as it is shown later.

B The spinor representation

We can perform a quantization of the phase space in (61) in terms of spinorial variables, providing a direct link between the spin network states and simplicial geometries. In the spinorial representation, we work in the Bargmann space of holomorphic square-integrable functions over the complex plane. There exists a unitary equivalence between this Bargmann space and the Hilbert space of the cotangent bundle of $SU(2)$, such that the integrals over the group elements can be performed as integrals over the complex plane [10].

Using a coordinate system on $T^*SU(2)$ given by the $\mathbb{C}^2$ spinors $(|z\rangle, |\tilde{z}\rangle)$ we can see these spinors as vectors in $\mathbb{R}^3$ and interpret them as the vectorial areas of the faces of elementary polyhedra, as mentioned before. Considering the spinorial variables now as the fundamental ones, flux and holonomy operators are derived from them as composite operators $(h(z,\tilde{z}), X(z,\tilde{z}))$. While the group elements are located on the edges, spinors live on the vertices, so for our proposes we only have one of them.

One can associate a pair of spinors $(|z\rangle, |\tilde{z}\rangle)$ with the initial and final vertex of an edge, respectively. Then, the standard symplectic structure in $\mathbb{C}^2 \times \mathbb{C}^2$ turns this space into a phase space, which is symplectomorphic to $T^*SU(2)$ after a gauge reduction by $U(1)$, which demands that both spinors have equal lengths.

The spinor $|z\rangle \in \mathbb{C}^2$

$$|z\rangle = \begin{pmatrix} z^0 \\ z^1 \end{pmatrix}, \quad \langle z| = (\bar{z}^0, \bar{z}^1) \tag{73}$$

transforms naturally under a representation of $SU(2)$

$$h : \mathbb{C}^2 \to \mathbb{C}^2; |z\rangle \to h|z\rangle \ \ \forall h \in SU(2). \tag{74}$$

Endowed with an inner product $\langle w|z\rangle = \bar{w}^0 z^0 + \bar{w}^1 z^1$ and the norm $||z|| = \sqrt{\langle z|z\rangle}$, we can construct the quantities $X(z,\bar{z})$ and $h(z,\tilde{z})$, being $\vec{X}$ invariant under the $U(1)$ transformation $|z\rangle \to e^{i\theta}|z\rangle$, and such that $g(z,\tilde{z})$ transforms exactly as the holonomy of an $SU(2)$-connection, i.e.

$$\begin{aligned} (|z\rangle, |\tilde{z}\rangle) &\to (h_1|z\rangle, h_2|\tilde{z}\rangle); \\ g(z,\tilde{z}) &\to h_1 g(z,\tilde{z}) h_2^{-1}, \ \forall\, h_1, h_2 \in SU(2) \end{aligned} \tag{75}$$

Because of this, in order to leave $g(|z\rangle, |\tilde{z}\rangle)$ invariant, we have to perform a twisted rotation on the spinors, instead of the usual one, which is defined as

$$\begin{pmatrix} |z\rangle \\ |\tilde{z}\rangle \end{pmatrix} \to \begin{pmatrix} h\,|z\rangle \\ g^{-1}hg\,|\tilde{z}\rangle \end{pmatrix}, \ \forall \ h \in SU(2). \tag{76}$$

With this in mind, we can construct an invariant Haar measure in terms of spinor variables, which is indeed a Gaussian measure on $\mathbb{C}^2$. Thus, the integrals are re-expressed as[27]

$$\int_{SU(2)} dh f(h) = \int_{\mathbb{C}^2\times\mathbb{C}^2} d\mu(z)d\mu(\tilde{z})f(h(z,\tilde{z})) \tag{77}$$

where $d\mu(z) = \frac{1}{\pi^2}dz^0dz^1e^{-\langle z|z\rangle}$ is the normalized Gaussian measure on $\mathbb{C}^2$, and we can compute the integrals in terms of complex numbers.

Let us analyze the spinorial phase space $\mathbb{C}^2 \times \mathbb{C}^2 \simeq \mathbb{C}^4$, equipped with a symplectic structure of the form

$$\{\bar{z}^i, z^j\} = i\delta^{ij}, \ \{\bar{\tilde{z}}^i, \tilde{z}^j\} = i\delta^{ij} \tag{78}$$

and consider the constraint

$$\mathfrak{h} = ||z||^2 - ||\tilde{z}||^2. \tag{79}$$

This constraint generates $U(1)$ gauge transformations on the spinors but the group and algebra elements are unaffected. When $\mathfrak{h} = 0$, z and $\tilde{z}$ have equal length, and we recover the same symplectic structure for h and X written in (60), now in terms of the spinors.

Each edge e connects two vertices v_I and v_J, associating a spinor $|z\rangle$ with v_I and $|\tilde{z}\rangle$ with v_J. Thus, a 4-valent vertex carries 4 spinors $\{|z_1\rangle, \ldots, |z_4\rangle\}$ which transform under $SU(2)$ transformations in the usual way on each spinor separately, and the $SU(2)$ invariance on each vertex can be understood again as arising from the constraint

$$\vec{\mathfrak{c}} = \sum_{i=1}^{n} \vec{X}_i. \tag{80}$$

From a geometric point of view, imposing $\mathfrak{c} = 0$ is the same as demanding closure. This probes the same arguments made above can be applied in the spinor representation.

Finally, we want to find a unitary map between the spaces $\mathcal{H}_e^{\text{spin}} = L^2_{\text{hol}}(\mathbb{C}^4, d\mu)/U(1)$ of holomorphic square-integrable complex functions and $\mathcal{H}_e = L^2(SU(2), dh)$ in order to probe their equivalence. $\mathcal{H}_e^{\text{spin}} = L^2_{\text{hol}}(\mathbb{C}^4, d\mu)/U(1)$ is isomorphic to $(\mathcal{F}_2\otimes\mathcal{F}_2)/U(1)$, with $\mathcal{F}_2 = L^2_{\text{hol}}(\mathbb{C}^2, d\mu(z))$, where we can define an orthonormal basis

$$e^{j_1}_m(z) \otimes e^{j_2}_n(\tilde{z}) = \frac{(z^0)^{j_1+m}(z^1)^{j_1-m}(\tilde{z}^0)^{j_2+n}(\tilde{z}^1)^{j_2-n}}{\sqrt{(j_1+m)!(j_1-m)!(j_2+n)!(j_2-n)!}}.$$

This basis transforms under $U(1)$ as $e^{j_1}_m(z) \otimes e^{j_2}_n(\tilde{z}) \to e^{i[2(j_2-j_1)\phi]}e^{j_1}_m(z) \otimes e^{j_2}_n(\tilde{z})$, so the states are only invariant if $j_1 = j_2$, and we can finally define the orthonormal basis of $\mathcal{H}_e^{\text{spin}}$ as

$$\mathcal{P}^j_{mn}(z,\tilde{z}) = e^j_m(z) \otimes e^j_n(\tilde{z}). \tag{81}$$

[27]For a proof of this result see section 2 of [10].

Using this basis and the standard Peter-Weyl decomposition of $\mathcal{H}_e$ we can probe the existence of this unitary equivalence. The unitary map is defined as[28]

$$\begin{aligned} \mathcal{T} : \mathcal{H}_e &\to \mathcal{H}_e^{\text{spin}}; \\ D^j_{mn}(h) \to (\mathcal{T} D^j_{mn})(z, \tilde{z}) &= \frac{1}{\sqrt{d_j}} \mathcal{P}^j_{mn}(z, \tilde{z}). \end{aligned} \tag{82}$$

Also, we can express this map in terms of an internal kernel $\mathcal{K}_h(z, \tilde{z})$, writing

$$(\mathcal{T} f)(z, \tilde{z}) = \int dh \mathcal{K}_h(z, \tilde{z}) f(h) \tag{83}$$

$$\mathcal{K}_h(z, \tilde{z}) = \sum_{j \in \mathbb{N}/2} \sum_{m,n=-j}^{j} \sqrt{d_j}\, \overline{D^j_{mn}(h)} \mathcal{P}^j_{mn}(z, \tilde{z}). \tag{84}$$

This way, the scalar product between two spinor basis states is exactly dual to the scalar product between group representation matrices used in the Peter-Weyl theorem.

V Conclusions

From the arguments developed above, we can conclude first that when we want to describe the quantum representation of a system in the momentum space, given by the Lie algebra of a classical phase space of the form $T^*G \simeq G \times \mathfrak{g}^*$, since the coordinate functions do not commute we have to introduce a deformation product, specified only by the choice of a quantization map, and then with it we can define the kernel of a unitary transformation between the Hilbert spaces $L^2(G)$ and $L^2(\mathfrak{g}^*)$. This allows us to have at our disposal quantum operators that are directly related to its classical coordinate versions. This fact, proven in section II, can then be applied to define a NCFT between the space $\mathcal{H}_\gamma$, of holonomies of a connection on a graph $h[A]$, and the space $\mathcal{H}_{\star,\gamma}$, of fluxes E across a surface. This is useful to describe the quantization of the phase space of GR, as is done in part A of section IV. In part B of this section we also study the spinor polarization, taking a pair of spinors $(z, \tilde{z})$, and using the Peter-weyl decomposition to introduce a unitary map in (82) between the spin network space of an edge $\mathcal{H}_e$ and the Hilbert space in this spinor representation $\mathcal{H}_e^{\text{spin}}$. This probes that the description in terms of group variables or holonomies, the one in terms of continuous fluxes or its spin network discrete representation, and the spinor representation are unitarily equivalent and they have the same physical information.

Finally, we have seen in (49), (51), (72) and (80) different forms to define the closure constraint. This implies a geometric interpretation of the quantum spin network graphs as the analog to the quantization of a discrete triangulation of the space manifold S with polyhedrons glued together. The first form of the constraint condition in (49) just tells us that if we define a tetrahedron by their normals to its faces, they all sum up to zero, and in the spin network basis this implies that the total spin on a vertex v is conserved. In (51) we consider the same form of the constraint, with the particular identification $\vec{E}_e = j_e \vec{N}_e(v)$ that confirms the interpretation of j_e as the eigenvalue or norm of the flux. In the other two quantum representations considered, the constraints are expressed in a similar way, with the peculiarity that in the non-commutative flux version in (72) this constraint is probed to arrive as a gauge-invariant condition on the vertex.

[28] The factor $\frac{1}{\sqrt{d_j}}$, where $d_j = 2j + 1$, have to be included in order to preserve the orthogonality of the group basis and ensure the unitarity of $\mathcal{T}$.

References

[1] Daniele Oriti and Matti Raasakka. Quantum mechanics on so (3) via noncommutative dual variables. *Physical Review D—Particles, Fields, Gravitation, and Cosmology*, 84(2):025003, 2011.

[2] Carlos Guedes, Daniele Oriti, and Matti Raasakka. Quantization maps, algebra representation, and non-commutative fourier transform for lie groups. *Journal of Mathematical Physics*, 54(8), 2013.

[3] Aristide Baratin, Bianca Dittrich, Daniele Oriti, and Johannes Tambornino. Non-commutative flux representation for loop quantum gravity. *Classical and quantum gravity*, 28(17):175011, 2011.

[4] Andrea Barbieri. Quantum tetrahedra and simplicial spin networks. *Nuclear Physics B*, 518(3):714–728, 1998.

[5] John C. Baez. Spin networks in nonperturbative quantum gravity, 2010.

[6] Laurent Freidel and Simone Speziale. Twisted geometries: A geometric parametrization of su (2) phase space. *Physical Review D—Particles, Fields, Gravitation, and Cosmology*, 82(8):084040, 2010.

[7] John C Baez and Javier P Muniain. *Gauge fields, knots and gravity*, volume 4. World Scientific Publishing Company, 1994.

[8] John C Baez and John W Barrett. The quantum tetrahedron in 3 and 4 dimensions. *arXiv preprint gr-qc/9903060*, 1999.

[9] Carlo Rovelli. *Quantum gravity*. Cambridge university press, 2004.

[10] Etera Livine and Johannes Tambornino. Spinor representation for loop quantum gravity. *Journal of mathematical physics*, 53(1), 2012.

[11] Pietropaolo Frisoni. Introduction to loop quantum gravity: Rovelli's lectures on lqg. *arXiv preprint arXiv:2305.12215*, 2023.

[12] Laurent Freidel and Jonathan Ziprick. Spinning geometry= twisted geometry. *Classical and Quantum Gravity*, 31(4):045007, 2014.