Optimal existence classes and nonlinear–like dynamics in the linear heat equation in $\mathbb{R}^d$

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June 4, 2018

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Abstract

We analyse the behaviour of solutions of the linear heat equation in $\mathbb{R}^d$ for initial data in the classes $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ of Radon measures with $\int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2}\,\mathrm{d}|u_0| < \infty$. We show that these classes are in some sense optimal for local and global existence of non-negative solutions: in particular $\mathcal{M}_0(\mathbb{R}^d) = \cap_{\varepsilon>0}\mathcal{M}_\varepsilon(\mathbb{R}^d)$ consists precisely of those initial data for which the a solution of the heat equation can be given for all time using the heat kernel representation formula. After considering properties of existence, uniqueness, and regularity for such initial data, which can grow rapidly at

*Partially supported by Project MTM2016-75465, MICINN and GR58/08 Grupo 920894, UCM, Spain
†Partially supported by PRX17/00522 Programa Salvador de Madariaga MECyD, Spain, and by an EPSRC grant EP/R023778/1, UK.
[3]*Partially supported by ICMAT Severo Ochoa project SEV-2015-0554 (MINECO)

infinity, we go on to show that they give rise to properties associated more often with nonlinear models. We demonstrate the finite-time blowup of solutions, showing that the set of blowup points is the complement of a convex set, and that given any closed convex set there is an initial condition whose solutions remain bounded precisely on this set at the 'blowup time'. We also show that wild oscillations are possible from non-negative initial data as $t \to \infty$ (in fact we show that this behaviour is generic), and that one can prescribe the behaviour of $u(0,t)$ to be any real-analytic function $\gamma(t)$ on $[0,\infty)$.

1 Introduction

In this paper we consider the linear heat equation posed on the whole space $\mathbb{R}^d$, with very general initial data, which may be either only locally integrable or even a Radon measure. For an appropriate class of initial data u_0, see e.g. [26], it is well known that solutions to this equation,

$$u_t - \Delta u = 0, \; x \in \mathbb{R}^d, \; t > 0, \qquad u(x,0) = u_0(x), \tag{1.1}$$

can be written using the heat kernel as

$$u(x,t) = S(t)u_0(x) := \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/4t} u_0(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d, \; t > 0. \tag{1.2}$$

It turns out that the behaviour of solutions in (1.2) is significantly affected by the way the mass of the initial data is distributed in space.

If the mass as $|x| \to \infty$ is not too large it is well known that the 'mass' of the initial data moves to infinity and the solutions decay to zero in suitable norms. For example, if $u_0 \in L^p(\mathbb{R}^d)$ for some $1 \le p < \infty$ then classical estimates ensure that

$$\|u(t)\|_{L^q(\mathbb{R}^d)} \le (4\pi t)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p(\mathbb{R}^d)}, \quad \text{for every } t > 0 \text{ and } q \text{ with } p \le q \le \infty, \tag{1.3}$$

which in particular implies that all solutions converge uniformly to zero on the whole of $\mathbb{R}^d$. In particular, for $u_0 \in L^1(\mathbb{R}^d)$ since we also have

$$\int_{\mathbb{R}^d} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^d} u_0(y)\,\mathrm{d}y, \quad t > 0,$$

it follows that for such u_0 the total mass is preserved but (from (1.3)) the supremum tends to zero, i.e. the mass moves to infinity.

It is also known that as $t \to \infty$, solutions asymptotically resemble the heat kernel

$$K(x,t) = (4\pi t)^{-d/2} \mathrm{e}^{-|x|^2/4t},$$

see for example Section 1.1.4 in [11]. The faster the initial data decays as $|x| \to \infty$ the higher the order of the asymptotics of the solution that are described by the heat kernel, see e.g. [9].

When the initial data is bounded, $u_0 \in L^\infty(\mathbb{R}^d)$, the decay described above does not necessarily take place. In fact (1.3) reduces to

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}, \quad t > 0,$$

which does not in general imply any decay. For example, if $u_0 \equiv 1$ then $u(x,t) = 1$ for every $t > 1$; for any $R > 0$ we can write

$$1 = u(t, \mathcal{X}_{B(0,R)}) + u(t, \mathcal{X}_{\mathbb{R}^d \setminus B(0,R)}),$$

where $\mathcal{X}_A$ denotes the characteristic function of the set A. Since $\mathcal{X}_{B(0,R)} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ the mass of $0 \le u(t, \mathcal{X}_{B(0,R)})$ escapes to infinity but, on the other hand, the mass of $\mathcal{X}_{\mathbb{R}^d \setminus B(0,R)}$, diffused by $u(t, \mathcal{X}_{\mathbb{R}^d \setminus B(0,R)})$, moves 'inwards' from infinity and both balance precisely at every time.

Hence, it turns out the dynamics of the solutions (1.2) of the heat equation (1.1) for bounded initial data is much richer than for initial data with small mass at infinity. For example, the existence of one-dimensional bounded oscillations was proved in Section 8 in [6], while bounded 'wild' oscillations in any dimensions were shown to exist in [27] by a scaling method. It is worth noting that this scaling argument is also applied in [27] to some nonlinear equations (porous medium, p-Laplacian, and scalar conservation laws). Indeed, this scaling argument allows one to show that for $L^p(\mathbb{R}^d)$ initial data, $1 \le p < \infty$, the solution of (1.1) asymptotically approaches the heat kernel. The scaling argument was later extended to some nonlinear dissipative reaction diffusion equations in [5].

In this paper our goal is to consider some (optimal) classes of unbounded data that possess large mass at infinity. In such a situation we show how the mechanism of mass moving inwards from infinity plays a dominant role on the structure and properties of solutions of (1.1). It turns out that in this setting, solutions of (1.1) show surprising dynamical behaviours more akin to what is expected in nonlinear equations.

For example, in our class of 'large' initial data finite-time blowup is possible. We completely characterise (non-negative) initial data for which the solution ceases to exist in some finite time; we determine the maximal existence time and characterise the blow-up points, which are the complement of a convex set. Hence we are able to construct non-negative initial data for which the solution exhibits regional, or complete blow–up. One can even find solutions with a finite pointwise limit at every point in $\mathbb{R}^d$ at the maximal existence time, but that can not be continued beyond this maximal time ('finite existence time without blowup'). In particular, we prove that given any closed convex set in $\mathbb{R}^d$, there exists an initial condition such that the solution remains bounded at the maximal existence time precisely on this set. Observe that most of this behaviour is characteristic of nonlinear non-dissipative problems, see e.g. [20]. Our analysis includes and extends the classical example $u_0(x) = \mathrm{e}^{A|x|^2}$, with $A > 0$ for which the solution is given by

$$u(x,t) = \frac{T^{d/2}}{(T-t)^{d/2}} \mathrm{e}^{\frac{|x|^2}{4(T-t)}},$$

with $T = \frac{1}{4A}$ which blows up at every point $x \in \mathbb{R}^d$ as $t \to T$.

For those solutions that exist globally in time we characterise those that are unbounded and also construct (non-negative) initial data such that the solution displays wild unbounded oscillations (cf. [27]). For this, given any sequence of nonnegative numbers $\{\alpha_k\}_k$ we construct initial data such that there exists a sequence of times $t_k \to \infty$ such that for any $k \in \mathbb{N}$ there exists a subsequence $\{t_{k_j}\}_j$ such that

$$u(0, t_{k_j}) \to \alpha_k \quad \text{as} \quad j \to \infty.$$

We also show that this oscillatory behaviour is generic within a suitable (optimal) class of solutions. Notice that unbounded oscillatory behaviour is an outstanding feature of some nonlinear non-dissipative equations, see, for example, Theorem 6.2 in [21] where some solutions are shown to satisfy

$$\liminf_{t\to\infty} \|u(t;u_0)\|_{L^\infty(\mathbb{R}^d)} = 0 \quad \text{and} \quad \limsup_{t\to\infty} \|u(t;u_0)\|_{L^\infty(\mathbb{R}^d)} = \infty.$$

All the nonlinear-like behaviour described above is caused by the large mass of the initial data at infinity that is diffused by the solution of the heat equation and is moved inwards bounded regions in $\mathbb{R}^d$, so that its effect is felt at later times.

Throughout the paper our analysis is based on the following spaces: we define the subclass $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ of Radon measures $\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ by setting

$$\mathcal{M}_\varepsilon(\mathbb{R}^d) := \left\{ \mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2} \, \mathrm{d}|\mu(x)| < \infty \right\};$$

where $|\mu|$ denotes the total variation of μ, with the norm

$$\|\mu\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} := \left(\frac{\varepsilon}{\pi}\right)^{d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2} \, \mathrm{d}|\mu(x)|;$$

i.e. $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ consists of Radon measures for which $\mathrm{e}^{-\varepsilon|x|^2} \in L^1(\mathrm{d}|\mu|)$ and is a Banach space. [This set of measures was briefly mentioned in [2], which considered only non-negative weak solutions of parabolic problems.] Since any locally integrable function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ defines the Radon measure $f \, \mathrm{d}x \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ the class above contains

$$L^1_\varepsilon(\mathbb{R}^d) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2} |f(x)| \, \mathrm{d}x < \infty \right\}.$$

These classes turn out to be optimal in several ways for non-negative solutions (1.2) of (1.1) which are now given by

$$u(x,t) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/4t} \, \mathrm{d}u_0(y). \tag{1.4}$$

First an initial condition in $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ gives rise to a (classical) solution of (1.1) defined for $0 < t < T(\varepsilon) = \frac{1}{4\varepsilon}$. Conversely for any non-negative solution (1.2) of (1.1) that is finite at some (x,t) then the initial data must belong to $\mathcal{M}_{1/4t}(\mathbb{R}^d)$. As a consequence a non-negative initial condition in $\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ gives rise to a globally defined solution if and only if it belongs to

$$\mathcal{M}_0(\mathbb{R}^d) := \bigcap_{\varepsilon>0} \mathcal{M}_\varepsilon(\mathbb{R}^d).$$

Within this class of initial data we also show that a non-negative solution is bounded for some $t_0 > 0$ (and hence for all $t > 0$) if and only if the initial data is a uniform measure in the sense that

$$\sup_{x\in\mathbb{R}^d} \int_{B(x,1)} \mathrm{d}|u_0(y)| < \infty.$$

Finally we show that a non-negative solution is bounded on sets of the form $|x|^2/t \le R$, with $R > 0$, if and only if

$$\sup_{\varepsilon>0} \|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} < \infty.$$

In contrast, if

$$\varepsilon_0(u_0) = \inf\{\varepsilon > 0 : \ 0 \le u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)\} > 0$$

then the solution will exists only up to $T = T(u_0) = \frac{1}{4\varepsilon_0}$ and cannot be continued beyond this time at any point. The points x at which the solution has a finite limit as $t \to T$ are characterised by a condition on the translated measure, namely $\tau_{-x}u_0 \in \mathcal{M}_{\varepsilon_0}(\mathbb{R}^d)$, and they must form a convex set. Conversely, as mentioned above, at any chosen closed convex subset of $\mathbb{R}^d$, there exist some $u_0 \ge 0$ such that the limit as $t \to T$ of the solution is finite precisely at this set. In particular there are initial conditions such that $\lim_{t\to T} u(x,t) < \infty$ for every $x \in \mathbb{R}^d$ but the solution cannot be defined past time T.

Large initial data can also exhibit other unusual properties not normally associated with the heat equation. For example, observe that for any $\omega \in \mathbb{R}^d$ the function $\varphi(x) = \mathrm{e}^{\omega x} \in L^1_0(\mathbb{R}^d) := \bigcap_{\varepsilon>0} L^1_\varepsilon(\mathbb{R}^d)$ satisfies $-\Delta\varphi = -|\omega|^2\varphi$, while $\phi(x) = \mathrm{e}^{\mathrm{i}\omega x} \in L^1_0(\mathbb{R}^d)$ satisfies $-\Delta\phi = |\omega|^2\phi$. It follows that the spectrum of the Laplacian satisfies in this setting is the whole of $\mathbb{R}$,

$$\sigma_{L^1_0(\mathbb{R}^d)}(-\Delta) = \mathbb{R},$$

and that for any $\omega \in \mathbb{R}^d$ the function

$$u(x,t) = \mathrm{e}^{|\omega|^2 t + \omega x} \quad x \in \mathbb{R}^d, \ t > 0$$

is a globally-defined solution of (1.1) in $L^1_0(\mathbb{R}^d)$; the exponential growth rate of such solutions can be arbitrarily large.

The paper is organized as follows. In Section 2 we recall some basic properties of Radon measures. In Section 3 we show that for an initial condition in $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ the integral expression (1.4) defines a classical solution of the heat equation that attains the initial data in the sense of measures. Conversely, we show that if (1.4) is finite at some (x,t) for some non–negative measure u_0, then it must be in some $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ space. In Section 4 we tackle the problem of uniqueness. In Section 5 we discuss and characterise the non-negative solutions that cease to exist in finite time, determining both the blow-up time T and the points at which the solution has a finite limit as $t \to$T. In Section 6 we discuss the long-time behaviour of global solutions showing, in particular, wild unbounded oscillations for some initial data; we show that this behaviour is generic (in an appropriate sense). Allowing for sign-changing solutions we also show there how to obtain solutions with any prescribed behavior in time at $x = 0$. Finally, in Section 7, we briefly discuss other problems that can be dealt with the same techniques. Appendix A contains some required technical results.

2 Radon measures on $\mathbb{R}^d$

In this section we will recall some basic results on Radon measures that will be used throughout the rest of the paper; details can be found in [4, 10, 12, 14]. A Radon measure in $\mathbb{R}^d$ is a regular Borel measure assigning finite measure to each compact set. The set of all Radon measures in $\mathbb{R}^d$ is denoted $\mathcal{M}_{\rm loc}(\mathbb{R}^d)$.

Radon measures arise as the natural representation of linear functionals on the set $C_c(\mathbb{R}^d)$ of real-valued functions of compact support in two distinct settings.

Theorem 2.1. *If $L\colon C_c(\mathbb{R}^d) \to \mathbb{R}$ is linear and positive, i.e. $L(\varphi) \geq 0$ for $0 \leq \varphi \in C_c(\mathbb{R}^d)$, then there exists a (unique) non-negative Radon measure $\mu \in \mathcal{M}_{\rm loc}(\mathbb{R}^d)$ such that*

$$L(\varphi) = \int_{\mathbb{R}^d} \varphi \, \mathrm{d}\mu \qquad \textit{for every} \quad \varphi \in C_c(\mathbb{R}^d).$$

A similar result holds if positivity is replaced by continuity, in the following sense: we equip $C_c(\mathbb{R}^d)$ with the final (linear) topology associated with the inclusions

$$C_c(K) \hookrightarrow C_c(\mathbb{R}^d), \qquad K \subset\subset \mathbb{R}^d$$

where, for each compact set $K \subset \mathbb{R}^d$ we consider the sup norm in $C_c(K)$. More concretely, a sequence $\{\varphi_j\}_j$ in $C_c(\mathbb{R}^d)$ converges to $\varphi \in C_c(\mathbb{R}^d)$, iff there exists a compact $K \subset \mathbb{R}^d$ such that $\mathrm{supp}(\varphi_j) \subset K$ for all $j \in \mathbb{N}$ and $\varphi_j \to \varphi$ uniformly in K. A linear map

$L\colon C_c(\mathbb{R}^N) \to \mathbb{R}$ is then continuous if for every compact set $K \subset \mathbb{R}^d$ there exists a constant C_K such that for every $\varphi \in C_c(\mathbb{R}^d)$ with support in K

$$|L(\varphi)| \le C_K \sup_{x\in K} |\varphi(x)|.$$

Theorem 2.2. *If $L\colon C_c(\mathbb{R}^d) \to \mathbb{R}$ is linear and continuous (in the sense described above) then there exists a (unique, signed) Radon measure $\mu \in \mathcal{M}_{\rm loc}(\mathbb{R}^d)$ such that*

$$L(\varphi) = \int_{\mathbb{R}^d} \varphi \,\mathrm{d}\mu \qquad \textit{for every} \quad \varphi \in C_c(\mathbb{R}^d). \tag{2.1}$$

As a consequence of this second theorem the set of Radon measures can be characterised as the dual space of $C_c(\mathbb{R}^d)$,

$$\mathcal{M}_{\rm loc}(\mathbb{R}^d) = \Big(C_c(\mathbb{R}^d)\Big)',$$

and we typically identify $L \in (C_c(\mathbb{R}^d))'$ with the corresponding Radon measure μ from (2.1). In this way we can write

$$\langle \mu, \varphi\rangle = \int_{\mathbb{R}^d} \varphi \,\mathrm{d}\mu \qquad \text{for every} \quad \varphi \in C_c(\mathbb{R}^d).$$

Notice that, in particular,

$$L^1_{\rm loc}(\mathbb{R}^d) \subset \mathcal{M}_{\rm loc}(\mathbb{R}^d)$$

as we identify $f \in L^1_{\rm loc}(\mathbb{R}^d)$ with the measure $f\,\mathrm{d}x \in \mathcal{M}_{\rm loc}(\mathbb{R}^d)$.

Any Radon measure $\mu \in \mathcal{M}_{\rm loc}(\mathbb{R}^d)$ can be (uniquely) split as the difference of two non-negative, mutually singular, Radon measures $\mu = \mu^+ - \mu^-$ (the 'Jordan decomposition' of μ). Then we can define the Radon measure $|\mu|$, the 'total variation of μ', by setting

$$|\mu| := \mu^+ + \mu^-.$$

Then for every $\varphi \in C_c(\mathbb{R}^d)$ and $\mu \in \mathcal{M}_{\rm loc}(\mathbb{R}^d)$ we have

$$\left|\int_{\mathbb{R}^d} \varphi \,\mathrm{d}\mu\right| \le \int_{\mathbb{R}^d} |\varphi| \,\mathrm{d}|\mu|. \tag{2.2}$$

Finally we recall the definition of measures of bounded total variation. Consider the space $C_0(\mathbb{R}^d)$ of continuous functions converging to 0 as $|x| \to \infty$ with the sup norm ($C_c(\mathbb{R}^d)$ is dense in this space).

Theorem 2.3. *A linear mapping $L\colon C_0(\mathbb{R}^d) \to \mathbb{R}$ is continuous, iff there exists a (signed) Radon measure $\mu \in \mathcal{M}_{\rm loc}(\mathbb{R}^d)$ such that $|\mu|(\mathbb{R}^d) < \infty$ and*

$$L(\varphi) = \int_{\mathbb{R}^d} \varphi \,\mathrm{d}\mu \qquad \textit{for every} \quad \varphi \in C_0(\mathbb{R}^d).$$

The quantity $\|\mu\|_{\mathrm{BTV}} = |\mu|(\mathbb{R}^d)$ is the *total variation* of μ and is the norm of the functional L. In other words

$$\mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^d) = \Big(C_0(\mathbb{R}^d)\Big)'$$

is the Banach space of Radon measures with bounded total variation. It is then immediate that $L^1(\mathbb{R}^d) \subset \mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^d)$, isometrically, and $\mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^d) \subset \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$. We discuss solutions of the heat equation with initial data in $\mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^d)$ in Lemma 5.3.

Note that the set of Radon measures is therefore distinct from the class of tempered distributions on $\mathbb{R}^d$, which are continuous linear functionals on the Schwarz class $\mathscr{S}(\mathbb{R}^{\mathrm{d}})$: such functions are smoother than functions in $C_c(\mathbb{R}^{\mathrm{d}})$ but satisfy less stringent growth conditions, so neither class is contained in the other. Recall that $\mathscr{S}(\mathbb{R}^d)$ is made up of $C^\infty(\mathbb{R}^d)$ functions such that for all multi-indices α, β

$$|x^\alpha||D^\beta \varphi(x)| \to 0 \quad \text{as} \quad |x| \to \infty.$$

The family of seminorms

$$p_{\alpha,\beta}(\varphi) = \sup_{x\in\mathbb{R}^d} (1 + |x^\alpha|)|D^\beta \varphi(x)|$$

defines a locally-convex topology on $\mathscr{S}(\mathbb{R}^d)$, and the tempered distributions are the dual space $\mathscr{S}'(\mathbb{R}^{\mathrm{d}})$.

A tempered distribution $L \in \mathscr{S}'(\mathbb{R}^d)$ has order $(m, n) \in \mathbb{N} \times \mathbb{N}$ if for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and some constant $c > 0$

$$|\langle L, \varphi\rangle| \le c p_{\alpha,\beta}(\varphi)$$

with $|\alpha| = m$ and $|\beta| = n$.

Since $(1 + x^\alpha)\varphi(x) \in \mathscr{S}(\mathbb{R}^d)$ for every $\varphi \in \mathscr{S}(\mathbb{R}^{\mathrm{d}})$ and multi-index α and $\mathscr{S}(\mathbb{R}^{\mathrm{d}})$ is dense in $C_0(\mathbb{R}^{\mathrm{d}})$, it follows that if $L \in \mathscr{S}'(\mathbb{R}^d)$ has order $(m, 0)$ then $(1+|x|^2)^{-m/2}L$ is an element of $\mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^d)$. That is, L can be identified with a measure $\mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (1 + |x|^2)^{-m/2}\, \mathrm{d}|\mu(x)| < \infty, \tag{2.3}$$

since

$$|\langle L, \varphi\rangle| = |\langle (1 + |x|^2)^{-m/2}L, (1 + |x|^2)^{m/2}\varphi\rangle| \le c p_{\alpha,0}(\varphi) \le c \sup_{x\in\mathbb{R}} |\xi(x)|,$$

with $\xi(x) = (1 + |x|^2)^{m/2}\varphi(x) \in C_0(\mathbb{R}^{\mathrm{d}})$ and $|\alpha| = m$.

Let us denote by $\mathscr{C}_m(\mathbb{R}^{\mathrm{d}})$ the collection of all measures μ that satisfy (2.3). Then any such μ defines a tempered distribution of order $(m, 0)$ since for any $\varphi \in \mathscr{S}(\mathbb{R}^{\mathrm{d}})$ we have

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \varphi(x)\, d\mu(x)\right| &= \left|\int_{\mathbb{R}^d} (1 + |x|^2)^{m/2}\varphi(x)\, (1 + |x|^2)^{-m/2}\, \mathrm{d}\mu(x)\right| \\ &\le p_{\alpha,0}(\varphi) \int_{\mathbb{R}^d} (1 + |x|^2)^{-m/2}\, \mathrm{d}|\mu(x)|\end{aligned}$$

with $|\alpha| = m$. Hence $\mathscr{C}_m(\mathbb{R}^{\mathrm{d}})$ is precisely the class of tempered distributions of order $(m, 0)$.

3 Initial data in $\mathcal{M}_\varepsilon(\mathbb{R}^d)$: existence and regularity

Throughout this paper we consider the Cauchy problem

$$u_t - \Delta u = 0,\ x \in \mathbb{R}^d,\ t > 0, \qquad u(x,0) = u_0(x), \tag{3.1}$$

whose solutions we expect to be given in terms of the heat kernel by

$$u(x,t;u_0) = S(t)u_0(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/4t} u_0(y)\,\mathrm{d}y,$$

if $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ or, more generally, if $u_0 \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ is a Radon measure, by

$$u(x,t;u_0) = S(t)u_0(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/4t}\,\mathrm{d}u_0(y). \tag{3.2}$$

Of course, it is entirely natural to consider sets of measures as initial conditions for the heat equation, since the heat kernel, which is smooth for all $t > 0$, is precisely the solution when u_0 is the δ measure.

Notice that from (3.2) and (2.2) we immediately obtain

$$|S(t)u_0| \le S(t)|u_0|, \qquad t > 0, \quad u_0 \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d).$$

We start with some estimates for the expression in (3.2) which show that the solution can be essentially estimated by its value at $x = 0$.

Lemma 3.1. *If $u_0 \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ and $u(x,t)$ is given by (3.2) then for any $a > 1$ we have*

$$|u(x,t,u_0)| \le c_{d,a}\, u(0,at,|u_0|)\, \mathrm{e}^{\frac{|x|^2}{4(a-1)t}} \qquad \textit{for all} \quad x \in \mathbb{R}^d,\ t > 0, \tag{3.3}$$

where $c_{d,z} := z^{d/2}$ for any $z > 0$.

If in addition $0 \le u_0 \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ then for any $0 < b < 1 < a$ we have

$$c_{d,b}\, u(0,bt)\, \mathrm{e}^{-\frac{|x|^2}{4(1-b)t}} \le u(x,t) \le c_{d,a}\, u(0,at)\, \mathrm{e}^{\frac{|x|^2}{4(a-1)t}} \qquad \textit{for all} \quad x \in \mathbb{R}^d,\ t > 0. \tag{3.4}$$

Proof. For the upper bound we use the fact that for any $0 < \delta < 1$,

$$|x-y|^2 \ge |y|^2 + |x|^2 - 2|y||x| \ge (1-\delta)|y|^2 + (1 - \frac{1}{\delta})|x|^2, \tag{3.5}$$

from which it follows that

$$|u(x,t,u_0)| \le \mathrm{e}^{(\frac{1}{\delta}-1)\frac{|x|^2}{4t}} \left(\frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-(1-\delta)\frac{|y|^2}{4t}}\, \mathrm{d}|u_0(y)| \right);$$

taking $a = \frac{1}{1-\delta} > 1$ yields (3.3).

For the lower bound when $u_0 \ge 0$, we argue similarly, now using the fact that for any $\delta > 0$,

$$|x-y|^2 \le |x|^2 + |y|^2 + 2|x||y| \le (1+\delta)|x|^2 + (1+\frac{1}{\delta})|y|^2;$$

we obtain

$$u(x,t) \ge \mathrm{e}^{-(1+\delta)\frac{|x|^2}{4t}} \left(\frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-(1+\frac{1}{\delta})\frac{|y|^2}{4t}}\, \mathrm{d}u_0(y) \right)$$

and then take $b = \frac{\delta}{1+\delta} < 1$. $\square$

We now introduce some classes of initial data that are particularly suited to an analysis of solutions of the heat equation: for $\varepsilon > 0$ we define

$$L^1_\varepsilon(\mathbb{R}^d) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2} |f(x)|\, \mathrm{d}x < \infty \right\}; \tag{3.6}$$

with the norm

$$\|f\|_{L^1_\varepsilon(\mathbb{R}^d)} := \left(\frac{\varepsilon}{\pi}\right)^{d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2} |f(x)|\, \mathrm{d}x \tag{3.7}$$

for which a positive constant function has norm equal to itself. For the case of measures for $\varepsilon > 0$ we define

$$\mathcal{M}_\varepsilon(\mathbb{R}^d) := \left\{ \mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2}\, \mathrm{d}|\mu(x)| < \infty \right\}; \tag{3.8}$$

i.e. $\mathrm{e}^{-\varepsilon|x|^2} \in L^1(d|\mu|)$, with the norm

$$\|\mu\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} := \left(\frac{\varepsilon}{\pi}\right)^{d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2}\, \mathrm{d}|\mu(x)|. \tag{3.9}$$

Obviously $L^1_\varepsilon(\mathbb{R}^d) \subset \mathcal{M}_\varepsilon(\mathbb{R}^d)$ isometrically, that is, if $f \in L^1_\varepsilon(\mathbb{R}^d)$ then $\|f\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} = \|f\|_{L^1_\varepsilon(\mathbb{R}^d)}$. Also note that $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ and $L^1_\varepsilon(\mathbb{R}^d)$ are increasing in $\varepsilon > 0$ and if $\varepsilon_1 < \varepsilon_2$ then for $\mu \in \mathcal{M}_{\varepsilon_1}(\mathbb{R}^d)$

$$\|\mu\|_{\mathcal{M}_{\varepsilon_2}(\mathbb{R}^d)} \le \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{d/2} \|\mu\|_{\mathcal{M}_{\varepsilon_1}(\mathbb{R}^d)}. \tag{3.10}$$

Finally $L^1_\varepsilon(\mathbb{R}^d)$ and $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ with the norms (3.7) and (3.8) respectively, are Banach spaces, see Lemma A.1.

The following simple lemma demonstrates the relevance of the spaces $L^1_\varepsilon(\mathbb{R}^d)$ and $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ to the heat equation. Note that the first part of the statement does not require that u_0 is non-negative. We will improve on the first part of this lemma in Proposition 3.7, obtaining bounds on $u(t)$ in the norm of $L^1_{\varepsilon(t)}(\mathbb{R}^d)$.

Lemma 3.2. *Let $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$, set $T(\varepsilon) = 1/4\varepsilon$, and let $u(x,t)$ be given by (3.2). Then for each $t \in (0, T(\varepsilon))$ we have $u(t) \in L^1_\delta(\mathbb{R}^d)$ for any $\delta > \varepsilon(t) := \frac{1}{4(T(\varepsilon)-t)} = \frac{\varepsilon}{1-4\varepsilon t}$.*

Conversely, if $0 \le u_0 \in \mathcal{M}_{\rm loc}(\mathbb{R}^d)$ and $u(x,t) < \infty$ for some $x \in \mathbb{R}^d$, $t > 0$ then

$$u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d) \quad \text{for every} \quad \varepsilon > 1/4t.$$

Proof. Taking $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ we use the upper bound (3.3) from Lemma 3.1 to obtain

$$\int_{\mathbb{R}^d} \mathrm{e}^{-\delta|x|^2} |u(x,t)| \, \mathrm{d}x \le c_{d,a} \, u(0, at, |u_0|) \int_{\mathbb{R}^d} \mathrm{e}^{-(\delta - \frac{1}{4(a-1)t})|x|^2} \, \mathrm{d}x,$$

where we choose any $1 < a < T(\varepsilon)/t$. Given such a choice of a, to ensure that the integral is finite we require $\delta > \frac{1}{4(a-1)t}$. Noting that the right-hand side of this expression can be made arbitrarily close to $\frac{1}{4(T(\varepsilon)-t)}$ it follows that $u(t) \in L^1_\delta(\mathbb{R}^d)$ for any $\delta > \varepsilon(t) := 1/4(T(\varepsilon) - t) = \varepsilon/(1 - 4\varepsilon t)$, as claimed.

Conversely, from the lower bound in (3.4), if $0 \le u(x,t) < \infty$ for some $x \in \mathbb{R}^d$, $t > 0$ then for any $0 < b < 1$

$$u(0, bt) = \frac{1}{(4\pi bt)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|y|^2/4bt} \, \mathrm{d}u_0(y) < \infty,$$

i.e. $u_0 \in \mathcal{M}_{1/4bt}(\mathbb{R}^d)$. Since we can take any $0 < b < 1$, it follows that $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ for any $\varepsilon > 1/4t$. □

We reserve the notation $T(\varepsilon)$ and $\varepsilon(t)$ in what follows for the functions defined in the statement of this lemma; for the latter this is something of an abuse of notation, since $\varepsilon(t)$ is really a function that depends on a particular choice of ε (as well as t):

$$T(\varepsilon) = \frac{1}{4\varepsilon} \qquad \text{and} \qquad \varepsilon(t) := \frac{1}{4(T(\varepsilon) - t)} = \frac{\varepsilon}{1 - 4\varepsilon t}, \quad 0 \le t < T(\varepsilon). \tag{3.11}$$

At something of an opposite extreme, the following lemma - which we will require many times in what follows - allows us to capture some of the ways in which any solution starting from a continuous function with compact support retains a trace of its initial data; more or less it satisfies the same decay as the heat kernel, $\sim t^{-d/2} \mathrm{e}^{-|x|^2/4t}$.

Lemma 3.3. *If $\varphi \in C_c(\mathbb{R}^d)$ with $\operatorname{supp} \varphi \subset B(0,R)$ then for any $0 < \delta < 1$ and $t > 0$*

(i) $$|S(t)\varphi(x)| \le \begin{cases} C_\varphi(t) \mathrm{e}^{-\gamma(t)|x|^2} & |x| \ge 2R/\delta \\ \|\varphi\|_{L^\infty(\mathbb{R}^d)} & |x| \le 2R/\delta, \end{cases}$$

where

$$C_\varphi(t) = \frac{\mathrm{e}^{-3(1-\delta)R^2/4\delta t}}{(4\pi t)^{d/2}} \|\varphi\|_{L^1(\mathbb{R}^d)} \qquad \text{and} \qquad \gamma(t) = \frac{(1-\delta)^2}{4t}.$$

(ii) $|S(t)\varphi(x)-\varphi(x)| \leq \begin{cases} C_\varphi(t)\mathrm{e}^{-\gamma(t)|x|^2} & |x| \geq 2R/\delta \\ \tilde{C}_\varphi(t) & |x| \leq 2R/\delta, \end{cases}$

with $C_\varphi(t)$ *and* $\gamma(t)$ *as above and* $\tilde{C}_\varphi(t) \to 0$ *as* $t \to 0$.

(iii) *In particular, for any* $\varepsilon > 0$ *and* $0 < T < T(\varepsilon) = \frac{1}{4\varepsilon}$ *there exists* $\gamma = \gamma(T, \varepsilon) > 0$ *such that*

$$\mathrm{e}^{\varepsilon|x|^2}|S(t)\varphi(x)| \leq C_{T,\varphi,\varepsilon}\mathrm{e}^{-\gamma|x|^2}, \quad x \in \mathbb{R}^d \quad \textit{for every } t \in [0, T].$$

In addition,

$$\mathrm{e}^{\varepsilon|x|^2}\big(S(t)\varphi(x)-\varphi(x)\big) \to 0 \quad \textit{uniformly in } \mathbb{R}^d \quad \textit{as} \quad t \to 0.$$

Proof. For any $\varphi \in C_c(\mathbb{R}^d)$ with support in the ball $B(0,R)$, we have

$$|S(t)\varphi(x)| \leq S(t)|\varphi(x)| = \frac{1}{(4\pi t)^{d/2}} \int_{B(0,R)} \mathrm{e}^{-\frac{|x-y|^2}{4t}} |\varphi(y)|\,\mathrm{d}y; \tag{3.12}$$

using again

$$|x-y|^2 \geq (1-\delta)|x|^2 - \left(\frac{1}{\delta}-1\right)|y|^2 \geq (1-\delta)|x|^2 - \left(\frac{1}{\delta}-1\right)R^2$$

for any $0 < \delta < 1$, it follows that

$$0 \leq S(t)|\varphi(x)| \leq \frac{\mathrm{e}^{-(1-\delta)\frac{|x|^2}{4t}+(\frac{1}{\delta}-1)\frac{R^2}{4t}}}{(4\pi t)^{d/2}} \int_{B(0,R)} |\varphi(y)|\,\mathrm{d}y = \frac{\mathrm{e}^{-\frac{(1-\delta)}{4t}(|x|^2-\frac{R^2}{\delta})}}{(4\pi t)^{d/2}} I(\varphi), \tag{3.13}$$

where $I(\varphi) = \|\varphi\|_{L^1(\mathbb{R}^d)}$.

Now note that

$$|x|^2 - \frac{R^2}{\delta} \geq (1-\delta)|x|^2 + \frac{3R^2}{\delta}$$

if $|x| \geq 2R/\delta$, and hence for any such x we obtain

$$0 \leq |S(t)\varphi(x)| \leq \frac{\mathrm{e}^{-3(1-\delta)R^2/4\delta t}}{(4\pi t)^{d/2}} \mathrm{e}^{-\frac{(1-\delta)^2}{4t}|x|^2} I(\varphi).$$

Since also $\|S(t)\varphi\|_{L^\infty(\mathbb{R}^d)} \leq \|\varphi\|_{L^\infty(\mathbb{R}^d)}$ for all $t \geq 0$, we get part (i).

Now, observe that for $|x| \geq 2R/\delta$ we get the same upper bound for $|S(t)\varphi(x)-\varphi(x)|$ as above and since as $\varphi \in \mathrm{BUC}(\mathbb{R}^d)$ we know from e.g. [18, 15, 17] that $S(t)\varphi - \varphi \to 0$ uniformly in $\mathbb{R}^d$ as $t \to 0$. Hence we get part (ii).

Now fix $\varepsilon > 0$ and $0 < T < T(\varepsilon)$; we choose $0 < \delta < 1$ such that

$$\gamma := \frac{(1-\delta)^2}{4T} - \varepsilon > 0,$$

i.e. so that for all $0 \le t \le T$ we have $(1-\delta)^2/4t \ge (1-\delta)^2/4T = \varepsilon + \gamma$; note that γ and δ can be chosen explicitly in such a way that they depend only on T and ε. Then parts (i) and (ii) give part (iii). □

Notice that in particular if $u_0 \in \mathcal{M}_\varepsilon$ and φ is as in the previous lemma then

$$\int_{\mathbb{R}^d} S(t)\varphi \, \mathrm{d}u_0 = \int_{\mathbb{R}^d} \mathrm{e}^{\varepsilon|x|^2} S(t)\varphi(x) \, \mathrm{e}^{-\varepsilon|x|^2} \, \mathrm{d}u_0(x) \tag{3.14}$$

is well defined for all $0 \le t \le T < T(\varepsilon)$.

The next preparatory result shows that the solution of the heat equation for an initial condition that decays like a quadratic exponential preserves this sort of decay, but with a rate that degrades in time.

Lemma 3.4. *If $\varphi \in C_0(\mathbb{R}^d)$ with $|\varphi(x)| \le A\mathrm{e}^{-\gamma|x|^2}$, $x \in \mathbb{R}^d$, then $u(t) = S(t)\varphi$ satisfies*

$$|u(x,t)| \le \frac{A}{(1+4\pi\gamma t)^{d/2}} \mathrm{e}^{-\frac{\gamma}{1+4\gamma t}|x|^2}, \qquad x \in \mathbb{R}^d, \quad t > 0.$$

Proof. Note that completing the square yields

$$\frac{|x-y|^2}{4t} + \gamma|y|^2 = \frac{1+4\gamma t}{4t}\left|y - \frac{1}{1+4\gamma t}x\right|^2 + \frac{\gamma|x|^2}{1+4\gamma t}$$

and then

$$|u(x,t)| \le \frac{A}{(4\pi t)^{d/2}} \mathrm{e}^{-\frac{\gamma|x|^2}{1+4\gamma t}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{1+4\gamma t}{4t}\left|y - \frac{1}{1+4\gamma t}x\right|^2} \, \mathrm{d}y = \frac{A}{(4\pi t)^{d/2}} \mathrm{e}^{-\frac{\gamma|x|^2}{1+4\gamma t}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{1+4\gamma t}{4t}|y|^2} \, \mathrm{d}y$$

and the estimate follows. □

As a consequence, for any $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ and φ that decays sufficiently fast, u_0 and $S(t)\varphi$ can be integrated against each other for some time, see (3.14). In fact the following symmetry property holds.

Lemma 3.5. *Assume that $\mu \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ and $\phi \in C_0(\mathbb{R}^d)$ is such that $|\phi(x)| \le A\mathrm{e}^{-\gamma|x|^2}$, $x \in \mathbb{R}^d$ with $\gamma > \varepsilon$.*

Then for every $0 < t < T(\varepsilon) - T(\gamma) = \frac{1}{4\varepsilon} - \frac{1}{4\gamma}$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x-y,t)|\phi(x)| \; \mathrm{d}|\mu(y)| \le (4\varepsilon t)^{-d/2} \|\mu\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} \int_{\mathbb{R}^d} \mathrm{e}^{\varepsilon(t)|x|^2} |\phi(x)| \, \mathrm{d}x.$$

where $K(x,t) = (4\pi t)^{-d/2}\mathrm{e}^{-\frac{|x|^2}{4t}}$ is the heat kernel and $\varepsilon(t) = \frac{1}{4(T(\varepsilon)-t)} = \frac{\varepsilon}{1-4\varepsilon t}$.

In particular, for $0 < t < T(\varepsilon) - T(\gamma) = \frac{1}{4\varepsilon} - \frac{1}{4\gamma}$

$$\int_{\mathbb{R}^d} \phi \, S(t)\mu = \int_{\mathbb{R}^d} S(t)\phi \, \mathrm{d}\mu. \tag{3.15}$$

Proof. Notice that

$$I = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x-y,t)|\phi(x)|\,\mathrm{d}x\,\mathrm{d}|\mu(y)| = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x-y,t)\mathrm{e}^{\varepsilon|y|^2}|\phi(x)|\mathrm{e}^{-\varepsilon|y|^2}\,\mathrm{d}x\,\mathrm{d}|\mu(y)|$$

and completing the square

$$\frac{|x-y|^2}{4t} - \varepsilon|y|^2 = \frac{1-4\varepsilon t}{4t}\left|y - \frac{1}{1-4\varepsilon t}x\right|^2 - \frac{\varepsilon|x|^2}{1-4\varepsilon t}.$$

Hence

$$\begin{aligned} I &\le (4\pi t)^{-d/2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{1-4\varepsilon t}{4t}\left|y-\frac{1}{1-4\varepsilon t}x\right|^2}\mathrm{e}^{\frac{\varepsilon|x|^2}{1-4\varepsilon t}}|\phi(x)|\mathrm{e}^{-\varepsilon|y|^2}\,\mathrm{d}x\,\mathrm{d}|\mu(y)| \\ &\le (4\varepsilon t)^{-d/2}\|\mu\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)}\int_{\mathbb{R}^d}\mathrm{e}^{\frac{\varepsilon|x|^2}{1-4\varepsilon t}}|\phi(x)|\,\mathrm{d}x, \end{aligned}$$

which is finite as long as $\varepsilon(t) < \gamma$, that is $0 < t < T(\varepsilon) - T(\gamma) = \frac{1}{4\varepsilon} - \frac{1}{4\gamma}$.

The rest follows from Fubini's theorem. □

We can now show that for $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ [there is no requirement for u_0 to be non-negative] the function defined in (3.2) is indeed the solution of the heat equation on the time interval $(0, 1/4\varepsilon)$, and satisfies the initial data in the sense of measures. There are, of course, many classical results on the validity of the heat kernel representation, but the proof that follows has to be particularly tailored to $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ initial data, since this allows for significant growth at infinity.

Theorem 3.6. *Suppose that $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$, set $T(\varepsilon) = 1/4\varepsilon$, and let $u(x,t)$ be given by (3.2). Then*

(i) *$u(t) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ for $t \in (0, T(\varepsilon))$. Also $u \in C^\infty(\mathbb{R}^d \times (0, T(\varepsilon)))$ and satisfies*

$$u_t - \Delta u = 0 \qquad \text{for all} \quad x \in \mathbb{R}^d,\ 0 < t < T(\varepsilon).$$

(ii) *For every $\varphi \in C_c(\mathbb{R}^d)$ and $0 \le t < T(\varepsilon)$*

$$\int_{\mathbb{R}^d} \varphi\, u(t) = \int_{\mathbb{R}^d} S(t)\varphi\,\mathrm{d}u_0.$$

In particular, $u(t) \to u_0$ as $t \to 0^+$ as a measure, i.e.

$$\int_{\mathbb{R}^d} \varphi\, u(t) \to \int_{\mathbb{R}^d} \varphi\,\mathrm{d}u_0 \qquad \text{for every} \quad \varphi \in C_c(\mathbb{R}^d).$$

(iii) *If $0 \le u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ is non-zero then $u(x,t) > 0$ for all $x \in \mathbb{R}^d$, $t \in (0, T(\varepsilon))$, i.e. the Strong Maximum Principle holds.*

Proof. (i) If $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$, then for any $a > 1$

$$u(0, at, |u_0|) = \frac{1}{(4\pi at)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|y|^2/4at}\, \mathrm{d}|u_0(y)| < \infty$$

provided that $\frac{1}{4at} \geq \varepsilon$, that is $t \leq \frac{1}{4a\varepsilon} < T(\varepsilon)$. Hence by (3.3) from Lemma 3.1, we have $u(t) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ for $t \in (0, T(\varepsilon))$.

The rest of part (i) follows from the regularity of the heat kernel, since for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and $n \in \mathbb{N}$, the derivatives satisfy

$$D^{\alpha,n}_{x,t} K(x,t) = \frac{p_{\alpha,n}(x,t)}{t^{d/2+|\alpha|+2n}}\, \mathrm{e}^{-|x|^2/4t},$$

where $p_{\alpha,n}(x,t)$ is a polynomial of degree not exceeding $|\alpha| + 2n$. For t bounded away from zero and $\delta > 0$ this can be bounded by a constant times $\mathrm{e}^{(-\frac{1}{4t}+\delta)|x|^2}$. Therefore for $0 < s \leq t \leq \tau < T(\varepsilon)$

$$\int_{\mathbb{R}^d} |D^{\alpha,n}_{x,t} K(x-y,t)|\, \mathrm{d}|u_0(y)| \leq C_{s,\tau} \int_{\mathbb{R}^d} \mathrm{e}^{(-\frac{1}{4\tau}+\delta)|x-y|^2}\, \mathrm{d}|u_0(y)|$$

with $0 < \delta < \frac{1}{4\tau}$. Proceeding as in the upper bound in Lemma 3.1 the above integral is bounded, for x in compact sets and $0 < \alpha < 1$, by a multiple of

$$\int_{\mathbb{R}^d} \mathrm{e}^{(-\frac{1}{4\tau}+\delta)(1-\alpha)|y|^2}\, \mathrm{d}|u_0(y)| = \int_{\mathbb{R}^d} \mathrm{e}^{(\varepsilon+(-\frac{1}{4\tau}+\delta)(1-\alpha))|y|^2} \mathrm{e}^{-\varepsilon|y|^2}\, \mathrm{d}|u_0(y)|$$

which is finite as long as we chose δ, α small such that $\varepsilon < (1-\alpha)(\frac{1}{4\tau} - \delta)$. For this it suffices that $\frac{1}{4T(\varepsilon)(1-\alpha)} = \frac{\varepsilon}{1-\alpha} < \frac{1}{4\tau} - \delta$ which is possible since $\tau < T(\varepsilon)$. Hence $u \in C^\infty(\mathbb{R}^d \times (0, T(\varepsilon)))$ and satisfies the heat equation pointwise.

For (ii), i.e. to show that the initial data is attained in the sense of measures, notice first that it is enough to consider non-negative test functions in $C_c(\mathbb{R}^d)$. Now, from Lemma 3.3 and (3.15) in Lemma 3.5, we get for t small

$$\int_{\mathbb{R}^d} \varphi u(t) = \int_{\mathbb{R}^d} S(t)\varphi\, \mathrm{d}u_0.$$

Since Lemma 3.3 also guarantees that $\mathrm{e}^{\varepsilon|x|^2}\big(S(t)\varphi(x) - \varphi(x)\big) \to 0$ uniformly in $\mathbb{R}^d$ as $t \to 0$, we can take $t \to 0$ in (3.14) and obtain

$$\int_{\mathbb{R}^d} \varphi\, u(t) = \int_{\mathbb{R}^d} S(t)\varphi\, \mathrm{d}u_0 = \int_{\mathbb{R}^d} \varphi\, \mathrm{d}u_0 + \int_{\mathbb{R}^d} \mathrm{e}^{\varepsilon|x|^2}\big(S(t)\varphi - \varphi\big)\, \mathrm{e}^{-\varepsilon|x|^2}\, \mathrm{d}u_0 \to \int_{\mathbb{R}^d} \varphi\, \mathrm{d}u_0$$

and (ii) is proved.

Part (iii) is a consequence of the lower bound in (3.4) from Lemma 3.1. □

Now we derive some estimates on the solution in the $L^1_\varepsilon(\mathbb{R}^d)$ spaces introduced in (3.6), using the norm from (3.7). We also discuss the continuity of the solutions in time. Note that part (i) shows that in fact whenever $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ we have $u(t) \in L^1_{\varepsilon(t)}(\mathbb{R}^d)$; in part (iii) we obtain a similar result for the derivatives of u, but with some loss in the allowed growth (in $L^1_\delta(\mathbb{R}^d)$ only for $\delta > \varepsilon(t)$).

Recalling the notations in (3.11), we have the following result.

Proposition 3.7. *Suppose that $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ and let $u(x,t)$ be given by (3.2).*

(i) *For $0 < t < T(\varepsilon)$ we have $u(t) \in L^1_\delta(\mathbb{R}^d)$ for any $\delta \geq \varepsilon(t)$. Moreover*

$$\|u(t)\|_{L^1_{\varepsilon(t)}(\mathbb{R}^d)} \leq \|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)}. \tag{3.16}$$

(ii) *For $0 \leq s < t < T(\varepsilon)$*

$$u(t) = S(t-s)u(s). \tag{3.17}$$

(iii) *For any multi-index $\alpha \in \mathbb{N}^d$, for $0 < t < T(\varepsilon)$ we have $D^\alpha_x u(t) \in L^1_\delta(\mathbb{R}^d)$ for any $\delta > \varepsilon(t)$. Moreover for any $\gamma > 1$ we have*

$$\|D^\alpha_x u(t)\|_{L^1_{\delta(t)}(\mathbb{R}^d)} \leq \frac{c_{\alpha,\gamma}}{t^{\frac{|\alpha|}{2}}}\|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} \qquad \textit{for all} \quad 0 < t < \frac{T(\varepsilon)}{\gamma}, \tag{3.18}$$

where $\delta(t) := \frac{1}{4(T(\varepsilon)-\gamma t)} = \frac{\varepsilon}{(1-4\varepsilon\gamma t)}$.

(iv) *For any multi-index $\alpha \in \mathbb{N}^d$, $m \in \mathbb{N}$ and for each $t_0 \in (0, T(\varepsilon))$ there exists $\delta(t_0) > \varepsilon$ such that the mapping $(0, T(\varepsilon)) \ni t \mapsto D^{\alpha,m}_{x,t} u(t)$ is continuous in $L^1_{\delta(t_0)}(\mathbb{R}^d)$ at $t = t_0$.*

Proof. (i) Setting $\delta = \frac{1}{4\tau}$

$$\int_{\mathbb{R}^d} \mathrm{e}^{-|x|^2/4\tau}|u(x,t)|\,\mathrm{d}x \leq \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{-|x|^2/4\tau}\mathrm{e}^{-|x-z|^2/4t}\,\mathrm{d}|u_0(z)|\,\mathrm{d}x.$$

Notice that completing the square we obtain

$$\frac{|x|^2}{\tau} + \frac{|x-z|^2}{t} = \frac{t+\tau}{t\tau}\left|x - \frac{\tau}{t+\tau}z\right|^2 + \frac{|z|^2}{t+\tau} \tag{3.19}$$

and so

$$\int_{\mathbb{R}^d} \mathrm{e}^{-|x|^2/4\tau}|u(x,t)|\,\mathrm{d}x \leq \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|z|^2}{4(t+\tau)}}\,\mathrm{d}|u_0(z)|\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{t+\tau}{4t\tau}|x-\frac{\tau}{t+\tau}z|^2}\,\mathrm{d}x.$$

Since

$$\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{t+\tau}{4t\tau}|x-\frac{\tau}{t+\tau}z|^2}\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{t+\tau}{4t\tau}|x|^2}\,\mathrm{d}x = \left(\frac{4\pi t\tau}{t+\tau}\right)^{d/2}$$

it follows that

$$\int_{\mathbb{R}^d} \mathrm{e}^{-|x|^2/4\tau} |u(x,t)| \,\mathrm{d}x \leq \left(\frac{\tau}{t+\tau}\right)^{d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|z|^2}{4(t+\tau)}} \,\mathrm{d}|u_0(z)|.$$

Now given t with $0 < t < T(\varepsilon)$, choose $\tau = T(\varepsilon) - t = (1 - 4\varepsilon t)/4\varepsilon$; then $1/4\tau = \varepsilon(t)$, $1/4(t+\tau) = \varepsilon$, and this estimate becomes

$$\varepsilon(t)^{d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon(t)|x|^2} |u(x,t)|, \mathrm{d}x \leq \varepsilon^{d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|z|^2} \,\mathrm{d}|u_0(z)|,$$

which is precisely (3.16) up to a constant multiple of both sides.

(ii) Now

$$S(t-s)u(s)(x) = \frac{1}{(4\pi(t-s))^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/4(t-s)} u(y,s) \,\mathrm{d}y$$

and

$$u(y,s) = \frac{1}{(4\pi s)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|y-z|^2/4s} \,\mathrm{d}u_0(z).$$

Notice that completing the square as in (3.19) with $x - y$ replacing x, $z - y$ replacing z and $t - s$ replacing τ and s replacing t, we get

$$S(t-s)u(s)(x) = \frac{1}{(4\pi(t-s))^{d/2}} \frac{1}{(4\pi s)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-z|^2/4t} \,\mathrm{d}u_0(z) \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{t}{4s(t-s)}|(y-z)-\frac{s}{t}(x-z)|^2} \,\mathrm{d}y$$

and

$$\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{t}{4s(t-s)}|(y-z)-\frac{s}{t}(x-z)|^2} \,\mathrm{d}y = \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{t}{4s(t-s)}|y|^2} \,\mathrm{d}y = \left(\frac{4\pi s(t-s)}{t}\right)^{d/2}$$

and the result is proved.

(iii) Notice that for any multi-index $\alpha \in \mathbb{N}^d$

$$D_x^\alpha u(x,t) = \int_{\mathbb{R}^d} D_x^\alpha K(x-y,t) \,\mathrm{d}u_0(y) = \frac{1}{t^{d/2+|\alpha|/2}} \int_{\mathbb{R}^d} p_\alpha(x-y,t) \,\mathrm{e}^{-|x-y|^2/4t} \,\mathrm{d}u_0(y)$$

with $p_\alpha(x-y,t)$ is a polynomial of degree $|\alpha|$ in powers of $\frac{x-y}{t^{1/2}}$. Hence for any $0 < \beta < 1$

$$\begin{aligned} |D_x^\alpha u(x,t)| &\leq \frac{c_{\alpha,\beta}}{t^{d/2+|\alpha|/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\beta\frac{|x-y|^2}{4t}} \,\mathrm{d}|u_0(y)| \qquad (3.20) \\ &= \frac{\tilde{c}_{\alpha,\beta}}{t^{|\alpha|/2}} \, v(x,\gamma t), \end{aligned}$$

where $v(x,t)$ is the solution with initial data $|u_0|$ and $\gamma = 1/\beta > 1$ is arbitrary. The estimate in (3.18) follows using part (i).

(iv) Note that we can argue as we did for (3.3), and use (3.20) to obtain, for $0 < \gamma < 1$,

$$|D_x^\alpha u(x,t)| \leq \frac{c_{\alpha,\beta}}{t^{d/2+|\alpha|/2}} \mathrm{e}^{(\frac{1}{\gamma}-1)(1-\beta)\frac{|x|^2}{4t}} \int_{\mathbb{R}^d} \mathrm{e}^{-(1-\gamma)(1-\beta)\frac{|y|^2}{4t}} \,\mathrm{d}|u_0(y)|, \tag{3.21}$$

which is finite provided we choose β, γ such that $(1-\gamma)(1-\beta)\frac{1}{4t} > \varepsilon$ i.e. provided that $t < T = (1-\gamma)(1-\beta)T(\varepsilon)$.

From the regularity of u in Theorem 3.6 we know that, as $t \to t_0$,

$$D_x^\alpha u(t) \to D_x^\alpha u(t_0) \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}^d).$$

Now, if $\alpha = 0$, (3.3) implies that for $\varepsilon(t_0) = \frac{\gamma}{4(a-1)t_0}$ and $a, \gamma > 1$ we have a uniform quadratic exponential bound for $u(t)$ for all t close enough to t_0. For nonzero α, (3.21) implies that for $0 < \beta, \gamma < 1$ and $\delta(t_0) = (1-\gamma)(1-\beta)\frac{1}{4t_0} > \varepsilon$ we have again a uniform quadratic exponential bound for $D_x^\alpha u(t)$ for all t close enough to t_0. Now, for $n \in \mathbb{N}$,

$$\|D_x^\alpha u(t) - D_x^\alpha u(t_0)\|_{L^1_{\delta(t_0)}(\mathbb{R}^d)} = c \int_{|x|\leq n} \mathrm{e}^{-\delta(t_0)|x|^2} |D_x^\alpha u(t) - D_x^\alpha u(t_0)|(x) \,\mathrm{d}x$$

$$+ c \int_{|x|\geq n} \mathrm{e}^{-\delta(t_0)|x|^2} |D_x^\alpha u(t) - D_x^\alpha u(t_0)|(x) \,\mathrm{d}x.$$

From the uniform quadratic exponential bound, the second term is arbitrarily small for sufficiently large n, uniformly in t close to t_0, while the first term is small, with fixed n and t close enough to t_0.

For time derivatives just note that for $m \in \mathbb{N}$, $\partial_t^m u(t) = (-\Delta)^{2m} u(t)$, and then

$$D_{x,t}^{\alpha,m} u(t) = \partial_t^m D_x^\alpha u(t) = (-\Delta)^{2m} D_x^\alpha u(t)$$

and we apply the argument above. □

We now discuss further the sense in which the initial data is attained (improving on part (ii) of Theorem 3.6). First we show that $u(t) = S(t)u_0$ with $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ attains the initial data against any test function that decays fast enough.

Corollary 3.8. *If $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ and $\varphi \in C_0(\mathbb{R}^d)$ is such that $|\varphi(x)| \leq A\mathrm{e}^{-\gamma|x|^2}$, $x \in \mathbb{R}^d$, with $\gamma > \varepsilon$, then $u(t) = S(t)u_0$ satisfies*

$$\int_{\mathbb{R}^d} u(t)\varphi \to \int_{\mathbb{R}^d} \varphi \,\mathrm{d}u_0 \qquad \textit{as} \quad t \to 0.$$

Proof. For $0 \leq t < T(\varepsilon)$ small and $\varepsilon(t) = \frac{\varepsilon}{1-4\varepsilon t}$ we have $\gamma > \varepsilon(t)$ and then from (3.15) in Lemma 3.5 $\int_{\mathbb{R}^d} u(t)\varphi = \int_{\mathbb{R}^d} S(t)\varphi \,\mathrm{d}u_0$. Now, from Lemma 3.4 it follows that for t sufficiently small

$$|S(t)\varphi|(x) \leq C\mathrm{e}^{-\gamma(t)|x|^2}, \quad \text{with} \quad \gamma(t) = \frac{\gamma}{1+4\gamma t} > \varepsilon,$$

and then $|S(t)\varphi|(x) \le C\mathrm{e}^{-\varepsilon|x|^2} \in L^1(\mathrm{d}|u_0|)$. Also, $S(t)\varphi(x) \to \varphi(x)$ for $x \in \mathbb{R}^d$ and then Lebesgue's theorem gives the result. □

Assuming the initial data is a pointwise defined function, we get the following result.

Corollary 3.9. *Suppose that $u_0 \in L^1_\varepsilon(\mathbb{R}^d)$, set $T(\varepsilon) = 1/4\varepsilon$, and let $u(x,t)$ be given by (3.2). Then*

(i) *$u(t) \to u_0$ in $L^1_\delta(\mathbb{R}^d)$ as $t \to 0^+$ for any $\delta > \varepsilon$;*

(ii) *if $u_0 \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ with $1 \le p < \infty$ then*

$$u(t) \to u_0 \quad \textit{in} \quad L^p_{\mathrm{loc}}(\mathbb{R}^d) \qquad \textit{as} \quad t \to 0^+; \qquad \textit{and}$$

(iii) *if $u_0 \in C(\mathbb{R}^d)$ then $u(t) \to u_0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ as $t \to 0^+$.*

Proof. (i) Note that for any $\varphi \in C_c(\mathbb{R}^d)$ we have

$$\|S(t)u_0 - u_0\|_{L^1_\delta(\mathbb{R}^d)} \le \|S(t)u_0 - S(t)\varphi\|_{L^1_\delta(\mathbb{R}^d)} + \|S(t)\varphi - \varphi\|_{L^1_\delta(\mathbb{R}^d)} + \|\varphi - u_0\|_{L^1_\delta(\mathbb{R}^d)}.$$

Let $\gamma > 0$ and take $\varphi \in C_c(\mathbb{R}^d)$ such that

$$\|u_0 - \varphi\|_{L^1_\varepsilon(\mathbb{R}^d)} = \int_{\mathbb{R}^\mathrm{d}} \mathrm{e}^{-\varepsilon|x|^2}|u_0(x) - \varphi(x)|\,\mathrm{d}x < \gamma.$$

To see this note that for $R > 0$, if $\mathrm{supp}\,(\varphi) \subset B(0,R)$ then

$$\int_{\mathbb{R}^\mathrm{d}} \mathrm{e}^{-\varepsilon|x|^2}|u_0(x) - \varphi(x)|\,\mathrm{d}x = \int_{|x|\le R} \mathrm{e}^{-\varepsilon|x|^2}|u_0(x) - \varphi(x)|\,\mathrm{d}x + \int_{|x|>R} \mathrm{e}^{-\varepsilon|x|^2}|u_0(x)|\,\mathrm{d}x.$$

The second term is small for R large and so is the first one if we approach u_0 by φ in $L^1(B(0,R))$.

Now for any $\delta > \varepsilon$ and all sufficiently small $t > 0$ we have $\tilde{\delta}(t) \le \delta$, where $\tilde{\delta}(t) := \frac{\varepsilon}{(1-4\varepsilon t)}$. Then from (3.10) and (3.16) we have

$$\|S(t)(u_0-\varphi)\|_{L^1_\delta(\mathbb{R}^d)} \le \left(\frac{\delta}{\varepsilon}\right)^{d/2} \|S(t)(u_0-\varphi)\|_{L^1_{\tilde{\delta}(t)}(\mathbb{R}^d)} \le \left(\frac{\delta}{\varepsilon}\right)^{d/2} \|u_0-\varphi\|_{L^1_\varepsilon(\mathbb{R}^d)} < \left(\frac{\delta}{\varepsilon}\right)^{d/2} \gamma.$$

Finally, as in Lemma 3.3 we have $S(t)\varphi - \varphi \to 0$ uniformly in $\mathbb{R}^d$ as $t \to 0$. Hence $\|S(t)\varphi - \varphi\|_{L^1_\delta(\mathbb{R}^d)} \to 0$ as $t \to 0$, which proves (i).

(ii) and (iii). Fix $x_0 \in \mathbb{R}^d$ and $\delta > 0$ and take $0 \le \varphi \in C_c(\mathbb{R}^d)$ such that $0 \le \varphi \le 1$, $\varphi = 1$ on $B(x_0,\delta)$, and $\mathrm{supp}(\varphi) \subset B(x_0, 2\delta)$. Decompose $u_0 = \varphi u_0 + (1-\varphi)u_0$ and write

$$u(t,u_0) = u(t,\varphi u_0) + u(t,(1-\varphi)u_0).$$

Then, if $u_0 \in L^p_{\rm loc}(\mathbb{R}^d)$ with $1 \le p < \infty$ we have $\varphi u_0 \in L^p(\mathbb{R}^d)$ then, as $t \to 0$,

$$u(t, \varphi u_0) \to \varphi u_0 \quad \text{in } L^p(\mathbb{R}^d).$$

In particular $u(t, \varphi u_0) \to u_0$ in $L^p(B(x_0, \delta))$. If $u_0 \in C(\mathbb{R}^d)$ then $\varphi u_0 \in \mathrm{BUC}(\mathbb{R}^d)$ then, as $t \to 0$,

$$u(t, \varphi u_0) \to \varphi u_0 \quad \text{in } L^\infty(\mathbb{R}^d).$$

In particular $u(t, \varphi u_0) \to u_0$ in $L^\infty(B(x_0, \delta))$.

Now we prove that, as $t \to 0$, $u(t, (1-\varphi)u_0) \to 0$ uniformly in a ball $B(x_0, \tilde\delta)$ for some $\tilde\delta < \delta$, independent of x_0; this will conclude the proof of (ii) and (iii).

For this notice that for $x \in B(x_0, \delta/2)$

$$u(t, (1-\varphi)u_0)(x) = \frac{1}{(4\pi t)^{d/2}} \int_{|y-x_0|\ge\delta} \mathrm{e}^{-\frac{|x-y|^2}{4t}} (1-\varphi)(y) u_0(y)\, \mathrm{d}y.$$

Then $|x-y| \ge |x_0-y| - |x-x_0| \ge \delta - \delta/2 = \delta/2$. Hence for $0 < t < t_0$ and $0 < \alpha < 1$, $|x-y|^2 \ge \alpha|x-y|^2 + (1-\alpha)\frac{\delta^2}{4}$ and we obtain

$$|u(t, (1-\varphi)u_0)(x)| \le \frac{\mathrm{e}^{-(1-\alpha)\frac{\delta^2}{16t}}}{(4\pi t)^{d/2}} \int_{|y-x_0|\ge\delta} \mathrm{e}^{-\frac{\alpha|x-y|^2}{4t}} |u_0(y)|\, \mathrm{d}y.$$

Now we look for a uniform estimate in $x \in B(x_0, \tilde\delta)$ for the right-hand side above. For this note that for $0 < \beta < 1$,

$$|x-y|^2 \ge |y-x_0|^2 + |x-x_0|^2 - 2|y-x_0||x-x_0| \ge (1-\beta)|y-x_0|^2 + (1-\frac{1}{\beta})|x-x_0|^2,$$

thus for $x \in B(x_0, \tilde\delta)$ and $0 < t < t_0 = \frac{\alpha(1-\beta)}{8\varepsilon}$ we have

$$\begin{aligned} |u(t, (1-\varphi)u_0)(x)| &\le \frac{\mathrm{e}^{-(1-\alpha)\frac{\delta^2}{16t}}}{(4\pi t)^{d/2}} \mathrm{e}^{(\frac{1}{\beta}-1)\frac{|x-x_0|^2}{4t}} \int_{|y-x_0|\ge\delta} \mathrm{e}^{-\frac{\alpha(1-\beta)|x_0-y|^2}{4t}} |u_0(y)|\, \mathrm{d}y \\ &\le \frac{\mathrm{e}^{-(1-\alpha)\frac{\delta^2}{16t} + (\frac{1}{\beta}-1)\frac{\tilde\delta^2}{4t}}}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-2\varepsilon|x_0-y|^2} |u_0(y)|\, \mathrm{d}y; \end{aligned} \tag{3.22}$$

again $|x_0-y|^2 \ge |y|^2 + |x_0|^2 - 2|y||x_0| \ge 1/2|y|^2 - |x_0|^2$ gives

$$\int_{\mathbb{R}^d} \mathrm{e}^{-2\varepsilon|x_0-y|^2} |u_0(y)|\, \mathrm{d}y \le \mathrm{e}^{2\varepsilon|x_0|} \int_{\mathbb{R}^N} \mathrm{e}^{-\varepsilon|y|^2} |u_0(y)|\, \mathrm{d}y$$

and so (3.22) tends to 0 as $t \to 0$ uniformly in $x \in B(x_0, \tilde\delta)$ if $(\frac{1}{\beta}-1)\tilde\delta^2 < (1-\alpha)\frac{\delta^2}{4}$. Notice, finally, that $\tilde\delta$ does not depend on x_0. □

Notice that by comparing $\mathrm{e}^{-\varepsilon|x|^2}$ and $(1+|x|^2)^{-m/2}$ it follows that the class of tempered distributions of class $(m, 0)$ as introduced at the end of Section 2, satisfies, for all $\varepsilon > 0$,

$$\mathscr{C}_m(\mathbb{R}^d) \subset \mathscr{C}_{m+1}(\mathbb{R}^d) \subset \mathcal{M}_\varepsilon(\mathbb{R}^d).$$

4 Initial data in $\mathcal{M}_\varepsilon(\mathbb{R}^d)$: uniqueness

Now we prove a uniqueness result for heat solutions with initial data $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$. Observe that uniqueness for *non-negative* weak solutions of (3.1) can be found in [2]. On the other hand, one can find a proof of the uniqueness of classical solutions with no sign assumptions but with bounded continuous initial data in [26] in dimension one and in e.g. [16] (Chapter 7, page 176) in arbitrary dimensions, provided that they satisfy the pointwise bound

$$|u(x,t)| \leq M\mathrm{e}^{a|x|^2}, \ x \in \mathbb{R}^d, \ 0 < t < T. \tag{4.1}$$

Here we prove a uniqueness result adapted to initial data in $\mathcal{M}_\varepsilon(\mathbb{R}^d)$, with no sign condition imposed. Observe that if $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ for some $\varepsilon > 0$ and $u(t) = S(t)u_0$ is as in (3.2), then assumptions (4.2), (4.4), (4.5), (4.6) and (4.7) below are ensured by parts (i) and (iii) in Proposition 3.7, Corollary 3.8 and part (iii) in Theorem 3.6. Also (4.3) is ensured by part (ii) in Proposition 3.7.

Theorem 4.1. *Suppose that u, defined in $\mathbb{R}^d \times (0,T]$, is such that for some $\delta > 0$ and for each $0 < t < T$, $u(t) \in L^1_\delta(\mathbb{R}^d)$.*

(i) *Suppose furthermore that*

$$u, \nabla u, \Delta u \in L^1_{\mathrm{loc}}((0,T), L^1_\delta(\mathbb{R}^d)) \tag{4.2}$$

and satisfies $u_t - \Delta u = 0$ almost everywhere in $\mathbb{R}^d \times (0,T)$. Then we have

$$u(t) = S(t-s)u(s) \tag{4.3}$$

for any $0 < s < t < T$.

Assume hereafter that u satisfies (4.3) *for any $0 < s < t < T$.*

(ii) *Then for each $0 < t < T$ and every $\varphi \in C_c(\mathbb{R}^d)$ the following limit exist*

$$\lim_{s\to 0} \int_{\mathbb{R}^d} u(s)S(t)\varphi = \int_{\mathbb{R}^d} u(t)\varphi.$$

(iii) *There exists $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ for some $\varepsilon > 0$ and such that $u(t) = S(t)u_0$ for $0 < t < T$ if and only if for every $\varphi \in C_c(\mathbb{R}^d)$ and t small enough*

$$\lim_{s\to 0} \int_{\mathbb{R}^d} u(s)S(t)\varphi = \int_{\mathbb{R}^d} S(t)\varphi \, \mathrm{d}u_0. \tag{4.4}$$

(iv) *Condition* (4.4) *is satisfied provided either one of the following holds:*

(iv-a) *For any function* $\phi \in C_0(\mathbb{R}^d)$ *such that* $|\phi(x)| \le A\mathrm{e}^{-\gamma|x|^2}$, $x \in \mathbb{R}^d$, *with* $\gamma > \varepsilon$ *we have, as* $t \to 0$

$$\lim_{t\to 0} \int_{\mathbb{R}^d} \phi u(t) \to \int_{\mathbb{R}^d} \phi \,\mathrm{d}u_0. \tag{4.5}$$

(iv-b) *For some* $\tau \le T$ *small and* $0 < t \le \tau$ *we have* $u(t) \in L^1_\varepsilon(\mathbb{R}^d)$ *with*

$$\int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2} |u(x,t)| \,\mathrm{d}x \le M \quad t \in (0,\tau]; \tag{4.6}$$

i.e. $u \in L^\infty((0,\tau], L^1_\varepsilon(\mathbb{R}^d))$ *and for every* $\varphi \in C_c(\mathbb{R}^d)$, *as* $t \to 0$

$$\int_{\mathbb{R}^d} \varphi \, u(t) \to \int_{\mathbb{R}^d} \varphi \,\mathrm{d}u_0. \tag{4.7}$$

Proof. For (i) the key to the proof is to show that for every $\varphi \in C_c(\mathbb{R}^d)$ and $0 < s < t < T$ one has

$$\int_{\mathbb{R}^d} u(t)\varphi = \int_{\mathbb{R}^d} u(s) S(t-s)\varphi \tag{4.8}$$

for some small enough T depending only on $\delta > 0$ in (4.2). In such a case, we then apply (3.15) with $\mu = u(s) \in L^1_\delta(\mathbb{R}^d)$ and $\phi = S(t-s)\varphi$, provided $0 \le t-s \le T$ is small enough (depending on $\delta > 0$) such that from Lemma 3.3, ϕ satisfies the assumption in Lemma 3.5 to obtain that the right hand side of (4.8) equals $\int_{\mathbb{R}^d} S(t-s)u(s)\varphi$. Hence, we get (4.3) for $0 < s < t < T$. Then for $0 < t_0 < T$ consider $v(t) = u(t+t_0)$ for $0 \le t \le T$ which satisfies the assumptions in (i). Hence (4.3) implies in particular $u(t+t_0) = S(t)u(t_0)$ for $0 \le t_0, t \le T$ which combined with (3.17) gives (4.3) for $0 < s < t < 2T$. In a finite numer of steps we obtain this property on any finite time interval.

For the proof of (4.8) we fix $0 < t < T$ and differentiate

$$I(s) := \int_{\mathbb{R}^d} u(s) S(t-s)\varphi \qquad s \in (0,t)$$

to obtain

$$I'(s) = \int_{\mathbb{R}^d} \partial_s u(s) S(t-s)\varphi - \int_{\mathbb{R}^d} u(s) \partial_s S(t-s)\varphi = \int_{\mathbb{R}^d} \Delta u(s) S(t-s)\varphi - \int_{\mathbb{R}^d} u(s) \Delta S(t-s)\varphi. \tag{4.9}$$

For this observe that from Lemma 3.3 and decreasing T if necessary but depending only on δ, we have that for all $0 < t < T$

$$|S(t)\varphi(x)| \le c\mathrm{e}^{-\alpha|x|^2}, \quad x \in \mathbb{R}^d, \quad 0 < t < T,$$

with $\alpha = \alpha(T) > \delta$, with δ as in (4.2). Also, by (3.12) and proceeding as in (3.20) and as in (3.13) we obtain

$$|\Delta S(t)\varphi(x)| \le \frac{c}{t^{d/2+1}} \mathrm{e}^{-(1-\delta)^2 \frac{|x|^2}{4t} + \frac{(1-\delta)^2}{\delta}\frac{R^2}{4t}} \int_{B(0,R)} |\varphi(y)| \,\mathrm{d}y \quad x \in \mathbb{R}^d, \quad 0 < t < T,$$

and again as in Lemma 3.3, we obtain

$$|\Delta S(t)\varphi(x)| \le c\mathrm{e}^{-\alpha|x|^2} \quad x \in \mathbb{R}^d, \quad 0 < t < T,$$

with $\alpha = \alpha(T) > \delta$ and some $c = c(\varphi, R, T)$ with δ as in (4.2).

With these, using (4.2), the integrand on the right-hand side of (4.9) has a bound

$$|\Delta u(\cdot)||S(t-\cdot)\varphi| + |u(\cdot)||\Delta S(t-\cdot)\varphi| \in L^1_{loc}((0,t), L^1(\mathbb{R}^d)).$$

Hence, by differentiation inside the integral, (4.9) is proved.

Now observe that (4.2) and the upper bounds above for $S(t)\varphi$, $\Delta S(t)\varphi$ mean that we can use a.e. $s \in (0,t)$, Lemma A.3 below to integrate by parts in (4.9) to get $I'(s) = 0$ for $s \in (0,t)$ which gives that $I(s)$ is constant in $(0,t)$.

Now we show that, as $s \to t$ we have $I(s) \to I(t) = \int_{\mathbb{R}^d} u(t)\varphi$. For this, write

$$I(s) = \int_{\mathbb{R}^d} u(s)\varphi + \int_{\mathbb{R}^d} u(s)\big(S(t-s)\varphi - \varphi\big)$$

and observe that from the assumptions $\partial_t u = \Delta u \in L^1_{\mathrm{loc}}((0,T), L^1_\delta(\mathbb{R}^d))$ and in particular, $u(s)$ is continuous as $s \to t$ in $L^1_\delta(\mathbb{R}^d)$. On the other hand from Lemma 3.3 we have $\mathrm{e}^{\delta|x|^2}\big(S(t-s)\varphi(x) - \varphi(x)\big) \to 0$ uniformly in $\mathbb{R}^d$ as $s \to t$. Hence (4.8) and part (i) are proved.

Now we prove (ii). For fixed $0 < t < T$ and s small, from Lemma 3.3, we get that $u(t)S(s)\varphi$ is integrable and then, from (4.3), using Lemma 3.5 again (with $\mu = u(s) \in L^1_\delta(\mathbb{R}^d)$ and $\phi = S(s)\varphi$) and (3.17), we get

$$\int_{\mathbb{R}^d} u(t)S(s)\varphi = \int_{\mathbb{R}^d} S(t-s)u(s)S(s)\varphi = \int_{\mathbb{R}^d} u(s)S(t-s)S(s)\varphi = \int_{\mathbb{R}^d} u(s)S(t)\varphi.$$

Now, from Lemma 3.3 we have that, as $s \to 0$,

$$\Big|\int_{\mathbb{R}^d} u(t)\big(S(s)\varphi - \varphi\big)\Big| \le \int_{\mathbb{R}^d} \mathrm{e}^{\delta|x|^2}|u(t)|\big(S(s)\varphi - \varphi\big)\,\mathrm{e}^{-\delta|x|^2} \to 0$$

and then the following limit exists

$$\lim_{s\to 0}\int_{\mathbb{R}^d} u(s)S(t)\varphi = \lim_{s\to 0}\int_{\mathbb{R}^d} u(t)S(s)\varphi = \int_{\mathbb{R}^d} u(t)\varphi.$$

This concludes the proof of part (ii).

To prove (iii) notice that from Lemma 3.3 with t small and Corollary 3.8 we have that condition (4.4) is necessary for $u(t)$ to be equal to $S(t)u_0$. Conversely if (4.4) is satisfied then for t small and $\varphi \in C_c(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} u(t)\varphi = \lim_{s\to 0}\int_{\mathbb{R}^d} u(s)S(t)\varphi = \int_{\mathbb{R}^d} S(t)\varphi\,\mathrm{d}u_0$$

Then by (3.15) in Lemma 3.5 with $\mu = u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$, $\phi = \varphi$, we get

$$\int_{\mathbb{R}^d} u(t)\varphi = \int_{\mathbb{R}^d} S(t)\varphi \, \mathrm{d}u_0 = \int_{\mathbb{R}^d} \varphi S(t) u_0$$

for every $\varphi \in C_c(\mathbb{R}^d)$ and then $u(t) = S(t)u_0$ for t small. This and (4.3) proves part (iii).

For part (iv-a) it is now clear that if u satisfies (4.5) then Lemma 3.3 and t small allows to take $\phi = S(t)\varphi$ in (4.5) to get that (4.4) satisfied.

Finally, assuming (4.6) and (4.7) we prove part (iv-b). For this consider a sequence of smooth functions $0 \le \phi_n \le 1$ with $supp(\phi_n) \subset B(0, 2n)$ and $\phi_n = 1$ in $B(0, n)$. Then we write

$$\int_{\mathbb{R}^d} u(s)S(t)\varphi = \int_{\mathbb{R}^d} u(s)\phi_n S(t)\varphi + \int_{\mathbb{R}^d} u(s)(1-\phi_n)S(t)\varphi$$

and then

$$\int_{\mathbb{R}^d} u(s)S(t)\varphi - \int_{\mathbb{R}^d} S(t)\varphi \, \mathrm{d}u_0 = I_1 + I_2 + I_3 =$$

$$\Big(\int_{\mathbb{R}^d} u(s)\phi_n S(t)\varphi - \int_{\mathbb{R}^d} \phi_n S(t)\varphi \, \mathrm{d}u_0\Big) + \int_{\mathbb{R}^d} (\phi_n - 1)S(t)\varphi \, \mathrm{d}u_0 + \int_{\mathbb{R}^d} u(s)(1-\phi_n)S(t)\varphi.$$

Now I_3 goes to zero with $n \to \infty$ uniformly in $0 < s < \tau$. To see this, observe that by Lemma 3.3, for some $t_0 > 0$ small and $0 < t < t_0$ we have $0 \le \mathrm{e}^{\varepsilon|x|^2}(1-\phi_n)|S(t)\varphi| \le c\mathrm{e}^{-\gamma|x|^2}(1-\phi_n)$, $\gamma > 0$ and $0 \le 1 - \phi_n \to 0$ uniformly in compact sets as $n \to \infty$. Hence $\mathrm{e}^{\varepsilon|x|^2}(1-\phi_n)S(t)\varphi \to 0$ as $n \to \infty$, uniformly in $\mathbb{R}^d$ and uniformly in $0 < t < t_0$. Thus by (4.6),

$$I_3 = \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2} u(s) \mathrm{e}^{\varepsilon|x|^2}(1-\phi_n)S(t)\varphi \to 0, \quad n \to \infty$$

uniformly for $0 < s < \tau$ and $0 < t < t_0$.

With the same argument, I_2 goes to zero with $n \to \infty$ uniformly in $0 < t < t_0$ since $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$. Finally, by (4.7), for any fixed n and $0 < t < t_0$, $I_1 \to 0$ as $s \to 0$. Hence, for any $0 < t < t_0$ we get $\lim_{s\to 0} \int_{\mathbb{R}^d} u(s)S(t)\varphi = \int_{\mathbb{R}^d} S(t)\varphi \, \mathrm{d}u_0$ and part (iv-b) is proved. □

Notice that condition (4.5) is precisely the definition of "initial data" for the weak solutions considered in [1], page 319.

Also observe that, for t small, from Lemma 3.5, condition (4.8) is indeed equivalent to (4.3) provided $u(s) \in L^1_\delta(\mathbb{R}^d)$ for some $\delta > 0$.

Finally, observe that if we assume $u \in C(\mathbb{R}^d \times (0, T])$ is such that for any $0 < s < T$ there exists $M, a > 0$ such that

$$|u(x,t)| \le M\mathrm{e}^{a|x|^2}, \ x \in \mathbb{R}^d, \ s \le t < T.$$

from the results in in [26] and [16] (Chapter 7, page 176), then (4.3) is satisfied. Also, observe that from Lemma 3.1 implies that $S(t)u_0$ satisfies the quadratic exponential bound above. Therefore, if additionally u satisfies (4.5) or (4.6) and (4.7) then we have $u(t) = S(t)u_0$. These contitions are slightly weaker than the classical Tychonov condition (4.1).

5 Global existence versus finite-time blowup

From the results in Section 3 it is natural to set

$$L^1_0(\mathbb{R}^d) = \bigcap_{\varepsilon>0} L^1_\varepsilon(\mathbb{R}^d)$$

and

$$\mathcal{M}_0(\mathbb{R}^d) = \bigcap_{\varepsilon>0} \mathcal{M}_\varepsilon(\mathbb{R}^d).$$

Clearly $L^1_0(\mathbb{R}^d) \subset \mathcal{M}_0(\mathbb{R}^d)$.

It is a simple consequence of Lemma 3.2 that these are precisely the collections of initial data for which (non-negative) solutions exist for all time.

Proposition 5.1. *If $u_0 \in \mathcal{M}_0(\mathbb{R}^d)$ then $u(x,t)$, given by (3.2), is well defined for all $x \in \mathbb{R}^d$ and $t > 0$; in particular $u(t) \in L^1_0(\mathbb{R}^d)$ for every $t > 0$. Conversely, if $u_0 \in \mathcal{M}_{\rm loc}(\mathbb{R}^d)$ with $u_0 \geq 0$ and $u(x,t)$ is defined for all $t > 0$ then $u_0 \in \mathcal{M}_0(\mathbb{R}^d)$.*

Note that $L^1_0(\mathbb{R}^d)$ is a natural space of functions in which to study the heat semigroup, since $S(t)\colon L^1_0(\mathbb{R}^N) \to L^1_0(\mathbb{R}^N)$ for every $t \geq 0$; this form the main topic of our paper [23]. For the time being, one can note that if $u_0 \in \mathcal{M}_0(\mathbb{R}^N)$ then the estimate from Proposition 3.7 can be reinterpreted as

$$\|u(t)\|_{L^1_\delta} \leq \|u_0\|_{\mathcal{M}_{\delta(t)}}, \qquad \text{where} \quad \delta(t) := \frac{\delta}{1+4\delta t}.$$

The collection $L^1_0(\mathbb{R}^d)$ is a large set of functions: it contains $L^p(\mathbb{R}^d)$ and $L^p_U(\mathbb{R}^d)$ for every $1 \leq p \leq \infty$, and (for example) any function that satisfies

$$|f(x)| \leq M \mathrm{e}^{k|x|^\alpha}, \quad x \in \mathbb{R}^d$$

for some $M > 0$ and $\alpha < 2$. It also contains functions that are not bounded by any quadratic exponential, such as

$$f(x) = \sum_k \alpha_k \, \mathcal{X}_{B(x_k,r_k)}(x), \quad \alpha_k = \mathrm{e}^{|x_k|^3}$$

with $|x_k| \to \infty$ and $r_k \to 0$ such that $r_k^d \leq \frac{1}{\alpha_k k^2}$.

We show below that $\mathcal{M}_0(\mathbb{R}^d)$ also contains the space of uniform measures $\mathcal{M}_U(\mathbb{R}^d)$ defined as the set of measures $\mu \in \mathcal{M}_{\rm loc}(\mathbb{R}^d)$ such that

$$\sup_{x\in\mathbb{R}^d} \int_{B(x,1)} \mathrm{d}|\mu(y)| < \infty \tag{5.1}$$

with norm

$$\|\mu\|_{\mathcal{M}_U(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \int_{B(x,1)} \mathrm{d}|\mu(y)|. \tag{5.2}$$

This is a Banach space, see Lemma A.2.

In fact, as a consequence of Theorem 3.6, we can show that the uniform space $\mathcal{M}_U(\mathbb{R}^d)$ is precisely the set of initial data for which non-negative solutions of the heat equation given by (3.2) remain bounded in $\mathbb{R}^d$ for positive times. See [3] for results of the heat equation between uniform spaces $L^p_U(\mathbb{R}^d)$, $1 \leq p < \infty$, the collection of all functions $\phi \in L^p_{\rm loc}(\mathbb{R}^d)$ such that

$$\sup_{x\in\mathbb{R}^d} \int_{B(x,1)} |\phi(y)|^p \, \mathrm{d}y < \infty$$

with norm $\|\phi\|_{L^p_U(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \|\phi\|_{L^p(B(x,1))}$. For $p=\infty$ we have $L^\infty_U(\mathbb{R}^d) = L^\infty(\mathbb{R}^d)$ with norm $\|\phi\|_{L^\infty_U(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \|\phi\|_{L^\infty(B(x,1))} = \|\phi\|_{L^\infty(\mathbb{R}^d)}$.

Proposition 5.2. (i) *If $u_0 \in \mathcal{M}_U(\mathbb{R}^d)$ then $u_0 \in \mathcal{M}_0(\mathbb{R}^d)$ and $u(t) \in L^\infty(\mathbb{R}^d)$ for all $t > 0$ and for every $1 \leq q \leq \infty$*

$$\|u(t)\|_{L^q_U(\mathbb{R}^d)} \leq M_0\Big(t^{-\frac{d}{2}(1-\frac{1}{q})} + 1\Big)\|u_0\|_{\mathcal{M}_U(\mathbb{R}^d)}.$$

(ii) *Conversely, assume that $0 \leq u_0 \in \mathcal{M}_0(\mathbb{R}^d)$. If $0 \leq u(t_0) \in L^\infty(\mathbb{R}^d)$ for some $t_0 > 0$ then*

$$u_0 \in \mathcal{M}_U(\mathbb{R}^d);$$

hence $u(t) \in L^\infty(\mathbb{R}^d)$ for all $t > 0$.

Proof. (i) Note first that from (3.2) we have $|S(t)u_0| \leq S(t)|u_0|$ and, by definition, that $u_0 \in \mathcal{M}_U(\mathbb{R}^d)$ iff $|u_0| \in \mathcal{M}_U(\mathbb{R}^d)$. Hence, using Proposition 5.1, it is enough to prove the result for non-negative u_0.

Let us consider a cube decomposition of $\mathbb{R}^d$ as follows. For any index $i \in \mathbb{Z}^d$, denote by Q_i the open cube in $\mathbb{R}^d$ of center i with all edges of length 1 and parallel to the axes. Then $Q_i \cap Q_j = \emptyset$ for $i \neq j$ and $\mathbb{R}^d = \cup_{i\in\mathbb{Z}^d}\overline{Q_i}$. For a given $i \in \mathbb{Z}^d$ let us denote by $N(i)$ the set of indexes near i, that is, $j \in N(i)$ if and only if $\overline{Q_i} \cap \overline{Q_j} \neq \emptyset$. Obviously

$$d_{ij} := \inf\{\mathrm{dist}(x,y),\, x \in Q_i, y \in Q_j\} \tag{5.3}$$

satisfies $d_{ij}=0$, if $j\in N(i)$, $d_{ij}\geq 1$, if $j\notin N(i)$, and as a matter of fact it is not difficult to see that $d_{ij}\geq \|i-j\|_\infty-1$. Let us denote by $Q_i^{\rm near}=\cup_{j\in N(i)}Q_i$ and $Q_i^{\rm far}=\mathbb{R}^d\setminus\overline{Q_i^{\rm near}}$.

Assume that $u_0\in\mathcal{M}_U(\mathbb{R}^d)$ and, for a fixed i, decompose

$$u_0=u_0\mathcal{X}_{Q_i^{\rm near}}+u_0\mathcal{X}_{Q_i^{\rm far}};$$

by applying the linear semigroup $S(t)$ to each term in this equality we obtain the decomposition

$$u(t)=u_i^{\rm near}(t)+u_i^{\rm far}(t).$$

The result will follow from the following estimates of the two terms of the decomposition. First,

$$\|u_i^{\rm near}(t)\|_{L^q(Q_i)}\leq (4\pi t)^{-\frac{d}{2}(1-\frac{1}{q})}\|u_0\|_{\mathcal{M}(Q_i^{\rm near})}\quad t>0, \tag{5.4}$$

for $1\leq q\leq\infty$ and, second,

$$\|u_i^{\rm far}(t)\|_{L^\infty(Q_i)}\leq c(t)\|u_0\|_{\mathcal{M}_U(Q_i^{\rm far})},\quad t\geq 0. \tag{5.5}$$

for some bounded monotonic function $c(t)$ such that $c(0)=0$ and $0\leq c(t)\leq Ct^{-d/2}\mathrm{e}^{-\alpha/t}$ as $t\to 0$, where C and $\alpha>0$ depend only on N.

Then since the constant for the embedding $L^\infty(Q_i)\hookrightarrow L^q(Q_i)$ is 1, independent of q and i, (5.5),(5.4) imply

$$\|u(t)\|_{L^q(Q_i)}\leq ((4\pi t)^{-\frac{d}{2}(1-\frac{1}{q})}+c(t))\|u_0\|_{\mathcal{M}_U(\mathbb{R}^d)}\qquad i\in\mathbb{Z}^d.$$

Since the $L^q_U(\mathbb{R}^N)$ norm can be bounded by a constant, only depending on N, times the supremum of the $L^q(Q_i)$ norms, (i) follows.

Now observe that (5.4) follows from "standard" estimates for the heat equation, since in fact, $u_0\mathcal{X}_{Q_i^{\rm near}}$ is a measure of bounded total variation and then for $t>0$,

$$\|u_i^{\rm near}(t)\|_{L^q(Q_i)}\leq\|S(t)(u_0\mathcal{X}_{Q_i^{\rm near}})\|_{L^q(\mathbb{R}^d)}\leq (4\pi t)^{-\frac{d}{2}(1-\frac{1}{q})}\|u_0\|_{\mathcal{M}(Q_i^{\rm near})} \tag{5.6}$$

since $\|u_0\mathcal{X}_{Q_i^{\rm near}}\|_{\mathcal{M}_{BTV}(\mathbb{R}^d)}=\|u_0\|_{\mathcal{M}(Q_i^{\rm near})}$ see Lemma 5.3 below.

We now prove (5.5). Observe that $u_0\mathcal{X}_{Q_i^{\rm far}}=\sum_{j\in\mathbb{Z}^d\setminus N(i)}u_0^j$ where $u_0^j=u_0\mathcal{X}_{Q_j}$; for each j we have

$$S(t)u_0^j(x)=(4\pi t)^{-d/2}\int_{\mathbb{R}^d}\mathrm{e}^{-\frac{|x-y|^2}{4t}}\,\mathrm{d}u_0^j(y)=(4\pi t)^{-d/2}\int_{Q_j}\mathrm{e}^{-\frac{|x-y|^2}{4t}}\,\mathrm{d}u_0^j(y)$$

which implies that for $j\notin N(i)$

$$\|S(t)u_0^j\|_{L^\infty(Q_i)}\leq(4\pi t)^{-d/2}\mathrm{e}^{\frac{-d_{ij}^2}{4t}}\|u_0^j\|_{\mathcal{M}(Q_j)}\leq(4\pi t)^{-d/2}\mathrm{e}^{\frac{-d_{ij}^2}{4t}}\|u_0\|_{\mathcal{M}_U(Q_i^{\rm far})},$$

where d_{ij} is defined above in (5.3). Hence,

$$\|u_i^{\text{far}}(t)\|_{L^\infty(Q_i)} \leq \sum_{j\in\mathbb{Z}^d\setminus N(i)} \|S(t)u_0^j\|_{L^\infty(Q_i)} \quad \leq (4\pi t)^{-d/2}\|u_0\|_{\mathcal{M}_U(Q_i^{\text{far}})} \sum_{j\in\mathbb{Z}^d\setminus N(i)} \mathrm{e}^{\frac{-d_{ij}^2}{4t}}.$$

But, using that $\#\{j \in \mathbb{Z},\ d_{ij} = k\} \leq Ck^{d-1}$, we obtain

$$\sum_{j\in\mathbb{Z}^d\setminus N(i)} \mathrm{e}^{\frac{-d_{ij}^2}{4t}} \leq C\sum_{k=1}^{\infty} k^{d-1}\mathrm{e}^{\frac{-k^2}{4t}}$$

which has the same character as the integral

$$\int_1^\infty r^{d-1}\mathrm{e}^{-\frac{r^2}{4t}}\,\mathrm{d}r = (4t)^{d/2}\int_{\frac{1}{\sqrt{4t}}}^\infty s^{d-1}\mathrm{e}^{-s^2}\,\mathrm{d}s = t^{d/2}c(t)$$

with $c(t)$ as claimed in (5.5).

(ii) If for some $t_0 > 0$ we have $u(t_0) \in L^\infty(\mathbb{R}^d)$ then from (3.2) we get for all $x \in \mathbb{R}^d$ and any $R > 0$

$$\begin{aligned}\infty > M \geq u(x,t_0) &\geq \frac{1}{(4\pi t_0)^{d/2}}\int_{B(x,R)} \mathrm{e}^{-\frac{|x-y|^2}{4t_0}}\,\mathrm{d}u_0(y)\\ &\geq \frac{1}{(4\pi t_0)^{d/2}}\inf_{z\in B(0,R)} \mathrm{e}^{-\frac{|z|^2}{4t_0}}\int_{B(x,R)}\mathrm{d}u_0(y)\end{aligned}$$

that is

$$0 \leq \int_{B(x,R)}\mathrm{d}u_0(y) \leq M\mathrm{e}^{\frac{R^2}{4t_0}}(4\pi t_0)^{d/2}, \quad x\in\mathbb{R}^d$$

i.e. $0 \leq u_0 \in \mathcal{M}_U(\mathbb{R}^d)$. From part (i) we obtain $u(t) \in L^\infty(\mathbb{R}^d)$ for all $t > 0$. □

Now we prove the result used above in (5.6). Note that from (3.8), $\mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^d) \subset \mathcal{M}_U(\mathbb{R}^d) \subset \mathcal{M}_0(\mathbb{R}^d)$. The following lemma shows that $\mathcal{M}_{\mathrm{BTV}}$ is invariant under the heat equation, and gives bound on the rate of decay in L^q of solutions when $u_0 \in \mathcal{M}_{\mathrm{BTV}}$.

Lemma 5.3. *For $\mu \in \mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^d)$ the solution of the heat equation given by (3.2) satisfies*

$$\|S(t)\mu\|_{\mathrm{BTV}} \leq \|\mu\|_{\mathrm{BTV}}, \quad t>0,$$

and for every $1 \leq q \leq \infty$

$$\|S(t)\mu\|_{L^q(\mathbb{R}^d)} \leq (4\pi t)^{-\frac{d}{2}(1-\frac{1}{q})}\|\mu\|_{\mathrm{BTV}}, \quad t>0.$$

Proof. Observe that since for every $\varphi \in C_c(\mathbb{R}^d)$ and $0 \leq t < \infty$, $u(t) = S(t)\mu$ satisfies (3.15) that is,

$$\int_{\mathbb{R}^d} u(t)\varphi = \int_{\mathbb{R}^d} S(t)\varphi \,\mathrm{d}\mu$$

then

$$\left| \int_{\mathbb{R}^d} u(t)\varphi \right| \leq \|S(t)\varphi\|_{L^\infty(\mathbb{R}^d)} \|\mu\|_{\mathrm{BTV}}.$$

Therefore the estimates (1.3) give, for every $1 \leq q \leq \infty$,

$$\left| \int_{\mathbb{R}^d} u(t)\varphi \right| \leq (4\pi t)^{-\frac{d}{2q'}} \|\varphi\|_{L^{q'}(\mathbb{R}^d)} \|\mu\|_{\mathrm{BTV}}$$

and the claims follow. Note that in particular, for $q = 1$ since $u(t) \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^d)$ then $u(t) \in L^1(\mathbb{R}^d)$. □

5.1 Finite-time blowup for non-negative initial data

Now we turn to non-negative solutions that may not exist for all time, that is, according to Proposition 5.1, $0 \leq u_0 \notin \mathcal{M}_0(\mathbb{R}^d)$. Lemma 3.2 shows that the maximal existence time for the solution arising from the non-negative initial condition $0 \leq u_0 \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ will be determined by its 'optimal index'

$$\varepsilon_0(\mu) := \inf\{\varepsilon : \ \mu \in \mathcal{M}_\varepsilon(\mathbb{R}^d)\} = \sup\{\varepsilon : \ \mu \notin \mathcal{M}_\varepsilon(\mathbb{R}^d)\} \leq \infty. \tag{5.7}$$

The simplest example is to take $A > 0$ and consider $u_0(x) = \mathrm{e}^{A|x|^2}$; then $u_0 \in L^1_\varepsilon(\mathbb{R}^d)$ if and only if $\varepsilon > A$, so in this case $\varepsilon_0(u_0) = A$ but $u_0 \notin L^1_A(\mathbb{R}^d)$. If we set $T = 1/4A$ then the integral in (3.2) can be computed explicitly and one gets

$$u(x,t) = \frac{T^{d/2}}{(T-t)^{d/2}} \mathrm{e}^{\frac{|x|^2}{4(T-t)}}, \tag{5.8}$$

which satisfies the heat equation for $t \in (0,T)$, has $u(x,0) = u_0(x)$, and blows up at every point $x \in \mathbb{R}^d$ as $t \to T$. At the other extreme is an initial condition like

$$u_0(x) = \mathrm{e}^{A|x|^2 - \gamma|x|^\alpha}$$

for some $\gamma > 0$ and $1 < \alpha < 2$, which we treat as Example 5.11, below. In this case $\lim_{t \to 1/4A} u(x,t)$ exists for every $x \in \mathbb{R}^d$, but the solution cannot be extended past $t = T$.

Below we analyse the behaviour for a general non-negative initial condition u_0 that is not an element of $\mathcal{M}_0(\mathbb{R}^d)$. While the time span of the solution does not depend specifically on any fine properties of the initial data, but only its asymptotic growth as $|x| \to \infty$ (in terms of its optimal index), the existence or otherwise of a finite limit as

$t \to T$ is more delicate. We will see below that at the maximal existence time a number of different behaviours are possible: from complete blowup, as in the example (5.8) above, to the existence of a finite limit at all points in space. We will show that by 'tuning' the initial data it is possible to obtain solutions with a finite limit only at any chosen convex subset of $\mathbb{R}^d$. These results, in turn, will depend on the integrability at the optimal index of the translate of the initial data.

In the case of pointwise-defined functions, any translation $\tau_y f(x) := f(x-y)$ has the same optimal index as f, since whenever $f \in L^1_\varepsilon(\mathbb{R}^d)$ we have $\tau_y f \in L^1_\delta(\mathbb{R}^d)$ for any $\delta > \varepsilon$:

$$\int_{\mathbb{R}^d} \mathrm{e}^{-\delta|x|^2}|f(x-y)|\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{e}^{-\delta|z+y|^2}|f(z)|\,\mathrm{d}z \le \mathrm{e}^{\delta(\frac{1}{\alpha}-1)|y|^2}\int_{\mathbb{R}^d} \mathrm{e}^{-\delta(1-\alpha)|z|^2}|f(z)|\,\mathrm{d}z < \infty$$

for $\delta(1-\alpha) \ge \varepsilon$, using (3.5). However, whether or not $\tau_y f \in L^1_{\varepsilon_0(f)}(\mathbb{R}^d)$ depends strongly on the decay at infinity of "lower order terms" of f as the examples below will show.

For the case of measures, observe that we can define translations of measures via the formula that would hold for a locally integrable f and $\varphi \in C_c(\mathbb{R}^d)$, namely

$$\int_{\mathbb{R}^d} \tau_y f(x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(z)\varphi(z+y)\,\mathrm{d}z = \int_{\mathbb{R}^d} f(z)\tau_{-y}\varphi(z)\,\mathrm{d}z.$$

That is, for $y \in \mathbb{R}^d$ and $\mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ we define $\tau_y\mu$ by setting

$$\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}\tau_y\mu(x) := \int_{\mathbb{R}^d} \tau_{-y}\varphi(z)\,\mathrm{d}\mu(z) \quad \text{for every} \quad \varphi \in C_c(\mathbb{R}^d).$$

Hence $\tau_y\mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ and is a positive measure whenever μ is. It also follows that if $\mu = \mu^+ - \mu^-$ then

$$\tau_y\mu = \tau_y\mu^+ - \tau_y\mu^- \qquad \text{and} \qquad |\tau_y\mu| = \tau_y|\mu|.$$

Lemma 5.4. *For $\mu \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, we have $\tau_y\mu \in \mathcal{M}_\delta(\mathbb{R}^d)$ for any $\delta > \varepsilon$ and*

$$\int_{\mathbb{R}^d} \mathrm{e}^{-\delta|x|^2}\,\mathrm{d}|\tau_y\mu(x)| = \int_{\mathbb{R}^d} \mathrm{e}^{-\delta|x+y|^2}\,\mathrm{d}|\mu(x)|.$$

In particular, $\tau_y\mu$ has the same optimal index as μ.

Proof. Take $\phi_k \in C_c(\mathbb{R}^d)$ such that $0 \le \phi_k \le 1$ and $\phi_k \to 1$ as $k \to \infty$ monotonically in compact sets of $\mathbb{R}^d$. Then

$$\int_{\mathbb{R}^d} \phi_k(x)\mathrm{e}^{-\delta|x|^2}\,\mathrm{d}|\tau_y\mu(x)| = \int_{\mathbb{R}^d} \phi_k(x)\mathrm{e}^{-\delta|x|^2}\,\mathrm{d}\tau_y|\mu(x)| = \int_{\mathbb{R}^d} \phi_k(x+y)\mathrm{e}^{-\delta|x+y|^2}\,\mathrm{d}|\mu(x)|.$$

Using (3.5), the right-hand side above is bounded by

$$\mathrm{e}^{\delta(\frac{1}{\alpha}-1)|y|^2}\int_{\mathbb{R}^d} \mathrm{e}^{-\delta(1-\alpha)|x|^2}\,\mathrm{d}|\mu(x)| < \infty$$

for $\delta(1-\alpha) \geq \varepsilon$. Then Fatou's Lemma gives $\displaystyle\int_{\mathbb{R}^d} \mathrm{e}^{-\delta|x|^2}\,\mathrm{d}|\tau_y\mu(x)| < \infty$. Now the Monotone Convergence Theorem gives the result. □

Now we can prove the following result on the pointwise behaviour, as $t \to T$ of the solution of the heat equation (3.2) with initial data $0 \leq u_0 \notin \mathcal{M}_0(\mathbb{R}^d)$.

Theorem 5.5. *Assume that* $0 \leq u_0 \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ *and that the optimal index* $\varepsilon_0 = \varepsilon_0(u_0)$, *see* (5.7), *satisfies* $0 < \varepsilon_0 < \infty$. *Then the solution* u *of the heat equation given by* (3.2) *is not defined (at any point* $x \in \mathbb{R}^d$*) beyond* $T := 1/4\varepsilon_0$. *Furthermore as* $t \to T$

$$u(x,t) \to \begin{cases} u(x,T) & \textit{if } \tau_{-x}u_0 \in \mathcal{M}_{\varepsilon_0}(\mathbb{R}^d) \\ \infty & \textit{if } \tau_{-x}u_0 \notin \mathcal{M}_{\varepsilon_0}(\mathbb{R}^d). \end{cases}$$

Proof. It follows from Lemma 3.2 that if $u(x,t)$ is finite for some $x \in \mathbb{R}^d$ and $t > T$ then $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ for some ε with $\varepsilon_0 > \varepsilon > \frac{1}{4t}$, which is impossible.

To analyse the limiting behaviour as $t \to T$, using Lemma 5.4 we first write, for $t < T$,

$$u(x,t) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-y|^2}{4t}}\,\mathrm{d}u_0(y) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|z|^2}{4t}}\,\mathrm{d}\tau_{-x}u_0(z).$$

Now, if $\tau_{-x}u_0 \notin \mathcal{M}_{\varepsilon_0}(\mathbb{R}^d)$ then by Fatou's Lemma

$$u(x,t) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|y|^2}{4t}}\,\mathrm{d}\tau_{-x}u_0(y) \to \infty, \quad t \to T.$$

On the other hand, if $\tau_{-x}u_0 \in \mathcal{M}_{\varepsilon_0}(\mathbb{R}^d)$ then $\mathrm{e}^{-\frac{|y|^2}{4t}} \leq \mathrm{e}^{-\varepsilon_0|y|^2}$ and using the Monotone Convergence Theorem it follows that as $t \to T$,

$$u(x,t) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|y|^2}{4t}}\,\mathrm{d}\tau_{-x}u_0(y) \to \frac{1}{(4\pi T)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon_0|y|^2}\,\mathrm{d}\tau_{-x}u_0(y) < \infty. \qquad \square$$

For a given initial data $0 \leq u_0 \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ with optimal index $0 < \varepsilon_0 = \varepsilon_0(u_0) < \infty$ we now analyse the 'regular set' of points $x \in \mathbb{R}^d$ such that the solution of the heat equation has a finite limit as $t \to T := 1/4\varepsilon_0$ as in Theorem 5.5. For short we define $A := \varepsilon_0(u_0) > 0$. Then observe that if no translation of u_0 satisfies $\tau_{-x}u_0 \in \mathcal{M}_{\varepsilon_0}(\mathbb{R}^d)$ then the solution $u(x,t)$ of the heat equation diverges to infinity at every point in $\mathbb{R}^d$ as $t \to T = 1/4A$. Otherwise, assume that $u_0 \in \mathcal{M}_{\varepsilon_0}(\mathbb{R}^d)$; then

$$u_0(x) = \mathrm{e}^{A|x|^2} v(x), \quad x \in \mathbb{R}^d \tag{5.9}$$

where $0 \leq v \in \mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^d)$. If, on the contrary $u_0 \notin \mathcal{M}_{\varepsilon_0}(\mathbb{R}^d)$ then for some $x_0 \in \mathbb{R}^d$ we have $v_0 := \tau_{-x_0}u_0 \in \mathcal{M}_{\varepsilon_0}(\mathbb{R}^d)$ and then v_0 is as in (5.9), while from Lemma 5.4 we obtain

$$u(x,t,u_0) = u(x - x_0, t, v_0), \quad x \in \mathbb{R}^d$$

so it suffices to study the 'regular set' of an initial data as in (5.9) with $0 \le v \in \mathcal{M}_{\rm BTV}(\mathbb{R}^d)$. For simplicity in the exposition we will restrict to the case $0 \le v \in L^1(\mathbb{R}^d)$. In such a case we have the following result that shows that the 'regular set' of $x \in \mathbb{R}^d$ at which $u(x,t)$ has a finite limit as $t \to T$ must be a convex set. For a converse result see Proposition 5.13 below.

Lemma 5.6. *Assume u_0 is as in (5.9) with $0 \le v \in L^1_{loc}(\mathbb{R}^d)$, so that $\varepsilon_0(u_0) = A$. Then*

(i) $\tau_{-x}u_0 \in L^1_{\varepsilon_0}(\mathbb{R}^d)$ *iff* $I_v(x) := \displaystyle\int_{\mathbb{R}^d} \mathrm{e}^{2A\langle x,z\rangle} v(z)\,\mathrm{d}z < \infty$.

(ii) *If moreover $0 \le v \in L^1(\mathbb{R}^d)$ then the set of $x \in \mathbb{R}^d$ such that $I_v(x) < \infty$ is a convex set that contains $x = 0$.*

Proof. Since $\mathrm{e}^{-A|y|^2}\tau_{-x}u_0(y) = \mathrm{e}^{A|x|^2}\mathrm{e}^{2A\langle x,y\rangle}v(x+y) = \mathrm{e}^{-A|x|^2}\mathrm{e}^{2A\langle x,x+y\rangle}v(x+y)$ part (i) follows.

For part (ii) note that since $v \in L^1(\mathbb{R}^d)$, it is always the case that

$$\int_{\langle x,y\rangle \le 0} \mathrm{e}^{\lambda 2A\langle x,y\rangle} v(y)\,\mathrm{d}y < \infty$$

whenever $\lambda \ge 0$.

Now observe that if $I_v(x) = \infty$ then for any $\lambda \ge 1$

$$\int_{\langle x,y\rangle > 0} \mathrm{e}^{\lambda 2A\langle x,y\rangle} v(y)\,\mathrm{d}y = \infty,$$

i.e. $I_v(\lambda x) = \infty$ for any $\lambda \ge 1$.

Consider now x_1, x_2 such that $I_v(x_i) < \infty$ and take $\theta \in (0,1)$. Then $I_v(\theta x_i) < \infty$ and $I_v((1-\theta)x_i) < \infty$ and

$$I_v(\theta x_1 + (1-\theta)x_2) = \int_{\mathbb{R}^d} \mathrm{e}^{2A\theta\langle x_1,y\rangle}\mathrm{e}^{2A(1-\theta)\langle x_2,y\rangle} v(y)\,\mathrm{d}y.$$

Observe that the integral in the regions where either $\langle x_1,y\rangle \le 0$ or $\langle x_2,y\rangle \le 0$ is finite while

$$\int_{\{\langle x_1,y\rangle > 0,\ \langle x_2,y\rangle > 0\}} \mathrm{e}^{2A\theta\langle x_1,y\rangle}\mathrm{e}^{2A(1-\theta)\langle x_2,y\rangle} v(y)\,\mathrm{d}y$$

can be split in the regions $\langle x_1,y\rangle \ge \langle x_2,y\rangle$ and $\langle x_1,y\rangle < \langle x_2,y\rangle$. In both of these the integral is finite, which completes the proof. □

Now we give some examples of initial data as in (5.9) and explicitly compute its regular set. In our first example $I_v(x)$ is never finite so we obtain complete blowup, generalising the example in (5.8).

Example 5.7. *If we take $v(x) \geq c > 0$ for all $x \in \mathbb{R}^d$ then $\varepsilon_0(u_0) = A$ and in* (5.9) *we have $\tau_{-x}u_0 \notin L^1_A(\mathbb{R}^d)$ for every $x \in \mathbb{R}^d$.*

In this case the solution u of the heat equation given by (3.2) and initial data (5.9) blows up at every point in $\mathbb{R}^d$ at time $T = \frac{1}{4A}$.

In our next example $I_v(x)$ is only finite at $x = 0$ so the regular set consists of a single point at the origin. Thus the solution u of the heat equation given by (3.2) has a finite limit at $x = 0$ as $t \to T$, but blows up at all other points of $\mathbb{R}^d$.

Example 5.8. *When $v(x) = (1+|x|^2)^{-\alpha/2}$ with $\alpha > d/2$ we have $\varepsilon_0(u_0) = A$ and in (5.9) we have $\tau_{-x}u_0 \in L^1_A(\mathbb{R}^d)$ only when $x = 0$.*

To see this we write $y = sx + y'$ with $y' \perp x$ to get

$$I(x) = \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{2A|x|^2 s}}{(1 + s^2|x|^2 + |y'|^2)^{\alpha/2}} \, \mathrm{d}s \, \mathrm{d}y'$$

If $x \neq 0$ then the integral in s is infinity for each $y' \in \mathbb{R}^{N-1}$.

In the next example $I_v(x)$ is finite only in the open ball $|x| < \gamma/2A$; the solution u of the heat equation given by (3.2) has a finite limit here as $t \to T$, but blows up at all other points of $\mathbb{R}^d$.

Example 5.9. *Take $v(x) = \mathrm{e}^{-\gamma|x|}$ with $\gamma > 0$. Then $\varepsilon_0(u_0) = A$ and in (5.9) we have $\tau_{-x}u_0 \in L^1_A(\mathbb{R}^d)$ if and only if $|x| < \frac{\gamma}{2A}$.*

To see this note first that

$$0 \leq \mathrm{e}^{2A\langle x,y\rangle} v(y) \leq \mathrm{e}^{(2A|x|-\gamma)|y|}$$

which is integrable if $|x| < \frac{\gamma}{2A}$. On the other hand writing $y = sx + y'$ with $y' \perp x$

$$I(x) = \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} \mathrm{e}^{2A|x|^2 s} \mathrm{e}^{-\gamma\sqrt{s^2|x|^2+|y'|^2}} \, \mathrm{d}s \, \mathrm{d}y'.$$

If $2A|x|^2 - \gamma|x| \geq 0$, that is, $|x| \geq \frac{\gamma}{2A}$, then the integral in s is infinite for each $y' \in \mathbb{R}^{d-1}$.

It is also possible to make $I_v(x)$ finite only on a closed ball.

Example 5.10. *Take $v(x) = \mathrm{e}^{-\gamma|x|}(1+|x|^2)^{-\alpha/2}$ with $\alpha > d/2$ and $\gamma > 0$. Then $\varepsilon_0(u_0) = A$ and in (5.9) we have $\tau_{-x}u_0 \in L^1_A(\mathbb{R}^d)$ if and only if $|x| \leq \frac{\gamma}{2A}$.*

To see this note first that

$$0 \leq \mathrm{e}^{2A\langle x,y\rangle} v(y) \leq \mathrm{e}^{(2A|x|-\gamma)|y|}(1 + |y|^2)^{-\alpha/2}$$

which is integrable if $|x| \le \frac{\gamma}{2A}$. On the other hand, if $x \neq 0$ writing $y = sx + y'$ with $y' \perp x$

$$I_v(x) = \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} \mathrm{e}^{2A|x|^2 s} \mathrm{e}^{-\gamma\sqrt{s^2|x|^2+|y'|^2}} \frac{1}{(1+s^2|x|^2+|y'|^2)^{\alpha/2}} \, \mathrm{d}s \, \mathrm{d}y'.$$

If $2A|x|^2 - \gamma|x| > 0$, that is, $|x| > \frac{\gamma}{2A}$, then the integral in s is infinite for each $y' \in \mathbb{R}^{d-1}$.

Our last example is perhaps the most striking: here $I_v(x)$ is finite for all $x \in \mathbb{R}^d$.

Example 5.11. *Take* $v(x) = \mathrm{e}^{-\gamma|x|^\alpha}$ *with* $\gamma > 0$, $1 < \alpha < 2$. *Then* $\varepsilon_0(u_0) = A$ *and in (5.9) we have* $\tau_{-x}u_0 \in L^1_A(\mathbb{R}^d)$ *for any* $x \in \mathbb{R}^d$.

To see this note that

$$0 \le \mathrm{e}^{2A\langle x,y\rangle} v(y) = \mathrm{e}^{2A<x,y>-\gamma|y|^\alpha} \le \mathrm{e}^{2A|x||y|-\gamma|y|^\alpha} \in L^1(\mathbb{R}^d).$$

Thus for the initial data $u_0(x) = \mathrm{e}^{A|x|^2-\gamma|x|^\alpha}$ the solution $u(x,t)$ of the heat equation takes a finite value at every point in $\mathbb{R}^d$ at $T = 1/4A$, but cannot be continued beyond this time.

We now show that in fact we can arrange for the regular set, $\{x: \ I_v(x) < \infty\}$, to be any chosen closed convex subset of $\mathbb{R}^d$. First we recall the following characterisation of such sets.

Lemma 5.12. *Any closed convex set with* $0 \in K$ *is of the form*

$$K = \bigcap_{j\in J} \{x: \ \langle x, n_j\rangle \le c_j\}$$

for some unit vectors n_j *and* $c_j \ge 0$, *where* J *is at most countable.*

Proof. Note first that K is the intersection of all closed half spaces containing K. Then observe that $K = \bigcap_{y\in\partial K}\{x: \ \langle x, n(y)\rangle \le c(y)\}$ for some unit vectors $n(y)$ and constants $c(y) \ge 0$, see e.g. [24].

This implies that, $\bigcup_{y\in\partial K}\{x: \ \langle x, n(y)\rangle > c(y)\}$ is an open covering of the open set $\mathbb{R}^d \setminus K$. Thus we can extract an, at most, countable covering. □

Note that the form of K_0 in the following results is more general than that given by the previous lemma; in particular it allows for any closed convex set.

Proposition 5.13. *Assume that* $K \subset \mathbb{R}^d$ *is a convex set given by the intersection of at most a countable number of half spaces, that is,* $K = x_0 + K_0$ *with*

$$0 \in K_0 = \bigcap_{j\in J_1}\{x: \ \langle x, n_j\rangle \le c_j\} \cap \bigcap_{j\in J_2}\{x: \ \langle x, n_j\rangle < c_j\} \tag{5.10}$$

for some unit vectors n_j *and* $c_j \geq 0$ *for* $j \in J_1$ *and* $c_j > 0$ *for* $j \in J_2$*, where* J_1 *and* J_2 *are at most countable and* $\inf_{j\in J_2} c_j > 0$.

Then there exist $0 \leq v \in L^1_{\rm loc}(\mathbb{R}^d)$ *such that*

$$I_v(x) < \infty \quad \textit{if and only if } x \in K:$$

the solution u *of the heat equation with initial data* $u_0(x) = \mathrm{e}^{A|x|^2} v(x)$ *has a finite limit at every point* $x \in K$ *but blows up at every other point in* $\mathbb{R}^d$ *as* $t \to \frac{1}{4A}$.

Proof. Assume first that $x_0 = 0$. Take any orthonormal basis $\mathcal{B} = \{e_j\}_j$ of $\mathbb{R}^d$. Using coordinates with respect to this basis, write $x \in \mathbb{R}^d$ as (x_1, x') and $y = (y_1, y')$. We choose $\eta_0 \geq 0$ and set

$$0 \leq v(x) = \begin{cases} \chi(x')\phi(x_1)\mathrm{e}^{-\eta_0 x_1}, & x_1 > 0 \\ 0 & x_1 < 0, \end{cases} \qquad 0 \leq w(x) = \begin{cases} \chi(x')\mathrm{e}^{-\eta_0 x_1}, & x_1 > 0 \\ 0 & x_1 < 0, \end{cases}$$

where χ is the characteristic function of the unit ball in $\mathbb{R}^{d-1}$ and $\phi(s) = \frac{1}{1+s^2}$. Note that $v \in L^1(\mathbb{R}^d)$ with $\|v\|_{L^1(\mathbb{R}^d)} \leq M$ and if $\eta_0 > 0$ then $w \in L^1(\mathbb{R}^d)$ with $\|w\|_{L^1(\mathbb{R}^d)} \leq \frac{M}{\eta_0}$ with M independent of η_0. Also, for $x \in \mathbb{R}^d$

$$\begin{aligned} I_v(x) = \int_{\mathbb{R}^d} \mathrm{e}^{2A\langle x,y\rangle} v(y)\,\mathrm{d}y &= \int_0^\infty \int_{|y'|\leq 1} \mathrm{e}^{(2Ax_1-\eta_0)y_1} \mathrm{e}^{2A\langle x',y'\rangle} \phi(y_1)\,\mathrm{d}y'\,\mathrm{d}y_1 \\ &= \left(\int_{|y'|\leq 1} \mathrm{e}^{2A\langle x',y'\rangle}\,\mathrm{d}y' \right) \left(\int_0^\infty \mathrm{e}^{(2Ax_1-\eta_0)y_1} \phi(y_1)\,\mathrm{d}y_1 \right). \end{aligned}$$

The first factor is always finite, and the second is finite if $x_1 \leq \frac{\eta_0}{2A}$ and infinite if $x_1 > \frac{\eta_0}{2A}$. So choosing $\eta_0 \geq 0$ appropriately, for any given $c \geq 0$ we can ensure that $I_v(x) < \infty$ iff $x \in \{z : \ z_1 \leq c\}$.

An analogous computation with w gives that for any given $c > 0$ we obtain $I_w(x) < \infty$ iff $x \in \{z : \ z_1 < c\}$.

Hence for any given unit vector n and $c \geq 0$ (respectively $c > 0$) we can find an integrable function $v = v(n, c)$ ($w = w(n, c)$ respectively) such that

$$I_v(x) < \infty \quad \text{only in the half space } \langle x, n\rangle \leq c \quad (\langle x, n\rangle < c \text{ respectively})$$

with $\|v\|_{L^1(\mathbb{R}^d)}$ bounded independent of n and $c \geq 0$ ($\|w\|_{L^1(\mathbb{R}^d)} \leq \frac{M}{c}$, M independent of n).

Based on the assumed form of K_0 in (5.10) we set

$$v_0(x) = \sum_{j\in J_1} j^{-2} v_j(x) + \sum_{j\in J_2} j^{-2} w_j(x)$$

where $v_j(x) = v(n_j, c_j)(x)$ and $w_j(x) = w(n_j, c_j)(x)$ are constructed as above. Since $\inf_{j\in J_2} c_j > 0$ then $v_0 \in L^1(\mathbb{R}^d)$ and clearly $I_{v_0}(x) < \infty$ iff $x \in K_0$.

Now, for $x_0 \neq 0$, define

$$v(x) = e^{-2A\langle x_0, x\rangle} v_0(x), \quad x \in \mathbb{R}^d.$$

Then $I_v(x) = \displaystyle\int_{\mathbb{R}^d} \mathrm{e}^{2A\langle x-x_0, y\rangle} v_0(y)\,\mathrm{d}y < \infty$ iff $x - x_0 \in K_0$. □

5.2 Continuation of signed solutions

For signed solutions the maximal existence time of the solution may not be given by $T = \frac{1}{4\varepsilon_0(u_0)}$ as in Theorem 5.5; see Section 6.4. However, we can establish the following continuation result.

Proposition 5.14. *Assume that $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ and that $u(t; u_0) = S(t)u_0$ given by (3.2) is defined on $[0, T)$ but cannot be defined any time after. Then for any $\delta > 0$*

$$\limsup_{t\to T} \|u(t, u_0)\|_{L^1_\delta(\mathbb{R}^d)} = \infty.$$

Proof. Assume otherwise that for some $\delta > 0$ (which, without loss of generality, we can take such that $\delta > \varepsilon$)

$$\|u(t, u_0)\|_{L^1_\delta(\mathbb{R}^d)} \leq M, \quad 0 \leq t < T.$$

Take $t_0 < T$ such that $T < t_0 + T(\delta)/2$ and define $v_0 = S(t_0)u_0 \in L^1_\delta(\mathbb{R}^d)$. Then define

$$U(t) = \begin{cases} S(t)u_0 & 0 \leq t < t_0 \\ S(t-t_0)v_0 & t_0 \leq t < t_0 + T(\delta), \end{cases} \qquad 0 \leq t < t_0 + T(\delta).$$

Then we claim that U satisfies assumptions (4.2), (4.6), (4.7) in $[0, t_0 + T(\delta))$. Hence, Theorem 4.1 implies $U(t) = S(t)u_0$ for $0 \leq t < t_0 + T(\delta)$ which contradicts the maximality of T.

To prove the claim, notice that (4.7) is satisfied (using Theorem 3.6). Also, (4.6) holds because of the assumption on u and by part (i) in Proposition 3.7 applied to $S(t - t_0)v_0$ for $t_0 \leq t \leq t_0 + \tau$ for any $\tau < T(\delta)$. Finally (4.2) follows from (3.3) and (3.21) applied to $S(t)u_0$ with $0 \leq t < t_0$ and $S(t - t_0)v_0$ with $t_0 \leq t \leq t_0 + \tau$ for any $\tau < T(\delta)$. □

6 Long-time behaviour of heat solutions

We now discuss the asymptotic behaviour as $t \to \infty$ of solutions when $u_0 \in \mathcal{M}_0(\mathbb{R}^d)$. For this Lemma 3.1 will be a central tool. We start with some simple consequences of this result, which show that the asymptotic behaviour is largely determined by the behaviour at $x = 0$. Observe that the converse of parts (i), (ii), (iii), and (v) are obviously true.

Proposition 6.1. *Assume that $u_0 \in \mathcal{M}_0(\mathbb{R}^d)$.*

(i) *If $u(0,t,|u_0|)$ is bounded for $t > 0$ then $u(t,u_0)$ remains uniformly bounded in sets $\frac{|x|}{\sqrt{t}} \leq R$. In particular, $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d \times (0,\infty))$.*

(ii) *If $u(0,t,|u_0|) \to 0$ as $t \to \infty$ then $u(t,u_0) \to 0$ uniformly in sets $\frac{|x|}{\sqrt{t}} \leq R$.*

Assume in addition that $u_0 \geq 0$. Then

(iii) *If $\lim_{t\to\infty} u(0,t,u_0) = L \in (0,\infty)$ exists, then $u(x,t,u_0) \to L$ uniformly in compact sets of $\mathbb{R}^d$.*

(iv) *If $u(0,t,u_0)$ is unbounded for $t > 0$ then $u(t,u_0)$ is unbounded in sets $\frac{|x|}{\sqrt{t}} \leq R$, and so in particular unbounded in any compact subset of $\mathbb{R}^d$.*

(v) *If $u(0,t,u_0) \to \infty$ as $t \to \infty$ then $u(t,u_0) \to \infty$ uniformly in sets $\frac{|x|}{\sqrt{t}} \leq R$.*

Proof. (i) and (ii). From the upper bound in Lemma 3.1 in sets with $\frac{|x|^2}{t} \leq R$ we get

$$|u(x,t)| \leq c_{d,a} u(0,at,|u_0|) \mathrm{e}^{\frac{|x|^2}{4(a-1)t}} \leq c_{d,a} u(0,at,|u_0|) \mathrm{e}^{\frac{R^2}{4(a-1)}}.$$

Assume furthermore that $0 \leq u_0 \in \mathcal{M}_0(\mathbb{R}^d)$. Then

(iii) Using the lower and upper bounds in Lemma 3.1 if $|x|^2 \leq R$ we get for every $b < 1 < a$

$$b^{d/2} u(0,bt) \mathrm{e}^{-\frac{R^2}{4(1-b)t}} \leq u(x,t) \leq a^{d/2} u(0,at) \mathrm{e}^{\frac{R^2}{4(a-1)t}}.$$

(iv) and (v) From the lower bound in Lemma 3.1 in sets with $\frac{|x|^2}{t} \leq R$ we get

$$c_{d,b} u(0,bt) \mathrm{e}^{-\frac{R^2}{4(1-b)}} \leq \inf_{\frac{|x|^2}{t} \leq R} u(x,t).$$ □

Recalling the definition of the $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ norm (3.9) observe that

$$u(0,t,|u_0|) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|y|^2}{4t}} \, \mathrm{d}|u_0(y)| = \|u_0\|_{\mathcal{M}_{1/4t}(\mathbb{R}^d)} \tag{6.1}$$

hence Proposition 6.1 could easily be restated in terms of the behavior of the norms $\|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)}$ as $\varepsilon \to 0$. In particular $u(t, u_0)$ remains uniformly bounded in sets $\frac{|x|}{\sqrt{t}} \le R$ if and only if

$$\sup_{\varepsilon>0} \|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} < \infty.$$

Also notice that part (iii) implies that there are no other stationary solutions of (1.1) other than constants. In other words, a harmonic function in $\mathcal{M}_0(\mathbb{R}^d)$ must be constant.

6.1 Sufficient conditions for decay

As observed above, solutions converge to zero as $t \to \infty$ if and only if

$$\lim_{\varepsilon\to 0} \|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} = 0.$$

We now give some (non-sharp) conditions to ensure this, in terms of the distribution of mass of the initial condition measured in terms of the averages over balls. Note that from Proposition 6.1 this behaviour is determined by the value of the solution at $x = 0$.

Theorem 6.2. *Suppose that $u_0 \in \mathcal{M}_0(\mathbb{R}^d)$.*

(i) *If*

$$\frac{1}{R^d} \int_{R/2\le|x|\le R} \mathrm{d}|u_0(x)| \le M$$

then u remains uniformly bounded in sets of the form $\frac{|x|}{\sqrt{t}} \le R$ for any $R > 0$; in particular, $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d \times (0, \infty))$.

(ii) *If*

$$\lim_{R\to\infty} \frac{1}{R^d} \int_{R/2\le|x|\le R} \mathrm{d}|u_0(x)| = 0;$$

then $u(0, t) \to 0$ as $t \to \infty$ and hence $u(t) \to 0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ and uniformly in sets $\frac{|x|}{\sqrt{t}} \le R$.

Assume in addition that $u_0 \ge 0$. Then

(iii) *If*

$$\liminf_{R\to\infty} \frac{1}{R^d} \int_{|x|\le R} \mathrm{d}u_0(x) > 0,$$

then $\liminf_{t\to\infty} u(0, t) > 0$.

(iv) *If*

$$\lim_{R\to\infty} \frac{1}{R^d} \int_{|x|\le R} \mathrm{d}u_0(x) = \infty$$

then $u(0,t) \to \infty$ *as* $t \to \infty$.

Proof. (i) Consider

$$|u(0,t)| = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|x|^2/t}\, \mathrm{d}|u_0(x)| = \frac{1}{(4\pi t)^{d/2}} \sum_{k=-\infty}^{\infty} \int_{2^k\le|x|\le 2^{k+1}} \mathrm{e}^{-|x|^2/t}\, \mathrm{d}|u_0(x)|.$$

Note that

$$\begin{aligned}\int_{R\le|x|\le 2R} \mathrm{e}^{-|x|^2/4t}\, \mathrm{d}|u_0(x)| &\le \mathrm{e}^{-R^2/4t} \int_{R\le|x|\le 2R} \mathrm{d}|u_0(x)| \\ &\le (2R)^d M \mathrm{e}^{-R^2/4t} \le 2^{d+1} M \int_{R/2\le|x|\le R} \mathrm{e}^{-|x|^2/4t}\, \mathrm{d}x,\end{aligned}$$

and so

$$\begin{aligned}|u(0,t)| &\le 2^{d+1} M \frac{1}{(4\pi t)^{d/2}} \sum_{k=-\infty}^{\infty} \int_{2^{k-1}\le|x|\le 2^k} \mathrm{e}^{-|x|^2/4t}\, \mathrm{d}x \\ &= 2^{d+1} M \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|x|^2/4t}\, \mathrm{d}x = 2^{d+1} M.\end{aligned}$$

(ii) Given $\varepsilon > 0$ take $k_0 > 0$ such that

$$\frac{1}{R^d} \int_{R/2\le|x|\le R} \mathrm{d}|u_0(x)| < \varepsilon$$

for all $R \ge R_0 := 2^{k_0}$.

Then we can split the domain of integration in the integral expression for $|u(0,t)|$ into $|x| \le R_0$ and $|x| > R_0$. The above argument shows that the integral over the unbounded region contributes at most $2^{d+1}\varepsilon$ for all $t > 0$, while the integral over the bounded region contributes no more than

$$\frac{1}{(4\pi t)^{d/2}} \int_{|x|\le R_0} \mathrm{d}|u_0(x)| \le \frac{c}{t^{d/2}}.$$

It follows that $|u(0,t)| \to 0$ as $t \to \infty$.

(iii) There exists an R_0 and $m > 0$ such that

$$\frac{1}{R^d} \int_{|x|\le R} \mathrm{d}u_0(x) \ge m > 0$$

for all $R \geq R_0$. Then for t sufficiently large such that $\sqrt{t} > R_0$ we have

$$u(0,t) \geq \frac{1}{(4\pi t)^{d/2}} \int_{|x|\leq\sqrt{t}} \mathrm{e}^{-|x|^2/4t}\, \mathrm{d}u_0(x) \geq \frac{1}{(4\pi t)^{d/2}} \mathrm{e}^{-1/4} m t^{d/2} = \frac{1}{(4\pi)^{d/2}} \mathrm{e}^{-1/4} m.$$

(iv) We repeat the above argument, taking m arbitrary. □

6.2 Wild behaviour of solutions as $t \to \infty$.

Theorem 6.2 gives conditions on the averages over balls to distinguish between various time-asymptotic regimes; but in the case that

$$\liminf_{R\to\infty} \frac{1}{R^d} \int_{|x|\leq R} \mathrm{d}u_0(x) > 0$$

and

$$\frac{1}{R^d} \int_{|x|\leq R} \mathrm{d}u_0(x) \not\to \infty \qquad \text{as} \quad R \to \infty$$

some very rich behaviour is possible.

In the following theorem we show that there is initial data in $L_0^1(\mathbb{R}^d)$ that gives rise to unbounded oscillating solutions. For bounded oscillations in the case of bounded initial data and solutions, see also Section 6.3 below and [27].

Theorem 6.3. *For any sequence of non-negative numbers $\{\alpha_k\}_k$ there exists a non-negative $u_0 \in L_0^1(\mathbb{R}^d)$ and a sequence $t_n \to \infty$ such that for every k there exists a subsequence $t_{k,j}$ with $u(0, t_{k,j}) \to \alpha_k$ as $j \to \infty$.*

In fact it is enough to prove the following (apparently weaker) result.

Proposition 6.4. *Given any sequence of non-negative numbers $\{b_k\}_k$ there exists a non-negative $u_0 \in L_0^1(\mathbb{R}^d)$ and a sequence $t_k \to \infty$ such that, as $k \to \infty$*

$$|u(0, t_k) - b_k| \to 0.$$

Indeed, given a sequence $\{\alpha_k\}_k$ as in the statement of Theorem 6.3 construct the sequence $\{b_k\}_k$ as

$$\alpha_1|\alpha_1, \alpha_2|\alpha_1, \alpha_2, \alpha_3|\alpha_1, \ldots, \alpha_4|\alpha_1, \ldots, \alpha_5| \ldots.$$

Now apply Proposition 6.4 and note that for any $k \in \mathbb{N}$ there is a subsequence k_j such that $b_{k_j} = \alpha_k$.

We now prove Proposition 6.4, inspired by the proof of Lemma 6 in [27].

Proof. Observe, with Proposition 6.1 in mind, that we can write

$$u(0,t) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|y|^2}{4t}} u_0(y)\,\mathrm{d}y = \frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|z|^2} u_0(2\sqrt{t}z)\,\mathrm{d}z.$$

So if $\lambda_n \to \infty$,

$$u(0,t_n) = \frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|z|^2} u_0(\lambda_n z)\,\mathrm{d}z. \tag{6.2}$$

with $\lambda_n = 2\sqrt{t_n}$. We set $c_d =: \frac{1}{\pi^{d/2}}$.

Consider for $r > 1$ the annulus $A(r) = \{y,\ r^{-1} < |y| < r\}$ and given the sequence $\{b_k\}_k$ consider a function

$$u_0(x) = \sum_j b_j \mathcal{X}_{\lambda_j A(r_j)}(x)$$

for some increasing and divergent sequences $\{\lambda_k\}_k$, $\{r_k\}_k$ chosen recursively as follows: first we choose r_k large with respect to the sequence $\{b_k\}_k$, according to

$$2^k \beta_{k-1} < r_{k-1}^d \tag{6.3}$$

where $\beta_k = \max_{1\le j\le k}\{b_k\}$ and

$$b_k c_d \int_{\mathbb{R}^d\setminus A(r_k)} \mathrm{e}^{-|x|^2}\,\mathrm{d}x < 2^{-k}. \tag{6.4}$$

Then choose λ_k sufficiently large that

$$2^k b_k r_k^{3d} < \lambda_k^d \tag{6.5}$$

and

$$b_k \exp\left(-\frac{\lambda_k^2}{\lambda_{k-1}^2 r_k^2}\right)\lambda_k^d r_k^d < 2^{-k}. \tag{6.6}$$

Finally choose the next value λ_{k+1} large enough that

$$\lambda_k r_k < \frac{\lambda_{k+1}}{r_{k+1}}. \tag{6.7}$$

Step 1. Observe that from (6.7) the scaled annulae $\lambda_j A(r_j)$ are disjoint and increasing.

Step 2. Now we prove that from (6.5), we get $u_0 \in L^1_0(\mathbb{R}^d)$. For this take any $\varepsilon > 0$ and then

$$\int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2} u_0(x)\,\mathrm{d}x = \sum_j b_j \int_{\lambda_j A(r_j)} \mathrm{e}^{-\varepsilon|x|^2}\,\mathrm{d}x \le \sum_j b_j \mathrm{e}^{-\varepsilon\lambda_j^2/r_j^2}\lambda_j^d r_j^d.$$

Now for any $m \in \mathbb{N}$ there exist $R_\varepsilon, c_\varepsilon$ (depending on m as well) such that if $z \ge R_\varepsilon$ then

$$\mathrm{e}^{-\varepsilon z} \le \frac{c_\varepsilon}{z^m}.$$

Since from (6.7) we get $\frac{\lambda_k}{r_k} \to \infty$, then for some $j_0 \in \mathbb{N}$, we get

$$\sum_{j \geq j_0} b_j \mathrm{e}^{-\varepsilon \lambda_j^2/r_j^2} \lambda_j^d r_j^d \leq c_\varepsilon \sum_{j \geq j_0} b_j \frac{r_j^{2m+d}}{\lambda_j^{2m-d}}.$$

For example with $m = d$ we get, by (6.5),

$$c_\varepsilon \sum_{j \geq j_0} b_j \frac{r_j^{3d}}{\lambda_j^d} \leq c_\varepsilon \sum_{j \geq j_0} 2^{-j} < \infty.$$

Step 3. Now we prove that from (6.3), (6.4) and (6.6) then $|u(0,t_k) - b_k| \to 0$.

For this observe that for any $\lambda > 0$

$$u_0(\lambda x) = \sum_j b_j \mathcal{X}_{\lambda_j \lambda^{-1} A(r_j)}(x)$$

and then for each k we have in (6.2)

$$\int_{\mathbb{R}^d} \mathrm{e}^{-|z|^2} u_0(\lambda_k z)\, \mathrm{d}z = \sum_{1 \leq j \leq k-1} b_j \int_{\lambda_j \lambda_k^{-1} A(r_j)} \mathrm{e}^{-|z|^2}\, \mathrm{d}z + b_k \int_{A(r_k)} \mathrm{e}^{-|z|^2}\, \mathrm{d}z$$
$$+ \sum_{j \geq k+1} b_j \int_{\lambda_j \lambda_k^{-1} A(r_j)} \mathrm{e}^{-|z|^2}\, \mathrm{d}z. \tag{6.8}$$

Then the first term in (6.8) is bounded by

$$\beta_{k-1} \frac{\lambda_{k-1}^d}{\lambda_k^d} r_{k-1}^d < \frac{\beta_{k-1}}{r_{k-1}^d} < 2^{-k}$$

by (6.3), where we used (6.7).

For the second term in (6.8) observe that by (6.4)

$$\left| b_k \int_{A(r_k)} \mathrm{e}^{-|z|^2}\, \mathrm{d}z - b_k \right| < \frac{1}{2^k c_d}.$$

Finally, observe that the third term in (6.8) is bounded by

$$\sum_{j \geq k+1} b_j \exp\left(-\frac{\lambda_j^2}{\lambda_k^2 r_j^2}\right) \frac{\lambda_j^d}{\lambda_k^d} r_j^d \leq \sum_{j \geq k+1} b_j \exp\left(-\frac{\lambda_j^2}{\lambda_{j-1}^2 r_j^2}\right) \lambda_j^d r_j^d \leq \sum_{j \geq k+1} 2^{-j}$$

by (6.6).

Then, from (6.2) and the bounds above on the three terms in (6.8) we get, with $\lambda_k = 2\sqrt{t_k}$

$$\left| u(0,t_k) - b_k \right| \leq \frac{c_d}{2^k} + \frac{1}{2^k} + c_d \sum_{j \geq k+1} 2^{-j} \to 0, \qquad k \to \infty. \qquad \square$$

The next result shows that the oscillatory behavior in Theorem 6.3 is somehow generic for heat solutions. For this, given a sequence of positive numbers $\alpha = \{\alpha_k\}_k$ denote $\mathscr{O}_\alpha$ the nonempty family of $0 \le u_0 \in L^1_0(\mathbb{R}^d)$ that satisfy the statement in Theorem 6.3.

We use the topology on $L^1_0(\mathbb{R}^d)$ generated by the family of $L^1_\varepsilon(\mathbb{R}^d)$ norms defined in (3.7), which makes $L^1_0(\mathbb{R}^d)$ into a Fréchet space (see [23] for more details); the following more explicit definition is sufficient for our statement of the following theorem: we say that $u_n \to u_0$ in $L^1_0(\mathbb{R}^d)$ if and only if $u_n \to u_0$ in $L^1_\varepsilon(\mathbb{R}^d)$ for every $\varepsilon > 0$. Note that, in particular, such convergence implies that $u_n \to u_0$ in $L^1_{\text{loc}}(\mathbb{R}^d)$.

Theorem 6.5. *For any sequence of positive numbers $\alpha = \{\alpha_k\}_k$, $\mathscr{O}_\alpha$ is dense in $L^1_0(\mathbb{R}^d)$.*

Proof. Denote by U_0 the initial data constructed in Theorem 6.3. Then $U_0 \in \mathscr{O}_\alpha$ and for any $n \in \mathbb{N}$, $U_0\mathcal{X}_{\mathbb{R}^d\setminus B(0,n)} \in \mathscr{O}_\alpha$ since we are only suppressing a finite number of annulae in U_0.

Then for given $0 \le v_0 \in L^1_0(\mathbb{R}^d)$ define

$$v_0^n = v_0\mathcal{X}_{B(0,n)} + U_0\mathcal{X}_{\mathbb{R}^d\setminus B(0,n)} \in \mathscr{O}_\alpha.$$

Take $\varepsilon > 0$; since

$$v_0^n - v_0 = (v_0 - U_0)\mathcal{X}_{\mathbb{R}^d\setminus B(0,n)}$$

we have

$$\|v_0^n - v_0\|_{L^1_\varepsilon(\mathbb{R}^d)} = \left(\frac{\varepsilon}{\pi}\right)^{d/2} \int_{|x|\ge n} \mathrm{e}^{-\varepsilon|x|^2}|v_0 - U_0|;$$

since $v_0, U_0 \in L^1_0(\mathbb{R}^d) \subset L^1_\varepsilon(\mathbb{R}^d)$, it follows that $v_0^n \to v_0$ in $L^1_0(\mathbb{R}^d)$. □

The following result shows that any heat solution can be "shadowed" as close as we want, in any large time interval and any large compact set by an oscillatory solution of the heat equation.

Theorem 6.6. *For any sequence of positive numbers $\alpha = \{\alpha_k\}_k$ and any $0 \le v_0 \in L^1_0(\mathbb{R}^d)$, any $\delta > 0$ and $T > 0$ and any compact set $K \subset \mathbb{R}^d$, there exists $u_0 \in \mathscr{O}_\alpha$ such that*

$$\sup_{K\times[0,T]} |u(x,t,v_0) - u(x,t,u_0)| \le \delta.$$

Proof. Observe that it is enough to find $u_0 \in \mathscr{O}_\alpha$ such that

$$\sup_{[0,T+1]} u(t,0,|v_0 - u_0|) \le \delta. \tag{6.9}$$

In such a case, from (3.3) we would get, for any $a > 1$

$$\sup_{K\times[0,T]} |u(x,t,v_0) - u(x,t,u_0)| \le c_{d,a} \sup_{K\times[0,T]} u(0,at,|v_0 - u_0|)\mathrm{e}^{\frac{|x|^2}{4(a-1)t}} \le C(K,T)\delta.$$

Denote by $U_0 \in \mathscr{O}_\alpha$ the initial data constructed in Theorem 6.3. Then for given $0 \le v_0 \in L^1_0(\mathbb{R}^d)$ define

$$u_0 = v_0 \mathcal{X}_{B(0,R)} + U_0 \mathcal{X}_{\mathbb{R}^d \setminus B(0,R)} \in \mathscr{O}_\alpha.$$

Then we show that (6.9) holds provided we take R large enough. For this, observe that

$$|v_0 - u_0| \le |v_0 - U_0| \mathcal{X}_{\mathbb{R}^d \setminus B(0,R)}$$

hence

$$0 \le u(0,t,|v_0 - u_0|) \le \frac{1}{(4\pi t)^{d/2}} \int_{|y| \ge R} \mathrm{e}^{-\frac{|y|^2}{4t}} |v_0(y) - U_0(y)| \, \mathrm{d}y. \tag{6.10}$$

Taking $R > 1$ we have, for any given $0 < t_0 < T+1$ and $0 < \alpha < 1$, $|y|^2 \ge \alpha |y|^2 + (1-\alpha)$ and then for $0 < t < t_0$ we obtain in (6.10)

$$0 \le u(0,t,|v_0 - u_0|) \le \frac{1}{(4\pi t)^{d/2}} \mathrm{e}^{-(1-\alpha)\frac{1}{4t}} \int_{|y| \ge 1} \mathrm{e}^{-\alpha \frac{|y|^2}{4t_0}} |v_0(y) - U_0(y)| \, \mathrm{d}y \le \frac{\delta}{2}$$

provided t_0 is small enough since $\frac{1}{(4\pi t)^{d/2}} \mathrm{e}^{-(1-\alpha)\frac{1}{4t}} \to 0$ as $t \to 0$.

Now for $t_0 < t < T+1$ we obtain in (6.10)

$$0 \le u(0,t,|v_0 - u_0|) \le \frac{1}{(4\pi t_0)^{d/2}} \int_{|y| \ge R} \mathrm{e}^{-\frac{|y|^2}{4(T+1)}} |v_0(y) - U_0(y)| \, \mathrm{d}y \le \frac{\delta}{2}$$

provided R is sufficiently large. □

6.3 The rescaling approach of Vázquez & Zuazua

For the case of solutions of the heat equation that remain locally bounded, the results in Propositions 6.1 and 6.4 and Theorem 6.3 can be revisited in terms of the rescaling argument of [27] as follows. As we now show, it is relatively straightforward to extend their approach from $L^\infty(\mathbb{R}^d)$ initial data to more general measure-valued data that leads to globally bounded solutions.

(i) We can define dilatations of measures through the analogous result holding for a locally integrable f and $\varphi \in C_c(\mathbb{R}^d)$, namely $f_\lambda(x) = f(\lambda x)$ satisfies

$$\int_{\mathbb{R}^d} f_\lambda(x) \varphi(x) \, \mathrm{d}x = \frac{1}{\lambda^d} \int_{\mathbb{R}^d} f(z) \varphi(\frac{z}{\lambda}) \, \mathrm{d}z = \frac{1}{\lambda^d} \int_{\mathbb{R}^d} f(z) \varphi_{\frac{1}{\lambda}}(z) \, \mathrm{d}z.$$

That is, for $\lambda > 0$ and $\mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \varphi(z) \, \mathrm{d}\mu_\lambda(z) := \frac{1}{\lambda^d} \int_{\mathbb{R}^d} \varphi_{\frac{1}{\lambda}}(z) \, \mathrm{d}\mu(z).$$

Hence $\mu_\lambda \in \mathcal{M}_{\rm loc}(\mathbb{R}^d)$ and is a positive measure whenever μ is. Then it follows that

$$\int_{\mathbb{R}^d} \varphi(z)\,\mathrm{d}|\mu_\lambda(z)| := \frac{1}{\lambda^d}\int_{\mathbb{R}^d} \varphi_{\frac{1}{\lambda}}(z)\,\mathrm{d}|\mu(z)|.$$

These extend, by density, to $\varphi \in L^1(\mathrm{d}\mu) = L^1(\mathrm{d}\mu_\lambda)$.

(ii) For $\mu \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ and $\varepsilon > 0$, $\lambda > 0$

$$\|\mu_\lambda\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} = \Big(\frac{\varepsilon}{\pi}\Big)^{d/2}\int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2}\,\mathrm{d}|\mu_\lambda(x)| = \Big(\frac{\varepsilon}{\pi\lambda^2}\Big)^{d/2}\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{\varepsilon}{\lambda^2}|y|^2}\,\mathrm{d}|\mu(y)| = \|\mu\|_{\mathcal{M}_{\frac{\varepsilon}{\lambda^2}}(\mathbb{R}^d)}.$$

Therefore, $\{\mu_\lambda\}_{\lambda>0}$ is bounded in $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ if and only if $\mu \in \mathcal{M}_{0,B}(\mathbb{R}^d)$, that is,

$$|||\mu|||_{\mathcal{M}_{0,B}(\mathbb{R}^d)} := \sup_{\varepsilon>0}\|\mu\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} < \infty.$$

According to (6.1) and part (i) in Proposition 6.1, this is equivalent to the solution of the heat equation $u(x,t,\mu)$ being uniformly bounded in sets $\frac{|x|}{\sqrt{t}} \le R$.

(iii) We also get for $u_0 \in \mathcal{M}_{0,B}(\mathbb{R}^d)$ and $\lambda > 0$

$$\begin{aligned} S(t)u_{0,\lambda}(x) = u(x,t,u_{0,\lambda}) &= \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-y|^2}{4t}}\,\mathrm{d}u_{0,\lambda}(y) = \frac{1}{(4\pi t\lambda^2)^{d/2}}\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-\frac{y}{\lambda}|^2}{4t}}\,\mathrm{d}u_0(y) \\ &= \frac{1}{(4\pi t\lambda^2)^{d/2}}\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|\lambda x-y|^2}{4t\lambda^2}}\,\mathrm{d}u_0(y) = u(\lambda x,\lambda^2 t,u_0) = S(\lambda^2 t)u_0(\lambda x). \end{aligned}$$

In particular, with $t = 1$

$$S(1)u_{0,\lambda}(x) = S(\lambda^2)u_0(\lambda x)$$

(iv) As a consequence of Lemma A.1 it follows that $\mathcal{M}_\varepsilon(\mathbb{R}^d) = (C_{-\varepsilon,0}(\mathbb{R}^d))'$, where

$$f \in C_{-\varepsilon,0}(\mathbb{R}^d) \qquad \text{if and only if} \qquad \mathrm{e}^{\varepsilon|x|^2}f(x) \in C_0(\mathbb{R}^d)$$

and the norm is $\|f\|_{C_{-\varepsilon,0}(\mathbb{R}^d)} := \sup_{x\in\mathbb{R}^d} \mathrm{e}^{\varepsilon|x|^2}|f(x)|$.

Now, if $u_0 \in \mathcal{M}_{0,B}(\mathbb{R}^d)$ then $\{u_{0,\lambda}\}_{\lambda>0}$ is sequentially weak-$*$ compact in $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ for any $\varepsilon > 0$. Taking subsequences $\lambda_n \to \infty$, we can assume that u_{0,λ_n} converges weakly-$*$ to $\mu \in \mathcal{M}_{0,B}(\mathbb{R}^d)$ with $|||\mu|||_{\mathcal{M}_{0,B}(\mathbb{R}^d)} \le |||u_0|||_{\mathcal{M}_{0,B}(\mathbb{R}^d)}$ and, by smoothing, $S(1)u_{0,\lambda_n}$ converges in $L^\infty_{\rm loc}(\mathbb{R}^d)$ to $v = S(1)\mu$. Hence, setting $t_n = \lambda_n^2$ (so $t_n \to \infty$) we obtain

$$S(t_n)u_0(\sqrt{t_n}x) = u(\sqrt{t_n}x, t_n, u_0) \to S(1)\mu(x), \quad L^\infty_{\rm loc}(\mathbb{R}^d).$$

In particular, with $x = 0$ we have

$$u(0, t_n, u_0) \to S(1)\mu(0), \quad L^\infty_{\mathrm{loc}}(\mathbb{R}^d).$$

This relates the results in Propositions 6.1, 6.4 and Theorem 6.3 to the set of weak-$*$ sequential limits of $\{u_{0,\lambda}\}_{\lambda>0}$ in $\mathcal{M}_\varepsilon(\mathbb{R}^d)$. Notice that for all such μ

$$|S(1)\mu(0)| = \left| \frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|y|^2}{4}} \,\mathrm{d}\mu(y) \right| \leq \|\mu\|_{\mathcal{M}_{\frac{1}{4}}(\mathbb{R}^d)} \leq |||u_0|||_{\mathcal{M}_{0,B}(\mathbb{R}^d)};$$

thus the above argument only applies when $u(0, t, u_0)$ is bounded for large times.

6.4 Prescribed behaviour at $x = 0$

We now show that if we drop the restriction that the solutions are non-negative then any (sufficiently smooth) behaviour of the solution at $x = 0$ can be obtained with an appropriate choice of initial condition. We use a construction inspired by the Tychonov example of an initial condition that leads to non-uniqueness with zero initial data (see [26] and Chapter 7, pages 171-172 in [16], for example).

Proposition 6.7. *Let γ be any real analytic function on $[0, T)$ with $T \leq \infty$. Then there exists $u_0 \in L^1_\varepsilon(\mathbb{R}^d)$ for some $\varepsilon > 0$ such that $u(t) = S(t)u_0$ given by (3.2) is defined for all $t \in [0, T)$ and such that*

$$u(0, t) = \gamma(t), \quad 0 \leq t < T.$$

Proof. We seek a solution of the one-dimensional heat equation in the form

$$u(x, t) = \sum_{k=0}^{\infty} g_k(t) x^k, \quad x \in \mathbb{R}$$

converging for every $x \in \mathbb{R}$ (for each t in some range), such that $u(0, t) = \gamma(t)$ for all $t \in [0, T)$. Substituting this expression into the PDE gives

$$g_k'(t) = (k + 2)(k + 1)g_{k+2}(t), \quad t \in [0, T).$$

Assume that $u_x(0, t) = 0$; then $g_1(t) = 0$ and so $g_{2m+1}(t) = 0$ for all t and $m \in \mathbb{N}$. Solving the recurrence for even powers gives $g_0(t) = \gamma(t)$ and

$$g_{2m}(t) = \frac{\gamma^{(m)}(t)}{(2m)!}$$

and therefore

$$u(x, t) = \sum_{k=0}^{\infty} \frac{\gamma^{(k)}(t)}{(2k)!} x^{2k}. \tag{6.11}$$

As γ is real analytic it follows that for each $t \in [0,T)$ there exist constants $C, \tau > 0$ (potentially depending on t) such that

$$|\gamma^{(k)}(t)| \le C k! \tau^{-k} \tag{6.12}$$

(see Exercise 15.3 in [22], for example; in fact the constants C and τ can be chosen uniformly on any compact subinterval of $[0,T)$). It follows that the series in (6.11) converges for all $x \in \mathbb{R}^d$ and for every $t \in [0,T)$, and given (6.12) we have then

$$|u(x,t)| \le \sum_{k=0}^{\infty} C \frac{k!}{(2k)!} x^{2k} \tau^{-k} \le C \sum_{k=0} \frac{(x/\sqrt{\tau})^2}{k!} = C\mathrm{e}^{|x|^2/\tau}.$$

In particular u satisfies (4.1) on any compact time interval of $[0,T)$. Also it is easy to see that $u(x,t)$ satisfies (4.2) in compact intervals of $(0,T)$ and (4.7). By Theorem 4.1, we get

$$u(t) = S(t)u_0, \quad t \in [0,T).$$

We can embed this solution in $\mathbb{R}^d$ by setting $u(x_1, \dots, x_d, t) = u(x_1, t)$. $\square$

When $T = \infty$ this provides an example showing how the condition $u_0 \in L^1_0(\mathbb{R}^d)$ is not required to ensure global existence for initial data that is not required to be non-negative: the solution $u(x,t)$ satisfies the heat equation for all time, remains in one of the $L^1_{\varepsilon(t)}(\mathbb{R}^d)$ spaces for each $t \ge 0$, but does not necessarily satisfy $u_0 \in L^1_0(\mathbb{R})$.

The non-uniqueness example of Tychonov uses precisely the above construction, but based on a function such as $\gamma(t) = \mathrm{e}^{-1/t^2}$ whose radius of analyticity shrinks as $t \to 0^+$, see [16, pg 172]. For such a case, we have that the heat solution in (6.11) satisfies $u(x,t) \to u_0(x) = 0$ uniformly in compact sets as $t \to 0^+$. In the language of this paper, $u(t) \in L^1_{\varepsilon(t)}(\mathbb{R}^d)$ for every $t > 0$, but as $t \to 0^+$ we have $\varepsilon(t) \to \infty$ and (4.6) is not satisfied. In this way this classic non-uniqueness example does not contradict the uniqueness result of Theorem 4.1.

It would be interesting to find conditions on $\gamma(t)$ that ensure the positivity of u_0 (and hence of $u(x,t)$). Certainly positivity of γ itself is not sufficient; indeed, note that

$$\gamma(t) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} t^k \qquad \Rightarrow \qquad u_0(x) = \sum_{k=0}^{\infty} \frac{\alpha_k}{(2k)!} x^{2k}.$$

The simple choice $\alpha_0 = 1$, $\alpha_1 = -2$, $\alpha_2 = 2$ yields

$$\gamma(t) = 1 - 2t + t^2 \ge 0 \qquad \text{but} \qquad u_0(x) = 1 - x + \frac{x^2}{24}$$

and $u_0(2) < 0$.

7 Extension to other problems

First note that by simple reflection arguments, we can also consider the heat equation in the half space $\mathbb{R}^d_+$, that is

$$\begin{cases} u_t - \Delta u = 0, & x \in \mathbb{R}^d_+,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d_+, \\ B(u)(x) = 0, & x \in \partial\mathbb{R}^d_+ \end{cases} \tag{7.1}$$

where $B(u)$ denotes boundary conditions of Dirichlet type, i.e. $B(u) = u$ or Neumann, i.e. $B(u) = \frac{\partial u}{\partial \vec{n}} = -\partial_{x_d} u(x', 0)$. Indeed, performing odd or even reflection respectively we extend (7.1) to the heat equation in $\mathbb{R}^d$ for solutions with odd or even symmetry. Hence, the arguments in previous sections apply.

Also note that a basic ingredient in the proofs above is the gaussian structure of the heat kernel. Hence, the same results apply to any parabolic operator with a similar gaussian bound for the kernel, see [7]. In particular, our results apply for differential operators of the form

$$L(u) = -\sum_{i=1}^{N} \partial_i \Big(a_{i,j}(x) \partial_j u + a_i(x) u \Big) + b_i(x) \partial_i u + c_0(x) u$$

with real coefficients $a_{i,j}, a_i, b_i, c_0 \in L^\infty(\mathbb{R}^d)$ and satisfies the ellipticity condition

$$\sum_{i,j=1}^{N} a_{i,j}(x) \xi_i \xi_j \geq \alpha_0 |\xi|^2$$

for some $\alpha_0 > 0$ and for every $\xi \in \mathbb{R}^d$. In such a case the fundamental solution of the parabolic problem $u_t + Lu = 0$ in $\mathbb{R}^d$ satisfies a gaussian bound

$$0 \leq k(x,y,t,s) \leq C(t-s)^{-d/2} \mathrm{e}^{\omega(t-s)} \mathrm{e}^{-c\frac{|x-y|^2}{(t-s)}}$$

for $t > s$ and $x, y \in \mathbb{R}^d$ where C, c, ω depend on the L^∞ norm of the coefficients. The gaussian bounds are obtained from [7] while the positivity of the kernel comes from the maximum principle, see [13], chapter 8. Therefore the analysis of previous sections, applies to solutions of the form

$$u(x,t) = S_L(t) u_0 = \int_{\mathbb{R}^d} k(x,y,t,0)\, \mathrm{d}u_0(y).$$

Other results on Gaussian upper bounds can be found in [1, 8, 19, 25].

A Some auxiliary results

Here we prove several technical results used above. First, we prove that certain spaces of functions or measures used above are Banach spaces.

Lemma A.1. *The sets* $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ *and* $L^1_\varepsilon(\mathbb{R}^d)$ *in* (3.8) *and* (3.6) *with the norms* (3.9) *and* (3.7) *respectively, are Banach spaces.*

Proof. For $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ we proceed as follows. Given $\mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ we define the Borel measure such that for all Borel sets $A \subset \mathbb{R}^d$

$$\Phi_\varepsilon(\mu)(A) = \int_A \rho_\varepsilon(x)\,\mathrm{d}\mu(x)$$

where $\rho_\varepsilon(x) = \left(\frac{\varepsilon}{\pi}\right)^{d/2} \mathrm{e}^{-\varepsilon|x|^2}$. Then $\Phi_\varepsilon(\mu) \in \mathcal{M}_{loc}(\mathbb{R}^d)$, is clearly absolutely continuous with respect to μ and for all $\varphi \in C_c(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}\Phi_\varepsilon(\mu)(x) = \int_{\mathbb{R}^d} \varphi(x)\rho_\varepsilon(x)\,\mathrm{d}\mu(x).$$

Now we claim that the total variation of $\Phi_\varepsilon(\mu)$ satisfies

$$|\Phi_\varepsilon(\mu)|(A) = \int_A \rho_\varepsilon(x)\,\mathrm{d}|\mu(x)|$$

for all Borel sets $A \subset \mathbb{R}^d$, which would imply that for all $\varphi \in C_c(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}|\Phi_\varepsilon(\mu)(x)| = \int_{\mathbb{R}^d} \varphi(x)\rho_\varepsilon(x)\,\mathrm{d}|\mu(x)|.$$

To prove the claim, observe that the positive part in the Jordan decomposition satisfies

$$\Phi_\varepsilon(\mu)^+(A) = \sup_{B\subset A} \int_B \rho_\varepsilon(x)\,\mathrm{d}\mu(x) = \sup_{B\subset A} \int_{B^+} \rho_\varepsilon(x)\,\mathrm{d}\mu^+(x) - \int_{B^-} \rho_\varepsilon(x)\,\mathrm{d}\mu^-(x)$$

$$= \sup_{B\subset A} \int_{B^+} \rho_\varepsilon(x)\,\mathrm{d}\mu^+(x) = \sup_{B\subset A} \int_B \rho_\varepsilon(x)\,\mathrm{d}\mu^+(x) = \int_A \rho_\varepsilon(x)\,\mathrm{d}\mu^+(x)$$

where we have used that $B = B^+ \cup B^-$ are the positive and negative parts of a set B, according to the Hahn decomposition of the measure μ; see Theorem 3.3 in [10]. Analogously, $\Phi_\varepsilon(\mu)^-(A) = \int_A \rho_\varepsilon(x)\mathrm{d}\mu^-(x)$ and with this the claim follows.

Now if, $\mu \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ we have

$$|\Phi_\varepsilon(\mu)|(\mathbb{R}^d) = \int_{\mathbb{R}^d} \rho_\varepsilon(x)\,\mathrm{d}|\mu(x)| < \infty$$

that is

$$\Phi_\varepsilon : \mathcal{M}_\varepsilon(\mathbb{R}^d) \to \mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^d)$$

is an isometry. To prove the result it remains to show that Φ_ε is onto. In fact if $\sigma \in \mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^d)$ we define μ such that for all borel sets $A \subset \mathbb{R}^d$

$$\mu(A) = \int_A \frac{\mathrm{d}\sigma(x)}{\rho_\varepsilon(x)}$$

and thus for all $\varphi \in C_c(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}^d} \frac{\varphi(x)}{\rho_\varepsilon(x)}\,\mathrm{d}\sigma(x).$$

Clearly $\mu \in \mathcal{M}_{loc}(\mathbb{R}^d)$ and arguing as above we get for all borel sets $A \subset \mathbb{R}^d$

$$|\mu|(A) = \int_A \frac{\mathrm{d}|\sigma(x)|}{\rho_\varepsilon(x)}$$

and for all $\varphi \in C_c(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}|\mu(x)| = \int_{\mathbb{R}^d} \frac{\varphi(x)}{\rho_\varepsilon(x)}\,\mathrm{d}|\sigma(x)|.$$

Now take an increasing sequence $0 \leq \varphi_n \in C_c(\mathbb{R}^d)$ such that $\varphi_n \to \rho_\varepsilon$ pointwise in $\mathbb{R}^d$ and then

$$\int_{\mathbb{R}^d} \varphi_n(x)\,\mathrm{d}|\mu(x)| = \int_{\mathbb{R}^d} \frac{\varphi_n(x)}{\rho_\varepsilon(x)}\,\mathrm{d}|\sigma(x)| \leq \int_{\mathbb{R}^d} \mathrm{d}|\sigma(x)| = |\sigma(\mathbb{R}^d)| < \infty$$

and by Fatou's lemma we get $\int_{\mathbb{R}^d} \rho_\varepsilon(x)\,\mathrm{d}|\mu(x)| < \infty$, that is, $\mu \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$. Clearly $\Phi_\varepsilon(\mu) = \sigma$ and we conclude the proof.

On the other hand, note that $L^1_\varepsilon(\mathbb{R}^d) = L^1(\mathbb{R}^d, \rho_\varepsilon\,\mathrm{d}x)$ and so is a Banach space. Also, note that along the lines of the proof above it is easy to see that the operator $\Phi_\varepsilon(f) = \rho_\varepsilon f$, $\Phi_\varepsilon : L^1_\varepsilon(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is an isometric isomorphism. □

Lemma A.2. *The space of uniform measures* $\mathcal{M}_U(\mathbb{R}^d)$ *defined in* (5.1) *with the norm* (5.2) *is a Banach space.*

Proof. Clearly a Cauchy sequence in the norm (5.2) is a Cauchy sequence in $\mathcal{M}_{\mathrm{BTV}}(\overline{B(x,1)})$ uniformly for $x \in \mathbb{R}^d$, that is, the dual of $C(\overline{B(x,1)})$ with the uniform convergence. Therefore, it converges in $\mathcal{M}_{\mathrm{BTV}}(\overline{B(x,1)})$ uniformly for $x \in \mathbb{R}^d$. Hence it converges in $\mathcal{M}_U(\mathbb{R}^d)$. □

Lemma A.3. (Green's formulae)

(i) Assume that $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ *satisfies* $u, \nabla u \in L^1_\varepsilon(\mathbb{R}^d)$ *and that* ξ *is a smooth function such that* $|\nabla\xi(x)|, |\Delta\xi(x)| \leq c\mathrm{e}^{-\alpha|x|^2}$ *for* $x \in \mathbb{R}^d$ *and* $\alpha \geq \varepsilon$*. Then*

$$\int_{\mathbb{R}^d} u(-\Delta\xi) = \int_{\mathbb{R}^d} \nabla u \nabla \xi.$$

(ii) Assume that $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ *satisfies* $\Delta u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ *and* $\nabla u, \Delta u \in L^1_\varepsilon(\mathbb{R}^d)$ *and that* ξ *is a smooth function such that* $|\xi(x)|, |\nabla\xi(x)| \leq c\mathrm{e}^{-\alpha|x|^2}$ *for* $x \in \mathbb{R}^d$ *and* $\alpha \geq \varepsilon$*. Then*

$$\int_{\mathbb{R}^d} \nabla u \nabla \xi = \int_{\mathbb{R}^d} (-\Delta u)\xi.$$

Proof. (i) Observe that for any $R > 0$

$$\int_{B(0,R)} u(-\Delta\xi) = \int_{B(0,R)} \nabla u \nabla \xi - \int_{\partial B(0,R)} u \frac{\partial \xi}{\partial \vec{n}} \,\mathrm{d}S.$$

Thanks to the Dominated Convergence Theorem it is enough to prove that the last term above converges to zero, as $R \to \infty$, since we can write

$$\int_{B(0,R)} u(-\Delta\xi) = \int_{B(0,R)} \mathrm{e}^{-\varepsilon|x|^2} u \,\mathrm{e}^{\varepsilon|x|^2}(-\Delta\xi)$$

and

$$\int_{B(0,R)} \nabla u \nabla \xi = \int_{B(0,R)} \mathrm{e}^{-\varepsilon|x|^2} \nabla u \,\mathrm{e}^{\varepsilon|x|^2} \nabla\xi$$

and pass to the limit in $R \to \infty$ in both terms.

For this observe that

$$\int_0^\infty \int_{\partial B(0,R)} |u \frac{\partial \xi}{\partial \vec{n}}| \,\mathrm{d}S \,\mathrm{d}R \leq \int_{\mathbb{R}^d} |u|||\nabla\xi| < \infty.$$

Hence for some sequence $R_n \to \infty$ we have $\int_{\partial B(0,R_n)} |u \frac{\partial \xi}{\partial \vec{n}}| \to 0.$

Part (ii) is obtained in a similar fashion. □

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