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Use of biorthogonal functions for the modal decomposition of multimode beams

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Abstract

From a recently proposed technique, used to determine the modal weights of beams made up of incoherent superpositions of Hermite–Gaussian (HG) modes, we derive the analytical expression of a family of functions which are biorthogonal to the squared HG functions. The knowledge of such functions enables the reconstruction of the modal content of these beams by means of the scalar-product with the intensity profile across a transverse plane. © 2001 Elsevier Science B.V. All rights reserved.

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The need of characterizing laser beams in a quantitative and rigorous way continues representing a problem of interest for researchers in optics [1–4]. In particular, the determination of the transverse-mode content of a real laser beam is an attractive subject of study, both from theoretical and applicative viewpoints. Indeed, such problem is strictly connected to the recovering of the information content and the degrees of freedom carried out by partially coherent fields [5–10]. From a more applicative point of view, laser beams are often required that present peculiar features as far as their coherence properties are concerned. This, in turn, implies a precise knowledge of the power distribution among the modes oscillating inside the laser cavity and building up the partially coherent beam. A lot of work has been done in this sense, and many techniques have been developed for tackling and solving the modal reconstruction problem, both in CW and pulsed regimes [11–19].

Recently, a new technique has been proposed [20–22] for the recovering of the mode distribution of a particular, but yet very important, class of partially coherent beams: the ones obtained as the output of a stable multimode laser cavity with spherical mirrors. In such a case, if the mirrors are large enough and the elements present inside the

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cavity do not perturb significantly the geometry of the resonator, it is well known that the radiated beam can be modeled as a superposition of higher-order Hermite–Gaussian (HG) modes [23]. Under the hypotheses that they oscillate at different frequencies and that measurements on the resulting beam are performed on time intervals longer than the beat period, the HG modes can be considered as completely uncorrelated from one another. An attempt of generalization to the case of partially correlated HG modes was recently presented in Ref. [24].

The above mentioned technique requires the knowledge of the spot size of the modes and leads to the evaluation of the modal coefficients through the Fourier transform (FT) of the intensity profile, across a typical transverse plane, and the subsequent scalar-product with a suitable set of orthogonal functions. In the present letter, we propose an alternative way to obtain the modal coefficients which does not require the FT of the intensity profile. This could represent an improvement of the above technique, especially in view of experimental applications. We show that the method proposed in Ref. [20] is equivalent to performing the scalar-product of the intensity profile itself with the functions belonging to a suitable family. More precisely, such functions, whose analytical expression will be given in the paper, constitute the set of functions which are biorthogonal to the squares of the HG functions. The feasibility of the new modal reconstruction process will be tested for Gaussianly and rectangularly shaped intensity distributions, for which analytical expressions of the modal coefficients will be obtained.

In the following we will consider, for simplicity, the one-dimensional case. Extension to (rectangular) two-dimensional geometries is straightforward. For the case of uncorrelated HG modes, the time-averaged intensities are additive, so that the intensity profile, say $I(x)$, of the beam at its waist plane, which is supposed to be the plane $z = 0$ of a suitable reference frame (x, z), can be written as

$$I(x) = \sum_{n=0}^{\infty} c_n G_n^2(x; v_0). \tag{1}$$

Here, c_n $(n = 0, 1, \ldots)$ are positive coefficients representing the power content carried by the nth-order HG mode, whose optical field distribution $G_n(x; v_0)$ is [23]

$$G_n(x; v_0) = \left(\frac{2}{\pi v_0^2}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\frac{x\sqrt{2}}{v_0}\right) \exp\left(-\frac{x^2}{v_0^2}\right), \tag{2}$$

where H_n is the Hermite polynomial [25] of order n and v_0 is the spot size of the modes.

The reconstruction problem consists in finding the c_n coefficients, starting from measurements on the intensity $I(x)$. In doing so, we assume that the spot size of the modes is known. It should be noted that the G_n^2 functions are not mutually orthogonal, so that the usual scalar-product rule cannot be used in Eq. (1).

Let us consider the set of functions, say $\mu_m(x; v_0)$ $(m = 0, 1, \ldots)$, which are orthogonal to the $G_n^2(x; v_0)$ functions, i.e., such that

$$\int_{-\infty}^{+\infty} G_n^2(x; v_0) \mu_m(x; v_0) \, dx = \delta_{n,m}, \tag{3}$$

where $\delta_{n,m}$ is the Kronecker symbol. Due to the parity of the G_n^2 functions, μ_ms can be chosen as even. In terms of these functions, the expanding coefficients c_m (see Eq. (1)) are expressed by the scalar-product

$$c_m = \int_{-\infty}^{+\infty} I(x) \mu_m(x; v_0) \, dx. \tag{4}$$

Before going on, let us analyze in more detail the meaning of the coefficients c_m given by the scalar-product in Eq. (4). Together with Eq. (1), they provide a representation of the function $I(x)$ in the non-orthogonal basis of the squared HG functions. This is formally analogous to the well-known procedure used for the case of a set of orthogonal functions [26]. We stress that our procedure leads to a non-ambiguous determination of the actual values of the c_m coefficients.

It is worthwhile to hint to a rather different mathematical problem. Suppose a certain function is expanded into a series of orthogonal functions. Then it is well known that the expansion coefficients evaluated through the scalar-product rule also furnish the best coefficients to be used in order to minimize the mean squared error if the series is truncated to a finite number of terms. This is no longer true if the given function is expanded into a series of non-orthogonal functions. In that case, the coefficients that minimize the mean squared error for a truncated series are different from those appearing in the infinite series. Although in this paper we are not involved in such an approximation problem we think it useful to quote that an efficient and elegant solution has been found by the group of Siegman [27].

In order to find the expression of μ_m, we firstly recall that the FT of the G_n^2s is given by $\Psi_n(\pi^2 v_0^2 p^2)$ [20,21], where

$$\Psi_n(t) = L_n(t)\exp\left(-\frac{t}{2}\right), \tag{5}$$

with $t = \pi^2 v_0^2 p^2$ and L_n is the nth-order Laguerre polynomial [25].

As is well known, the functions $\Psi_n(t)$ are mutually orthogonal in the half-space $t \geqslant 0$. Accordingly, the following relationship holds:

$$\int_0^\infty \Psi_n(\pi^2 v_0^2 p^2)\Psi_m(\pi^2 v_0^2 p^2)\,\mathrm{d}(\pi^2 v_0^2 p^2) = \delta_{n,m}. \tag{6}$$

Furthermore, since $G_n^2(x; v_0)$ is an even function, we can write

$$\Psi_n(\pi^2 v_0^2 p^2) = 2\int_0^\infty G_n^2(x; v_0)\cos(2\pi p x)\,\mathrm{d}x. \tag{7}$$

On substituting Eq. (7) into Eq. (6) and interchanging the integration order with respect to x and p, we obtain

$$2\int_0^\infty \mathrm{d}x\, G_n^2(x; v_0)\int_0^\infty \cos(2\pi p x)\Psi_n(\pi^2 v_0^2 p^2) \times \mathrm{d}(\pi^2 v_0^2 p^2) = 2\int_0^\infty G_n^2(x; v_0)\mu_m(x; v_0)\,\mathrm{d}x = \delta_{n,m}. \tag{8}$$

Eq. (8) is equivalent to Eq. (3), once the parity of $\mu_m(x; v_0)$ is taken into account, so that

$$\mu_m(x; v_0) = \int_0^\infty \cos(2\pi p x) L_m(\pi^2 v_0^2 p^2) \times \exp\left(-\frac{\pi^2 v_0^2 p^2}{2}\right)\mathrm{d}(\pi^2 v_0^2 p^2). \tag{9}$$

As we shall see in a moment, functions μ_m can be given in a closed form. In fact, on letting $\xi^2 = \pi^2 v_0^2 p^2$ into Eq. (9), we have

$$\mu_m(x; v_0) = 2\int_0^\infty \cos(2x\xi/v_0) L_m(\xi^2) \times \exp(-\xi^2/2)\xi\,\mathrm{d}\xi, \tag{10}$$

which can be written in the following form:

$$\mu_m(x; v_0) = 2\left[\frac{\mathrm{d}}{\mathrm{d}\beta}\int_0^\infty \sin(\beta\xi) L_m(\xi^2) \times \exp(-\xi^2/2)\,\mathrm{d}\xi\right]_{\beta = 2x/v_0}. \tag{11}$$

On using formula 7.418.1 of Ref. [28], after simple algebra we obtain

$$\mu_m(x; v_0) = \frac{4(-1)^{m+1} m!}{\sqrt{2\pi}}\mathrm{Re}\left\{D_{-(m+1)}\left(2\mathrm{i}\frac{x}{v_0}\right) \times D'_{-(m+1)}\left(2\mathrm{i}\frac{x}{v_0}\right)\right\}, \tag{12}$$

where $D_\nu(\cdot)$ is the cylinder parabolic function of index ν [25] and Re is the real-part operator. Eq. (12) gives the analytical expression of the set of functions which are biorthogonal to the squares of the HG modes.

In Fig. 1 behaviors of (a) the G_m^2 and (b) the μ_m functions are plotted versus x/v_0, for some values of m. On comparing Eqs. (2) and (12) it can be noted that the v_0 parameter acts only as a scaling factor for μ_m, whereas in the case of G_n^2 it also affects the absolute value. For such reason, curves in Fig. 1(a) has been normalized to the value $(\pi v_0^2/2)^{1/2}$.

In order to test the method, we are going to consider two examples for which the biorthogonal decomposition (1) can be directly obtained, in analytical terms, from the scalar-product (4). Before doing so, we write Eq. (4) in a more convenient way. On using Eq. (12) in Eq. (4), we have

$$c_m = \frac{8(-1)^{m+1} m!}{\sqrt{2\pi}}\int_0^\infty I(x)\mathrm{Re}\left\{D_{-(m+1)}\left(2\mathrm{i}\frac{x}{v_0}\right) \times D'_{-(m+1)}\left(2\mathrm{i}\frac{x}{v_0}\right)\right\}\mathrm{d}x, \tag{13}$$

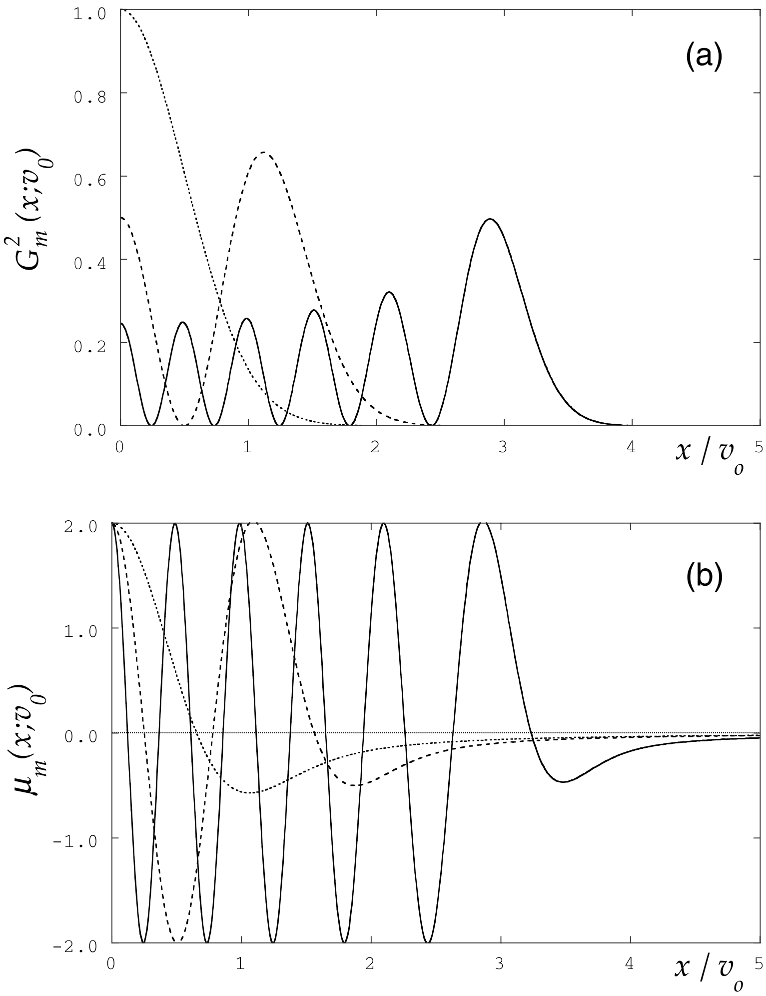

Fig. 1. Behavior of the functions (a) $G_m^2(x;v_0)$ normalized to the value $(\pi v_0^2/2)^{1/2}$, and (b) $\mu_m(x;v_0)$, versus x/v_0. Values of m are 0 (dotted lines), 2 (dashed lines), and 10 (solid lines).

where use has been made of the reality and the parity of $I(x)$. The last integral can be easily solved by parts, yielding

$$\int_0^\infty I(x)\,\mathrm{Re}\left\{D_{-(m+1)}\left(2\mathrm{i}\frac{x}{v_0}\right)D'_{-(m+1)}\left(2\mathrm{i}\frac{x}{v_0}\right)\right\}\mathrm{d}x = \left(\frac{v_0}{8\mathrm{i}}\right)\int_0^\infty I'(x)\left[D^2_{-(m+1)}\left(-2\mathrm{i}\frac{x}{v_0}\right) - D^2_{-(m+1)}\left(2\mathrm{i}\frac{x}{v_0}\right)\right]\mathrm{d}x, \quad (14)$$

so that

$$c_m = \frac{8(-1)^{m+1}m!}{\sqrt{2\pi}}\left(\frac{v_0}{8\mathrm{i}}\right)\int_0^\infty I'(x)\left[D^2_{-(m+1)}\left(-2\mathrm{i}\frac{x}{v_0}\right) - D^2_{-(m+1)}\left(2\mathrm{i}\frac{x}{v_0}\right)\right]\mathrm{d}x. \quad (15)$$

In particular, the first example deals with a well-known class of sources, namely the Gaussian Schell-model ones [29], for which the modal decomposition is already known in analytical terms [30,31]. In such a case we have

$$I(x) = I_0\exp\left(-\frac{x^2}{2\sigma_I^2}\right), \quad (16)$$

where I_0 and σ_I are positive parameters.

On substituting from Eq. (16) into Eq. (15), and taking formula 2.11.8.4 of Ref. [32] into account, [1]

$$\int_0^\infty x\exp(-px^2)\left[D_{-n}^2(-\mathrm{i}cx)-D_{-n}^2(\mathrm{i}cx)\right]\mathrm{d}x = \frac{(-1)^{n-1}}{(n-1)!}\sqrt{\frac{2}{p}}\left[\frac{\mathrm{i}\pi c^{2n-1}}{(2p+c^2)^n}\right]\left(1-\frac{2p}{c^2}\right)^{n-1}, \quad (17)$$

we obtain

$$c_m = I_0\sqrt{\frac{\pi}{2}}\frac{v_0^2}{\sigma_I\left[1+\left(\frac{v_0}{2\sigma_I}\right)^2\right]}\left[\frac{1-\left(\frac{v_0}{2\sigma_I}\right)^2}{1+\left(\frac{v_0}{2\sigma_I}\right)^2}\right]^m,$$

$$m = 0,1,2\ldots, \quad (18)$$

which coincides with the expression obtained in Ref. [21], where the FT-based algorithm was used.

For the second example we consider an intensity profile of the form

$$I(x) = I_0\,\mathrm{rect}\left(\frac{x}{b}\right), \quad (19)$$

where I_0 and b are two positive parameters, and

$$\mathrm{rect}(x) = \begin{cases} 1, & |x| \leqslant b/2, \\ 0, & \text{otherwise.} \end{cases} \quad (20)$$

On substituting from Eq. (19) into Eq. (15), and using the fact that $I'(x) = -I_0\,\delta(x-b/2)$ in the interval $[0,+\infty)$, being $\delta(\cdot)$ the Dirac function, the expression for the c_ms turns out to be

$$c_m = I_0\,v_0\sqrt{\frac{2}{\pi}}(-1)^{(m+1)}m!\,\mathrm{Im}\left\{D_{-(m+1)}^2\left(\mathrm{i}\frac{b}{v_0}\right)\right\}. \quad (21)$$

In Fig. 2 the behavior of the coefficients c_m, normalized to the value $I_0(2v_0^2/\pi)^{1/2}$, is plotted versus the index m for some values of the ratio b/v_0. It can be noted that the behavior of the coefficients is somewhat scaled when b/v_0 changes. This is due to the fact that the number of HG modes which significantly contribute to the overall intensity varies approximately as the square root of b/v_0 [19]. Note that in such case negative values of the expansion coefficients occur. This means that an intensity profile of the form (19) cannot ever be produced by an incoherent superposition of HG modes, so that the reconstruction process gives rise to negative values of the power content of the modes.

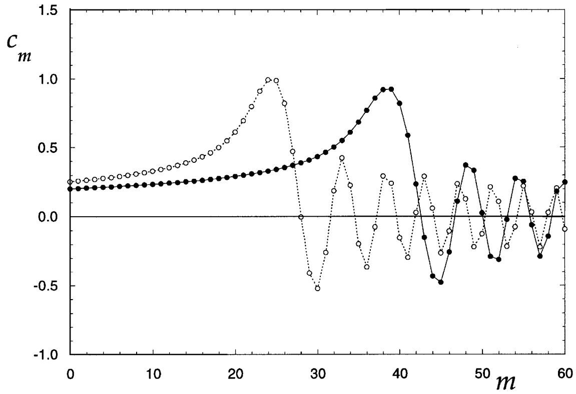

Fig. 2. Behavior of the c_ms, normalized to the value $I_0(2v_0^2/\pi)^{1/2}$, versus m for $b/v_0 = 10$ (circles) and $b/v_0 = 12.5$ (dots).

It is worthwhile to spend some more words about this case. Actually, the rect function can be viewed as the limiting case of flat-topped distributions when the flatness parameter assumes very large values. For instance, if the flattened Gaussian (FG) model [33] is used, the rect function corresponds to $N \to \infty$, where N is the order of the FG profile. For such reason, we expect the behavior of the coefficients c_m shown in Fig. 2 to be somewhat resembling that corresponding to a FG profile of high order. In Fig. 3(a) the FG intensity profile for $N = 200$ (solid curve) and the rect function (dotted) are shown, while corresponding values of the evaluated coefficients are plotted in Fig. 3(b). As can be easily seen, a qualitative agreement between the two behaviors is quite evident for small values of m. Moreover, the oscillating behavior of the coefficients for the rect function can be traced back to the finite extension of its support. Indeed, it is not possible to represent zero values by means of a superposition, with positive weights, of positive functions (the G_m^2s).

Although in the present work we are not focused on the reconstruction process of the original intensity profile when the non-orthogonal series

[1] It should be noticed that the solution of the integral of Eq. (17) does not coincide exactly with the referenced one, due to a missing factor in the latter.

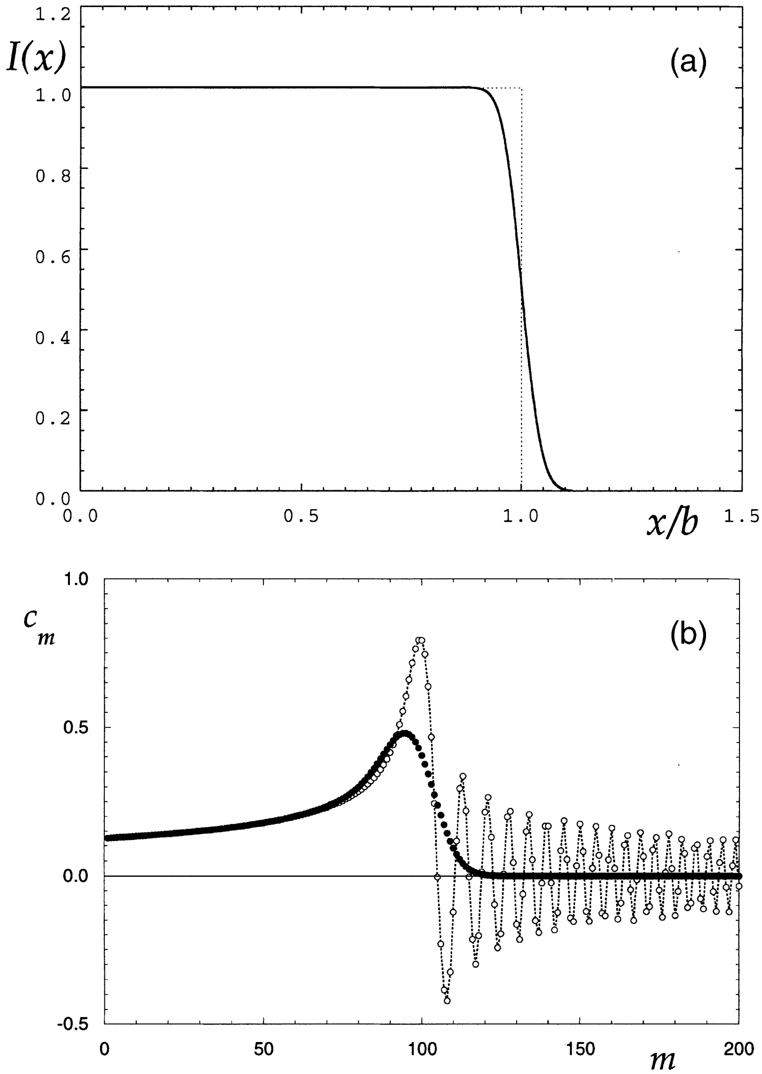

Fig. 3. (a) Intensity profiles for the FG function with $N = 200$ (solid) and the rect function (dotted); (b) behavior of the c_m coefficients for the previous profiles: FG (dots) and rect (circles). The spot-size v_0 of the HG modes is $v_0 = b\sqrt{2/(N+1)}$.

expansion (1) is truncated to a finite number of terms, it is worthwhile to see, for the 'rect' case shown in Fig. 3, how the reconstructed profile behaves when all the $N = 200$ coefficients of Fig. 3(b) (white circles) are used. This is indeed shown in Fig. 4, from which a good agreement between the reconstructed and the original 'rect' profile is evident.

To summarize, we have given the analytical expression of the functions $\{\mu_n\}$ which are orthogonal to the squares of the HG ones. As a consequence, the evaluation of the nth modal weight of a light beam made up by an incoherent superposition of HG modes is reduced to the scalar-product between a typical transverse intensity profile and the function μ_n. Furthermore, from the analysis of the behavior of the μ_n functions (see for instance Fig. 1(b)), it appears that the above scalar-product should be easily implementable, especially from a numerical point of view. This represents a considerable improvement of the FT-based algorithm proposed in Refs. [20,21]. It

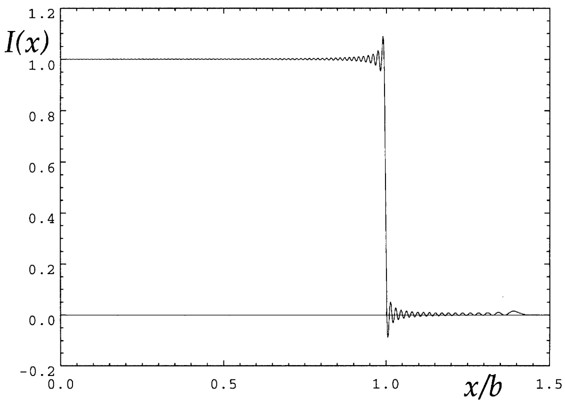

Fig. 4. Reconstructed intensity profile for the rect function in Fig. 3(a) through the expanding coefficients obtained by the biorthogonal approach (white circles in Fig. 3(b)).

should be pointed out that such an approach does not provide a complete description of the complex processes (such as for instance mode competition, mode distortion, cross saturation effects and so on) going on acting in a real cavity laser. Nonetheless, we think that our approach could be useful to furnish, in a simple way, at least a first estimate of the power distribution among the modes, which will turn out to be more or less accurate depending on the specific features of the laser under analysis. In any case it may represent a good starting point for further future developments.

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