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On Besov spaces of logarithmic smoothness and Lipschitz spaces

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ABSTRACT

We compare Besov spaces $B^{0,b}_{p,q}$ with zero classical smoothness and logarithmic smoothness b defined by using the Fourier transform with the corresponding spaces $\mathbf{B}^{0,b}_{p,q}$ defined by means of the modulus of smoothness. In particular, we show that $B^{0,b+1/2}_{2,2} = \mathbf{B}^{0,b}_{2,2}$ for $b > -1/2$. We also determine the dual of $\mathbf{B}^{0,b}_{p,q}$ with the help of logarithmic Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,q}$. Finally we show embeddings between spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,q}$ and $B^{1,b}_{p,q}$ which complement and improve embeddings established by Haroske (2000) [28].

1. Introduction

Besov spaces $B^s_{p,q}$ play a central role in the theory of function spaces as can be seen in the monographs by Triebel [39–41]. For the complete solution of some natural questions as compactness in limiting embeddings [33,11] or spaces on fractals [20,21], more general spaces have been introduced where smoothness of functions is considered in a more delicate manner than in $B^s_{p,q}$. These spaces of generalized smoothness have been studied for long and from different points of view. See, for example, the papers by DeVore, Riemenschneider and Sharpley [16], Brézis and Wainger [5], Gol'dman [26], Merucci [34], Kalyabin and Lizorkin [32], Cobos and Fernandez [9], Edmunds and Haroske [18], Haroske and Moura [30], Farkas and Leopold [24], Triebel [41, pp. 52–55] and the references given there.

As in the case of $B^s_{p,q}$, spaces of generalized smoothness on $\mathbb{R}^n$ can be introduced by following the Fourier analytic approach or by means of the modulus of smoothness. If we take classical smoothness s and additional logarithmic smoothness with exponent b, the first way leads to spaces $B^{s,b}_{p,q}$ and the second to spaces $\mathbf{B}^{s,b}_{p,q}$ (precise definitions are given in Section 2). If $1 \leq p \leq \infty$ and $s > 0$, it turns out that $B^{s,b}_{p,q} = \mathbf{B}^{s,b}_{p,q}$ with equivalence of norms (see [30, Theorem 2.5] and [39, 2.5.12]; but if $0 < p < 1$ and $0 < q \leq 1$ then

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$B^{n(1/p-1)}_{p,q} \neq \mathbf{B}^{n(1/p-1)}_{p,q}$ as it is shown in [37, Corollary 3.10]). However, the relation between these two kinds of spaces when $s = 0$ has not been described yet. This problem is stated in the report of Triebel [42, p. 6], where first results on this question have been shown: Working with spaces on the unit cube $\mathbb{Q}^n$ in $\mathbb{R}^n$ and using the Haar basis, Triebel established in [42, Proposition 9] that $\mathbf{B}^{0,b}_{p,p}(\mathbb{Q}^n) \hookrightarrow B^{0,b}_{p,p}(\mathbb{Q}^n)$ provided that $1 < p \le 2$ and $b \ge 0$ or $2 < p < \infty$ and $b > 1/2 - 1/p$. See also the paper by Besov [4], where spaces $B^0_{p,q}$, $1 \le p, q \le \infty$, are compared with certain spaces defined by first differences.

In this paper we compare spaces $B^{s,b}_{p,q}$ and $\mathbf{B}^{s,b}_{p,q}$ with the help of the limiting real method

$$(A_0, A_1)_{(\theta,\eta),q} = \left\{ a \in A_0 : \|a\|_{\bar{A}_{(\theta,\eta),q}} = \left(\int_0^1 \left(\frac{K(t,a)}{t^\theta (1 - \log t)^\eta} \right)^q \frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

Here $\theta = 0$ or 1, $A_1 \hookrightarrow A_0$ and $K(t,a)$ is the K-functional of Peetre (see Section 2). Among other things, for $b > -1/p$ we show that

$$B^{0,b+1/p}_{p,p} \hookrightarrow \mathbf{B}^{0,b}_{p,p} \hookrightarrow B^{0,b+1/2}_{p,p} \quad \text{if } 1 < p \le 2,$$
$$B^{0,b+1/2}_{p,p} \hookrightarrow \mathbf{B}^{0,b}_{p,p} \hookrightarrow B^{0,b+1/p}_{p,p} \quad \text{if } 2 \le p < \infty.$$

Therefore, $B^{0,b+1/2}_{2,2} = \mathbf{B}^{0,b}_{2,2}$. In particular, this implies that the classical space $\mathbf{B}^0_{2,2}$, defined by the modulus of continuity, coincides with the space $B^{0,1/2}_{2,2}$, defined by the Fourier transform, with zero classical smoothness and logarithmic smoothness with exponent $1/2$.

We also consider embeddings between spaces $\mathbf{B}^{s,b}_{p,q}$. According to [38, Theorem 2.8.1] or [2, Corollary 5.4.21], if $1 \le p \le r < \infty$, $1 \le q \le \infty$ and $s > 0$, then $\mathbf{B}^{n(1/p-1/r)+s}_{p,q} \hookrightarrow \mathbf{B}^s_{r,q}$. Note that in the embedding the two spaces have the same differential dimension. The limit case where $s = 0$ has been studied by DeVore, Riemenschneider and Sharpley [16, Corollary 5.3(ii)], where they showed that the embedding holds with a loss of a unit in the exponent of the logarithmic smoothness. To be more precise, if $1 \le p \le r \le \infty$, $1 \le q \le \infty$ and $b > -1/q$ then $\mathbf{B}^{n(1/p-1/r),b+1}_{p,q} \hookrightarrow \mathbf{B}^{0,b}_{r,q}$. This result has been improved recently by Gogatishvili, Opic, Tikhonov and Trebels [25, Corollary 2.8] by showing that the embedding holds with the loss of only $1/\min\{q,r\}$ in the exponent of the logarithmic smoothness. In this paper, we use limiting interpolation to derive the embedding $\mathbf{B}^{n(1/p-1/r),b+1/\min\{q,r\}}_{p,q} \hookrightarrow \mathbf{B}^{0,b}_{r,q}$ following a more simple approach than in [25].

In addition we determine the dual of $\mathbf{B}^{0,b}_{p,q}$ for $1 < p < \infty$, $1 \le q < \infty$ and $b > -1/q$. This is done with the help of logarithmic Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,q}$ introduced by Haroske in [28] (see also [29, Definition 2.16], [17, p. 149] and the references given there).

Finally we study embeddings between Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,q}$ and Besov spaces $B^{1,b}_{p,q}$. This problem was considered by Haroske [28,29] and Neves [35] among other authors. Our approach allows us to cover some critical cases which come up for the techniques used in [28]. As a consequence, we complement and improve several results of Haroske [28].

2. Preliminaries

Subsequently, given two quasi-Banach spaces X, Y, we put $X \hookrightarrow Y$ to mean that X is continuously embedded in Y.

If U, V are non-negative quantities depending on certain parameters, we write $U \lesssim V$ if there is a constant $c > 0$ independent of the parameters in U and V such that $U \le cV$. We put $U \sim V$ if $U \lesssim V$ and $V \lesssim U$.

Let $\bar{A} = (A_0, A_1)$ be a quasi-Banach couple, that is to say, two quasi-Banach spaces A_0, A_1 which are continuously embedded in some Hausdorff topological vector space. The *Peetre's K-functional* is given by

$$K(t,a) = K(t,a;A_0,A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1}\}, \quad t>0,\ a \in A_0+A_1,$$

where the infimum is taken over all representations $a = a_0 + a_1$ with $a_0 \in A_0$ and $a_1 \in A_1$.

For $0<\theta<1$ and $0<q\le\infty$, the *real interpolation space* $\bar{A}_{\theta,q} = (A_0,A_1)_{\theta,q}$ is formed by all $a \in A_0+A_1$ having a finite quasi-norm

$$\|a\|_{\bar{A}_{\theta,q}} = \left(\int_0^\infty \big(t^{-\theta}K(t,a)\big)^q \frac{dt}{t}\right)^{1/q}$$

(as usual, when $q=\infty$ the integral should be replaced by the supremum). See [3,6] or [38].

For $\mathbb{A} = (\alpha_0,\alpha_\infty) \in \mathbb{R}^2$, let $\ell(t) = 1 + |\log t|$ and

$$\ell^{\mathbb{A}}(t) = \begin{cases} \ell^{\alpha_0}(t) & \text{for } t \in (0,1], \\ \ell^{\alpha_\infty}(t) & \text{for } t \in (1,\infty). \end{cases}$$

Replacing t^θ by $t^\theta/\ell^{\mathbb{A}}(t)$ we obtain the spaces

$$\bar{A}_{\theta,q,\mathbb{A}} = (A_0,A_1)_{\theta,q,\mathbb{A}} = \left\{a \in A_0+A_1 : \|a\|_{\bar{A}_{\theta,q,\mathbb{A}}} = \left(\int_0^\infty \big(t^{-\theta}\ell^{\mathbb{A}}(t)K(t,a)\big)^q \frac{dt}{t}\right)^{1/q} < \infty\right\}$$

(see [22,23]). Under suitable assumptions on $\mathbb{A}$ and q, spaces $(A_0,A_1)_{\theta,q,\mathbb{A}}$ are well-defined even if $\theta=0$ or $\theta=1$. In the special case $\alpha_0=\alpha_\infty=\alpha$, we simply write $(A_0,A_1)_{\theta,q,\alpha}$ instead of $(A_0,A_1)_{\theta,q,(\alpha,\alpha)}$.

We shall also need the following *limiting real spaces*. Let $A_1 \hookrightarrow A_0$, $0<q\le\infty$ and $-\infty<\eta<\infty$. For $\theta=1$ or $\theta=0$, the space $\bar{A}_{(\theta,\eta),q} = (A_0,A_1)_{(\theta,\eta),q}$ consists of all those $a \in A_0$ with

$$\|a\|_{\bar{A}_{(\theta,\eta),q}} = \left(\int_0^1 \left(\frac{K(t,a)}{t^\theta(1-\log t)^\eta}\right)^q \frac{dt}{t}\right)^{1/q} < \infty$$

(see [27,10,12]). To avoid that $\bar{A}_{(1,\eta),q} = \{0\}$, if $\theta=1$, we assume that $\eta > 1/q$ if $q<\infty$, and $\eta \ge 0$ if $q=\infty$.

The following result is established in [8, Lemma 2.5] by using the connection between limiting real spaces $\bar{A}_{(\theta,\eta),q}$ and logarithmic spaces $\bar{A}_{\theta,q,\mathbb{A}}$ [19, Proposition 1], and reiteration results for logarithmic spaces [22, Theorems 5.9*, 4.7*, 5.7 and 4.7]. It will be important in our later considerations (note that our notation here follows [22] and so it is slightly different from [8]).

Lemma 2.1. *Let A_0, A_1 be quasi-Banach spaces with $A_1 \hookrightarrow A_0$. Assume that $0<\theta<1$, $0<p,q\le\infty$ and $\gamma < -1/q < \eta$. The following continuous embeddings hold:*

(a) $(A_0,A_1)_{\theta,q,\gamma+1/\min\{p,q\}} \hookrightarrow \big(A_0,(A_0,A_1)_{\theta,p}\big)_{(1,-\gamma),q} \hookrightarrow (A_0,A_1)_{\theta,q,\gamma+1/\max\{p,q\}}$,

(b) $(A_0,A_1)_{\theta,q,\eta+1/\min\{p,q\}} \hookrightarrow \big((A_0,A_1)_{\theta,p},A_1\big)_{(0,-\eta),q} \hookrightarrow (A_0,A_1)_{\theta,q,\eta+1/\max\{p,q\}}$.

Remark 2.2. Note that if $\gamma=-1/q$ then $(A_0,(A_0,A_1)_{\theta,p})_{(1,1/q),q} = \{0\}$, so none of the embeddings in statement (a) of Lemma 2.1 hold in this case. As for statement (b) when $\eta=-1/q$, if $p=q$ we can determine explicitly $((A_0,A_1)_{\theta,p},A_1)_{(0,1/p),p}$. Indeed, by Holmstedt's formula [31, Remark 2.1]

$$K\big(t,a;(A_0,A_1)_{\theta,p},A_1\big) \sim \left(\int_0^{t^{1/(1-\theta)}} \left(\frac{K(s,a;A_0,A_1)}{s^\theta}\right)^p \frac{ds}{s}\right)^{1/p}.$$

Hence, we obtain

$$\|a\|_{((A_0,A_1)_{\theta,p},A_1)_{(0,1/p),p}} \sim \left(\int_0^1 \frac{1}{1-\log t}\int_0^{t^{1/(1-\theta)}} \left(\frac{K(s,a;A_0,A_1)}{s^\theta}\right)^p \frac{ds}{s}\frac{dt}{t}\right)^{1/p}$$
$$= \left(\int_0^1 \left(\frac{K(s,a;A_0,A_1)}{s^\theta}\right)^p \int_{s^{1-\theta}}^1 \frac{1}{1-\log t}\frac{dt}{t}\frac{ds}{s}\right)^{1/p}$$
$$\sim \left(\int_0^1 \left(\frac{K(s,a;A_0,A_1)}{s^\theta}\big(\log(1-\log s)\big)^{1/p}\right)^p \frac{ds}{s}\right)^{1/p}.$$

Therefore, if $\eta = -1/q$ and $p = q$ we still have the embedding of the right-hand side in statement (b) of Lemma 2.1 because $\eta + 1/\max\{p,q\} = 0$ and so $(A_0,A_1)_{\theta,q,\eta+1/\max\{p,q\}} = (A_0,A_1)_{\theta,q}$. But the embedding of the left-hand side in (b) fails.

Other kind of limiting reiteration formulae can be seen in [13].

Let $\mathcal{S}$ and $\mathcal{S}'$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$, and the space of tempered distributions on $\mathbb{R}^n$, respectively. By $\mathcal{F}$ we denote the Fourier transform on $\mathcal{S}'$ and by $\mathcal{F}^{-1}$ the inverse Fourier transform.

Take $\varphi_0 \in \mathcal{S}$ such that

$$\operatorname{supp}\varphi_0 \subset \{x \in \mathbb{R}^n : |x| \le 2\} \quad \text{and} \quad \varphi_0(x) = 1 \quad \text{if } |x| \le 1.$$

For $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$ let $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$. Then the sequence $(\varphi_j)_{j=0}^\infty$ forms a dyadic resolution of unity, $\sum_{j=0}^\infty \varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$.

For $1 \le p \le \infty$, $0 < q \le \infty$ and $s, b \in \mathbb{R}$, the space $B^{s,b}_{p,q}$ consists of all $f \in \mathcal{S}'$ having a finite quasi-norm

$$\|f\|_{B^{s,b}_{p,q}} = \left(\sum_{j=0}^\infty \left(2^{js}(1+j)^b \left\|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\right\|_{L_p}\right)^q\right)^{1/q}$$

(with the usual modification if $q = \infty$). See [34,9,33,30]. Note that if $b = 0$ then $B^{s,0}_{p,q}$ coincides with the usual Besov space $B^s_{p,q}$.

Besov spaces of generalized smoothness can be also introduced by using the modulus of smoothness as we recall next. Let f be a function on $\mathbb{R}^n$, let $h \in \mathbb{R}^n$ and $k \in \mathbb{N}$. We put

$$(\Delta^1_h f)(x) = f(x+h) - f(x) \quad \text{and} \quad (\Delta^{k+1}_h f)(x) = \Delta^1_h(\Delta^k_h f)(x).$$

The *k-th order modulus of smoothness* of a function $f \in L_p$ is defined by

$$\omega_k(f,t)_p = \sup_{|h|\le t} \left\|\Delta^k_h f\right\|_{L_p}, \quad t > 0.$$

If $k = 1$ we simply write $\omega(f,t)_p$ instead of $\omega_1(f,t)_p$.

For $1 \le p \le \infty$, the following connection holds between the K-functional for the couple (L_p, W^k_p) and the k-th order modulus of smoothness: There are positive constants c_1 and c_2 such that

$$c_1 K(t^k, f; L_p, W^k_p) \le \min(1,t^k)\|f\|_{L_p} + \omega_k(f,t)_p \le c_2 K(t^k, f; L_p, W^k_p) \tag{2.1}$$

for all $f \in L_p$ and $t > 0$ (see [2, Theorem 5.4.12]).

For $1 \le p \le \infty$, $0 < q \le \infty$, $-\infty < b < \infty$, $s \ge 0$ and $k \in \mathbb{N}$ with $k > s$, the space $\mathbf{B}^{s,b}_{p,q}$ consists of all $f \in L_p$ such that

$$\|f\|_{\mathbf{B}^{s,b}_{p,q}} = \|f\|_{L_p} + \left(\int_0^1 \left[t^{-s}(1 - \log t)^b \omega_k(f,t)_p \right]^q \frac{dt}{t} \right)^{1/q} < \infty.$$

See [16,30]. Note that if $s = 0$ and $b < -1/q$, then $\mathbf{B}^{0,b}_{p,q} = L_p$.

3. Besov spaces with logarithmic smoothness

If $s > 0$ it is well-known that the definition of $\mathbf{B}^{s,b}_{p,q}$ does not depend on the choice of $k > s$ (see [30, Theorem 2.5]). Next we show that the same property holds for $\mathbf{B}^{0,b}_{p,q}$. We also characterize spaces $\mathbf{B}^{0,b}_{p,q}$ by interpolation.

Theorem 3.1. *Let $1 \le p \le \infty$, $0 < q \le \infty$, $-\infty < b < \infty$ and $k \in \mathbb{N}$.*

(a) *The space $\mathbf{B}^{0,b}_{p,q}$ does not depend on the choice of $k \in \mathbb{N}$.*

(b) *We have $\mathbf{B}^{0,b}_{p,q} = \left(L_p, W^k_p\right)_{(0,-b),q}$ with equivalence of quasi-norms.*

Proof. Let $k \in \mathbb{N}, k > 1$. Put

$$\|f\|_{\mathbf{B}^{0,b}_{p,q}} = \|f\|_{L_p} + \left(\int_0^1 \left((1 - \log t)^b \omega(f,t)_p \right)^q \frac{dt}{t} \right)^{1/q} \tag{3.1}$$

and

$$\|f\|^{(k)}_{\mathbf{B}^{0,b}_{p,q}} = \|f\|_{L_p} + \left(\int_0^1 \left((1 - \log t)^b \omega_k(f,t)_p \right)^q \frac{dt}{t} \right)^{1/q}.$$

Our aim is to show the equivalence between the quasi-norms $\|\cdot\|_{\mathbf{B}^{0,b}_{p,q}}$ and $\|\cdot\|^{(k)}_{\mathbf{B}^{0,b}_{p,q}}$. Since $\omega_k(f,t)_p \le 2^{k-1}\omega(f,t)_p$, it is clear that $\|f\|^{(k)}_{\mathbf{B}^{0,b}_{p,q}} \lesssim \|f\|_{\mathbf{B}^{0,b}_{p,q}}$. Let us check the converse inequality. Using Marchaud's inequality [2, Theorem 5.4.4], for $0 < t \le 1$ we obtain

$$\begin{aligned} \frac{\omega(f,t)_p}{t} &\lesssim \int_t^\infty \frac{\omega_k(f,s)_p}{s} \frac{ds}{s} \\ &\lesssim \int_t^1 \frac{\omega_k(f,s)_p}{s} \frac{ds}{s} + \|f\|_{L_p} \int_1^\infty s^{-1} \frac{ds}{s} \\ &\sim \int_t^1 \frac{\omega_k(f,s)_p}{s} \frac{ds}{s} + \|f\|_{L_p}. \end{aligned}$$

Therefore, since $\omega_k(f,s)_p/s^k$ is equivalent to a decreasing function, applying Hardy's inequality [1, Theorem 6.4], we get

$$\begin{aligned}
&\left(\int_0^1 \left((1-\log t)^b \omega(f,t)_p\right)^q \frac{dt}{t}\right)^{1/q} \\
&\lesssim \left(\int_0^1 \left(t(1-\log t)^b\right)^q \frac{dt}{t}\right)^{1/q} \|f\|_{L_p} \\
&\quad + \left(\int_0^1 \left[t(1-\log t)^b \int_t^1 \frac{\omega_k(f,s)_p}{s^k} s^{(k-1)-1} ds\right]^q \frac{dt}{t}\right)^{1/q} \\
&\lesssim \|f\|_{L_p} + \left(\int_0^1 \left[t^2(1-\log t)^b \frac{\omega_k(f,t)_p}{t^2}\right]^q \frac{dt}{t}\right)^{1/q} \\
&= \|f\|^{(k)}_{\mathbf{B}^{0,b}_{p,q}}.
\end{aligned}$$

This proves statement (a).

As for (b), using (2.1), we obtain

$$\begin{aligned}
\|f\|_{(L_p,W^k_p)_{(0,-b),q}} &= \left(\int_0^1 \left[(1-\log t)^b K\left(t,f;L_p,W^k_p\right)\right]^q \frac{dt}{t}\right)^{1/q} \\
&\sim \left(\int_0^1 \left[(1-\log t)^b K\left(t^k,f;L_p,W^k_p\right)\right]^q \frac{dt}{t}\right)^{1/q} \\
&\sim \left(\int_0^1 \left[(1-\log t)^b \left(t^k\|f\|_{L_p} + \omega_k(f,t)_p\right)\right]^q \frac{dt}{t}\right)^{1/q} \\
&\sim \|f\|_{L_p} + \left(\int_0^1 \left[(1-\log t)^b \omega_k(f,t)_p\right]^q \frac{dt}{t}\right)^{1/q} \\
&\sim \|f\|_{\mathbf{B}^{0,b}_{p,q}}
\end{aligned}$$

where we have used (a) in the last equivalence. This completes the proof. □

Remark 3.2. For Besov spaces defined over $\mathbb{T}^n$, the fact that the definition of $\mathbf{B}^{0,b}_{p,q}(\mathbb{T}^n)$ is independent of $k \in \mathbb{N}$ has been proved in [25, pp. 1041–1043] by using Jackson inequality and Bernstein inequality.

In what follows we assume that $\mathbf{B}^{0,b}_{p,q}$ is quasi-normed by (3.1).

Next we compare $B^{0,b}_{p,q}$ and $\mathbf{B}^{0,b}_{p,q}$.

Theorem 3.3. *Let $1 < p < \infty$, $0 < q \le \infty$ and $b > -1/q$. Then*

$$B^{0,b+1/\min\{2,p,q\}}_{p,q} \hookrightarrow \mathbf{B}^{0,b}_{p,q} \hookrightarrow B^{0,b+1/\max\{2,p,q\}}_{p,q}.$$

Proof. Recall that

$$B^s_{p,\min\{p,q\}} \hookrightarrow F^s_{p,q} \hookrightarrow B^s_{p,\max\{p,q\}}$$

where $F^s_{p,q}$ stands for the Triebel–Lizorkin space (see [39, Proposition 2.3.2/2(iii)]). Moreover, $F^s_{p,2} = H^s_p$ [39, Theorem 2.5.6(i)] and so $F^0_{p,2} = L_p$. According to Theorem 3.1(b), Lemma 2.1 and [9, Theorem 5.3 and Remark 5.4], we derive

$$\begin{aligned}\mathbf{B}^{0,b}_{p,q} &= \left(L_p, W^1_p\right)_{(0,-b),q} \hookrightarrow \left(B^0_{p,\max\{2,p\}}, H^1_p\right)_{(0,-b),q}\\ &= \left(\left(H^{-1}_p, H^1_p\right)_{1/2,\max\{2,p\}}, H^1_p\right)_{(0,-b),q}\\ &\hookrightarrow \left(H^{-1}_p, H^1_p\right)_{1/2,q,b+1/\max\{2,p,q\}}\\ &= B^{0,b+1/\max\{2,p,q\}}_{p,q}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}B^{0,b+1/\min\{2,p,q\}}_{p,q} &= \left(H^{-1}_p, H^1_p\right)_{1/2,q,b+1/\min\{2,p,q\}}\\ &\hookrightarrow \left(\left(H^{-1}_p, H^1_p\right)_{1/2,\min\{2,p\}}, H^1_p\right)_{(0,-b),q}\\ &= \left(B^0_{p,\min\{2,p\}}, H^1_p\right)_{(0,-b),q}\\ &\hookrightarrow \left(L_p, W^1_p\right)_{(0,-b),q}\\ &= \mathbf{B}^{0,b}_{p,q}. \qquad \square\end{aligned}$$

Remark 3.4. In general $\mathbf{B}^{0,b}_{p,q} \neq B^{0,b+1/\max\{2,p,q\}}_{p,q}$ because in any of the cases

$$\begin{cases} 1 < p < \infty, \quad 0 < q \le \min\{2,p\}, \quad b + \dfrac{1}{\max\{2,p\}} < 0 < b + \dfrac{1}{q},\\ 1 < p \le 2, \quad p < q \le \infty, \quad 0 < b + \dfrac{1}{q} \le \dfrac{1}{p} - \dfrac{1}{\max\{2,q\}},\\ 2 < p < \infty, \quad 2 < q \le \infty, \quad 0 < b + \dfrac{1}{q} \le \dfrac{1}{2} - \dfrac{1}{\max\{p,q\}},\end{cases}$$

the space $B^{0,b+1/\max\{2,p,q\}}_{p,q}$ does not contain only regular distributions (see [7, Theorem 4.3]).

Corollary 3.5. *Let $1 < p < \infty$ and $b > -1/p$.*

(a) *If $1 < p \le 2$ then $B^{0,b+1/p}_{p,p} \hookrightarrow \mathbf{B}^{0,b}_{p,p} \hookrightarrow B^{0,b+1/2}_{p,p}$.*
(b) *If $2 \le p < \infty$ then $B^{0,b+1/2}_{p,p} \hookrightarrow \mathbf{B}^{0,b}_{p,p} \hookrightarrow B^{0,b+1/p}_{p,p}$.*

In particular, for $b > -1/2$ we obtain with equivalence of norms

$$B^{0,b+1/2}_{2,2} = \mathbf{B}^{0,b}_{2,2}.$$

Remark 3.6. As in Remark 3.4, note that

$$\mathbf{B}^{0,b}_{p,p} \neq B^{0,b+1/2}_{p,p} \quad \text{if } 1 < p < 2 \quad \text{and} \quad b + \frac{1}{2} < 0 < b + \frac{1}{p}$$

and

$$\mathbf{B}^{0,b}_{p,p} \neq B^{0,b+1/p}_{p,p} \quad \text{if } 2 < p < \infty \quad \text{and} \quad 0 < b + \frac{1}{p} \le \frac{1}{2} - \frac{1}{p}.$$

We finish this section with an embedding result into spaces $\mathbf{B}^{0,b}_{p,q}$. This problem has been considered in [16, Corollary 5.3(ii)] and [25, Corollary 2.8]. We follow a more simple approach than in [25] based on limiting interpolation.

Theorem 3.7. *Let $1 \leq p < r < \infty$, $0 < q \leq \infty$, $b > -1/q$ and $\alpha = n(1/p - 1/r)$. Then*

$$\mathbf{B}^{\alpha,b+1/\min\{q,r\}}_{p,q} \hookrightarrow \mathbf{B}^{0,b}_{r,q}.$$

Proof. According to [2, Corollary 5.4.20], we have

$$\mathbf{B}^{\alpha}_{p,r} \hookrightarrow L_r. \tag{3.2}$$

On the other hand, let $k \in \mathbb{N}$ such that $k > \alpha$ and $0 < \theta < 1$ such that $\theta k > \alpha$. By [2, Corollaries 5.4.13 and 5.4.21], we derive

$$W^k_p \hookrightarrow \left(L_p, W^k_p\right)_{\theta,p} = \mathbf{B}^{\theta k}_{p,p} \hookrightarrow \mathbf{B}^{\theta k-\alpha}_{r,p}. \tag{3.3}$$

Interpolating embeddings (3.2) and (3.3) by the limiting real method we get

$$\left(\mathbf{B}^{\alpha}_{p,r}, W^k_p\right)_{(0,-b),q} \hookrightarrow \left(L_r, \mathbf{B}^{\theta k-\alpha}_{r,p}\right)_{(0,-b),q}.$$

The target space in this embedding can be determined by using [8, Lemma 2.2(b)] and Theorem 3.1(b). Indeed,

$$\begin{aligned}\left(L_r, \mathbf{B}^{\theta k-\alpha}_{r,p}\right)_{(0,-b),q} &= \left(L_r, \left(L_r, W^k_r\right)_{\frac{\theta k-\alpha}{k},p}\right)_{(0,-b),q} \\ &= \left(L_r, W^k_r\right)_{(0,-b),q} = \mathbf{B}^{0,b}_{r,q}.\end{aligned}$$

As for the domain space, Lemma 2.1(b) yields

$$\begin{aligned}\mathbf{B}^{\alpha,b+1/\min\{q,r\}}_{p,q} &= \left(L_p, W^k_p\right)_{\alpha/k,q,b+1/\min\{q,r\}} \\ &\hookrightarrow \left(\left(L_p, W^k_p\right)_{\alpha/k,r}, W^k_p\right)_{(0,-b),q} \\ &= \left(\mathbf{B}^{\alpha}_{p,r}, W^k_p\right)_{(0,-b),q}.\end{aligned}$$

This completes the proof. □

4. Duality

Let $1 < p < \infty$, $1 \leq q < \infty$ and $-\infty < b < \infty$. Since $B^{0,b}_{p,q} = (H^{-1}_p, H^1_p)_{1/2,q,b}$, using the duality formula for spaces $(A_0, A_1)_{\theta,q,\mathbb{A}}$ (see [15, Theorem 3.1] or [36, Theorem 2.4]) and that $(H^s_p)' = H^{-s}_{p'}$ [38, Theorem 2.6.1], it follows that

$$\left(B^{0,b}_{p,q}\right)' = B^{0,-b}_{p',q'} \quad \text{where } 1/p + 1/p' = 1 = 1/q + 1/q'$$

(see also [24, Theorem 3.1.10]).

In order to determine the dual space of $\mathbf{B}^{0,b}_{p,q}$, we first establish an auxiliary result and recall the definition of logarithmic Lipschitz spaces (see [28] and [29]).

Lemma 4.1. *Let A_0, A_1 be Banach spaces with A_1 continuously and densely embedded in A_0. Assume that $1 \le q < \infty$, $1/q + 1/q' = 1$, and $\eta > -1/q$. Then we have with equivalence of norms*

$$(A_0, A_1)'_{(0,-\eta),q} = \big(A'_1, A'_0\big)_{(1,\eta+1),q'}.$$

Proof. Since $A_1 \hookrightarrow A_0$, we have that $K(t, a; A_0, A_1) \sim \|a\|_{A_0}$ for $t \ge 1$. Take any $\tau < -1/q$. It follows that

$$\left(\int_1^\infty \big[(1+\log t)^\tau K(t, a; A_0, A_1)\big]^q \frac{dt}{t}\right)^{1/q} \sim \left(\int_1^\infty (1+\log t)^{\tau q} \frac{dt}{t}\right)^{1/q} \|a\|_{A_0} \sim \|a\|_{A_0} \lesssim \|a\|_{(A_0,A_1)_{(0,-\eta),q}}.$$

This yields that

$$(A_0, A_1)_{(0,-\eta),q} = (A_0, A_1)_{0,q,(\eta,\tau)} = (A_1, A_0)_{1,q,(\tau,\eta)}.$$

Since $\tau + 1/q < 0 < \eta + 1/q$, we can apply the duality formula established in [14, Theorem 5.6] to derive

$$(A_0, A_1)'_{(0,-\eta),q} = (A_1, A_0)'_{1,q,(\tau,\eta)} = \big(A'_1, A'_0\big)_{1,q',(-\eta-1,-\tau-1)}.$$

Density of the embedding $A_1 \hookrightarrow A_0$ implies that $A'_0 \hookrightarrow A'_1$. So $K(t, g; A'_1, A'_0) \sim \|g\|_{A'_1}$ for $t \ge 1$. Now, using that $K(t, g)/t$ is a decreasing function we get

$$\left(\int_1^\infty \big[t^{-1}(1+\log t)^{-\tau-1} K\big(t, g; A'_1, A'_0\big)\big]^{q'} \frac{dt}{t}\right)^{1/q'} \sim \|g\|_{A'_1} \sim K\big(1, g; A'_1, A'_0\big) \left(\int_0^1 (1-\log t)^{(-\eta-1)q'} \frac{dt}{t}\right)^{1/q'} \le \|g\|_{(A'_1,A'_0)_{(1,\eta+1),q'}}.$$

Consequently, $(A_0, A_1)'_{(0,-\eta),q} = (A'_1, A'_0)_{(1,\eta+1),q'}$. □

Definition 4.2. Let $1 \le p \le \infty$, $0 < q \le \infty$ and $\alpha > 1/q$ ($\alpha \ge 0$ if $q = \infty$). The space $\mathrm{Lip}^{(1,-\alpha)}_{p,q}$ is formed by all functions $f \in L_p$ having a finite quasi-norm

$$\|f\|_{\mathrm{Lip}^{(1,-\alpha)}_{p,q}} = \|f\|_{L_p} + \left(\int_0^1 \left[\frac{\omega(f,t)_p}{t(1-\log t)^\alpha}\right]^q \frac{dt}{t}\right)^{1/q}.$$

Now we are ready to describe the dual space of $\mathbf{B}^{0,b}_{p,q}$. Recall that the usual lift operator I_s is defined by

$$I_s f = \mathcal{F}^{-1}\big(1+|x|^2\big)^{s/2}\mathcal{F}f, \quad -\infty < s < \infty.$$

Theorem 4.3. *Let $1 < p < \infty$, $1 \le q < \infty$ and $b > -1/q$. The space $(\mathbf{B}^{0,b}_{p,q})'$ consists of all $f \in H^{-1}_{p'}$ such that $I_{-1}f \in \mathrm{Lip}^{(1,-b-1)}_{p',q'}$ with $1/p + 1/p' = 1 = 1/q + 1/q'$. Moreover,*

$$\|f\|_{(\mathbf{B}^{0,b}_{p,q})'} \sim \|I_{-1}f\|_{\mathrm{Lip}^{(1,-b-1)}_{p',q'}}.$$

Proof. By Theorem 3.1(b) and Lemma 4.1, we derive

$$(\mathbf{B}^{0,b}_{p,q})' = \big((L_p, W^1_p)_{(0,-b),q}\big)' = \big(H^{-1}_{p'}, L_{p'}\big)_{(1,b+1),q'}.$$

On the other hand, lift operators

$$I_{-1} : H^{-1}_{p'} \longrightarrow L_{p'}, \qquad I_{-1} : L_{p'} \longrightarrow W^1_{p'}$$

are bijective and bounded. Hence

$$\begin{aligned} K\big(t, f; H^{-1}_{p'}, L_{p'}\big) &\sim K\big(t, I_{-1}f; L_{p'}, W^1_{p'}\big) \\ &\sim \min(1,t)\|I_{-1}f\|_{L_{p'}} + \omega(I_{-1}f, t)_{p'} \end{aligned}$$

where we have used (2.1) for the last equivalence. Consequently

$$\begin{aligned} \|f\|_{(\mathbf{B}^{0,b}_{p,q})'} &\sim \left(\int_0^1 (1-\log t)^{(-b-1)q'} \frac{dt}{t}\right)^{1/q'} \|I_{-1}f\|_{L_{p'}} \\ &\quad + \left(\int_0^1 \left[\frac{\omega(I_{-1}f, t)_{p'}}{t(1-\log t)^{b+1}}\right]^{q'} \frac{dt}{t}\right)^{1/q'} \\ &\sim \|I_{-1}f\|_{\mathrm{Lip}^{(1,-b-1)}_{p',q'}}. \qquad \square \end{aligned}$$

5. Embeddings between Besov and Lipschitz spaces

We start by showing that Lipschitz spaces can be generated by interpolation from the couple (L_p, W^1_p).

Lemma 5.1. *Let $1 \le p \le \infty$, $0 < q \le \infty$ and $\alpha > 1/q$ ($\alpha \ge 0$ if $q = \infty$). Then*

$$\big(L_p, W^1_p\big)_{(1,\alpha),q} = \mathrm{Lip}^{(1,-\alpha)}_{p,q}$$

with equivalent quasi-norms.

Proof. Using (2.1) we derive

$$\begin{aligned} \|f\|_{(L_p, W^1_p)_{(1,\alpha),q}} &= \left(\int_0^1 \left[\frac{K(t, f; L_p, W^1_p)}{t(1-\log t)^{\alpha}}\right]^q \frac{dt}{t}\right)^{1/q} \\ &\sim \left(\int_0^1 (1-\log t)^{-\alpha q} \frac{dt}{t}\right)^{1/q} \|f\|_{L_p} + \left(\int_0^1 \left[\frac{\omega(f,t)_p}{t(1-\log t)^{\alpha}}\right]^q \frac{dt}{t}\right)^{1/q} \\ &\sim \|f\|_{\mathrm{Lip}^{(1,-\alpha)}_{p,q}}. \qquad \square \end{aligned}$$

The next result describes the position of Lipschitz spaces between Besov spaces with classical smoothness 1 and additional logarithmic smoothness.

Theorem 5.2. *Let $1 < p < \infty$, $0 < q \le \infty$ and $\alpha > 1/q$. Then*

$$B^{1,-\alpha+1/\min\{2,p,q\}}_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,q} \hookrightarrow B^{1,-\alpha+1/\max\{2,p,q\}}_{p,q}.$$

Proof. By Lemmata 5.1, 2.1(a) and [9, Theorem 5.3 and Remark 5.4], we obtain

$$\begin{aligned}\mathrm{Lip}^{(1,-\alpha)}_{p,q} &= \big(L_p, W^1_p\big)_{(1,\alpha),q} \hookrightarrow \big(L_p, B^1_{p,\max\{2,p\}}\big)_{(1,\alpha),q}\\ &= \big(L_p, \big(L_p, W^2_p\big)_{1/2,\max\{2,p\}}\big)_{(1,\alpha),q}\\ &\hookrightarrow \big(L_p, W^2_p\big)_{1/2,q,-\alpha+1/\max\{2,p,q\}}\\ &= B^{1,-\alpha+1/\max\{2,p,q\}}_{p,q}.\end{aligned}$$

Similarly, we derive

$$\begin{aligned}B^{1,-\alpha+1/\min\{2,p,q\}}_{p,q} &= \big(L_p, W^2_p\big)_{1/2,q,-\alpha+1/\min\{2,p,q\}}\\ &\hookrightarrow \big(L_p, \big(L_p, W^2_p\big)_{1/2,\min\{2,p\}}\big)_{(1,\alpha),q}\\ &= \big(L_p, B^1_{p,\min\{2,p\}}\big)_{(1,\alpha),q}\\ &\hookrightarrow \big(L_p, W^1_p\big)_{(1,\alpha),q}\\ &= \mathrm{Lip}^{(1,-\alpha)}_{p,q}. \qquad \square\end{aligned}$$

Corollary 5.3. *Let $1 < p < \infty$ and $\alpha > 1/p$.*

(a) *If $1 < p \le 2$ then $B^{1,-\alpha+1/p}_{p,p} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,p} \hookrightarrow B^{1,-\alpha+1/2}_{p,p}$.*
(b) *If $2 \le p < \infty$ then $B^{1,-\alpha+1/2}_{p,p} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,p} \hookrightarrow B^{1,-\alpha+1/p}_{p,p}$.*

In particular, if $\alpha > 1/2$ we have

$$B^{1,-\alpha+1/2}_{2,2} = \mathrm{Lip}^{(1,-\alpha)}_{2,2}.$$

Next we recall a result of Haroske [28, Proposition 16].

Proposition 5.4. *Let $1 \le p \le \infty$, $0 < q, v \le \infty$, $\alpha > 1/q$ and $\beta > 1/v$. Then*

$$\mathrm{Lip}^{(1,-\alpha)}_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\beta)}_{p,v} \quad \text{if, and only if,} \quad \begin{cases}\beta - \dfrac{1}{v} \ge \alpha - \dfrac{1}{q} & \text{and} \quad v \ge q,\\[2mm] \beta - \dfrac{1}{v} > \alpha - \dfrac{1}{q} & \text{and} \quad v < q.\end{cases}$$

In the remaining part of this section we show that combining Proposition 5.4 with the previous results we can derive some complements and improvements of the results of [28].

Theorem 5.5. *Let* $1 < p < \infty$, $0 < q, v \le \infty$ *and* $\alpha > 1/v$. *Then*

$$B^1_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,v} \quad \textit{if} \begin{cases} 0 < q \le \min\{2,p\}, \\ \min\{2,p\} < q, v < q \quad \textit{and} \quad \alpha > \dfrac{1}{v} + \dfrac{1}{\min\{2,p\}} - \dfrac{1}{q}, \\ \min\{2,p\} < q \le v \quad \textit{and} \quad \alpha \ge \dfrac{1}{v} + \dfrac{1}{\min\{2,p\}} - \dfrac{1}{q}. \end{cases}$$

Proof. If $0 < q \le \min\{2,p\}$, we obtain

$$\begin{aligned} B^1_{p,q} &\hookrightarrow B^1_{p,\min\{2,p\}} \hookrightarrow \left(L_p, B^1_{p,\min\{2,p\}}\right)_{(1,\alpha),v} \\ &\hookrightarrow \left(L_p, W^1_p\right)_{(1,\alpha),v} = \mathrm{Lip}^{(1,-\alpha)}_{p,v}. \end{aligned}$$

If $\min\{2,p\} < q$, let $\beta = 1/\min\{2,p\}$. By Theorem 5.2 and Proposition 5.4, we derive

$$B^1_{p,q} = B^{1,-\beta+\frac{1}{\min\{2,p,q\}}}_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\beta)}_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,v}. \qquad \square$$

Remark 5.6. Theorem 5.5 confirms a conjecture of Haroske in [28, Remark 12] and closes a problem also mentioned in [29, p. 115] by showing that if $1 < p < \infty$ the embedding $B^1_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,v}$ holds not only for $\alpha = 1/q' + 1/v$ but even for smaller values of α.

Remark 5.7. Note that the argument given in Theorem 5.5 when $0 < q \le \min\{2,p\}$ works for any $\alpha \ge 0$ if $v = \infty$. So when $q = \min\{2,p\}$ we recover a result proved by Neves [35, Proposition 5.6] using different techniques.

Next for $1 \le q < \infty$ we cover a limit case left open in [28, Corollary 23(i)] (see also [29, Corollary 7.20(i)]).

Corollary 5.8. *Let* $1 < p < \infty$, $1 \le q \le \infty$ *and* $0 < \beta = \alpha - 1/q$. *Then*

$$B^{1,-\beta}_{p,1} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,q}.$$

Proof. Using again Theorem 5.2 and Proposition 5.4 we obtain

$$B^{1,-\beta}_{p,1} \hookrightarrow \mathrm{Lip}^{(1,-\beta-1)}_{p,1} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,q}. \qquad \square$$

Proceeding as in Corollary 5.8, we can also derive the embedding

$$B^{1,-\alpha+1/q}_{p,\min\{2,p,q\}} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,q}$$

provided that $1 < p < \infty$, $0 < q \le \infty$ and $\alpha > 1/q$. This improves [28, embedding (29), p. 793] because

$$B^{1,-\alpha+1/q}_{p,\min\{1,q\}} \hookrightarrow B^{1,-\alpha+1/q}_{p,\min\{2,p,q\}}.$$

Note also that from Theorem 5.2 we can recover [28, embeddings (41), p. 796] for $1 < p < \infty$. Besides, Theorem 5.2 also yields that if $\alpha > 1/q$ then

$$\mathrm{Lip}^{(1,-\alpha)}_{p,q} \hookrightarrow B^{1,-\alpha+1/q}_{p,q} \quad \text{if } \max\{2,p\} \le q,$$

and

$$B^{1,-\alpha+1/q}_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,q} \quad \text{if } q \le \min\{2,p\}.$$

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References

[1] C. Bennett, K. Rudnick, On Lorentz–Zygmund spaces, Dissertationes Math. 175 (1980) 1–72.
[2] C. Bennett, R. Sharpley, Interpolation of Operators, Academic Press, Boston, 1988.
[3] J. Bergh, J. Löfström, Interpolation Spaces. An Introduction, Springer, Berlin, 1976.
[4] O.V. Besov, On spaces of functions of smoothness zero, Sb. Math. 203 (2012) 1077–1090.
[5] H. Brézis, S. Wainger, A note on limiting cases of Sobolev embeddings and convolution inequalities, Comm. Partial Differential Equations 5 (1980) 773–789.
[6] Yu.A. Brudnyĭ, N.Ya. Krugljak, Interpolation Functors and Interpolation Spaces, vol. 1, North-Holland, Amsterdam, 1991.
[7] A.M. Caetano, H.-G. Leopold, On generalized Besov and Triebel–Lizorkin spaces of regular distributions, J. Funct. Anal. 264 (2013) 2676–2703.
[8] F. Cobos, Ó. Domínguez, Approximation spaces, limiting interpolation and Besov spaces, J. Approx. Theory 189 (2015) 43–66.
[9] F. Cobos, D.L. Fernandez, Hardy–Sobolev spaces and Besov spaces with a function parameter, in: Function Spaces and Applications, in: Lecture Notes in Math., vol. 1302, Springer, Berlin, 1988, pp. 158–170.
[10] F. Cobos, L.M. Fernández-Cabrera, T. Kühn, T. Ullrich, On an extreme class of real interpolation spaces, J. Funct. Anal. 256 (2009) 2321–2366.
[11] F. Cobos, T. Kühn, Approximation and entropy numbers in Besov spaces of generalized smoothness, J. Approx. Theory 160 (2009) 56–70.
[12] F. Cobos, T. Kühn, Equivalence of K- and J-methods for limiting real interpolation spaces, J. Funct. Anal. 261 (2011) 3696–3722.
[13] F. Cobos, A. Segurado, Some reiteration formulae for limiting real methods, J. Math. Anal. Appl. 411 (2014) 405–421.
[14] F. Cobos, A. Segurado, Description of logarithmic interpolation spaces by means of the J-functional and applications, preprint, Madrid, 2014.
[15] M. Cwikel, J. Peetre, Abstract K and J spaces, J. Math. Pures Appl. 60 (1981) 1–50.
[16] R.A. DeVore, S.D. Riemenschneider, R.C. Sharpley, Weak interpolation in Banach spaces, J. Funct. Anal. 33 (1979) 58–94.
[17] D.E. Edmunds, W.D. Evans, Hardy Operators, Function Spaces and Embeddings, Springer, Berlin, 2004.
[18] D.E. Edmunds, D.D. Haroske, Spaces of Lipschitz type, embeddings and entropy numbers, Dissertationes Math. 380 (1999) 1–43.
[19] D.E. Edmunds, B. Opic, Limiting variants of Krasnosel'skiĭ's compact interpolation theorem, J. Funct. Anal. 266 (2014) 3265–3285.
[20] D.E. Edmunds, H. Triebel, Spectral theory for isotropic fractal drums, C. R. Acad. Sci. Paris 326 (1998) 1269–1274.
[21] D.E. Edmunds, H. Triebel, Eigenfrequencies of isotropic fractal drums, Oper. Theory Adv. Appl. 110 (1999) 81–102.
[22] W.D. Evans, B. Opic, Real interpolation with logarithmic functors and reiteration, Canad. J. Math. 52 (2000) 920–960.
[23] W.D. Evans, B. Opic, L. Pick, Real interpolation with logarithmic functors, J. Inequal. Appl. 7 (2002) 187–269.
[24] W. Farkas, H.-G. Leopold, Characterisations of function spaces of generalised smoothness, Ann. Mat. Pura Appl. 185 (2006) 1–62.
[25] A. Gogatishvili, B. Opic, S. Tikhonov, W. Trebels, Ulyanov-type inequalities between Lorentz–Zygmund spaces, J. Fourier Anal. Appl. 20 (2014) 1020–1049.
[26] M.L. Gol'dman, A covering method for describing general spaces of Besov type, Proc. Steklov Inst. Math. 156 (1983) 51–87, translation from Tr. Mat. Inst. Steklova 156 (1980) 47–81.
[27] M.E. Gomez, M. Milman, Extrapolation spaces and almost-everywhere convergence of singular integrals, J. Lond. Math. Soc. 34 (1986) 305–316.
[28] D.D. Haroske, On more general Lipschitz spaces, Z. Anal. Anwend. 19 (2000) 781–799.
[29] D.D. Haroske, Envelopes and Sharp Embeddings of Function Spaces, Chapman & Hall/CRC Res. Notes Math., vol. 437, Chapman & Hall/CRC, Boca Raton, FL, 2007.
[30] D.D. Haroske, S.D. Moura, Continuity envelopes of spaces of generalized smoothness, entropy and approximation numbers, J. Approx. Theory 128 (2004) 151–174.
[31] T. Holmstedt, Interpolation of quasi-normed spaces, Math. Scand. 26 (1970) 177–199.
[32] G.A. Kalyabin, P.I. Lizorkin, Spaces of functions of generalized smoothness, Math. Nachr. 133 (1987) 7–32.
[33] H.-G. Leopold, Embeddings and entropy numbers in Besov spaces of generalized smoothness, in: Function Spaces, in: Lect. Notes Pure Appl. Math., vol. 213, Marcel Dekker, New York, 2000, pp. 323–336.
[34] C. Merucci, Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces, in: Interpolation Spaces and Allied Topics in Analysis, in: Lecture Notes in Math., vol. 1070, Springer, Berlin, 1984, pp. 183–201.
[35] J.S. Neves, Extrapolation results on general Besov–Hölder–Lipschitz spaces, Math. Nachr. 230 (2001) 117–141.
[36] L.E. Persson, Interpolation with a parameter function, Math. Scand. 59 (1986) 199–222.
[37] C. Schneider, On dilation operators in Besov spaces, Rev. Mat. Complut. 22 (2009) 111–128.
[38] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland, Amsterdam, 1978.

[39] H. Triebel, Theory of Function Spaces, Birkhäuser, Basel, 1983.
[40] H. Triebel, Theory of Function Spaces II, Birkhäuser, Basel, 1992.
[41] H. Triebel, Theory of Function Spaces III, Birkhäuser, Basel, 2006.
[42] H. Triebel, Comments on tractable embeddings and function spaces of smoothness near zero, report, Jena, 2012.