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GIT CHARACTERIZATIONS OF HARDER-NARASIMHAN FILTRATIONS

CARACTERIZACIONES GIT DE FILTRACIONES DE HARDER-NARASIMHAN

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ALFONSO ZAMORA SAIZ

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TOMÁS L. GÓMEZ

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Alfonso Zamora Saiz

Dirigida por Tomás L. Gómez

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To my mother

TOMÁS.- *Entonces...*
ASEL.- *¡Entonces hay que salir a la otra cárcel!*
¡Y cuando estés en ella, salir a otra, y de ésta, a otra!
La verdad te espera en todas, no en la inacción. Te esperaba
aquí, pero sólo si te esforzabas en ver la mentira de la
Fundación que imaginaste. Y te espera en el esfuerzo de ese
oscuro túnel del sótano... En el holograma de esa evasión.

La Fundación, Antonio Buero Vallejo

THOMAS.- *Then...*
ASEL.- *Then you must exit into the other jail!*
And when you're in it, go into another, and from it, to still another!
Truth awaits you in all of them, not in inaction. You found it
here, but only by making yourself see the lie of the
Foundation you imagined. And it waits for you in the effort of that
dark tunnel in the basement... in the hologram of that escape.

The Foundation, Antonio Buero Vallejo, Trans. Marion Peter Holt

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Contents

Introducción (en español)

La presente tesis doctoral está dedicada al estudio de la relación entre la inestabilidad maximal en el sentido de la Teoría Geométrica de Invariantes (que abreviaremos por sus siglas en inglés, GIT) y la filtración de Harder-Narasimhan para diferentes problemas de espacios de móduli. Muchos de los problemas de móduli en geometría hacen uso de la Teoría Geométrica de Invariantes para la construcción de espacios de móduli. Imponemos, de inicio, una noción de estabilidad en los objetos para los cuales queremos construir un espacio de móduli y, mediante la Teoría Geométrica de Invariantes, un objeto estable (resp. semiestable, inestable) se corresponde con un punto GIT estable (resp. GIT semiestable, inestable) en cierto espacio, estableciendo una correspondencia entre ambas nociones de estabilidad. El concepto de máxima inestabilidad en el sentido GIT ha sido estudiado por diferentes autores, y para el presente propósito consideraremos el tratamiento de Kempf, cuyo artículo [Ke] lo explora. Por otra parte, la filtración de Harder-Narasimhan, ampliamente usada en muchos problemas en geometría algebraica, es el objeto geométrico que representa la idea de inestabilidad maximal para la condición de estabilidad impuesta de inicio sobre los objetos.

En esta tesis se demuestra que ambas nociones de inestabilidad maximal coinciden, y se muestra una correspondencia entre ellas en diferentes casos. El primer capítulo contiene nociones generales sobre problemas de móduli, la Teoría Geométrica de Invariantes y la filtración de Harder-Narasimhan que usaremos en los capítulos 2 y 3, además de un ejemplo de construcción de un espacio de móduli para tensores. En el segundo capítulo estudiamos diferentes problemas de móduli relacionados con haces, o con haces con estructura adicional. Desarrollamos una técnica para probar la mencionada correspondencia para haces coherentes sin torsión sobre variedades proyectivas de dimensión arbitraria, pares holomorfos, haces de Higgs, tensores de rango 2, y hacemos algunos comentarios acerca de los tensores de rango 3, siendo el primer caso de tensores para el cual la técnica desarrollada no funciona. En el tercer capítulo estudiamos representaciones de un carcaj, demostrando un resultado similar para representaciones en la categoría de espacios vectoriales y, nuevamente, haces coherentes vistos como representaciones de un carcaj de un solo vértice en la categoría de haces coherentes.

Inestabilidad maximal

Considérese un problema de móduli en el que queremos clasificar una clase de objetos algebro-geométricos módulo una relación de equivalencia. Usualmente, tenemos que imponer una condición de estabilidad en los objetos que clasificamos para obtener un espacio de móduli con buenas propiedades donde cada punto corresponda a una clase de equivalencia de objetos. Entonces, añadiendo un dato adicional a los objetos (lo que se suele llamar en la literatura en inglés *to rigidify the data*) los incluímos en un espacio de parámetros con el que nos es más sencillo trabajar (como pueda ser un esquema afín o proyectivo). Para deshacernos de este nuevo dato añadido, tenemos que tomar el cociente por la acción de un grupo que está precisamente codificando los cambios en este dato adicional. Y para tomar este cociente usamos la Teoría Geométrica de Invariantes de Mumford, (véase [Mu] para la primera edición, [MF] y [MFK] para la segunda y tercera ediciones) para obtener un espacio de móduli proyectivo clasificando los objetos en el problema de móduli. El estudio de las órbitas de la acción de este grupo nos lleva a la noción de estabilidad GIT definiendo puntos, en el espacio de parámetros, que son GIT estables y otros que son GIT inestables.

En cada problema de móduli en el que usamos GIT, llegado cierto momento tenemos que demostrar que ambas nociones de estabilidad coinciden, con lo que los objetos estables corresponden a los puntos GIT estables y los objetos inestables corresponden a los puntos GIT inestables. A tal efecto, Mumford enuncia un criterio numérico (c.f. [Mu, Theorem 2.1]) basado en ideas de Hilbert en [Hi]. El Teorema 1.1.14, conocido como el criterio de Hilbert-Mumford, caracteriza la estabilidad GIT a través de subgrupos uniparamétricos, donde una función numérica (que llamamos el *mínimo exponente relevante*) toma un valor positivo o negativo dependiendo de que el subgrupo uniparamétrico desestabilice un punto o no, en el sentido GIT. Además, si un punto es GIT inestable, podemos hablar de *grados de inestabilidad*, o de ciertos subgrupos uniparamétricos que son más desestabilizantes que otros.

Basado en el trabajo de Mumford, Tits y otros autores, podemos medir esto mediante una función racional en el espacio de subgrupos uniparamétricos, cuyo numerador es la función numérica del criterio de Hilbert-Mumford y cuyo denominador es una **norma** del subgrupo uniparamétrico que escogemos para evitar reescalar la función numérica (véase sección 1.4). La **conjetura del centro de Tits** (c.f. [Mu, p. 64]) establece que existe un único subgrupo uniparamétrico maximizando esta función, que representa la inestabilidad maximal en el sentido GIT. Kempf explora estas ideas en un artículo en 1978 (c.f. [Ke]), resolviendo lo que él llama **la conjetura de Mumford-Tits** (refiriéndose a la conjetura del centro de Tits, tal como aparece en [Mu]), demostrando que existe un único subgrupo uniparamétrico con estas propiedades en [Ke, Theorem 2.2, Theorem 3.4] (para una correspondencia entre las definiciones en [Mu] y [Ke] véase [MFK, Appendix 2B]).

Un objeto inestable proporciona un punto GIT inestable para el cual existe un único subgrupo uniparamétrico máximamente desestabilizante en el sentido GIT. Los diferentes subgrupos uniparamétricos producen, de forma natural, banderas formando el **complejo de banderas** de un grupo G, estudiado por Tits y Mumford. Por tanto, nos gustaría considerar la bandera asociada a ese único subgrupo uniparamétrico máximamente desestabilizante en el sentido GIT y construir, a partir de él, una filtración de subobjetos del objeto inestable original. Los principales resultados de esta tesis consisten en la traducción de este subgrupo uniparamétrico a una filtración del objeto, demostrar que esta traducción está bien y unívocamente definida (es decir, que no depende de diversas elecciones hechas durante el proceso) y, finalmente, demostrar que coincide con la filtración de Harder-Narasimhan en los casos en los que esta filtración es ya conocida, o proporciona una nueva definición de una tal filtración en otro caso.

Teoría Geométrica de Invariantes y espacio de móduli de tensores

En el primer capítulo se recogen las nociones preliminares acerca de espacios de móduli y la Teoría Geométrica de Invariantes para presentar el problema estudiado.

La sección 1.1 contiene una descripción de lo que es un problema de móduli, con su formulación rigurosa. Se proporcionan ejemplos básicos como el espacio de móduli de las cúbicas complejas no singulares o la formulación del problema del espacio de móduli de las curvas algebraicas de género g. Entonces, recuperamos las nociones de la Teoría Geométrica de Invariantes que necesitaremos en lo sucesivo, los diferentes tipos de cocientes, linearizaciones de acciones de grupos y el criterio de Hilbert-Mumford, esencial para tratar con la estabilidad GIT. También damos un ejemplo, el Ejemplo 1.1.4, para resaltar el concepto de S-equivalencia para las órbitas de la acción de un grupo, o el ejemplo de combinaciones de puntos en $\mathbb{P}^1_{\mathbb{C}}$ (véase Ejemplos 1.1.5 y 1.1.15) para aplicar el criterio de Hilbert-Mumford.

La sección 1.2 está dedicada a presentar un ejemplo completo de construcción de un espacio de móduli usando la Teoría de Invariantes Geométricos: el espacio de móduli de tensores sobre una variedad proyectiva. Esta construcción fue primeramente estudiada para tensores sobre curvas por Schmitt (c.f. [Sch]) y después por Gómez y Sols (c.f. [GS1]). En la sección, seguimos la referencia [GS1] pero usando el embebimiento de Gieseker (c.f. subsección 1.2.3) en vez de el de Simpson, como se hace en [GS1]. Con este embebimiento, los tensores se meten en un espacio de parámetros donde actúa un grupo. El Teorema 1.2.31 establece que los tensores semiestables corresponden a las órbitas GIT semiestables bajo la acción del grupo, por lo tanto el conciente GIT será el espacio de móduli de los tensores semiestables. En este ejemplo aparecen muchos de los elementos que suelen encontrarse en las construcciones GIT de espacios de móduli, tales como la dependencia de la estabilidad con un parámetro, la necesidad de probar la acotación del conjunto de objetos semiestables (una prueba que exige un gran esfuerzo, basada en resultados de [Ma1, Ma2]), o la identificación de tensores S-equivalentes (véase la Proposición 1.2.35) como puntos semiestables.

Una vez que hemos discutido la correspondencia entre estabilidad y estabilidad GIT con el ejemplo del móduli de tensores, recordamos la noción de la filtración de Harder-Narasimhan en la sección 1.3. Explicamos por qué esta filtración captura la idea de la máxima forma de desestabilizar un objeto a través de casos sencillos (como en el Ejemplo 1.3.8) y probamos su existencia y unicidad para el caso de haces coherentes sin torsión. Entonces explicamos la noción de la filtración de Harder-Narasimhan en el contexto abstracto de una categoría abeliana, como aparece en [Ru].

Finalmente, la sección 1.4 finaliza el capítulo explicando las ideas de [Ke]. En ese artículo, Kempf prueba la conjetura de Mumford-Tits, enunciando que un punto GIT inestable tiene un único subgrupo uniparamétrico que lo desestabiliza máximamente, en el sentido de GIT, maximizando una función, la **función de Kempf**, en el Teorema 1.4.6. Por tanto, teniendo este subgrupo uniparamétrico dando la máxima forma de desestabilización GIT, y la filtración de Harder-Narasimhan, podemos conjeturar que ambas corresponden a la misma noción, y formular la pregunta

¿Existe una relación entre la filtración de Harder-Narasimhan y el subgrupo uniparamétrico dado por Kempf?

En los capítulos 2 y 3 respondemos positivamente la anterior pregunta en diferentes casos.

Correspondencia entre las filtraciones de Kempf y Harder-Narasimhan

Primero, resumimos cómo demostrar la correspondencia para el caso principal, haces coherentes sin torsión sobre variedades proyectivas. Los casos restantes serán probados de forma análoga a este caso principal, basado en las mismas ideas y técnicas.

• Haces coherentes sin torsión sobre variedades proyectivas

Sea X una variedad proyectiva compleja no singular y sea $\mathcal{O}_X(1)$ un fibrado de línea amplio en X. Si E es un haz coherente en X, sea P_E su polinomio de Hilbert con respecto a $\mathcal{O}_X(1)$, es decir, $P_E(m) = \chi(E \otimes \mathcal{O}_X(m))$. Si P y Q son polinomios, escribimos $P \leq Q$ si $P(m) \leq Q(m)$ para $m \gg 0$.

Un haz sin torsión E sobre X se llama **semiestable** si para todo subhaz propio $F \subset E$, se verifica

$$\frac{P_F}{\operatorname{rk} F} \leq \frac{P_E}{\operatorname{rk} E} \ .$$

Si no es semiestable, se llama **inestable**, y posee una filtración canónica

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E \ ,$$

que satisface las siguientes propiedades, donde $E^i := E_i/E_{i-1}$:

1. Los polinomios de Hilbert verifican

$$\frac{P_{E^1}}{\operatorname{rk} E^1} > \frac{P_{E^2}}{\operatorname{rk} E^2} > \ldots > \frac{P_{E^{t+1}}}{\operatorname{rk} E^{t+1}}$$

2. Cada E^i es semiestable

que es llamada la **filtración de Harder-Narasimhan** de E (véase Teorema 1.3.5).

La construcción del espacio de móduli para estos objetos es originalmente debida a Gieseker para superficies, y generalizada a dimensión superior por Maruyama (c.f. [Gi1, Ma1, Ma2]). Para construir el espacio de móduli de haces sin torsión con polinomio de Hilbert fijo P, elegimos un cierto entero grande m y consideramos el esquema Quot (siguiendo la nomenclatura de Grothendieck) que parametriza cocientes

$$V \otimes \mathcal{O}_X(-m) \longrightarrow E \ , \tag{0.0.1}$$

donde V es un espacio vectorial fijo de dimensión $P(m)$ y E es un haz con $P_E = P$. El esquema de cocientes tiene una acción canónica de $\mathrm{SL}(V)$. Gieseker (c.f. [Gi1]) da una linearización de esta acción en un cierto fibrado de línea amplio, para usar la Teoría Geométrica de Invariantes para cocientar por la acción. El espacio de móduli de haces semiestables se obtiene como el cociente GIT.

Sea E un haz sin torsión que es inestable. Eligiendo m suficientemente grande (dependiendo de E), y eligiendo un isomorfismo $V \cong H^0(E(m))$, obtenemos un cociente como en (0.0.1). El correspondiente punto en el esquema Quot será GIT inestable y, por el criterio de Hilbert-Mumford, habrá al menos un subgrupo uniparamétrico de $\mathrm{SL}(V)$ que lo **desestabilizará** en el sentido de GIT.

Un **subgrupo uniparamétrico** de $\mathrm{SL}(V)$ es un homomorfismo no trivial $\mathbb{C}^* \to \mathrm{SL}(V)$. A un subgrupo uniparamétrico le asociamos una filtración con pesos como sigue. Existe una base de V, $\{e_1, \dots, e_p\}$, para la cual el subgrupo uniparamétrico toma la forma diagonal

$$t \mapsto \mathrm{diag}\left(t^{\Gamma_1}, \dots, t^{\Gamma_1}, t^{\Gamma_2}, \dots, t^{\Gamma_2}, \dots, t^{\Gamma_{t+1}}, \dots, t^{\Gamma_{t+1}}\right)$$

donde $\Gamma_1 < \cdots < \Gamma_{t+1}$. Sobre todos los subgrupos uniparamétricos, Kempf muestra que existe una clase de conjugación de máximamente desestabilizantes (es decir, que maximicen la función de Kempf en la Definición 1.4.4), todos ellos dando una única filtración de V con pesos, $(V_\bullet, n_\bullet)$,

$$0 \subset V_1 \subset V_2 \subset \ \cdots \ \subset V_t \subset V_{t+1} = V, \tag{0.0.2}$$

y números positivos n_1, $n_2, \dots, n_t > 0$ (c.f. Teorema 2.1.5).

Esta filtración induce una filtración de haces de E, evaluando los espacios de secciones globales,

$$0 \subseteq E_1^m \subseteq E_2^m \subseteq \cdots \subseteq E_t^m \subseteq E_{t+1} = E \ ,$$

que llamaremos la m-**filtración de Kempf de** E. Esta filtración depende del entero m que usamos en la construcción del espacio de móduli y que asegura que los haces semiestables (resp. inestables) se corresponden con las órbitas GIT semiestables (resp. GIT inestables). Entonces, el punto principal es probar el siguiente

Teorema 0.0.1 (c.f. Teorema 2.1.6)**.** *Existe un entero $m' \gg 0$ tal que la m-filtración de Kempf es independiente de m, para $m' \geq m$.*

Dado un entero m, la m-filtración de Kempf maximiza una función, llamada la **función de Kempf**,

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^{t} n_i (r \dim V_i - r_i \dim V)}{\sqrt{\sum_{i=1}^{t+1} \dim V^i \Gamma_i^2}} \ ,$$

la cual identificamos con una función geométrica (véase Proposición 2.1.13)

$$\mu_v(\Gamma) = \frac{(\Gamma, v)}{\|\Gamma\|} \ ,$$

donde (,) es un producto escalar en $\mathbb{R}^{t+1}$ dado por una matriz diagonal con elementos $\dim V^i$ en la diagonal, y el vector v tiene coordenadas

$$v_i = \frac{1}{\dim V^i \dim V} \left[r^i \dim V - r \dim V^i \right] \ .$$

El vector v está relacionado con la filtración $V_\bullet \subset V$ y el vector Γ está relacionado con los números $n_\bullet$ (poniendo $n_i = \frac{\Gamma_{i+1} - \Gamma_i}{\dim V}$) en la filtración de Kempf (0.0.2). Entonces, fijando un vector v en un espacio Euclídeo, consideramos la función $\mu_v(\Gamma)$ y nos preguntamos por el vector Γ que proporciona el máximo para μ_v. Ocurre que la respuesta es dada por la envolvente convexa del grafo determinado por v (véase Teorema 2.1.9).

Las m-filtraciones de Kempf, (es decir, las filtraciones de E que obtenemos evaluando $(V_\bullet, n_\bullet)$ para diferentes enteros m, donde $V \simeq H^0(E(m))$) pueden diferir para diferentes valores de m. Sin embargo, por una parte son maximales con respecto al valor que la función de Kempf alcanza en ellas, y por otra parte verifican propiedades de convexidad con respecto a la función $\mu_v(\Gamma)$. A partir de esto, podemos probar diferentes propiedades satisfechas por los filtros que aparecen en las filtraciones, las cuales caracterizan la **filtración de Kempf** y muestran que es independiente de m (c.f. Teorema 2.1.6).

La filtración que obtenemos, la cual de hecho no depende del entero m, se llama la **filtración de Kempf** de E. Entonces, observamos que las dos propiedades de convexidad que estaban implícitas en los argumentos que condujeron a probar el Teorema

2.1.6, propiedades que quedan descritas en los Lemas 2.1.15 y 2.1.16, son igualmente satisfechas por la filtración de Kempf (c.f Proposiciones 2.1.28 y 2.1.29). Observamos que estas propiedades de convexidad, las pendientes descendentes y la semiestabilidad de los cocientes, son precisamente las propiedades de la filtración de Harder-Narasimhan para haces (c.f. Teorema 1.3.5). Por consiguiente, por unicidad de la filtración de Harder-Narasimhan, probamos el siguiente

Teorema 0.0.2 (c.f. Teorema 2.1.7)**.** *La filtración de Kempf de un haz coherente sin torsión E coincide con la filtración de Harder-Narasimhan de E.*

Si reemplazamos los polinomios de Hilbert por los grados de los haces, la noción de estabilidad se transforma en μ-estabilidad (también conocida como estabilidad de las pendientes) y obtenemos la μ-filtración de Harder-Narasimhan. En [Br, BT], Bruasse y Teleman dan una interpretación en términos de teoría gauge de la μ-filtración de Harder-Narasimhan para haces sin torsión y pares holomorfos. Ellos también usan las ideas de Kempf, pero en el marco del grupo gauge, por lo que tienen que generalizar los resultados de Kempf a grupos infinito dimensionales.

Una correspondencia similar ha sido recientemente probada por Hoskins y Kirwan (c.f. [HK]) usando un método diferente. En la referencia se comienza con una filtración que se encuentra en un estrato de tipo de Harder-Narasimhan fijado (lo que llamamos *m-type*, c.f. Definición 2.1.22). Una diferencia con nuestro tratamiento es que ellas usan la existencia previa de la filtración de Harder-Narasimhan, mientras que nosotros no lo hacemos.

A continuación, brevemente resumimos otros problemas de móduli para los cuales mostramos la correspondencia entre la filtración de Harder-Narasimhan y la inestabilidad maximal GIT, usando un método similar al de los haces sin torsión.

• Pares holomorfos

Sea X una variedad proyectiva compleja no singular. Consideramos **pares holomorfos**

$$(E, \varphi : E \to \mathcal{O}_X)$$

dados por un haz coherente sin torsión de rango r con determinante fijo $\det(E) \cong \Delta$ y un morfismo al haz de estructura $\mathcal{O}_X$. Obsérvese que se trata de un caso perticular de los tensores estudiados en la sección 1.2, particularizando para $c = 1$, $b = 0$ y $s = 1$.

Sea δ un polinomio de grado a lo sumo $\dim X - 1$ y coeficiente director positivo. Dado un subhaz $E' \subset E$, definimos $\epsilon(E') = 1$ si $\varphi|_{E'} \neq 0$ y $\epsilon(E') = 0$ en caso contrario. Un par holomorfo (E, φ) es **δ-semiestable** si para cada $E' \subset E$, se tiene

$$\frac{P_{E'} - \delta\epsilon(E')}{\operatorname{rk} E'} \leq \frac{P_E - \delta\epsilon(E)}{\operatorname{rk} E} .$$

Existe una construcción del espacio de móduli de pares holomorfos δ-semiestables fijando el polinomio de Hilbert P y el determinante $\det(E) \simeq \Delta$ en [HL1] siguiendo las ideas de Gieseker, y en [HL2] (donde dichos pares son llamados *framed modules*) siguiendo las ideas de Simpson.

Un par holomorfo δ-inestable da un punto GIT inestable para el cual obtenemos un subgrupo uniparamétrico máximamente desestabilizante y una filtración de subpares. Mostramos que esta filtración no depende del entero m usado en la construcción del espacio de móduli en el Teorema 2.2.9, y que coincide con la filtración de Harder-Narasimhan para pares holomorfos en el Teorema 2.2.23.

La noción de par holomorfo es dual a la de un par consistente en un haz coherente junto con una sección. En ambos casos, la condición de estabilidad depende de un parámetro que es un polinomio. Ésta fue la primera construcción de un espacio de móduli con una condición de estabilidad dependiente de parámetros.

• Haces de Higgs

Sea X una variedad proyectiva compleja no singular. Un **haz de Higgs** es un par (E, φ) donde E es un haz coherente sobre X y

$$\varphi : E \to E \otimes \Omega^1_X ,$$

verificando $\varphi \wedge \varphi = 0$, es un morfismo llamado el **campo de Higgs**. Si el haz E es localmente libre hablaremos de **fibrados de Higgs**. Decimos que un haz de Higgs es **semiestable** (en el sentido de Gieseker) si para todo subhaz propio $F \subset E$ preservado por φ (i.e. $\varphi\big|_F : F \to F \otimes \Omega^1_X$) tenemos

$$\frac{P_F}{\operatorname{rk} F} \leq \frac{P_E}{\operatorname{rk} E} ,$$

donde P_E y P_F son los respectivos polinomios de Hilbert.

Podemos pensar un haz de Higgs (E, φ) como un haz coherente $\mathcal{E}$ en el fibrado cotangente T^*X (c.f. Lema 2.3.1) de tal forma que $\pi_*\mathcal{E} = E$, donde $\pi : T^*X \to X$. Usamos la construcción de Simpson (c.f. [Si1, Si2]) del espacio de móduli de haces de Higgs con este punto de vista. Entonces, probamos que el subgrupo uniparamétrico que da la máxima inestabilidad GIT (en el sentido de [Ke]) produce una filtración de subhaces

$$0 \subset \pi_* \mathcal{E}_1 \subset \pi_* \mathcal{E}_2 \subset \cdots \subset \pi_* \mathcal{E}_t \subset \pi_* \mathcal{E}_{t+1} = E$$

que no depende de los enteros m y l usados en la construcción de Simpson (c.f. Teorema 2.3.8). Aplicando π_*, obtenemos una filtración de subhaces de Higgs

$$0 \subset (E_1, \varphi|_{E_1}) \subset (E_2, \varphi|_{E_2}) \subset \ \cdots \ \subset (E_t, \varphi|_{E_t}) \subset (E_{t+1}, \varphi|_{E_{t+1}}) = (E, \varphi)$$

y probamos que coincide con la filtración de Harder-Narasimhan para haces de Higgs (c.f. Corlario 2.3.23).

Este caso tiene la particularidad de usar el embebimiento de Simpson (que depende de dos enteros m y l) en vez de el de Gieseker, lo que hace que el método funcione, indistintamente, en ambos casos.

• Tensores de rango 2

Sea X una variedad proyectiva compleja no singular de dimensión n. Sea E un haz coherente sin torsión de rango 2 sobre X. Llamamos **tensor de rango** 2 al par

$$(E, \varphi : \overbrace{E \otimes \cdots \otimes E}^{\text{s veces}} \longrightarrow \mathcal{O}_X) \ .$$

Este es otro caso particular de tensores (c.f. sección 1.2), haciendo $c = 1$, $b = 0$, $r = 2$ y s arbitrario. Consideramos la construcción del espacio de móduli para tales tensores de rango 2 con determinante fijo $\det(E) \cong \Delta$. De modo similar, probamos que la filtración de Kempf no depende del entero m, para $m \gg 0$ (c.f. Teorema 2.4.4) para construir una filtración de Harder-Narasimhan para el tensor (E, φ), que en este caso es un subhaz de rango 1 L,

$$0 \subset (L, \varphi|_L) \subset (E, \varphi) \ .$$

Cuando la variedad X es una curva y el morfismo φ es simétrico, podemos interpretar esta noción en términos de recubrimientos. Mirando el lugar de anulación de φ podemos ver el tensor (E, φ) como un recubrimiento de grado s, $X' \to X$, dentro de la superficie reglada $\mathbb{P}(E)$, para definir una noción de recubrimiento estable y caracterizar geométricamente el subhaz máximamente desestabilizante $L \subset E$ en términos de teoría de intersección para superficies regladas. Ocurre que, la expresión de la condición de estabilidad para tales tensores (c.f. (2.4.9)), puede interpretarse como la estabilidad de Gieseker del haz E (c.f. Definición 2.1.1) y la estabilidad de una configuración de puntos en $\mathbb{P}^1_{\mathbb{C}}$ (c.f. Ejemplos 1.1.5 y 1.1.15), ponderadas por el parámetro de estabilidad.

• Tensores de rango 3 y más allá

El capítulo 2 finaliza con algunas observaciones acerca del caso de los tensores de rango 3. Poniendo $s = 2$, $r = 3$ en la Definición 1.2.3, obtenemos el caso más sencillo para el cual no podemos usar las ideas previas para demostrar que el subgrupo uniparamétrico máximamente desestabilizante produce una filtración de subtensores que no dependa del entero usado en la construcción del espacio de móduli, para valores grandes del entero.

La razón de esto es que los resultados en la subsección 2.1.2 no se pueden aplicar. No podemos ver la función de Kempf (c.f. Definición 1.4.4) como una función en el espacio Euclídeo tomando valores en los pesos de la filtración (c.f. Proposición 2.1.13) porque los pesos dependerán de la filtración. En este caso no podemos probar los análogos a los Lemas 2.2.7 y 2.4.6, por lo que el método que usamos no sirve en general.

La alternativa a esto es comparar filtraciones candidatas a ser la filtración de Harder-Narasimhan mirando los valores que toma la función de Kempf en ellas, por métodos elementales. La sección finaliza con la observación de que existe una clase restringida de tensores (aquéllos para los cuales no se producen estos hechos y podemos probar la independencia entre los pesos y las filtraciones), tal que los pasos de la prueba de la correspondencia pueden ser llevados a cabo (véase la Definición 2.5.1).

Representaciones de un carcaj

En el capítulo 3 exploramos las ideas desarrolladas en el capítulo 2 para representaciones de un carcaj. Probamos la correspondencia análoga para representaciones de un carcaj en espacios vectoriales finito dimensionales y usamos la construcción functorial de un espacio de móduli para haces coherentes en [ACK] para dar otra demostración del Teorema 2.1.7.

Sea Q un carcaj finito, dado por un conjunto finito de vértices y flechas entre ellos, y una representación de Q en k-espacios vectoriales finito dimensionales, donde k es un cuerpo algebraicamente cerrado de característica arbitraria. Existe una noción de estabilidad para tales representaciones (c.f. Definición 3.1.1) dada por King en [Ki] y, más en general, por Reineke en [Re] (ambas casos particulares de la noción abstracta de estabilidad para una categoría abeliana que podemos encontrar en [Ru]), y una noción de existencia de una única filtración de Harder-Narasimhan con respecto a la condición de estabilidad (c.f. Teorema 3.1.6).

En la sección 3.1 consideramos la construcción de un espacio de móduli para estos objetos dada por King (c.f. [Ki]), y asociamos a una representación inestable un punto inestable, en el sentido de la Teoría Geométrica de Invariantes, en un espacio de parámetros donde actúa un grupo. Entonces, el subgrupo uniparamétrico dado por Kempf (c.f. Teorema 3.1.13), que es máximamente desestabilizante en el sentido GIT, otorga una filtración de subrepresentaciones. Probamos que esta filtración coincide con la filtración de Harder-Narasimhan para la representación incial, en el Teorema 3.1.15.

Este caso es ligeramente diferente porque, en los anteriores, el grupo por el cual estábamos tomando el cociente GIT en la construcción del espacio de móduli, era $\mathrm{SL}(V)$, pero en este caso se trata de un producto de grupos generales lineales, uno para cada vértice del carcaj. Entonces, la **longitud** que elegimos en el espacio de subgrupos uniparamétricos al definir la función de Kempf (véase Definición 1.4.2) depende de ciertos parámetros (uno para cada factor simple en el grupo), y mostramos cómo tenemos que colocar los parámetros convenientemente eligiendo una longitud particular, para ser capaces de relacionar el subgrupo uniparamétrico GIT máximamente desestabilizante con la filtración de Harder-Narasimhan.

Finalmente, en la sección 3.2 definimos los Q-haces, que son representaciones de un carcaj en la categoría de haces coherentes, y damos una noción de estabilidad para ellos, siguiendo [AC, ACGP]. Para un carcaj de un sólo vértice, un Q-haz es lo mismo que un haz coherente y usamos la construcción functorial del espacio de móduli para haces dada en [ACK]. En esta construcción, se relacionan los haces con los módulos de Kronecker para reescribir la condición de estabilidad en términos de representaciones de un carcaj en espacios vectoriales. El Teorema 3.2.4 relaciona todas las diferentes nociones de estabilidad que aparecen envueltas en esta tesis, a saber, la estabilidad de un haz como Q-haz (que equivale a la estabilidad de Gieseker para haces), la estabilidad del módulo de Kronecker asociado, la estabilidad de la representación de otro carcaj asociado $\tilde{Q}$, y la estabilidad GIT del punto correspondiente en el espacio de parámetros. Usando la equivalencia de las diferentes estabilidades, podemos aplicar el teorema de Kempf (c.f. Teorema 3.1.13) para encontrar una filtración de Harder-Narasimhan para la representación asociada de un carcaj en espacios vectoriales y, desde aquí, obtener la filtración de Harder-Narasimhan para el Q-haz en el Teorema 3.2.11.

Conclusiones

Esta tesis contiene ideas en dos direcciones. Por una parte, exploramos la relación entre las condiciones de estabilidad y las nociones de estabilidad GIT en las construcciones de espacios de móduli. Por otra parte, relacionamos el concepto natural de máxima inestabilidad GIT (en el sentido de Kempf) y la filtración de Harder-Narasimhan, lo cual produce una correspondencia en varios casos donde la filtración de Harder-Narasimhan es previamente conocida, o da una nueva noción de una tal filtración en otros casos.

El esquema de la prueba es similar en todos los casos. Primero, obtener una filtración de subespacios vectoriales de secciones globales que maximice la función de Kempf (c.f. Definición 1.4.4) para el problema GIT considerado, entonces evaluar las secciones para conseguir una filtración de subobjetos, que llamamos la **m-filtración de Kempf**. Relacionamos la función de Kempf con una función en el espacio Euclídeo (c.f. Proposición 2.1.13) para aplicar los resultados de convexidad (véase subsección 2.1.2). Seguidamente, demostramos propiedades de la m-filtración de Kempf que la caracterizan y la harán independiente del entero m, por lo que obtendremos una filtración que llamamos la **filtración de Kempf**. Finalmente, en los casos donde la filtración de Harder-Narasimhan es previamente conocida, probamos que las propiedades de convexidad de la filtración de Kempf son, precisamente, las de la filtración de Harder-Narasimhan, luego por unicidad ambas filtraciones coinciden (c.f. Teorema 2.1.7). En otros casos, como los tensores de rango 2 en la sección 2.4, la filtración de Kempf (que es única) define una filtración de Harder-Narasimhan. Nótese que, en la sección 3.1, la construcción del móduli no depende de ningún entero, por tanto no tenemos que probar un análogo al Teorema 2.1.6, y la correspondencia es mucho más sencilla y rápida.

La noción de longitud (c.f. Definición 1.4.2) que necesitamos para definir la velocidad del subgrupo uniparamétrico (véase sección 1.4), juega un papel importante. En principio, diferentes longitudes darían diferentes subgrupos uniparamétricos GIT máximamente desestabilizantes, por tanto diferentes filtraciones de Kempf candidatas a ser la filtración de Harder-Narasimhan. En la sección 3.1 observamos cómo, diferentes elecciones de longitud corresponden a diferentes definiciones de estabilidad en la Definición 3.1.1. Por tanto, dada una noción de filtración de Harder-Narasimhan (dependiente de la noción de estabilidad), tenemos que colocar los parámetros en la linearización en la construcción del espacio de móduli, y en la definición de longitud en el conjunto de subgrupos uniparamétricos (c.f. Proposición 3.1.8), de forma conveniente, para lograr una correspondencia entre las filtraciones de Kempf y Harder-Narasimhan.

Potencialmente, podríamos esperar usar estas ideas para definir filtraciones de Harder-Narasimhan en casos donde no han sido estudiadas, y usarlas como poderosa herramienta para importantes aplicaciones donde ha sido usada en el pasado, tales como los teoremas de restricción o el cálculo de los números de Betti y los puntos racionales en espacios de móduli (c.f. [HN]).

Muchos de los casos donde hemos aplicado el método caen dentro de categorías abelianas, donde la filtración de Harder-Narasimhan verifica las propiedades de convexidad que se desprenden de su definición. Sería interesante entender si esto impone una condición para esperar una noción de filtración de Harder-Narasimhan, tal como la conocemos.

Los resultados principales de esta tesis están recogidos en los preprints [GSZ, Za]. El primero contiene la correspondencia entre la máxima forma de desestabilizar en el sentido GIT y la filtración de Harder-Narasimhan para haces coherentes sin torsión sobre variedades proyectivas y para pares holomorfos (secciones 2.1 y 2.2). El segundo contiene el resultado para representaciones de un carcaj en espacios vectoriales de dimensión finita (sección 3.1).

Introduction

This Ph.D. thesis is devoted to the study of the relation between the maximal unstability in the sense of Geometric Invariant Theory and the Harder-Narasimhan filtration in different moduli problems. Many of the moduli problems in geometry use Geometric Invariant Theory (abbreviated GIT) in the construction of a moduli space. We impose, from the beginning, a notion of stability on the objects for which we want to construct a moduli space and, by the Geometric Invariant Theory, we associate to a stable (resp. semistable, unstable) object, a GIT stable (resp. GIT semistable, GIT unstable) point in certain space, establishing a correspondence between both concepts of stability. The GIT concept of maximal unstability has been studied by several authors, and for our purposes we consider the work of Kempf, whose paper [Ke] explores it. On the other hand, the Harder-Narasimhan filtration, widely used in many problems in algebraic geometry, is the geometrical object which represents the idea of maximal unstability for the previous notion of stability imposed on the objects.

In this thesis we prove that both notions of maximal unstability do coincide, and show a correspondence between them in different cases. The first chapter contains general notions about moduli problems, Geometric Invariant Theory and the Harder-Narasimhan filtration we will use in chapters 2 and 3, apart from an example of the construction of a moduli space for tensors. In the second chapter we study different moduli problems in relation with sheaves, or sheaves with additional structure. We develop a technique to prove the mentioned correspondence for torsion free coherent sheaves over arbitrary dimensional projective varieties, holomorphic pairs, Higgs sheaves, rank 2 tensors, and we discuss rank 3 tensors as the first case of tensors for which the technique we use breaks down. In the third chapter we study representations of quivers, proving a similar result for representations on the category of vector spaces and, again, coherent sheaves seen as representations of a one vertex quiver on the category of coherent sheaves.

Maximal unstability

Consider a moduli problem where we try to classify a class of algebro-geometric objects modulo an equivalence relation. Usually, we have to impose a stability condition on the objects we classify, in order to obtain a moduli space with good properties where each point corresponds to an equivalence class of objects. Then, by adding additional data to the objects (what is usually called in the literature *to rigidify the data*) we include them in a parameter space easier to work with (an affine or projective scheme). To get rid of these new data we have to quotient by the action of a group which is precisely encoding the changes in the additional data. And to take this quotient we use Mumford's Geometric Invariant Theory (see [Mu] for the first edition, [MF] and [MFK] for second and third editions) to obtain a projective moduli space classifying the objects in the moduli problem. The study of the orbits of the action of this group leads to the notion of GIT stability defining points, in the parameter space, which are GIT stable and points which are GIT unstable.

In every moduli problem using GIT, at some point one has to prove that both notions of stability do coincide, then the stable objects correspond to the GIT stable points, and the unstable ones are related to the GIT unstable ones. To that purpose, Mumford states a numerical criterion (c.f. [Mu, Theorem 2.1]) based on ideas of Hilbert in [Hi]. Theorem 1.1.14, known as the **Hilbert-Mumford criterion**, characterizes the GIT stability through 1-parameter subgroups, where a numerical function (which we call the *minimal relevant weight*) turns out to be positive or negative whether the 1-parameter subgroup destabilizes a point or not, in the sense of GIT. Besides, when a point is GIT unstable, we are able to talk about *degrees of unstability*, or some 1-parameter subgroups which are more destabilizing than others.

Based on the work of Mumford, Tits and other authors, we can measure this notion by means of a rational function on the space of 1-parameter subgroups, whose numerator is the numerical function of the Hilbert-Mumford criterion and whose denominator is a **length** of the 1-parameter subgroup that we choose to avoid rescaling of the numerical function (c.f. section 1.4). The **center's conjecture of Tits** (c.f. [Mu, p. 64]) establishes that there exists a unique 1-parameter subgroup giving a maximum for this function, representing the GIT maximal unstability. Kempf explores these ideas in a paper in 1978 (c.f. [Ke]), solving what he calls **the Mumford-Tits conjecture** (referring to Tits center's conjecture as it appears on [Mu]) by proving that there exists a unique

1-parameter subgroup with these properties in [Ke, Theorem 2.2, Theorem 3.4] (for a correspondence between definitions in [Mu] and [Ke] see [MFK, Appendix 2B]).

An unstable object gives a GIT unstable point for which there exists a unique 1-parameter subgroup GIT maximally destabilizing. The different 1-parameter subgroups produce, in a natural way, flags (giving the **flag complex** of a group G studied by Tits and Mumford), hence we would like to consider the flag associated to that unique 1-parameter subgroup GIT maximally destabilizing, and construct a filtration by subobjects of the original unstable object, out of this 1-parameter subgroup. The main results of this Ph.D. thesis consist on translating this 1-parameter subgroup to a filtration of the object, proving that this translation is well and uniquely defined (i.e. it does not depend on several choices made during the process) and, finally, proving that it coincides with the Harder-Narasimhan filtration in cases where it is already known, or gives a new notion of such filtration in other cases.

Geometric Invariant Theory and moduli space of tensors

In the first chapter I collect the necessary background about moduli spaces and Geometric Invariant Theory to present the problem studied.

Section 1.1 contains a description of what a moduli problem is, with its rigorous formulation. We provide basic examples as the moduli space of non singular complex cubics or the formulation of the problem of the moduli space of algebraic curves of genus g. Then, we recall the notions of Geometric Invariant Theory we will need in the following, the different types of quotients, linearizations of actions of groups and the Hilbert-Mumford criterion, essential to deal with GIT stability. We also give an example (c.f. Example 1.1.4) to realize the concept of S-equivalence for the orbits of the action of a group, or the classical example of combinations of points in $\mathbb{P}^1_{\mathbb{C}}$ (c.f. Examples 1.1.5 and 1.1.15) to apply the Hilbert-Mumford criterion.

Section 1.2 is devoted to present a complete example of a construction of a moduli space using Geometric Invariant Theory: the moduli space of tensors over a projective variety. This construction was first studied for tensors over curves by Schmitt (c.f. [Sch]) and then by Gómez and Sols (c.f. [GS1]). In the section, we follow [GS1] but using the embedding of Gieseker (c.f. subsection 1.2.3) instead of Simpson's, as it is done in [GS1].

With this embedding, tensors are collected in a parameter space on which a group is acting. Theorem 1.2.31 establishes that the semistable tensors correspond to the GIT semistable orbits under the action of the group, hence the GIT quotient will be a moduli space for semistable tensors. In this example, many of the general features which appear in GIT constructions of moduli spaces take place, such as the dependence of the stability notion with a parameter, the necessity of proving the boundedness of the set of semistable objects (a proof which usually takes a big effort, based on results in [Ma1, Ma2]), or the identification of S-equivalent tensors (c.f. Proposition 1.2.35) as semistable points.

Once we have discussed the correspondence between stability and GIT stability with the example of the moduli of tensors, we recall the notion of the Harder-Narasimhan filtration in section 1.3. We explain why this filtration captures the idea of the maximal way of destabilizing an object through easy cases (as in Example 1.3.8) and prove its existence and uniqueness for the case of torsion free coherent sheaves. Then we explain the notion of the Harder-Narasimhan filtration in the abstract context of an abelian category, which appears in [Ru].

Finally, section 1.4 closes this chapter explaining the ideas of [Ke]. There, Kempf proves the Mumford-Tits conjecture, asserting that a GIT unstable point has a unique 1-parameter subgroup which maximally destabilizes it, in the sense of GIT, by maximizing a function, the **Kempf function** (c.f. Definition 1.4.4), in Theorem 1.4.6. Hence, having this 1-parameter subgroup giving GIT maximal way of destabilizing and the Harder-Narasimhan filtration, we can conjecture that they do correspond to the same notion, and formulate the question

Is the Harder-Narasimhan related to the 1-parameter subgroup given by Kempf?

In chapters 2 and 3 we answer positively the previous question for different cases.

Correspondence between Kempf and Harder-Narasimhan filtrations

First, we summarize how to prove the correspondence for the main case, torsion free sheaves over projective varieties. The rest of cases will be proven in an analogous way to this main case, based on the same ideas and techniques.

• Torsion free coherent sheaves over projective varieties

Let X be a smooth complex projective variety and let $\mathcal{O}_X(1)$ be an ample line bundle on X. If E is a coherent sheaf on X, let P_E be its Hilbert polynomial with respect to $\mathcal{O}_X(1)$, i.e., $P_E(m) = \chi(E \otimes \mathcal{O}_X(m))$. If P and Q are polynomials, we write $P \leq Q$ if $P(m) \leq Q(m)$ for $m \gg 0$.

A torsion free sheaf E on X is called **semistable** if for all proper subsheaves $F \subset E$, it is

$$\frac{P_F}{\operatorname{rk} F} \leq \frac{P_E}{\operatorname{rk} E} \; .$$

If it is not semistable, it is called **unstable**, and it has a canonical filtration

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E \; ,$$

satisfying the following properties, where $E^i := E_i/E_{i-1}$:

1. The Hilbert polynomials verify

$$\frac{P_{E^1}}{\operatorname{rk} E^1} > \frac{P_{E^2}}{\operatorname{rk} E^2} > \ldots > \frac{P_{E^{t+1}}}{\operatorname{rk} E^{t+1}}$$

2. Every E^i is semistable

which is called the **Harder-Narasimhan filtration** of E (c.f. Theorem 1.3.5).

The construction of the moduli space for these objects is originally due to Gieseker for surfaces, and generalized to higher dimension by Maruyama (c.f. [Gi1, Ma1, Ma2]). To construct the moduli space of torsion free sheaves with fixed Hilbert polynomial P, we choose a suitably large integer m and consider the Quot scheme parametrizing quotients

$$V \otimes \mathcal{O}_X(-m) \longrightarrow E \; , \qquad (0.0.3)$$

where V is a fixed vector space of dimension $P(m)$ and E is a sheaf with $P_E = P$. The Quot scheme has a canonical action by $\mathrm{SL}(V)$. Gieseker (c.f. [Gi1]) gives a linearization of this action on a certain ample line bundle, in order to use Geometric Invariant Theory to take the quotient by the action. The moduli space of semistable sheaves is obtained as the GIT quotient.

Let E be a torsion free sheaf which is unstable. Choosing m large enough (depending on E), and choosing an isomorphism $V \cong H^0(E(m))$, we obtain a quotient as in (0.0.3). The corresponding point in the Quot scheme will be GIT unstable and, by the Hilbert-Mumford criterion, there will be at least one 1-parameter subgroup of $\mathrm{SL}(V)$ which **destabilizes** the point in the sense of GIT.

A **1-parameter subgroup** of $\mathrm{SL}(V)$ is a non trivial homomorphism $\mathbb{C}^* \to \mathrm{SL}(V)$. To a 1-parameter subgroup we associate a weighted filtration as follows. There is a basis $\{e_1, \ldots, e_p\}$ of V where it has a diagonal form

$$t \mapsto \operatorname{diag}\left(t^{\Gamma_1}, \ldots, t^{\Gamma_1}, t^{\Gamma_2}, \ldots, t^{\Gamma_2}, \ldots, t^{\Gamma_{t+1}}, \ldots, t^{\Gamma_{t+1}}\right)$$

with $\Gamma_1 < \cdots < \Gamma_{t+1}$. Among all these 1-parameter subgroups, Kempf shows that there is a conjugacy class of maximally destabilizing 1-parameter subgroups (i.e. maximizing the Kempf function in Definition 1.4.4) all of them giving a unique weighted filtration $(V_\bullet, n_\bullet)$ of V,

$$0 \subset V_1 \subset V_2 \subset \ \cdots \ \subset V_t \subset V_{t+1} = V, \tag{0.0.4}$$

and positive numbers n_1, $n_2, \ldots,$ $n_t > 0$ (c.f. Theorem 2.1.5).

This filtration induces a sheaf filtration of E by evaluating the spaces of global sections,

$$0 \subseteq E_1^m \subseteq E_2^m \subseteq \cdots \subseteq E_t^m \subseteq E_{t+1} = E \ ,$$

which we call the **m-Kempf filtration of** E. This filtration depends on the integer m which we use to construct the moduli space and assure that the semistable sheaves (resp. unstable) correspond to GIT semistable (resp. GIT unstable) orbits. Then, the main point is to prove the following

Theorem 0.0.1 (c.f. Theorem 2.1.6)**.** *There exists an integer $m' \gg 0$ such that the m-Kempf filtration is independent of m, for $m' \geq m$.*

Given an integer m, the m-Kempf filtration maximizes a function, called **the Kempf function**,

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^{t} n_i(r \dim V_i - r_i \dim V)}{\sqrt{\sum_{i=1}^{t+1} \dim V^i \Gamma_i^2}} \ ,$$

which we identify with a geometrical function (c.f. Proposition 2.1.13)

$$\mu_v(\Gamma) = \frac{(\Gamma, v)}{\|\Gamma\|} \ ,$$

where $(\ , \ )$ is an inner product in $\mathbb{R}^{t+1}$ given by a diagonal matrix with elements $\dim V^i$ in the diagonal, and the vector v having coordinates

$$v_i = \frac{1}{\dim V^i \dim V}\left[r^i \dim V - r \dim V^i\right] .$$

The vector v is related with the flag $V_\bullet \subset V$ and the vector Γ is related with the numbers $n_\bullet$ (by setting $n_i = \frac{\Gamma_{i+1}-\Gamma_i}{\dim V}$) in the Kempf filtration (0.0.4). Then, fixing a vector v in an Euclidean space, consider the function $\mu_v(\Gamma)$ and ask for the vector Γ which gives maximum for μ_v. It turns out that the answer is given by the convex envelope of the graph produced by v (c.f. Theorem 2.1.9).

The m-Kempf filtrations, (i.e. the filtrations of E we obtain by evaluating $(V_\bullet, n_\bullet)$ for different integers m, where $V \simeq H^0(E(m)))$ can differ for different values of m. However, they are, on the one hand, maximal with respect to the value the Kempf function achieves on them and, on the other hand, verify convexity properties with respect to $\mu_v(\Gamma)$. From this, we can prove different properties satisfied by the filters appearing in the filtrations, which characterize the **Kempf filtration** and show that it is independent of m (c.f. Theorem 2.1.6).

The filtration we obtain, which does actually not depend on the integer m, is called the **Kempf filtration** of E. Then, we observe that the two convexity properties which were implicit in the arguments leading to prove Theorem 2.1.6, properties which are described by Lemmas 2.1.15 and 2.1.16, are also satisfied by the Kempf filtration (c.f Propositions 2.1.28 and 2.1.29). And we realize that these convexity properties are precisely the properties of the Harder-Narasimhan filtration for sheaves (c.f. Theorem 1.3.5), the descending slopes and the semistability of the quotients. Therefore, by uniqueness of the Harder-Narasimhan filtration, we prove the following

Theorem 0.0.2 (c.f. Theorem 2.1.7)**.** *The Kempf filtration of an unstable torsion free coherent sheaf E coincides with the Harder-Narasimhan filtration of E.*

If we replace Hilbert polynomials with degrees, the notion of stability becomes μ-stability (also known as slope stability) and we obtain the μ-Harder-Narasimhan filtration. In [Br, BT], Bruasse and Teleman give a gauge-theoretic interpretation of the μ-Harder-Narasimhan filtration for torsion free sheaves and for holomorphic pairs. They also use Kempf's ideas, but in the setting of the gauge group, so they have to generalize Kempf's results to infinite dimensional groups.

A similar correspondence has been proved recently by Hoskins and Kirwan (c.f. [HK]) by using a different method. They start with a filtration which lays on a stratum with fixed *Harder-Narasimhan type* (what we call m-**type**, c.f. Definition 2.1.22). One difference with our approach is that they use the previous existence of the Harder-Narasimhan filtration, whereas we do not use it.

Now, we briefly summarize other moduli problems for which we show the correspondence between the Harder-Narasimhan filtration and the GIT maximal unstability, using a similar method as in the case of torsion free sheaves.

• Holomorphic pairs

Let X be a smooth complex projective variety. Let us consider **holomorphic pairs**

$$(E, \varphi : E \to \mathcal{O}_X)$$

given by a coherent torsion free sheaf of rank r with fixed determinant $\det(E) \cong \Delta$ and a morphism to the structure sheaf $\mathcal{O}_X$. Observe that this is a particular case of the tensors studied in section 1.2, by setting $c = 1$, $b = 0$ and $s = 1$.

Let δ be a polynomial of degree at most $\dim X - 1$ and positive leading coefficient. Given a subsheaf $E' \subset E$, let $\epsilon(E') = 1$ if $\varphi|_{E'} \neq 0$ and $\epsilon(E') = 0$ otherwise. A holomorphic pair (E, φ) is δ-**semistable** if for every $E' \subset E$

$$\frac{P_{E'} - \delta\epsilon(E')}{\operatorname{rk} E'} \leq \frac{P_E - \delta\epsilon(E)}{\operatorname{rk} E} \ .$$

There is a construction of the moduli space of δ-semistable holomorphic pairs with fixed Hilbert polynomial P and fixed determinant $\det(E) \simeq \Delta$ in [HL1] following Gieseker's ideas, and in [HL2] (where these pairs are called framed modules) following Simpson's ideas.

A δ-unstable holomorphic pair give a GIT unstable point for which we obtain a 1-parameter subgroup GIT maximally destabilizing and a filtration of subpairs. We show that this filtration does not depend on the integer m used in the construction of the moduli space in Theorem 2.2.9, and that it coincides with the Harder-Narasimhan filtration for holomorphic pairs in Theorem 2.2.23.

This notion of holomorphic pair is dual to the pair consisting on a coherent sheaf together with a section. In both cases, the stability condition depends on a parameter which is a polynomial. This was the first construction of a moduli space with a stability condition depending on parameters.

• Higgs sheaves

Let X be a smooth complex projective variety. A **Higgs sheaf** is a pair (E,φ) where E is a coherent sheaf over X and

$$\varphi : E \to E \otimes \Omega^1_X \ ,$$

verifying $\varphi \wedge \varphi = 0$, a morphism called the **Higgs field**. If the sheaf E is locally free we talk about **Higgs bundles**. We say that a Higgs sheaf is **semistable** (in the sense of Gieseker) if for all proper subsheaves $F \subset E$ preserved by φ (i.e. $\varphi\big|_F : F \to F \otimes \Omega^1_X$) we have

$$\frac{P_F}{\operatorname{rk} F} \leq \frac{P_E}{\operatorname{rk} E} \ ,$$

where P_E and P_F are the respective Hilbert polynomials.

A Higgs sheaf (E,φ) can be thought as a coherent sheaf $\mathcal{E}$ on the cotangent bundle T^*X (c.f. Lemma 2.3.1) such that $\pi_*\mathcal{E} = E$, where $\pi : T^*X \to X$. We use the construction of Simpson (c.f. [Si1, Si2]) of a moduli space for Higgs sheaves with this point of view. Then, we prove that the 1-parameter subgroup giving the GIT maximal unstability (in the sense of [Ke]) provides a filtration of subsheaves

$$0 \subset \pi_*\mathcal{E}_1 \subset \pi_*\mathcal{E}_2 \subset \cdots \subset \pi_*\mathcal{E}_t \subset \pi_*\mathcal{E}_{t+1} = E$$

which does not depend on the integers m, l used in Simpson's construction (c.f. Theorem 2.3.8). By applying π_*, we get a filtration of Higgs subsheaves

$$0 \subset (E_1, \varphi|_{E_1}) \subset (E_2, \varphi|_{E_2}) \subset \ \cdots \ \subset (E_t, \varphi|_{E_t}) \subset (E_{t+1}, \varphi|_{E_{t+1}}) = (E,\varphi)$$

and we prove that it coincides with the Harder-Narasimhan filtration for Higgs sheaves (c.f. Corollary 2.3.23).

This case has the particularity of using the embedding of Simpson (which depends on two integers m and l) instead of Gieseker's, what makes the method work, indistinctly, both cases.

• Rank 2 tensors

Let X be a smooth complex projective variety of dimension n. Let E be a rank 2 coherent torsion free sheaf over X. We call a **rank** 2 **tensor** the pair

$$(E, \varphi : \overbrace{E \otimes \cdots \otimes E}^{\text{s times}} \longrightarrow \mathcal{O}_X) \ .$$

This is another particular case of tensors (c.f. section 1.2), by setting $c = 1$, $b = 0$, $r = 2$ and arbitrary s. Consider the given construction of the moduli space for such rank 2 tensors with fixed determinant $\det(E) \cong \Delta$. Similarly we prove that the Kempf filtration does not depend on the integer m, for $m \gg 0$ (c.f. Theorem 2.4.4) to construct a Harder-Narasimhan filtration for the tensor (E, φ), which in this case is a rank 1 subsheaf L,

$$0 \subset (L, \varphi|_L) \subset (E, \varphi) .$$

When the variety X is a curve and the morphism φ is symmetric, we can interpret this notion in terms of coverings. Looking at the vanishing locus of φ we can see the tensor (E, φ) as a degree s covering $X' \to X$ lying on the ruled surface $\mathbb{P}(E)$, to define a notion of stable covering and characterize geometrically the maximally destabilizing subsheaf $L \subset E$ in terms of intersection theory for ruled surfaces. It turns out that, the expression of the stability condition for such tensors (c.f. (2.4.9)), can be seen as the Gieseker's stability of the sheaf E (c.f. Definition 2.1.1) and the stability of a configuration of points in $\mathbb{P}^1_{\mathbb{C}}$ (c.f. Examples 1.1.5 and 1.1.15), pondered by the stability parameter.

• Rank 3 tensors and beyond

Chapter 2 finishes with some observations about the rank 3 tensors case. Setting $s = 2$, $r = 3$ on Definition 1.2.3 we obtain the easiest case for which we cannot use the previous ideas to prove that the 1-parameter subgroup GIT maximally destabilizing produces a filtration of subtensors which does not depend on some integer used in the construction of the moduli space, for large values of the integer.

The reason is that results on subsection 2.1.2 do not apply. We cannot see the Kempf function (c.f. Definition 1.4.4) as a function on the Euclidean space taking values on the weights of the filtration (c.f. Proposition 2.1.13) because the weights will depend on the filtration. In this case we cannot prove analogous to Lemmas 2.2.7 and 2.4.6, hence the method we use does not apply in general.

The alternative is to compare candidates to be the Harder-Narasimhan filtration by looking at the values the Kempf function takes at them, by elementary methods. The section finishes with the observation that there exists a restricted class of tensors (those for which the features discussed before do not apply and we can prove the independence between weights and filtrations), such that the steps of the proof of the correspondence do hold (c.f. Definition 2.5.1).

Representations of quivers

In chapter 3 we explore the ideas developed in chapter 2 for representations of quivers. We prove the analogous correspondence for representations of quivers on finite dimensional vector spaces and use the functorial construction of a moduli space for coherent sheaves in [ACK] to give another proof of Theorem 2.1.7.

Let Q be a finite quiver, given by a finite set of vertices and arrows between them, and a representation of Q on finite dimensional k-vector spaces, where k is an algebraically closed field of arbitrary characteristic. There exists a notion of stability for such representations (c.f. Definition 3.1.1) given by King in [Ki] and, more generally, by Reineke in [Re] (both particular cases of the abstract notion of stability for an abelian category that we can find in [Ru]), and a notion of the existence of a unique Harder-Narasimhan filtration with respect to that stability condition (c.f. Theorem 3.1.6).

In section 3.1 we consider the construction of a moduli space for these objects by King (c.f. [Ki]), and associate to an unstable representation an unstable point, in the sense of Geometric Invariant Theory, in a parameter space where a group acts. Then, the 1-parameter subgroup given by Kempf (c.f. Theorem 3.1.13), which is maximally destabilizing in the GIT sense, gives a filtration of subrepresentations. We prove that it coincides with the Harder-Narasimhan filtration for the initial representation, in Theorem 3.1.15.

This case is slightly different because, in the previous ones, the group we were taking the GIT quotient by in the construction of the moduli space, was $\mathrm{SL}(N)$, but in this case it is a product of general linear groups, one for each vertex of the quiver. Then, the **length** we choose in the space of 1-parameter subgroups when defining the Kempf function (c.f. Definition 1.4.2) depends on some parameters (one for each simple factor in the group), and we show how we have to set the parameters conveniently by choosing a particular length, to be able to relate the 1-parameter subgroup GIT maximally destabilizing with the Harder-Narasimhan filtration.

Finally, in section 3.2 we define Q-sheaves, which are representations of a quiver on the category of coherent sheaves, and give a stability notion for them, following [AC, ACGP]. For a one vertex quiver, a Q-sheaf is the same that a coherent sheaf and we use the functorial construction of a moduli space for sheaves given in [ACK]. In this construction, sheaves are related to Kronecker modules, then the stability condition turns out to be rewritten in terms of representations of quivers on vector spaces. Theorem 3.2.4

relates all different notions of stability involved in this thesis, say, stability of the sheaf as a Q-sheaf (equivalent to Gieseker's stability for sheaves), stability of the Kronecker module associated, stability of the representation of another quiver associated $\tilde{Q}$, and GIT stability of the corresponding point in the parameter space. Using the equivalence of different stabilities, we can apply the Kempf theorem (c.f. Theorem 3.1.13) to find a Harder-Narasimhan filtration for the associated representation of a quiver on vector spaces and, from it, obtain the Harder-Narasimhan filtration for the Q-sheaf in Theorem 3.2.11.

Conclusions

This thesis contains ideas in two directions. On the one hand, we explore the relation between stability conditions and GIT stability notions in constructions of moduli spaces. On the other hand, we relate a natural concept of GIT maximal unstability (in the sense of Kempf) and the Harder-Narasimhan filtration, which gives a correspondence in several cases where the Harder-Narasimhan filtration is previously known or gives a new notion of a such filtration in other cases.

The sketch of the proof is similar in all cases. First, obtaining a filtration of vector subspaces of global sections which maximizes the Kempf function (c.f. Definition 1.4.4) for the GIT problem considered, then evaluate the sections to get a filtration of subobjects, called **the m-Kempf filtration**. We relate the Kempf function with a function in the Euclidean space (c.f. Proposition 2.1.13) to apply results on convexity (c.f. subsection 2.1.2). Next, we prove properties of the m-Kempf filtration which characterize it and will make it not to depend on the integer m, hence we get a filtration called **the Kempf filtration**. Finally, in cases where the Harder-Narasimhan filtration is known, we prove that the convexity properties of the Kempf filtration are, precisely, the ones of the Harder-Narasimhan filtration, hence by uniqueness both filtrations do coincide (c.f. Theorem 2.1.7). In other cases, as rank 2 tensors in section 2.4, Kempf filtration (which is unique) defines a Harder-Narasimhan filtration. Note that, in section 3.1, the moduli construction does not depend on any integer, hence we do not have to prove an analogous to Theorem 2.1.6, and the correspondence is much easier and quicker.

The notion of **length** (c.f. Definition 1.4.2) we need to define the *speed* of the 1-parameter subgroups (c.f. section 1.4) plays an important role. In principal, different lengths would give different 1-parameter subgroups GIT maximally destabilizing, hence different Kempf filtrations candidates to be the Harder-Narasimhan filtration. In section 3.1 we observe how, different choices of length correspond to different definitions of stability in Definition 3.1.1. Hence, given a notion of Harder-Narasimhan filtration (depending on the notion of stability), we have to set the parameters in the linearization in the construction of the moduli space, and in the definition of the length in the set of 1-parameter subgroups (c.f. Proposition 3.1.8), conveniently, in order to achieve a correspondence between the Kempf and the Harder-Narasimhan filtrations.

Potentially, we could expect to use these ideas to define Harder-Narasimhan filtrations in cases where it has not been studied, and use them as a powerful tool for very important applications where it has been used in the past, as restriction theorems or calculation of Betti numbers and rational points of moduli spaces (c.f. [HN]).

Many of the cases where we have applied the method fall into abelian categories, where the Harder-Narasimhan filtration verifies the convexity properties out of its definition. It would be interesting to understand if this imposes a condition to expect a notion of Harder-Narasimhan filtration, as we usually know.

The main results of this thesis are collected on the preprints [GSZ, Za]. First one contains the correspondence between GIT maximal way of destabilizing and the Harder-Narasimhan filtration for torsion free coherent sheaves over projective varieties and holomorphic pairs (sections 2.1 and 2.2). Second one contains the result for representations of quivers on finite dimensional vector spaces (section 3.1).

Chapter 1

Moduli spaces and maximal unstability

1.1 Constructions of moduli spaces using Geometric Invariant Theory

1.1.1 Moduli problems

Since decades, the study of moduli spaces seems to be the right answer to various classification problems in algebra and geometry. A general classification problem should consist on a collection of objects $\mathcal{A}$ and an equivalence relation $\sim$ on $\mathcal{A}$. The problem is, then, to describe the set of equivalence classes $\mathcal{A}/\sim$. We usually refer to $\mathcal{A}/\sim$ as the **quotient space**.

In principle, we can think of the solution to our problem as just the quotient set where each equivalence class corresponds to a point. But in the field of algebraic geometry, the objects we are dealing with have rich algebraic and geometric structures, so we would like this quotient set to have similar properties. Besides, it is usual to have *continuous families* of objects in $\mathcal{A}$ and we want to reflect this fact in the quotient space. In other words, if two objects in $\mathcal{A}$ are very close, or are very similar (more similar than other objects in $\mathcal{A}$), we want them to be also very close in the quotient.

Thus, the ingredients of a moduli problem are three: the class of objects $\mathcal{A}$ we are trying to classify, the equivalence relation $\sim$ and a notion of **family** and equivalence of families. The object of the theory of moduli spaces is to provide *good spaces* (to be defined later, meaning spaces with good algebraic and geometric properties) for the quotient space $\mathcal{A}/\sim$.

Sometimes there are discrete invariants which divide $\mathcal{A}/\sim$ into a countable number of subsets, but this does not give a complete solution, usually. However, in many cases, in order to consider a useful moduli space we fix these invariants and try to classify the subclass of these objects. Examples of this are fixing rank and degree when studying the moduli space of vector bundles over a Riemann surface, or fixing dimension and degree to consider the moduli space of hypersurfaces in a projective space.

First basic examples of moduli spaces can be the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ as the space of lines in $\mathbb{C}^{n+1}$ which pass through the origin or, more generally, the Grassmannian $\mathcal{GR}(k,n)$ as the moduli space of all k-dimensional linear subspaces of $\mathbb{C}^n$.

Another classic example is to construct a moduli space for the collection $\mathcal{A}$ of all the non-singular complex cubics. Two curves X and X' are equivalent, $X \sim X'$, if they are isomorphic. By a change of coordinates we consider that all of them are of the form $y^2 = x(x-1)(x-\lambda)$, where $\lambda \in \mathbb{C}$. Then we define

$$j(X) = j(\lambda) = 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2} ,$$

called the j-**invariant** of the curve X. It can be proved that two cubics are equivalent, i.e. $X \sim X'$, if and only if $j(X) = j(X')$ (c.f. [Ha, Theorem 4.1]). We see that all the non-singular complex cubics are parametrized by the affine complex line (an algebraic variety), the corresponding points in the line given by the j-invariant of each isomorphism class of curves. Hence, to classify cubics up to isomorphism is the same that to give a 1-dimensional variety where each point corresponds to a class of cubics.

The fact of having many non-trivial automorphisms for some of the objects being classified makes it difficult to have a moduli space as the set of isomorphism classes. This will be the object of study of the theory of stacks which we will not face here. Stacks can give a different answer for the classification problem. Indeed, a stack problem is formulated as a 2-functor problem, whose answer falls in a more general category of spaces. To avoid that, in many cases we restrict the class of objects $\mathcal{A}$ we are trying to classify to some subclass for which we will be able to give a moduli space. The best example of this is the notion of stability for vector bundles or sheaves, where we can give a solution for the moduli problem when restricting to the **semistable** objects.

In the same direction, it is often possible to consider a modified moduli problem, meaning to classify the original objects together with additional data, chosen in such a way that the identity is the only automorphism also respecting the additional data. This choice of additional data is usually called **to rigidify the objects** or **to rigidify the data**. With a suitable choice of the rigidifying data, the modified moduli problem will have a moduli space. One of the most successful approaches to construct moduli spaces is this rigidifying the data. Consider an object $A \in \mathcal{A}$ and suppose it to be enriched to (A, α), where α represents an additional data. In this situation, an action by a group G appears, taking (A, α) to (A, α'), this is, changing the additional data for a given object $A \in \mathcal{A}$. Hence, in our moduli problem, two objects will be equivalent if they lay on the same orbit by the action of the group G. Then, in order to get rid of this choice of data, we have to quotient by the action of G. This is the object of the Geometric Invariant Theory, developed by David Mumford, which provides moduli spaces as quotients of affine or projective spaces by the action of groups.

The origin of the theory of moduli spaces started with the theory of elliptic functions, where one can show that there exists a continuous family of these functions parametrized by the complex numbers, as in the previous example. Riemann showed in a famous article in 1857 (c.f. [Ri]) that there is a $3g-3$ dimensional family of complex structures a compact topological surface of genus $g \geq 2$ can be endowed with. In this paper, it was coined the term *moduli*, referring to the number of parameters for the complex structure.

The modern formulation of moduli problems and definition of moduli spaces dates back to Alexander Grothendieck, (1960/1961), *"Techniques de construction en géométrie analytique. I. Description axiomatique de l'espace de Teichmüller et de ses variantes"* (c.f. [Gr]) in which he described the general framework, approaches and main problems using Teichmüller spaces in complex analytic geometry as an example. The text describes a general method to construct moduli spaces.

Another general approach is primarily associated with Michael Artin. Here the idea is to start with any object of the kind to be classified and study those objects which are closer to it, in the sense that they can be seen as deformations of the object. This is called **deformation theory**.

1.1.2 Formulation of moduli problems

Given a moduli problem, i.e. a class of objects $\mathcal{A}$, an equivalence relation $\sim$ between objects and a notion of family and equivalence of families, we want to give an algebraic structure or geometric structure to the set $\mathcal{A}/\sim$. This structure will depend on the category we are working on and the precise context (it can be an algebraic variety, an scheme or an algebraic space, for example). In the following, we will consider the category Sch_k of schemes over a field k, and recall that this category has fiber products. Let us denote by Sets the category of sets.

By a family of objects in $\mathcal{A}$ we understand a proper flat morphism of k-schemes $f : X \to S$, where fibers X_s (i.e. X_s is the pull back of f along the inclusion $s \in S$) of the morphism f are objects in $\mathcal{A}$. We say that X is a family of objects in $\mathcal{A}$ parametrized by S.

To formulate a moduli problem we need that the equivalence relation $\sim$ verifies certain conditions (c.f. [Ne, Conditions 1.4])

- A family parameterized by a one point scheme $\{p\}$ is a single object of $\mathcal{A}$.
- There exists a notion of equivalence between families reducing to $\sim$ for single objects in $\mathcal{A}$. Then equivalence of objects turns out to be equivalence of families parametrized by $\{p\}$.
- The equivalence for families is functorial, i.e. for any morphism $\varphi : S' \to S$ and a family X parameterized by S (i.e. $f : X \to S$), there is an induced family $\varphi^* X$ parameterized by S' and this operation satisfies functorial properties.

Definition 1.1.1. *Let $\mathcal{A}$ be a class and let $\sim$ be an equivalence relation for families in $\mathcal{A}$. A* ***moduli functor*** *is a contravariant functor*

$$\mathcal{F} : \mathrm{Sch}_k \to \mathrm{Sets}$$

where $\mathcal{F}(S)$ denotes the set of equivalence classes of families parameterized by S. The triple $(\mathcal{A}, \sim, \mathcal{F})$ is called a ***moduli problem****.*

Suppose that M is a k-scheme with underlying set $\mathcal{A}/\sim$. To have a family X of objects in $\mathcal{A}/\sim$ parameterized by a k-scheme S is the same that a map $\nu_{[X]} : S \to M$ and we would like all the different morphisms $\nu_{[X]} : S \to M$ to be in correspondence with the different equivalence classes of families $[X]$ parameterized by S. In the language of categories and functors this is expressed with the moduli functor in Definition 1.1.1. Let $\mathrm{Hom}(-, M)$ be the functor of points of M. Recall that the functor of points of a k-scheme M is the contravariant functor from the category of k-schemes to the category of sets, which sends a k-scheme S to the set of morphisms from S to M. There is a natural transformation

$$\Phi : \mathcal{F} \to \mathrm{Hom}(-, M)$$

where $\Phi_S : \mathcal{F}(S) \to \mathrm{Hom}(S, M)$ is the natural map given by $\Phi_S([X]) = \nu_{[X]}$.

To give a **moduli problem** is to give a functor $\mathcal{F}$ as in Definition 1.1.1 and ask if there exists any k-scheme M such that $\mathcal{F}$ and the functor of points of M are related, meaning that the set of equivalence classes of families parameterized by S, $\mathcal{F}(S)$, is related with the set of different morphisms from S to M. In particular, for a one point scheme $\{p\}$, $\mathcal{F}(p)$ will be the set of equivalence classes of objects, so will be in correspondence with the **points** of M, $\mathrm{Hom}(\{p\}, M)$. Hence, such M will be the moduli space we are seeking.

Definition 1.1.2. *A moduli functor $\mathcal{F} : \mathrm{Sch}_k \to \mathrm{Sets}$ is **representable** if there exists a k-scheme M such that $\mathcal{F}$ is isomorphic to the functor of points of M $\mathrm{Hom}(-, M)$. Denote such isomorphism by Φ and say that the pair (M, Φ) **represents** the functor $\mathcal{F}$. A **fine moduli space** for the moduli problem considered is a pair (M, Φ) which represents the functor $\mathcal{F}$.*

Note that, by Definition 1.1.2, if (M, Φ) represents $\mathcal{F}$ we have a natural bijection

$$\Phi(p) : \mathcal{A}/\sim = \mathcal{F}(p) \to \mathrm{Hom}(p, M) = M$$

where p is a one point k-scheme. Moreover, the identity morphism 1_M determines, up to equivalence, a family U parameterized by M such that every family X parameterized by a k-scheme S is equivalent to $\nu_X^* U$, where $\nu_X^* : S \to M$ is the morphism corresponding to the family. The family U is called a **universal family** for the moduli problem considered. Therefore, we can define a **fine moduli space** as a k-scheme M together with a **universal family** U parameterized by M such that every family is given as the pull back from U by the corresponding morphism.

Definition 1.1.3. *A moduli functor $\mathcal{F} : \mathrm{Sch}_k \to \mathrm{Sets}$ is **corepresentable** if there exists a k-scheme M and a natural transformation $\Phi : \mathcal{F} \to \mathrm{Hom}(-, M)$ to the functor of points of M such that, for every k-scheme N and a natural transformation $\Phi' : \mathcal{F} \to \mathrm{Hom}(-, N)$, there exists a unique natural transformation $\Psi : \mathrm{Hom}(-, M) \to \mathrm{Hom}(-N)$ such that Φ factors through Ψ. Such pair (Φ, M) is said to **corepresent** the functor $\mathcal{F}$ and, if it exists, it is unique up to unique isomorphism. If furthermore, $\Phi(p) : \mathcal{F}(p) \to M$ is bijective, where p is a one point k-scheme, we say that (M, Φ) is a **coarse moduli space** for the moduli problem considered.*

There are many moduli problems for which we cannot find a fine moduli space. The existence of a coarse moduli space turns out to be a weaker solution. Note that, if (M, Φ) is a fine moduli space, it is automatically a coarse moduli space.

One reason for the non existence of a moduli space with good properties, easy to explain, is the **jump phenomenon**. It happens when there exists a family X parametrized by a scheme S of dimension ≥ 0 for which there is a point $s_0 \in S$ such that

- $X_s \sim X_t$ for all $s, t \in S - \{s_0\}$
- $X_s \nsim X_{s_0}$ for all $s \in S - \{s_0\}$

With this feature, if we include in the hypothetical moduli space the equivalence or the isomorphism class of X_{s_0}, the moduli space would be non separated. This is the usual property shared by the **unstable** objects (those which behave like X_{s_0}). The notion of stability was introduced first by Mumford, in order to construct moduli spaces for the subclass of **semistable** objects.

As an example, we can formulate the problem of finding a moduli space of algebraic curves of genus g. Consider the class $\mathcal{A}$ of smooth projective curves of genus g over an algebraically closed field k, and the equivalence relation $\sim$ being the isomorphism between curves. A family of curves parametrized by S is a proper flat morphism $f : X \to S$ between algebraic varieties where fibers are curves of genus g. There exists a moduli space, denoted $\mathcal{M}_g$, for this moduli problem. Define a curve to be **stable** if it is complete, connected, has no singularities other than double points, and has only a finite group of automorphisms. The moduli space of stable curves of genus g is usually denoted by $\overline{\mathcal{M}_g}$. The space $\overline{\mathcal{M}_g}$ is projective and it is a compactification of $\mathcal{M}_g$.

1.1.3 Results on Geometric Invariant Theory

In this section we recall the basic results of Geometric Invariant Theory we need when taking quotients by the action of groups in moduli problems.

Let G be an algebraic group over an algebraically closed field k. A right action on an scheme X is a morphism $\sigma : X \times G \to X$, where $\sigma(x, g) = x \cdot g$, $\forall x \in X$, such that $x \cdot (gh) = (x \cdot g) \cdot h$ and $x \cdot e = x$, e being the identity element of G. A left action is defined by $(hg) \cdot x = h \cdot (g \cdot x)$.

We denote by $x \cdot G$ the orbit of $x \in X$ by a right action of G (resp. $G \cdot x$ for a left action). A morphism $f : X \to Y$ between two varieties endowed with G-actions is called **G-equivariant** if it commutes with the actions, that is $f(x) \cdot g = f(x \cdot g)$. In the case that the action on Y is trivial (i.e. $y \cdot g = y$, for all $g \in G$ and $y \in Y$), then a morphism f which is G-equivariant is called **G-invariant**.

If X is an affine scheme, to construct affine quotients is much simpler when the group G is reductive. Recall that G is **reductive** if its radical is isomorphic to a direct product of copies of k^*. On the other hand, G is **geometrically reductive** if, for every linear action of G on k^n, and every G-invariant point v of k^n, $v \neq 0$, there exists a G-invariant homogeneous polynomial f of degree ≥ 1 such that $f(v) \neq 0$. Due to results of Weil, Nagata, Mumford and Haboush, it turns out that every reductive group is geometrically reductive and, if a reductive group G is acting on a finitely generated k-algebra R (as it is the ring of functions of an affine variety X, $R = A(X)$), the ring of invariants R^G is finitely generated. Therefore, we define the quotient of an affine variety X by the action of a reductive group G, as the affine variety whose ring of functions is $A(X)^G$.

The following example shows that the quotient of an affine scheme X by the action of a reductive group G can differ of an orbit space (c.f. Definition 1.1.8), because the quotient $A(X)^G$ can possibly identify different orbits in the same point in the quotient space.

Example 1.1.4. *Consider the action*

$$\sigma : \mathbb{C}^* \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

$$(\lambda, (x, y)) \longmapsto (\lambda x, \lambda^{-1} y)$$

whose orbits are represented in Figure 1.1. The orbits are the hyperboles $xy = constant$,

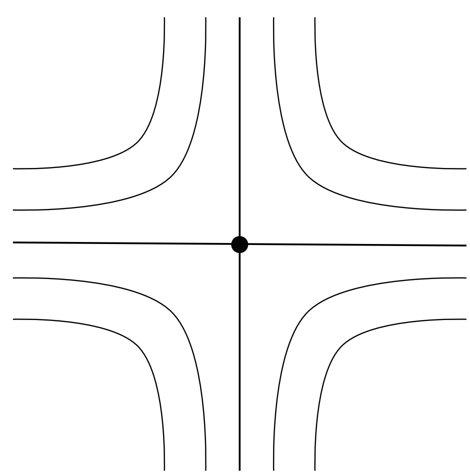


Figure 1.1: Orbits of the action in Example 1.1.4

plus three special orbits, the x-axis, the y-axis and the origin. Observe that the origin is in the closure of the x-axis and the y-axis.

The ring of functions of $\mathbb{C}^2$ is $\mathbb{C}[X,Y]$ and the ring of invariants is $\mathbb{C}[X,Y]^{\mathbb{C}^*} \simeq \mathbb{C}[XY] \simeq \mathbb{C}[Z]$. So, the ring of invariants does not distinguish between the three special orbits, and identifies them in a unique single point in the quotient space. Hence, the orbit space (the space where each point corresponds to an orbit) would be non separated, but the quotient space whose ring of functions is $\mathbb{C}[X,Y]^{\mathbb{C}^*} \simeq \mathbb{C}[Z]$ is the affine line, which is separated.

The case when G acts on a projective scheme X is more complicated. We call $\psi : G \times X \to X$, a **linearization** of the action on an ample line bundle $\mathcal{O}_X(1)$. It consists of giving an action on the total space L of the line bundle $\mathcal{O}_X(1)$, $\sigma : G \times L \longrightarrow L$, such that for every $g \in G$ and $x \in X$, there exists a isomorphism which takes one fiber onto another $L_x \longrightarrow L_{g \cdot x}$ (i.e. σ is linear along the fibers and the projection $L \to X$ is G-equivariant). A linearization is the same thing as giving, for each $g \in G$, an isomorphism of line bundles $\widetilde{g} : \mathcal{O}_X(1) \longrightarrow \varphi_g^* \mathcal{O}_X(1)$, $(\varphi_g = \psi(g,\cdot))$ which also satisfies the previous associative property. We say also that $\sigma = \widetilde{\psi}$ is a lifting to L of the action ψ:

$$\begin{tikzcd}
G \times L \arrow[r, "\sigma=\widetilde{\psi}"] \arrow[d] & L \arrow[d] \\
G \times X \arrow[r, "\psi"] & X
\end{tikzcd}$$

If $\mathcal{O}_X(1)$ is very ample, then a linearization is the same thing as a representation of G on the vector space $H^0(\mathcal{O}_X(1))$ such that the natural embedding

$$X \hookrightarrow \mathbb{P}(H^0(\mathcal{O}_X(1))^\vee)$$

is G-equivariant.

Then, if we have a group G acting on a projective scheme X and consider the set of orbits X/G, when can we define X/G as a scheme M, i.e., the points of X/G correspond, in a natural way, to the points of M?

The next example (c.f. [Gi2]) illustrates some of the features which can arise when trying to define X/G.

Example 1.1.5. *[Gi2] Let N be an integer and consider the set of all homogeneous polynomials of degree N in two variables, $V_N = \{\sum_i a_i X_0^i X_1^{N-i}\}$. Let $\mathbb{P}(V_N)$ be its projec-*

tivization. The group $G = SL(2,\mathbb{C})$ acts on V_N as

$$P^g(X_0, X_1) = P(g^{-1}\begin{pmatrix} X_0 \\ X_1 \end{pmatrix})$$

where $P \in V_N$. The vanishing locus of each $P \in V_N$ consists of a finite set of points in $\mathbb{P}^1$ where their multiplicities are the orders as zeroes of P. Then, we can think of $\{P = 0\}$ as a divisor D_f on $\mathbb{P}^1$, and $\mathbb{P}(V_N)$ as the space of divisors of degree N on $\mathbb{P}$. Observe that G acts on divisors moving them by linear fractional transformations.

The orbit space $\mathbb{P}(V_N)/G$ is not a variety, because it is not Hausdorff. To see this, let $\overline{P} \in \mathbb{P}(V_N)$ and let $P \in V_N$ be a polynomial in the corresponding line. We look for an element Q in the orbit of P so that X_1^N occurs in $Q(X_0, X_1)$ (i.e., $Q(X_0, X_1) = a_0X_0^N + a_1X_0^{N-1}X_1 + ... + a_{N-1}X_0X_1^{N-1} + X_1^N$). Let $Q_t(X_0, X_1) = t^N Q(tX_0, t^{-1}X_1)$ and note that $Q_t(X_0, X_1)$ lays in the orbit of P and Q for every $t \neq 0$, since $Q_t(X_0, X_1) = t^N \cdot Q^{g_t}(X_0, X_1)$, with $g_t = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$, and all of them give the same point in $\mathbb{P}(V_N)$. Therefore, $\mathbb{P}(V_N)/G$ cannot be given a Hausdorff topology so that $\phi : \mathbb{P}(V_N) \longrightarrow \mathbb{P}(V_N)/G$ is continuous. Indeed, if ϕ were continuous, it would be

$$\lim_{t\to 0} Q_t(X_0, X_1) = Q_0(X_0, X_1) = X_1^N,$$

we would have

$$\phi(P) = \phi(Q) = \lim_{t\to 0} \phi(Q_t) = \phi(X_1^N)$$

and the image of ϕ would be one single element. The reason of this is that the polynomial X_1^N is not in the orbit of f and g, but it is in its adherence. Then, when we try to define a continuous quotient map, the adherent orbits have to go to the same point.

As we have seen in Examples 1.1.4 and 1.1.5, in order to obtain a quotient space with good properties (for example, being Hausdorff), we have to make some considerations about the orbits of the action of the group G, putting together in the quotient space all orbits whose closures have non empty intersection. We will call two of these orbits **S-equivalent** (c.f. Remark 1.1.16).

Geometric Invariant Theory, abbreviate GIT, will be a technique to construct such quotients with good properties.

Definition 1.1.6. *Let X be a scheme endowed with a G-action. A* ***categorical quotient*** *is a scheme M with a G-invariant morphism $p : X \longrightarrow M$, such that for every scheme*

M', and every G-invariant morphism $p' : X \longrightarrow M'$, there is a unique morphism φ with $p' = \varphi \circ p$

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "p'"] & \\ M \arrow[r, dashed, "\exists!\,\varphi"'] & M' \end{tikzcd}$$

Definition 1.1.7. *Let X be a scheme endowed with a G-action. A* ***good quotient*** *is a scheme M with a G-invariant morphism $p : X \longrightarrow M$ such that*

1. *p is surjective and affine.*
2. *$p_*(\mathcal{O}_X^G) = \mathcal{O}_M$, where $\mathcal{O}_X^G$ is the sheaf of G-invariant functions on X.*
3. *If Z is a closed G-invariant subset of X, then $p(Z)$ is closed in M. Furthermore, if Z_1 and Z_2 are two closed G-invariant subsets of X with $Z_1 \cap Z_2 = \emptyset$, then $f(Z_1) \cap f(Z_2) = \emptyset$.*

Definition 1.1.8. *A* ***geometric quotient*** *is a good quotient $p : X \to M$ such that $p(x_1) = p(x_2)$ if and only if the orbit of x_1 is equal to the orbit of x_2.*

Note that a geometric quotient is a good quotient, and a good quotient is a categorical quotient.

Let X be a projective scheme, let G be a reductive algebraic group and an action $\sigma : G \times X \longrightarrow X$ of G on X. We call $\tilde{\sigma}$ a linearization of the action on an ample line bundle $\mathcal{O}_X(1)$.

Definition 1.1.9. *A closed point $x \in X$ is called* ***GIT semistable*** *if, for some $m > 0$, there is a G-invariant section s of $\mathcal{O}_X(m)$, $s \in H^0(X, \mathcal{O}_X(m))$, such that $s(x) \neq 0$. If, moreover, the orbit of x is closed in the open set of all GIT semistable points, it is called* ***GIT polystable*** *and, if furthermore, this closed orbit has the same dimension as G (i.e. if x has finite stabilizer), then x is called a* ***GIT stable*** *point. We say that a closed point of X is* ***GIT unstable*** *if it is not GIT semistable.*

With this definition, the stable points are precisely the polystable points with finite stabilizer.

Remark 1.1.10. *We consider X embedded in a projective space by the ample line bundle $\mathcal{O}_X(1)$,*

$$X \hookrightarrow \mathbb{P}(H^0(\mathcal{O}_X(1))^\vee) = \mathbb{P}(V) .$$

Then, we can see a section $s \in H^0(\mathcal{O}_X(m))$ as a homogeneous polynomial of degree m in V. Then, the GIT unstable points are those for which, for all $m > 0$, all G-invariant homogeneous polynomials vanish at the point. As all homogeneous polynomials (in particular the G-invariant ones) vanish at zero, the points which contains zero in the closure of their orbits will be GIT unstable.

*This idea of considering invariant homogeneous polynomials comes from Hilbert who calls them **nullforms** in [Hi].*

The central result of Mumford's Geometric Invariant Theory is the following theorem:

Theorem 1.1.11. *[Mu, Proposition 1.9, Theorem 1.10] Let X^{ss} (respectively, X^s) be the subset of GIT semistable points (respectively, GIT stable). Both X^{ss} and X^s are open subsets. There is a good quotient $X^{ss} \longrightarrow X^{ss}/\!\!/G$ (where closed points are in one-to-one correspondence to the orbits of GIT polystable points), the image $X^s/\!\!/G$ of X^s is open, $X/\!\!/G$ is projective, and the restriction $X^s \to X^s/\!\!/G$ is a geometric quotient.*

Hence, to construct good quotients, first we have to get rid of the unstable points. To find these unstable points there exists a numerical criterion based on the use of 1-parameter subgroups of G. It was first used by Hilbert and later by Mumford, to characterize the GIT stability.

Definition 1.1.12. *Let G be an algebraic group over the field k. A **1-parameter subgroup** of G, Γ, is a non-trivial algebraic homomorphism $\Gamma : k^* \longrightarrow G$.*

Let X be a projective scheme where the group G acts. Suppose that this action is linearized on a line bundle $\mathcal{O}_X(1)$ and call the linearization σ. Then, given Γ, a 1-parameter subgroup of G, and given $x \in X$, we can define $\Phi : k^* \longrightarrow X$ by $\Phi(t) = \Gamma(t) \cdot x$. We say $\lim\limits_{t\to 0} \Gamma(t) \cdot x = \infty$ if Φ cannot be extended to a map $\widetilde{\Phi} : k \longrightarrow X$. If Φ can be extended, we write $\lim\limits_{t\to 0} \Gamma(t) \cdot x = x_0$.

Then, the criterion is the following:

Theorem 1.1.13. *Let $\tilde{x}$ be a point in the affine cone over X, lying over $x \in X$. With the previous notations:*

- *x is **semistable** if for all 1-parameter subgroups Γ, $\exists \lim\limits_{t\to 0} \Gamma(t) \cdot \tilde{x} \neq 0$ or $\lim\limits_{t\to 0} \Gamma(t) \cdot \tilde{x} = \infty$.*

- *x is **polystable** if it is semistable and the orbit of $\tilde{x}$ is closed.*

- *x is **stable** if for all 1-parameter subgroups Γ, $\lim\limits_{t\to 0}\Gamma(t)\cdot\tilde{x}=\infty$ (then the stabilizer of x is finite).*
- *x is **unstable** if there exists a 1-parameter subgroup Γ such that $\lim\limits_{t\to 0}\Gamma(t)\cdot\tilde{x}=0$.*

The point x_0 is, clearly, a fixed point for the action of k^* on X induced by Γ. Thus, k^* acts on the fiber of $\mathcal{O}_X(1)$ over x_0, say, with weight γ. One defines the numerical function

$$\mu(\Gamma,x):=\gamma\ .$$

We will call this γ the *minimum relevant exponent* of the action of Γ over x.

With the definition of $\mu(\Gamma,x)$ we can state the Hilbert-Mumford criterion of GIT stability:

Theorem 1.1.14 (**Hilbert-Mumford criterion**). *[Mu, Theorem 2.1], [Ne, Theorem 4.9] With the previous notations:*

- *x is **semistable** if for all 1-parameter subgroups Γ, $\mu(\Gamma,x)\leq 0$.*
- *x is **stable** if for all 1-parameter subgroups Γ, $\mu(\Gamma,x)<0$.*
- *x is **unstable** if there exists a 1-parameter subgroup Γ such that $\mu(\Gamma,x)>0$.*

Example 1.1.15. *Returning to Example 1.1.5 (c.f. [Gi2]), we apply the Hilbert-Mumford criterion in Theorem 1.1.14. Let $G=SL(2,\mathbb{C})$ and $V_N=\{\sum\limits_i a_iX_0^iX_1^{N-i}\}$. Consider the following 1-parameter subgroup of G*

$$\Gamma(t)=\begin{pmatrix} t^{-r} & 0\\ 0 & t^r\end{pmatrix},r>0\ .$$

Let $P(X_0,X_1)=\sum a_{ij}X_0^iX_1^j$ be a polynomial in V_N. We want to know when $\lim\limits_{t\to 0}P^{\Gamma(t)}=\lim\limits_{t\to 0}\Gamma(t)\cdot P=0$. It is $P^{\Gamma(t)}(X_0,X_1)=\sum a_{ij}X_0^iX_1^jt^{r(i-j)}$, hence, $\lim\limits_{t\to 0}P^{\Gamma(t)}=0$ implies that $a_{ij}=0$, if $j\geq i$. This means that, if P has a factor of X_0^k with degree $k>\frac{N}{2}$, in that case, P is unstable. For a general 1-parameter subgroup, it turns out that P is semistable if and only if P has no linear factors of degree greater that $\frac{N}{2}$.

Remark 1.1.16. *A theorem of Geometric Invariant Theory (c.f. [Si1, Lemma 1.10]) says that, if $G\cdot v$ is the orbit of a point $v\in V$, in its closure $\overline{G\cdot v}$ there is a unique orbit*

*$Y \subset \overline{G \cdot v}$ such that Y is closed in $\overline{G \cdot v}$, so it is closed also in the whole space V. The GIT polystable points are in correspondence with these closed orbits. Two orbits, $G{\cdot}v$ and $G{\cdot}w$, with the same closed orbit Y in their closures $\overline{G \cdot v}$, $\overline{G \cdot w}$, are called **S-equivalent**. The points of the moduli space are in correspondence with these distinguished closed orbits, so the moduli space we obtain classifies polystable points, or points modulo S-equivalence.*

Remark 1.1.17. *Geometric Invariant Theory states that we can reach every point in the closure of an orbit through 1-parameter subgroups. It can be proved (c.f. [Ne, Proposition 4.3], [Mu, Proposition 2.2]) that a point x is GIT semistable if $0 \notin \overline{G \cdot \hat{x}}$, where $\hat{x}$ lies over x in the affine cone. Then, GIT stability measures whether 0 belongs to the closure of the lifted orbit or not, belonging which can be checked through 1-parameter subgroups.*

1.2 Example of a construction of a moduli space using GIT: Moduli of tensors

Here, we present a complete example of the construction of a moduli space through Geometric Invariant Theory. We construct a moduli space for tensors over higher dimensional projective varieties following the Gieseker-Maruyama method. This was constructed by Alexander Schmitt in [Sch] for curves.

This section follows the paper [GS1], where the authors carry out the same construction, but using the method of Simpson.

1.2.1 Definitions and stability of tensors

Let X be a smooth projective variety of dimension n over $\mathbb{C}$. Fix an ample line bundle $\mathcal{O}_X(1)$ on X. Fix a polynomial P of degree n, and integers s, c, b. Let R be an scheme and fix a locally free sheaf $\mathcal{D}$ on $X \times R$, i.e. a family $\{D_u\}_{u \in R}$ of locally free sheaves on X parametrized by R, where given a point $u \in R$, we denote by D_u the restriction of $\mathcal{D}$ to the slice $X \times u$.

Definition 1.2.1. *[GS1, Definition 1.1] A **tensor** is a triple (E, φ, u), where E is a coherent sheaf on X with Hilbert polynomial $P_E = P$, u is a point in R, and φ is a homomorphism*

$$\varphi : (E^{\otimes s})^{\oplus c} \longrightarrow (\det E)^{\otimes b} \otimes D_u \ ,$$

that is not identically zero. Let (E,φ,u) *and* (F,ψ,v) *be two tensors with* $P_E = P_F$, $\det E \simeq \det F$, *and* $u = v$. *A homomorphism between* (E,φ,u) *and* (F,ψ,v) *is a pair* (f,α) *where* $f : E \to F$ *is a homomorphism of sheaves,* $\alpha \in \mathbb{C}$, *and the following diagram commutes*

$$\begin{tikzcd} (E^{\otimes s})^{\oplus c} \arrow[r, "(f^{\otimes s})^{\oplus c}"] \arrow[d, "\varphi"] & (F^{\otimes s})^{\oplus c} \arrow[d, "\psi"] \\ (\det E)^{\otimes b} \otimes D_u \arrow[r, "\hat{f}\otimes\alpha"] & (\det F)^{\otimes b} \otimes D_v \end{tikzcd} \tag{1.2.1}$$

where $\hat{f} : \det E \longrightarrow \det F$ *is the homomorphism induced by* f. *In particular,* (E,φ,u) *and* $(E,\lambda\varphi,u)$ *are isomorphic for* $\lambda \in \mathbb{C}^*$.

Remark 1.2.2. *This notion of isomorphism can be restricted by considering only isomorphisms for which* $\alpha = 1$. *In this case we would obtain another category where, for example, if* E *is simple, the set of automorphism of* (E,φ,u) *is* $\mathbb{C}^*$, *but if* $\alpha = 1$, *the set of automorphisms is* $\mathbb{Z}/(rb-s)\mathbb{Z}$ *(provided* $rb - s \neq 0$*). If* $rb - s \neq 0$, *the set of isomorphism classes will be the same (changing* f *into* $\alpha^{1/(rb-s)}f$*)), and then the moduli spaces will be the same. If* $rb - s = 0$, *the set of isomorphism classes is not the same.*

Let δ be a polynomial with $\deg(\delta) < n = \dim X$

$$\delta(t) = \delta_1 t^{n-1} + \delta_2 t^{n-2} + \cdots + \delta_n \in \mathbb{Q}[t], \tag{1.2.2}$$

and $\delta(m) > 0$ for $m \gg 0$. We denote $\tau = \delta_j(n-j)!$ where δ_j is the leading coefficient of δ. We will define a notion of stability for these tensors, which depends on the polarization $\mathcal{O}_X(1)$ and δ, and we will construct, using Geometric Invariant Theory, a moduli space for semistable tensors.

A weighted filtration $(E_\bullet, n_\bullet)$ of a sheaf E is a filtration

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subsetneq E_{t+1} = E, \tag{1.2.3}$$

and rational positive numbers $n_1, n_2, \ldots, n_t > 0$. We denote $r_i = \operatorname{rk}(E_i)$. If $t = 1$ (what we will call **one-step filtration**), then we set $n_1 = 1$. The filtration is called **saturated** if all sheaves E_i are saturated in E, i.e. if E/E_i is torsion free for all i.

Let γ be a vector of $\mathbb{C}^r$ defined as $\gamma = \sum_{i=1}^t n_i \gamma^{(\operatorname{rk} E_i)}$ where

$$\gamma^{(k)} := \big(\overbrace{k-r,\ldots,k-r}^{k}, \overbrace{k,\ldots,k}^{r-k}\big) \qquad (1 \le k < r)\,.$$

Hence, the vector is of the form

$$\gamma = (\overbrace{\gamma_{r_1}, \ldots, \gamma_{r_1}}^{\operatorname{rk} E^1}, \overbrace{\gamma_{r_2}, \ldots, \gamma_{r_2}}^{\operatorname{rk} E^2}, \ldots, \overbrace{\gamma_{r_{t+1}}, \ldots, \gamma_{r_{t+1}}}^{\operatorname{rk} E^{t+1}})\ ,$$

where $n_i = \dfrac{\gamma_{r_{i+1}} - \gamma_{r_i}}{r}$.

Now let $\mathcal{I} = \{1, ..., t+1\}^{\times s}$ be the set of all multi-indexes $I = (i_i, ..., i_s)$ and define

$$\mu(\varphi, E_\bullet, n_\bullet) = \min_{I\in\mathcal{I}}\{\gamma_{r_{i_1}} + \cdots + \gamma_{r_{i_s}} : \varphi|_{(E_{i_1}\otimes\cdots\otimes E_{i_s})^{\oplus c}} \neq 0\}. \tag{1.2.4}$$

If P_1 and P_2 are two polynomials, we write $P_1 \prec P_2$ if $P_1(m) < P_2(m)$ for $m \gg 0$, and analogously for "$\leq$" and "$\preceq$".

Definition 1.2.3. *[GS1, Definition 1.3] Let δ be a polynomial as in (1.2.2). We say that (E, φ, u) is δ-**semistable** if for all weighted filtrations $(E_\bullet, n_\bullet)$ of E, it is*

$$\Big(\sum_{i=1}^{t} n_i(rP_{E_i} - r_iP_E)\Big) + \delta\mu(\varphi, E_\bullet, n_\bullet) \preceq 0 \tag{1.2.5}$$

*We say that (E, φ, u) is δ-**stable** if we have a strict inequality in (1.2.5) for every weighted filtration. If (E, φ, u) is not δ-semistable we say that it is δ-unstable.*

We assume that φ is not identically zero, then (1.2.4) is well defined.

Remark 1.2.4. *It is enough to consider saturated filtrations in Definition 1.2.3. Indeed, it is $P_{E_i} \leq P_{\overline{E_i}}$ for Hilbert polynomials, if $\overline{E_i}$ is the saturation of a subsheaf $E_i \subset E$.*

Also it suffices to consider filtrations with $\operatorname{rk}(E_i) < \operatorname{rk}(E_{i+1})$. If not, suppose $E_i \subsetneq E_{i+1}$ and $\operatorname{rk} E_i = \operatorname{rk} E_{i+1}$, then E_i is not saturated in E_{i+1} and E_{i+1}/E_i has torsion. Therefore E/E_i has torsion and E_i is not saturated in E. Note that the definition of (1.2.4) coincides for E_i and $\overline{E_i}$.

Definition 1.2.5. *[GS1, Definition 1.7] We say that (E, φ, u) is **slope-τ-semistable** if E is torsion free, and for all weighted filtrations $(E_\bullet, n_\bullet)$ of E, it is*

$$\Big(\sum_{i=1}^{t} n_i(r\deg E_i - r_i\deg E)\Big) + \tau\mu(\varphi, E_\bullet, n_\bullet) \leq 0 \tag{1.2.6}$$

*We say that (E, φ, u) is **slope-τ-stable** if we have a strict inequality in (1.2.6) for every weighted filtration. If (E, φ, u) is not slope-τ-semistable we say that it is slope-τ-unstable.*

Recall the relation

$$\tau = \delta_j (n-j)!$$

between the parameter τ and the leading coefficient of polynomial δ and note that we have the following implications

$$\text{slope} - \tau - \text{stable} \Rightarrow \delta - \text{stable} \Rightarrow \delta - \text{semistable} \Rightarrow \text{slope} - \tau - \text{semistable}$$

Note that, if the dimension of the variety X is $n = 1$, Definitions 1.2.3 and 1.2.5 do coincide.

Let $\mathcal{I} = \{1, ..., t+1\}^{\times s}$ be the set of all multi-indexes $I = (i_1, ..., i_s)$. Let us call $\nu^i(I)$ the number of times that i appears on the multi-index I and $\nu_i(I)$ the number of elements k in I with $k \leq i$. Note that $\nu^{i+1}(I) = \nu_{i+1}(I) - \nu_i(I)$. Given a multi-index $I \in \mathcal{I}$, we have

$$\gamma_{r_{i_1}} + \cdots + \gamma_{r_{i_s}} = \sum_{i=1}^{t} \gamma_i \nu^i(I) = \sum_{i=1}^{t} \gamma_{r_i} (\nu_{i+1}(I) - \nu_i(I))$$

$$= s\gamma_{r_{t+1}} - \sum_{i=1}^{t} (\gamma_{r_{i+1}} - \gamma_{r_i}) \nu_i(I) = s\gamma_r - \sum_{i=1}^{t} n_i r \nu_i(I)$$

$$= s(\sum_{i=1}^{t} n_i r_i) - \sum_{i=1}^{t} n_i r \nu_i(I) = \sum_{i=1}^{t} n_i (s r_i - \nu_i(I) r) \ .$$

Now let I_0 be the multi-index giving minimum in (1.2.4). We will denote by $\epsilon_i(\varphi, E_\bullet, n_\bullet)$ (or just $\epsilon_i(E_\bullet)$ if the rest of the data is clear from the context) the number of elements k of the multi-index I_0 such that $r_k \leq r_i$. Let us call $\epsilon^i(E_\bullet) = \epsilon_{i+1}(E_\bullet) - \epsilon_i(E_\bullet)$. Therefore, we can rewrite (1.2.4) as

$$\mu(\varphi, E_\bullet, n_\bullet) = \sum_{i=1}^{t} n_i (s r_i - \epsilon_i(E_\bullet) r) \ . \tag{1.2.7}$$

In the following we will consider the stability and slope-stability conditions, (1.2.5) and (1.2.6), with the calculation made in (1.2.7).

Remark 1.2.6. *Note that, if (E, φ, u) is δ-semistable, then it is torsion free. Indeed, consider the filtration $0 \subsetneq T(E) \subsetneq E$ where $T(E)$ is the torsion subsheaf, and apply (1.2.5). Then we obtain this inequality of polynomials*

$$rP_{T(E)} - \mathrm{rk}(T(E))P_E + \delta\mu(0 \subsetneq T(E) \subsetneq E) = rP_{T(E)} + \delta\mu(0 \subsetneq T(E) \subsetneq E) \preceq 0$$

which is a contradiction, because we have that the leading coefficient of $P_{T(E)}$ is positive and $\mu(0 \subsetneq T(E) \subsetneq E) = 0$ by (1.2.7). Then $T(E) = 0$ and E is torsion free.

Lemma 1.2.7. *[GS1, Lemma 1.4] There is an integer A_1 (depending only on P, s, c, b and $\mathcal{D}$) such that it is enough to check the stability condition (1.2.5) for weighted filtrations with $n_i \leq A_1$ for all i.*

Proof. Let $\mathcal{I} = \{1, \ldots, t+1\}^{\times s}$ be the set of multi-indexes $I = (i_1, \ldots, i_s)$ and note that (1.2.4) is a piece-wise linear function of $\gamma \in \mathcal{C}$, where $\mathcal{C} \subset \mathbb{Z}^r$ is the cone defined by $\gamma_1 \leq \ldots \leq \gamma_r$. This is due that it is defined as the minimum among a finite set of the linear functions $\gamma_{r_{i_1}} + \cdots + \gamma_{r_{i_s}}$ for those $I \in \mathcal{I}$ giving a non-zero restriction of morphism φ, i.e. $\varphi|_{(E_{i_1} \otimes \cdots \otimes E_{i_r})^{\oplus c}} \neq 0$. Decompose $\mathcal{C} = \bigcup_{I \in \mathcal{I}} \mathcal{C}_I$ into a finite number of subcones

$$\mathcal{C}_I := \{\gamma \in \mathcal{C} : \gamma_{r_{i_1}} + \cdots + \gamma_{r_{i_s}} \leq \gamma_{r_{i'_1}} + \cdots + \gamma_{r_{i'_s}} \text{ for all } I' \in \mathcal{I}\} ,$$

such that (1.2.4) is linear on each cone $\mathcal{C}_I$. Maybe some subcones I are irrelevant, meaning that φ vanishes on them, then we set $\mu(\varphi, E_\bullet, n_\bullet)|_I = 0$. Choose one vector $\gamma \in \mathbb{Z}^r$ in each edge of each cone $\mathcal{C}_I$ and multiply all these vectors by r, so that all their coordinates are divisible by r, and call this set of vectors S. The vectors in S come from weights $n_i > 0$, $i = 1, \ldots, t+1$, given by the formula $\gamma = \sum_{i=1}^{t} n_i \gamma^{(r_i)}$. Hence, to obtain the finite set S of vectors it is enough to consider a finite set of values for n_i, therefore there is a maximum value A_1.

Finally, we will show that it is enough to check (1.2.5) for the weights associated to the vectors in S. Indeed, since the first term in (1.2.5) is linear on $\mathcal{C}$, then it is also linear on each $\mathcal{C}_I$. Then the expression in the left side of (1.2.5) is linear on each subcone $\mathcal{C}_I$, and hence, it is enough to check its non-positivity on all the edges of all the cones $\mathcal{C}_I$, then it is enough to check it for weights associated to vectors in S. ■

Note that the reason why we have to consider filtrations instead of just subsheaves is that (1.2.4) is not linear as a function of the weights $\{n_i\}$. But, nevertheless, we can compare (1.2.4) for subfiltrations of a given filtration with the following Lemma. It will be used in the proof of Theorem 2.5.

Lemma 1.2.8. *[GS1, Lemma 1.6] Let $(E_\bullet, n_\bullet)$ be a weighted filtration and let $\mathcal{T}'$ be a subset of $\mathcal{T} = \{1, ..., t\}$. Let $(E'_\bullet, n'_\bullet)$ be the subfiltration obtained by considering only those terms E_i for which $i \in \mathcal{T}'$. Then*

$$\mu(\varphi, E_\bullet, n_\bullet) \leq \mu(\varphi, E'_\bullet, n'_\bullet) + \sum_{i \in \mathcal{T} - \mathcal{T}'} n_i s r_i .$$

Proof. We index the filtration $(E'_\bullet, n'_\bullet)$ with $\mathcal{T}'$. Let $I' = (i'_1, ..., i'_s) \in \{\mathcal{T}' \cup \{t+1\}\}^{\times s}$ be the multi-index giving minimum for the filtration $(E'_\bullet, n'_\bullet)$. In particular, we have $\varphi|_{(E_{i'_1}\otimes\cdots\otimes E_{i'_s})^{\oplus c}} \neq 0$. Then

$$\mu(\varphi, E_\bullet, n_\bullet) = \min_{I\in\mathcal{I}}\{\gamma_{r_{i_1}} + \cdots + \gamma_{r_{i_s}} : \varphi|_{(E_{i_1}\otimes\cdots\otimes E_{i_r})^{\oplus c}} \neq 0\} \leq$$

$$\gamma_{r_{i'_1}} + \cdots + \gamma_{r_{i'_s}} = \sum_{i=1}^{t} n_i(sr_i - \nu_i(I')r) = \sum_{i=1}^{t} n_i(sr_i - \epsilon_i(E'_\bullet)r) =$$

$$\sum_{i\in\mathcal{T}'} n_i(sr_i - \epsilon_i(E_\bullet)r) + \sum_{i\in\mathcal{T}-\mathcal{T}'} n_i(sr_i - \epsilon_i(E'_\bullet)r) \leq \mu(\varphi, E'_\bullet, n'_\bullet) + \sum_{i\in\mathcal{T}-\mathcal{T}'} n_i s r_i \ .$$

■

A **family of coherent sheaves parametrized by a scheme** T is a coherent sheaf E_T on $X\times T$ which is flat over T, such that, $E_t := E_T|_{X\times\{t\}}$ is a coherent sheaf over X for every point $t\in T$. Let us define the ingredients of our moduli problem.

A **family of δ-semistable tensors parametrized by a scheme** T is a tuple (E_T, φ_T, u_T, N), consisting of a torsion free sheaf E_T on $X\times T$, flat over T, that restricts to a torsion free sheaf with Hilbert polynomial P on every slice $X\times\{t\}$, a morphism $u_T : T \longrightarrow R$, a line bundle N on T and a homomorphism φ_T,

$$\varphi_T : (E_T^{\otimes s})^{\oplus c} \longrightarrow (\det E_T)^{\otimes b} \otimes \overline{u_T}^* \mathcal{D} \otimes \pi_T^* N \ , \tag{1.2.8}$$

(where we define $\overline{u_T} = \mathrm{id}_X \times u_T$) such that if we consider the restriction of this homomorphism on every slice $X\times\{t\}$,

$$\varphi_t : (E_t^{\otimes s})^{\oplus c} \longrightarrow (\det E_t)^{\otimes b} \otimes D_{u_T(t)} \ ,$$

the triple $(E_t, \varphi_t, u_T(t))$ is a δ-semistable tensor for every t. Particularly, φ_t is not identically zero. Two families (E_T, φ_T, u_T, N) and $(E'_T, \varphi'_T, u'_T, N')$ parametrized by T are **isomorphic** if $u_T = u_{T'}$ and there are isomorphisms $f : E_T \longrightarrow E'_T$, $\alpha : N \longrightarrow N'$, such that the induced diagram

$$\begin{tikzcd}
(E_T^{\otimes s})^{\oplus c} \arrow[r, "(f^{\otimes s})^{\oplus c}"] \arrow[d, "\varphi_T"] & (E_T'^{\otimes s})^{\oplus c} \arrow[d, "\varphi'_T"] \\
(\det E_T)^{\otimes b} \otimes \overline{u_T}^* \mathcal{D} \otimes \pi_T^* N \arrow[r, "\hat{f}\otimes\pi_T^*\alpha"] & (\det E'_T)^{\otimes b} \otimes \overline{u'_T}^* \mathcal{D} \otimes \pi_T^* N'
\end{tikzcd} \tag{1.2.9}$$

commutes, where $\pi_T : X\times T \to T$ is the natural projection.

Therefore, we have a category of objects, o notion of stability, a notion of isomorphism between objects and a notion of family and equivalence of families. We are ready to define the **functor** for our **moduli problem**.

Let $\mathcal{M}_\delta$ (respectively $\mathcal{M}_\delta^s$) be the contravariant functor from the category of schemes over $\mathbb{C}$, locally of finite type, $(\mathrm{Sch}/\mathbb{C})$ to the category of sets (Sets) which sends a scheme T to the set of isomorphism classes of families of δ-semistable (respectively stable) tensors parametrized by T, and send a morphism $f : T' \longrightarrow T$ to the morphism of sheaves $\tilde{f} : E'_{T'} \longrightarrow E_T$ given by the pullback diagram

$$\begin{tikzcd} E'_{T'} \arrow[r, "\tilde{f}"] \arrow[d] & E_T \arrow[d] \\ T' \times X \arrow[r, "f\times\mathrm{id}"] & T \times X \end{tikzcd}$$

and similarly for the morphism $\Phi : \varphi'_{T'} \longrightarrow \varphi_T$.

We will construct schemes $\mathfrak{M}_\delta$, $\mathfrak{M}_\delta^s$ corepresenting the functors $\mathcal{M}_\delta$ and $\mathcal{M}_\delta^s$ (c.f. Definition 1.1.3). In general $\mathfrak{M}_\delta$ will not be a coarse moduli space, because non-isomorphic tensors can correspond to the same point in $\mathfrak{M}_\delta$. Then, we will declare two such tensors S-equivalent, and $\mathfrak{M}_\delta$ will become a coarse moduli space for the functor of S-equivalence classes of tensors (c.f. Remark 1.1.16). This is the main theorem (c.f. Theorem [GS1, Theorem 1.8]):

Theorem 1.2.9. *Fix P, s, c, b and a family $\mathcal{D}$ of locally free sheaves on X parametrized by a scheme R. Let d be the degree of a coherent sheaf whose Hilbert polynomial is P. Let δ be a polynomial as in (1.2.2).*

There exists a coarse moduli space $\mathfrak{M}_\delta$, projective over $\mathrm{Pic}^d(X) \times R$, of S-equivalence classes of δ-semistable tensors. There is an open set $\mathfrak{M}_\delta^s$ corresponding to δ-stable tensors. Points in this open set correspond to isomorphism classes of δ-stable tensors.

In Proposition 1.2.33 we will give a criterion to decide when two tensors are S-equivalent. We will prove Theorem 1.2.9 in subsection 1.2.5.

Therefore, in the language of Section 1.1, we have our moduli problem stated where $\mathcal{A}$ is the class of δ-semistable (resp. δ-stable) tensors, the equivalence relation $\sim$ is given by the notion of S-equivalence (c.f. Remark 1.1.16) for which we will give a criterion in Proposition 1.2.33 (resp. isomorphism of tensors in Definition 1.2.1), and the notion of equivalence of families given by (1.2.9). See [GS1, Remark 1.9] and [Si1, p. 60] for a comment on the notion of equivalence of families giving, as a result, moduli functors which are not sheaves.

Remark 1.2.10. *The obtention of a fine moduli space also requires the existence of a universal family $E_{\mathfrak{M}_\delta}$ over $\mathfrak{M}_\delta$ such that for every family E_T of tensors over T, there is a unique morphism $E_T \longrightarrow \mathfrak{M}_\delta$ induced by pulling back the universal family (c.f. Definition 1.1.2). As we will see, this cannot be done for the case of tensors as in the case of sheaves.*

1.2.2 Results on boundedness

In this section we reformulate the stability for tensors using some boundedness results to prove Theorem 1.2.19. First we recall definitions and well known results by Simpson, Grothendieck and Maruyama.

Definition 1.2.11. *A set $\mathcal{E} = \{E_i\}_{i\in I}$ of coherent sheaves is **bounded** if there exists a family $E_T \longrightarrow X \times T$ parametrized by T, a scheme of finite type over $\mathbb{C}$, such that for every $i \in I$ there exists $t \in T$ with $E_i \simeq E_t$.*

Recall that a scheme T is **of finite type over** $\mathbb{C}$ if T can be covered by a finite number of open affine subsets $\operatorname{Spec} A_i$, where each A_i is a finitely generated $\mathbb{C}$-algebra.

Definition 1.2.12. *A sheaf E is called m-**regular** if $h^i E(m-i)) = 0$ for $i > 0$.*

Lemma 1.2.13. *If E is m-regular then the following holds*

1. *E is m'-regular for $m' > m$.*
2. *$E(m)$ is globally generated.*
3. *For all $m' \geq 0$ the following homomorphisms are surjective*

$$H^0(E(m)) \otimes H^0(\mathcal{O}_X(m')) \longrightarrow H^0(E(m+m')) .$$

Proposition 1.2.14. *The following properties for a family of sheaves $\mathcal{E} = \{E_i\}_{i\in I}$ are equivalent:*

1. *$\mathcal{E}$ is bounded.*
2. *The set of Hilbert polynomials $\{P_E\}_{E\in\mathcal{E}}$ is finite and there exists a uniform bound $m_0 \in \mathbb{Z}$ such that all $E \in \mathcal{E}$ is m_0-regular.*

3. *The set of Hilbert polynomials $\{P_E\}_{E\in\mathcal{E}}$ is finite and there is a coherent sheaf F such that all $E_i \in \mathcal{E}$ admit surjective homomorphisms $F \longrightarrow E_i$.*

Note that, given a bounded set $\mathcal{E}$ of coherent sheaves, the set of Hilbert polynomials $\{P_E\}_{E\in\mathcal{E}}$ is finite and hence, $\{\operatorname{rk} E\}_{E\in\mathcal{E}}$ and $\{\deg E\}_{E\in\mathcal{E}}$, are bounded as sets of numbers.

We denote

$$P_{\mathcal{O}_X}(m) = \frac{\alpha_n}{n!}m^n + \frac{\alpha_{n-1}}{(n-1)!}m^{n-1} + ... + \frac{\alpha_1}{1!}m + \frac{\alpha_0}{0!} \ , \tag{1.2.10}$$

the **Hilbert polynomial** of $\mathcal{O}_X$, where $\alpha_n = g = \deg \mathcal{O}_X(1)$, and

$$P_E(m) = \frac{rg}{n!}m^n + \frac{d + r\alpha_{n-1}}{(n-1)!}m^{n-1} + ... \ , \tag{1.2.11}$$

the **Hilbert polynomial** of a sheaf E with rank r and degree d. For the following Lemma, see [Si1, Lemma 1.5 and Corollary 1.7] and [LeP, Lemme 2.4].

Lemma 1.2.15. *[HL2, Lemma 2.2] Let $r > 0$ be an integer. Then there exists a constant B with the following property: for every torsion free sheaf E with $0 < \operatorname{rk}(E) \leq r$, we have*

$$h^0(E) \leq \frac{1}{g^{n-1}n!}\left((\operatorname{rk}(E)-1)([\mu_{max}(E)+B]_+)^n + ([\mu_{mim}(E)+B]_+)^n\right) \ ,$$

where $g = \deg \mathcal{O}_X(1)$, $[x]_+ = \max\{0, x\}$, and $\mu_{max}(E)$ (resp. $\mu_{min}(E)$) is the maximum (resp. minimum) slope of the semistable factors of the Mumford-Harder-Narasimhan filtration of E (c.f. Definition 1.3.3 and Remark 1.3.4).

Lemma 1.2.16. *[Gr, Lemma 2.5] Let $\mathcal{E}$ be a bounded set of sheaves E and fix a constant C. The set of torsion free quotients $E \twoheadrightarrow E''$ of the sheaves $E \in \mathcal{E}$, with $|\deg(E'')| \leq C$, is bounded.*

Theorem 1.2.17. *[Ma3] Fix a constant C. The family of sheaves E with fixed Hilbert polynomial P and such that $\mu_{\max}(E) \leq C$, is bounded.*

Recall that we denote by $\mu(E) = \frac{\deg E}{\operatorname{rk} E}$ the slope of a sheaf. As a consequence of Maruyama's result in Theorem 1.2.17, we can prove the boundedness of the set of δ-semistable tensors:

Corollary 1.2.18. *The set of δ-semistable tensors (E, φ, u) with fixed Hilbert polynomial P is bounded.*

Proof. Let (E,φ,u) a δ-semistable tensor. Then, we have seen that (E,φ,u) is τ-slope-semistable, where $\tau=\delta_j(n-j)!$. Then, for every weighted filtration and, in particular, for every one-step filtration $0\subset E'\subset E$, the expression (1.2.6) holds, hence

$$(r\deg(E')-\operatorname{rk}(E')\deg E)\big)+\tau(s\operatorname{rk}(E')-\epsilon(E'\subset E)r)\leq 0\ .$$

Then, dividing by $r\cdot\operatorname{rk}(E')$ we obtain this condition for the slopes

$$\mu(E')\leq\mu(E)\tau(\frac{s}{r}+\frac{\epsilon(E'\subset E)}{\operatorname{rk}E'})\leq\mu(E)-\tau(\frac{s}{r}+s)=C\ ,$$

where C is a constant depending only on P, τ and s, which are fixed. Hence, we apply Theorem 1.2.17, provided $\mu(E')\leq\mu_{\max}(E)$, for every subsheaf $E'\subset E$. ∎

This is the main theorem of this section, whose proof we will give after some preliminary results.

Theorem 1.2.19. *(c.f. [GS1, Theorem 2.5]) There is an integer N_0 such that if $m\geq N_0$, the following properties of tensors (E,φ,u), with E torsion free and $P_E=P$, are equivalent.*

1. *(E,φ,u) is semistable (resp. stable).*
2. *$P(m)\leq h^0(E(m))$ and for every weighted filtration $(E_\bullet,n_\bullet)$ as in (2.1.6),*
$$\big(\sum_{i=1}^{t}n_i\big(rh^0(E_i(m))-r_iP(m)\big)\big)+\delta(m)\mu(\varphi,E_\bullet,n_\bullet)\leq 0$$
(resp. $<$).
3. *For every weighted filtration $(E_\bullet,n_\bullet)$ as in (2.1.6),*
$$\big(\sum_{i=1}^{t}n_i(r^iP(m)-rh^0(E^i(m)))\big)+\delta(m)\mu(\varphi,E_\bullet,n_\bullet)\leq 0$$
(resp. $<$).

Furthermore, for any tensor (E,φ,u) satisfying these conditions, E is m-regular.

The set of tensors (E,φ,u), with E torsion free and $P_E=P$, satisfying the weak version of conditions $1-3$ will be called $\mathcal{S}^s$, $\mathcal{S}'_m$ and $\mathcal{S}''_m$, respectively.

Lemma 1.2.20. *There is an integer N_1 and a positive constant D, such that if (E,φ,u) belongs to $\mathcal{S} = \mathcal{S}^s \cup \bigcup_{m\geq N_1} \mathcal{S}''_m$, then for all saturated weighted filtrations $(E_\bullet, n_\bullet)$, the following holds for all i:*

$$\mu(E_i) \leq \mu(E) + D \tag{1.2.12}$$

and, either $\mu(E) - D \leq \mu(E_i)$, or

1. *$rh^0(E_i(m)) < r_i(P(m) - s\delta(m))$, if $(E,\varphi,u) \in \mathcal{S}^s$ and $m \geq N_1$*
2. *$rP_{E_i} - r_iP - r_is\delta \prec 0$, if $(E,\varphi,u) \in \bigcup_{m\geq N_1} \mathcal{S}''_m$*

Proof. Let $(E_\bullet, n_\bullet)$ be a weighted filtration of E and let B be as in Lemma 1.2.15. Let G be the following polynomial,

$$\begin{aligned} G(m) &= \frac{1}{g^{n-1}n!}\big((r-1)(\mu(E) + \tau s(1-\frac{1}{r}) + gm + B)^n + (\mu(E) - D + gm + B)^n\big) \\ &= \frac{1}{g^{n-1}n!}\big[rg^nm^n + ng^{n-1}(r\mu(E) + \tau s\frac{(r-1)^2}{r} - D + rB)m^{n-1} + \cdots\big] . \end{aligned}$$

Then, the leading coefficient of $G - (P - s\delta)$ (i.e. the coefficient of m^n) is

$$r\frac{g}{n!} - r\frac{g}{n!} = 0 ,$$

but the coefficient of m^{n-1} is

$$\big[\frac{1}{(n-1)!}(r\mu(E) + \tau s\frac{(r-1)^2}{r} - D + rB)\big] - \big[\frac{1}{(n-1)!}(d + r\alpha_{n-1}) + \tau s\big] ,$$

so we can choose D large enough so that the leading coefficient of $G - (P - s\delta)$ is negative. We choose D also to verify $D > \tau s$.

Let N_1 be large enough so that, for $m \geq N_1$, the following three expressions hold:

$$\delta(m) \geq 0 \tag{1.2.13}$$

$$\mu(E) - D + gm + B > 0 \tag{1.2.14}$$

$$G(m) - (P(m) - s\delta(m)) < 0 . \tag{1.2.15}$$

Given that the filtration is supposed to be saturated, and E to be torsion free, we have $0 < r_i < r$.

Case 1. Suppose that $(E,\varphi,u) \in \mathcal{S}^s$. Then, (E,φ,u) is τ-slope-semistable hence, for each i, consider the one-step filtration $E_i \subsetneq E$ and apply (1.2.6),

$$r \deg E_i - r_i \deg E + \tau(sr_i - \epsilon_i(E_i \subsetneq E)r) \leq 0 .$$

Dividing by $r_i \cdot r$ we get

$$\mu(E_i) \le \mu(E) - \tau(\frac{s}{r} + \frac{\epsilon_i(E_i \subsetneq E)}{r_i}) \le \mu(E) + \tau s(1 - \frac{1}{r}) < \mu(E) + D$$

using $D > \tau s$, hence (1.2.12).

Let $E_{i,\max} \subset E_i$ be the term in the Harder-Narasimhan filtration of E_i with maximal slope (c.f. Definition 1.3.3 and Remark 1.3.4). Then, the same argument as before, applied to the filtration $E_{i,\max} \subsetneq E$, gives

$$\mu_{\max}(E_i) = \mu(E_{i,\max}) \le \mu(E) + \tau s(1 - \frac{1}{r}). \tag{1.2.16}$$

Suppose that the first alternative does not hold, i.e.

$$\mu(E_i) < \mu(E) - D \ .$$

Then, by Lemma 1.2.15,

$$h^0(E_i(m)) \le \frac{1}{g^{n-1}n!}\big((r_i-1)([\mu_{max}(E_i)+gm+B]_+)^n+([\mu_{min}(E_i)+gm+B]_+)^n\big)\ , \tag{1.2.17}$$

where note that $\mu_{max}(E_i(m)) = \mu_{max}(E_i) + gm$ and $\mu_{min}(E_i(m)) = \mu_{min}(E_i) + gm$. Combining the hypothesis, the expression

$$\mu_{min}(E_i) \le \mu(E_i) < \mu(E) - D\ ,$$

and the expressions (1.2.16) and (1.2.14), the formula (1.2.17) becomes

$$h^0(E_i(m)) \le \frac{1}{g^{n-1}n!}\big((r_i - 1)(\mu(E) + \tau s(1 - \frac{1}{r}) + gm + B)^n + (\mu(E) - D + gm + B)^n\big)$$

$$\le \frac{r_i}{rg^{n-1}n!}\big((r - 1)(\mu(E) + \tau s(1 - \frac{1}{r}) + gm + B)^n + (\mu(E) - D + gm + B)^n\big) = \frac{r_i}{r}G(m)\ .$$

Now, by using (1.2.15), it is

$$rh^0(E_i(m)) < r_iG(m) < r_i(P(m) - s\delta(m))\ ,$$

hence we obtain condition 1.

Case 2. Suppose that $(E, \varphi, u) \in \mathcal{S}''_m$ for some $m \ge N_1$. For each i, consider the quotient $E^i = E/E_i$. Let $E^i_{\min}$ be the last factor of the Mumford-Harder-Narasimhan filtration of E^i (c.f. Definition 1.3.3 and Remark 1.3.4), i.e. $\mu(E^i_{\min}) = \mu_{\min}(E^i)$. Let E' be the kernel

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E^i_{\min} \longrightarrow 0 \;,$$

and consider (1.2.15) in the form

$$\frac{G(m)}{r} - \frac{P(m) - s\delta(m)}{r} < 0 \;.$$

Using (1.2.13), we apply condition 3 in Theorem 1.2.19 to the one-step filtration $E' \subsetneq E$ (note that $(E, \varphi, u) \in \mathcal{S}''_m$) to get

$$\frac{G(m)}{r} < \frac{h^0(E^i_{\min}(m))}{\operatorname{rk} E^i_{\min}} - \delta(m)\frac{(s \operatorname{rk} E' - \epsilon(E' \subsetneq E)r)}{r \operatorname{rk} E^i_{\min}} - \delta(m)\frac{s}{r}$$

$$\leq \frac{h^0(E^i_{\min}(m))}{\operatorname{rk} E^i_{\min}} + \delta(m)s(-\frac{1}{r} - \frac{1}{\operatorname{rk} E^i_{\min}}(\frac{\operatorname{rk} E'}{r} - 1)) = \frac{h^0(E^i_{\min}(m))}{\operatorname{rk} E^i_{\min}} \;.$$

Applying Lemma 1.2.15,

$$\frac{G(m)}{r} < \frac{1}{g^{n-1} n! \operatorname{rk} E^i_{\min}} \big((\operatorname{rk} E^i_{\min} - 1)[\mu_{max}(E^i_{\min}) + gm + B]^n_+ + [\mu_{\min}(E^i_{\min}) + gm + B]^n_+ \big) \;.$$

By definition, $E^i_{\min}$ is semistable, then

$$\mu(E^i_{\min}) = \mu_{max}(E^i_{\min}) = \mu_{min}(E^i_{\min}), \tag{1.2.18}$$

hence

$$\frac{G(m)}{r} < \frac{1}{g^{n-1} n!} \big([\mu(E^i_{\min}) + gm + B]^n_+ \big) \;.$$

By definition of G and (1.2.14), we have $0 < G(m)$, hence $\mu(E^i_{\min}) + gm + B > 0$ and, then,

$$\frac{G(m)}{r} < \frac{1}{g^{n-1} n!} \big(\mu(E^i_{\min}) + gm + B)^n \big) \;.$$

This inequality of polynomials holds for some $m \geq N_1$, therefore, it holds for larger values of m, and hence, we will have this inequality between the second coefficients (note that leading coefficients are equal),

$$\frac{1}{(n-1)!}(\mu(E) + \tau s\frac{(r-1)^2}{r^2} - \frac{D}{r} + B) < \frac{1}{(n-1)!}(\mu(E^i_{\min}) + B) \;.$$

Now, $\mu(E^i_{\min}) \leq \mu(E^i) = \frac{d - \deg E_i}{r - \operatorname{rk} E_i}$ and, using $\frac{r - \operatorname{rk} E_i}{\operatorname{rk} E_i} < r$ and $D > \tau s$, previous inequality gives (1.2.12).

Now, suppose that the first alternative does not hold, i.e.

$$\mu(E_i) < \mu(E) - D \ ,$$

then

$$\frac{\deg E_i}{\operatorname{rk} E_i} < \frac{d}{r} - D < \frac{d}{r} - \tau s < \frac{d - \tau s}{r} \ ,$$

which is equivalent to

$$r \deg E_i - \operatorname{rk} E_i r + \tau s \operatorname{rk} E_i < 0 \ ,$$

and hence, the leading coefficient of the polynomial $rP_{E_i} - r_i P - r_i s\delta$ is negative, therefore, condition 2 holds. ■

Lemma 1.2.21. *The set $\mathcal{S} = \mathcal{S}^s \cup \bigcup_{m \geq N_1} \mathcal{S}''_m$ is bounded.*

Proof. Let $(E, \varphi, u) \in \mathcal{S}$. Let E' be a subsheaf of E, and $\overline{E'}$ the saturation of E' on E. Then we have this exact sequence

$$0 \longrightarrow E' \longrightarrow \overline{E'} \longrightarrow T(E') \longrightarrow 0$$

and, by additivity of the Hilbert polynomial on exact sequences, and $\operatorname{rk} E' = \operatorname{rk} \overline{E'}$, we get $\deg(E') \leq \deg(\overline{E'})$. Then, Lemma 1.2.20 gives (1.2.12), then

$$\mu(E') \leq \mu(\overline{E'}) \leq \mu(E) + D$$

and, therefore, by Theorem 1.2.17, the set $\mathcal{S}$ is bounded. ■

Lemma 1.2.22. *Let $\mathcal{S}_0$ be the set of sheaves E' such that E' is a saturated subsheaf of E for some $(E, \varphi, u) \in \mathcal{S}$, and furthermore*

$$|\mu(E') - \mu(E)| \leq D. \tag{1.2.19}$$

Then, $\mathcal{S}_0$ is bounded.

Proof. Let $E' \in \mathcal{S}_0$. The sheaf $E'' = E/E'$ is torsion free and

$$|\deg(E'')| = |\deg(E) - \deg(E')| \leq |\deg(E)| + |\deg(E')| \leq 2|\deg(E)| + rD \ ,$$

where in last inequality we use (1.2.19). Then, as $\deg E$ is fixed, by Lemma 1.2.16, the set of sheaves E'' obtained in this way is bounded, and hence, also $\mathcal{S}_0$ is bounded. ■

Lemma 1.2.23. *There exists an integer N_2 such that for every weighted filtration $(E_\bullet, n_\bullet)$ as in (2.1.6), with $E_i \in \mathcal{S}_0$, $\forall i$, the inequality of polynomials (1.2.5) in Definition 1.2.3,*

$$\big(\sum_{i=1}^{t} n_i(rP_{E_i} - r_iP)\big) + \delta\mu(\varphi, E_\bullet, n_\bullet) \preceq 0 \ ,$$

holds if and only if it holds for a particular value of $m \geq N_2$. The same holds for $\prec$.

Proof. Since $\mathcal{S}_0$ is bounded by Lemma 1.2.21, the set of polynomials $P_{E'}$, for $E' \in \mathcal{S}_0$, is finite. Lemma 1.2.7 implies that we only need to consider a finite number of values for n_i, hence the result follows from it. ∎

Proof of Theorem 1.2.19. Given that $\mathcal{S}$ and $\mathcal{S}_0$ are bounded, let $N_0 > \max\{N_1, N_2\}$ (c.f. Lemmas 1.2.20 and 1.2.23) and such that all sheaves in $\mathcal{S}$ and $\mathcal{S}_0$ are N_0-regular, and $E_1 \otimes \cdots \otimes E_s$ is sN_0-regular for all $E_1, ..., E_s$ in $\mathcal{S}_0$. Let $m \geq N_0$.

(2. $\Rightarrow$ 3.) Let $(E, \varphi, u) \in \mathcal{S}'_m$ and consider a weighted filtration $(E_\bullet, n_\bullet)$ as in (2.1.6). Note that the functor of global sections is only left exact, then applying it to the exact sequence

$$0 \longrightarrow E_i(m) \longrightarrow E(m) \longrightarrow E^i(m) \longrightarrow 0 \ ,$$

we obtain

$$0 \longrightarrow H^0(E_i(m)) \longrightarrow H^0(E(m)) \longrightarrow H^0(E^i(m)) \ ,$$

and we have this inequality for the dimensions of the vector spaces

$$h^0(E(m)) \leq h^0(E_i(m)) + h^0(E^i(m)) \ . \tag{1.2.20}$$

Then, using hypothesis $P(m) \leq h^0(E(m))$ and (1.2.20), we get

$$\big(\sum_{i=1}^{t} n_i(r^iP(m) - rh^0(E^i(m)))\big) + \delta(m)\mu(\varphi, E_\bullet, n_\bullet) =$$

$$\big(\sum_{i=1}^{t} n_i(r(P(m) - h^0(E^i(m))) - r_iP(m))\big) + \delta(m)\mu(\varphi, E_\bullet, n_\bullet) \leq$$

$$\big(\sum_{i=1}^{t} n_i(r(h^0(E(m)) - h^0(E^i(m))) - r_iP(m))\big) + \delta(m)\mu(\varphi, E_\bullet, n_\bullet) \leq$$

$$\big(\sum_{i=1}^{t} n_i(rh^0(E_i(m)) - r_iP(m))\big) + \delta(m)\mu(\varphi, E_\bullet, n_\bullet) \leq 0 \ ,$$

therefore $(E,\varphi,u)\in\mathcal{S}''_m$, and similarly for the strict inequality.

(1. $\Rightarrow$ 2.) Let $(E,\varphi,u)\in\mathcal{S}^s$ and consider a saturated weighted filtration $(E_\bullet,n_\bullet)$. Since E is N_0-regular, $P(m)=h^0(E(m))$. If $E_i\in\mathcal{S}_0$ (meaning that (1.2.19) holds), by choice of N_0, it is also $P_{E_i}(m)=h^0(E_i(m))$. If $E_i\notin\mathcal{S}_0$ then, by definition of $\mathcal{S}_0$ (c.f. 1.2.19), the second alternative of Lemma 1.2.20 holds, hence, as $(E,\varphi,u)\in\mathcal{S}^s$, we have assertion 1.,

$$rh^0(E_i(m))<r_i(P(m)-s\delta(m))\ .$$

Let $\mathcal{T}'\subset\mathcal{T}=\{1,...,t\}$ be the subset of those i for which $E_i\in\mathcal{S}_0$. Let $(E'_\bullet,n'_\bullet)$ be the corresponding subfiltration. Hence, previous argument and Lemma 1.2.8 shows that

$$\begin{aligned}
&\Big(\sum_{i=1}^{t}n_i(rh^0(E_i(m))-r_iP(m))\Big)+\delta(m)\mu(\varphi,E_\bullet,n_\bullet)\leq \qquad (1.2.21)\\
&\Big(\sum_{i\in\mathcal{T}'}n'_i(rP_{E_i}(m)-r_iP(m))\Big)+\delta(m)\mu(\varphi,E'_\bullet,n'_\bullet)+\\
&\Big(\sum_{i\in\mathcal{T}-\mathcal{T}'}n_i(rh^0(E_i(m))-r_iP(m))+\delta(m)sr_i\Big)\leq\\
&\Big(\sum_{i\in\mathcal{T}'}n_i(rP_{E_i}(m)-r_iP(m))\Big)+\delta(m)\mu(\varphi,E'_\bullet,n'_\bullet)\leq 0\ ,
\end{aligned}$$

where last inequality follows from $(E,\varphi,u)\in\mathcal{S}^s$. The condition that E_i is saturated can be dropped, since $h^0(E_i(m))\leq h^0(\overline{E_i}(m))$ and $\mu(\varphi,E_\bullet,n_\bullet)=\mu(\varphi,\overline{E_\bullet},n_\bullet)$, where $\overline{E_i}$ is the saturated subsheaf generated by $\overline{E_i}$ in E. Therefore, $(E,\varphi,u)\in\mathcal{S}'_m$ and similarly for the strict inequality.

(3. $\Rightarrow$ 1.) Let $(E,\varphi,u)\in\mathcal{S}''_m$ and consider a saturated weighted filtration $(E_\bullet,n_\bullet)$. Since E is N_0-regular, $P(m)=h^0(E(m))$. If $E_i\in\mathcal{S}_0$, then also $P_{E_i}(m)=h^0(E_i(m))$. Hence, $h^1(E_i(m))=0$ and (1.2.20) becomes an equality. Then, using the additivity of the Hilbert polynomial on exact sequences, we get $h^0(E^i(m))=P_{E^i}(m)$. Now, hypothesis 3. applied to the subfiltration $(E'_\bullet,n_\bullet)$ consisting on those terms such that $E_i\in\mathcal{S}_0$, implies

$$\Big(\sum_{E_i\in\mathcal{S}_0}n_i(r^iP(m)-rP_{E^i}(m))\Big)+\delta(m)\mu(\varphi,E'_\bullet,n'_\bullet)\leq 0$$

and, using Lemma 1.2.23, this is equivalent to

$$\Big(\sum_{E_i\in\mathcal{S}_0}n_i(rP_{E_i}-r_iP)\Big)+\delta\mu(\varphi,E'_\bullet,n'_\bullet)\preceq 0\ .$$

If $E_i \notin \mathcal{S}_0$ then, by definition of $\mathcal{S}_0$ (c.f. (1.2.19)) the second alternative of Lemma 1.2.20 holds, hence, as $(E,\varphi,u)\in \mathcal{S}''_m$, we have

$$rP_{E_i} - r_i P + r_i s\delta \prec 0 \ .$$

Therefore, previous arguments together with Lemma 1.2.8 give

$$\Big(\sum_{i=1}^{t} n_i(rP_{E_i} - r_i P)\Big) + \delta\mu(\varphi, E_\bullet, n_\bullet) \preceq$$

$$\Big(\sum_{E_i\in\mathcal{S}_0} n'_i(rP_{E_i} - r_i P)\Big) + \delta\mu(\varphi, E'_\bullet, n'_\bullet) + \Big(\sum_{E_i\notin\mathcal{S}_0} n_i(rP_{E_i} - r_i P + \delta s r_i)\Big) \preceq 0 \ .$$

We proceed similarly for the strict inequality. As before, the condition that the filtration is saturated can be dropped, and this finishes the proof of the Theorem. ∎

Corollary 1.2.24. *Let (E,φ,u) be δ-semistable, $m\geq N_0$, and suppose that there exists a weighted filtration $(E_\bullet, n_\bullet)$ with*

$$\Big(\sum_{i=1}^{t} n_i(rh^0(E_i(m)) - r_i P(m))\Big) + \delta(m)\mu(\varphi, E_\bullet, n_\bullet) = 0 \ .$$

Then $E_i\in\mathcal{S}_0$ and, by choice of N_0, $h^0(E_i(m)) = P_{E_i}(m)$ for all i.

Proof. By the proof of the part (1. $\Rightarrow$ 2.) of Theorem 1.2.19, if we have this equality then all inequalities in (1.2.21) are equalities, hence $\mathcal{T}=\mathcal{T}'$ and $E_i\in\mathcal{S}_0$, for all i. ∎

Note that in Theorem 1.2.19 we are assuming that E is torsion free. To apply the Lemma in the general case, we will use the following Lemma.

Lemma 1.2.25. *[GS1, Lemma 2.11] Fix $u\in R$. Let (E,φ,u) be a tensor. Assume that there exists a family $(E_t,\varphi_t,u)_{t\in C}$ parametrized by a smooth curve C such that $(E_0,\varphi_0,u) = (E,\varphi,u)$ and E_t is torsion free for $t\neq 0$. Then there exists a tensor (F,ψ,u), a homomorphism*

$$(E,\varphi,u)\longrightarrow (F,\psi,u)$$

such that F is torsion free with $P_E = P_F$, and an exact sequence

$$0\longrightarrow T(E)\longrightarrow E\stackrel{\beta}{\longrightarrow} F \ ,$$

where $T(E)$ is the torsion subsheaf of E.

Proof. The family is given by a tuple (E_C, φ_C, u_C, N) as in (1.2.8), where u_C is the constant map from C to R with constant value u. Shrinking C if necessary, assume that N is trivial. Let $U = (X \times C) - \mathrm{Supp}(T(E_0))$. Let $F_C = j_*(E_C|_U)$. Since F_C has no elements of torsion supported on a fiber of the projection $X \times C \to C$, then F_C is flat over C (c.f. [Ha, Proposition 9.7]). The natural map $\tilde{\beta} : E_C \to F_C$ is an isomorphism on U, hence we have a homomorphism $\psi_U := \varphi_C|_U$ on U which extends to a homomorphism ϕ_C on $X \times C$ because $\overline{u_C}^*\mathcal{D}$ is locally free, where $\overline{u_C} = \mathrm{id}_X \times u_C$. Finally define $(F, \psi) = (F_0, \psi_0)$, and let β be the homomorphism induced by $\tilde{\beta}$. ■

1.2.3 Gieseker's embedding

In this subsection we are going to change the embedding used in [GS1] by the one given by Gieseker for the construction of the moduli space of sheaves over algebraic surfaces.

By Serre Vanishing Theorem, choose $N \geq N_0$ (c.f. proof of Theorem 1.2.19) to be large enough so that for all $m \geq N$, all $i > 0$, all line bundles L of degree d and all locally free sheaves $\{D_u\}_{u \in R}$, we have

$$h^i(L^{\otimes b} \otimes D_u(sm)) = 0 \ ,$$

and $L^{\otimes b} \otimes D_u(sm)$ is generated by global sections.

Fix $m \geq N$ and let V be a vector space of dimension $p := P(m)$. The choice of m implies that if (E, φ, u) is δ-semistable then, by Theorem 1.2.19, $E(m)$ is m-regular and hence, $h^i(E(m)) = 0 \ , \forall i > 0$, and it is generated by global sections. Consider a tuple (g, E, φ, u), where (E, φ, u) is a δ-semistable tensor and

$$g : V \longrightarrow H^0(E(m))$$

is an isomorphism. This induces a quotient, as morphism of sheaves, by the evaluation map

$$\begin{array}{rcccl} q: & V & \otimes & \mathcal{O}_X(-m) & \twoheadrightarrow & E \\ & s|_U & \otimes & a|_U & \mapsto & s \cdot a|_U \end{array} \qquad (1.2.22)$$

where s a section of E, given as the image by g of an element of V. Let $\mathcal{H}$ be Quot-scheme of Grothendieck which is the scheme of such quotients with Hilbert polynomial P,

$$\mathcal{H} := \mathrm{Quot}_{\mathcal{F},X,P} = \{q : \mathcal{F} \twoheadrightarrow E, P_E = P\} \ ,$$

where $\mathcal{F} = V \otimes \mathcal{O}_X(-m)$. Each quotient q induces the following homomorphisms

$$\begin{aligned} q(m) : V \otimes \mathcal{O}_X(-m) \otimes \mathcal{O}_X(m) &\longrightarrow E \otimes \mathcal{O}_X(m) \\ q(m) : V \otimes \mathcal{O}_X &\longrightarrow E(m) \end{aligned} \tag{1.2.23}$$

$$\begin{aligned} H^0(q(m)) : V \otimes H^0(\mathcal{O}_X) &\longrightarrow H^0(E(m)) \\ H^0(q(m)) : V \otimes \mathbb{C} &\longrightarrow H^0(E(m)) \\ H^0(q(m)) : V &\longrightarrow H^0(E(m)) \end{aligned} \tag{1.2.24}$$

$$\wedge^r H^0(q(m)) : \wedge^r V \longrightarrow \wedge^r H^0(E(m)) \longrightarrow H^0(\wedge^r E(m)) \simeq H^0(\det(E)(rm)) \tag{1.2.25}$$

where note that we have to choose an isomorphism between $\wedge^r E$ and $\det E$. We call $Q := \wedge^r H^0(q(m))$ and $A := H^0(\det(E)(rm))$, then we have

$$Q \in \mathrm{Hom}(\wedge^r V, A) \ .$$

Given that two of these points Q differing by a scalar correspond to the same morphism (because the isomorphism $\wedge^r E \simeq \det E$ is well defined up to a scalar) we get a well defined point in a projective space

$$\overline{Q} \in \mathbb{P}(\mathrm{Hom}(\wedge^r V, A)) \ .$$

Therefore we get a Grothendieck's embedding

$$\mathcal{H} \longrightarrow \mathbb{P}(\mathrm{Hom}(\wedge^r V, A))$$

and, hence, a very ample line bundle $\mathcal{O}_{\mathcal{H}}(1)$ on $\mathcal{H}$ (depending on m).

The tuple (g, E, φ, u), where recall

$$\varphi : (E^{\otimes s})^{\oplus c} \longrightarrow (\det E)^{\otimes b} \otimes D_u \ ,$$

also induces this linear map

$$\Phi : (V^{\otimes s})^{\oplus c} \longrightarrow H^0(E(m)^{\otimes s})^{\oplus c} \longrightarrow H^0((\det E)^{\otimes b} \otimes D_u(sm)) \ , \tag{1.2.26}$$

by tensoring each E with $\mathcal{O}_X(m)$ and taking global sections.

A Poincare bundle on $J \times X$, where $J = Pic^d(X)$, is a universal family such that

$$\begin{tikzcd} \det E \simeq \mathcal{P}|_{\{detE\}\times X} \arrow[r] \arrow[d] & \mathcal{P} \arrow[d] \\ \{\det E\} \times X \arrow[r, hook] & J \times X \end{tikzcd} \tag{1.2.27}$$

Then, we fix an isomorphism

$$\beta : \det E \longrightarrow \mathcal{P}|_{\{\det E\}\times X}$$

and hence, Φ induces a quotient

$$(V^{\otimes s})^{\oplus c} \otimes H^0(\mathcal{P}|^{\otimes b}_{\{\det E\}\times X} \otimes D_u(sm))^{\vee} \longrightarrow \mathbb{C} .$$

Note that, if we choose a different isomorphism β', this quotient will only change by a scalar, so we get a well defined point $[\Phi]$ in $\mathcal{W}$, where $\mathcal{W}$ is the projective bundle over $J \times R$ defined as

$$\mathcal{W} = \mathbb{P}\big(((V^{\otimes s})^{\oplus c})^{\vee} \otimes \pi_{J\times R,*}(\pi^*_{X\times J}\mathcal{P}^{\otimes b} \otimes \pi^*_{X\times R}\mathcal{D}(sm))\big) \longrightarrow J \times R ,$$

where $\pi_{X\times J}$ (resp. $\pi_{J\times R}$,...) denotes the natural projection from $X \times J \times R$ to $X \times J$ (resp. $J \times R$,...) and we denote $\mathcal{D}(m) := \mathcal{D} \otimes \pi^*_X\mathcal{O}_X(m)$. Note that $\pi_{J\times R,*}$ $(\pi^*_{X\times J}\mathcal{P}^{\otimes b} \otimes \pi^*_{X\times R}\mathcal{D}(sm))$ is locally free because of the choice of m. Replacing $\mathcal{P}$ with another Poincare bundle defined by tensoring with the pullback of a sufficiently positive line bundle on J, we can assume that $\mathcal{O}_{\mathcal{W}}(1)$ is very ample (this line bundle will also depend on m).

A point $(\overline{Q}, [\Phi]) \in \mathcal{H} \times \mathcal{W}$ associated to a tuple (g, E, φ, u) verifies that the homomorphism Φ in (1.2.26), composed with evaluation, factors as in the diagram

$$\begin{tikzcd} (V^{\otimes s})^{\oplus c} \otimes \mathcal{O}_X \arrow[r, "(q^{\otimes s})^{\oplus c}(m)"] \arrow[d, "\Phi"] & (E(m)^{\otimes s})^{\oplus c} \arrow[dd, bend left, "\varphi"] \\ H^0((detE)^{\otimes b} \otimes D_u(sm)) \otimes \mathcal{O}_X \arrow[d, "ev"] & \\ (detE)^{\otimes b} \otimes D_u(sm) & \end{tikzcd} \tag{1.2.28}$$

Consider the relative version of the homomorphisms in (1.2.28), i.e. the commutative diagram on $X \times \mathcal{H} \times \mathcal{W}$,

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[dr, "f"'] & (V^{\otimes s})^{\oplus c} \otimes \mathcal{O}_{X\times\mathcal{H}\times\mathcal{W}} \arrow[r] \arrow[d, "\Phi_{\mathcal{H}\times\mathcal{W}}"] & (\pi^*_{X\times\mathcal{H}} E_{\mathcal{H}}(m)^{\otimes s})^{\oplus c} \arrow[r] & 0 \\ & & \mathcal{A} := \pi^*_{X\times J}\mathcal{P}^{\otimes b} \otimes \pi^*_{X\times R}\mathcal{D} \otimes \pi^*_X \mathcal{O}_X(sm) & & \end{tikzcd} \tag{1.2.29}$$

where again $\pi_{X\times J}$ (resp. π_X,...) denotes the natural projection from $X \times \mathcal{H} \times \mathcal{W}$ to $X \times J$ (resp. X,...). We denote by $E_{\mathcal{H}}$ the tautological sheaf on $X \times \mathcal{H}$, and $\Phi_{\mathcal{H}\times\mathcal{W}}$ is the relative version of the composition $ev \circ \Phi$ in diagram (1.2.28).

The points $(\overline{Q}, [\Phi])$ where the restriction $\Phi_{\mathcal{H}\times\mathcal{W}}|_{X\times(\overline{Q},[\Phi])}$ factors through $(E(m)^{\otimes s})^{\oplus c}$ as in diagram (1.2.28) are exactly the points where $f_{X\times(\overline{Q},[\Phi])}$ is identically zero. Hence, points $(\overline{Q}, [\Phi])$ corresponding to tuples (g, E, φ, u) have to verify $f_{X\times(\overline{Q},[\Phi])} = 0$ identically, a closed condition, then we will look for them in a closed subscheme of $\mathcal{H} \times \mathcal{W}$. We will need the following technical Lemma.

Lemma 1.2.26. *[GS1, Lemma 3.1] Let Y be a scheme, and let $f : \mathcal{G} \longrightarrow \mathcal{F}$ be a homomorphism of coherent sheaves on $X \times Y$. Assume that $\mathcal{F}$ is flat over Y. Then there exists a unique closed subscheme $Z' \subset Y$ satisfying the following universal property: given a Cartesian diagram*

$$\begin{tikzcd} X \times Z' \arrow[r, "\bar{i}"] \arrow[d, "p_{Z'}"] & X \times Y \arrow[d, "p"] \\ Z' \arrow[r, "i"] & Y \end{tikzcd}$$

$\bar{i}^* f = 0$ if and only if h factors through Z.

Proof. See [GS1, Lemma 3.1]. ■

Let Z' be the scheme given by this lemma setting $Y = \mathcal{H}\times\mathcal{W}$ and the homomorphism $f : \mathcal{K} \to \mathcal{A}$. Let $i : Z' \hookrightarrow \mathcal{H}\times\mathcal{W}$ and $\bar{i} = id_X \times i$. Then $\bar{i}^* f = 0$, and we get a commutative diagram on $X \times Z'$,

$$\begin{tikzcd} \bar{i}^*\mathcal{K} \arrow[r] \arrow[dr, "\bar{i}^* f"'] & (V^{\otimes s})^{\oplus c} \otimes \mathcal{O}_{X\times Z'} \arrow[r] \arrow[d, "\bar{i}^*\tilde{\Phi}"] & (\bar{i}^* \pi^*_{X\times\mathcal{H}} E_{\mathcal{H}}(m)^{\otimes s})^{\oplus c} \arrow[r] \arrow[dl, "\tilde{\varphi}"] & 0 \\ & \bar{i}^*\mathcal{A} & & \end{tikzcd} \tag{1.2.30}$$

and, hence, there is a universal family of based tensors parametrized by Z',

$$\varphi_{Z'} : E_{Z'}^{\otimes s} \longrightarrow (\det E_{Z'})^{\otimes b} \otimes \pi^*_{Z'}\mathcal{D} \ . \tag{1.2.31}$$

Thanks to the tautological family (1.2.31), given a point $(\overline{Q},[\Phi])$ in Z', we get a tuple (g,E,φ,u) up to isomorphism. Moreover, if $H^0(q(m)) : V \to H^0(E(m))$ is an isomorphism, then we recover exactly the original tuple $(g = H^0(q(m)), E, \varphi, u)$ up to isomorphism, i.e. if (g',E',φ',u') is another tuple corresponding to the same point $(\overline{Q},[\Phi])$, then there exists an isomorphism (f,α) between (E,φ,u) and $(E'\varphi'u')$ as in (1.2.1), and $H^0(f(m))\circ q = q'$.

Let $Z \subset Z'$ be the Zariski closure of the points associated to δ-semistable tensors. Let $\pi_{\mathcal{H}}$ and $\pi_{\mathcal{W}}$ be the projections of Z to $\mathcal{H}$ and $\mathcal{W}$, and define a **polarization** on Z by

$$\mathcal{O}_Z(a_1,a_2) := \pi_{\mathcal{H}}^*\mathcal{O}_{\mathcal{H}}(a_1)\otimes \pi_{\mathcal{W}}^*\mathcal{O}_{\mathcal{W}}(a_2) \tag{1.2.32}$$

where we choose integers a_1 and a_2 for their ratio to verify

$$\frac{a_2}{a_1} = \frac{r\delta(m)}{P(m)-s\delta(m)}\ . \tag{1.2.33}$$

The projective scheme Z is preserved by the natural $SL(V)$ action, and this action has a natural linearization en $\mathcal{O}_Z(a_1,a_2)$, using the linearizations on $\mathcal{O}_{\mathcal{H}}(1)$ and $\mathcal{O}_{\mathcal{W}}(1)$.

Recall that the points of Z for which $H^0(q(m))$ is an isomorphism correspond (up to isomorphism) to the tuples (g,E,φ,u), where $g : V \simeq H^0(E(m))$. To get rid of the choice of g, we have to take the quotient by $GL(V)$, but if $\lambda\in\mathbb{C}^*$, (g,E,φ,u) and $(\lambda g,E,\varphi,u)$ correspond to the same point, and hence it suffices to take the quotient by the action of $SL(V)$. We construct this quotient by using Geometric Invariant Theory.

In the following, in Proposition 1.2.29 we identify the GIT semistable points in Z using the Hilbert-Mumford criterion (c.f. Theorem 1.1.14). In Theorem 1.2.31 we relate filtrations of sheaves with filtrations of the vector space V, to prove that GIT semistable points of $(\overline{Q},[\Phi])\in Z$ coincide with the points associated to δ-semistable tensors (E,φ,u) plus an isomorphism g. Therefore, we will have $Z^{ss} = Z$.

The moduli space of δ-semistable tensors, $\mathfrak{M}_\delta$, will be the GIT quotient of $Z^{ss}=Z$ by $SL(V)$,

$$\mathfrak{M}_\delta = Z/\!\!/SL(V)\ ,$$

which is **good quotient** by Theorem 1.1.11.

1.2.4 Application of Geometric Invariant Theory

Recall that a 1-parameter subgroup of G is a non-trivial homomorphism $\Gamma : \mathbb{C}^* \longrightarrow G$. In our case, the group is $G = SL(V) = SL(p,\mathbb{C})$. It follows from elementary representation theory that there exists a basis $v_1, ..., v_p$ of $\mathbb{C}^p$ and $\Gamma_i \in \mathbb{Z}$ such that

$$\Gamma(t) = \begin{pmatrix} t^{\Gamma_1} & & 0 \\ & \ddots & \\ 0 & & t^{\Gamma_p} \end{pmatrix} ,$$

so we will refer to Γ by giving the exponents of the diagonal, $(\Gamma_1, ..., \Gamma_p)$. Note that $\sum_{i=1}^{p} \Gamma_i = 0$ because $\Gamma(t) \in SL(V)$.

The group $SL(V)$ acts on Z and this action is linearized by means of the line bundle $\mathcal{O}_Z(a_1, a_2)$. The point z_0 is a fixed point for the $\mathbb{C}^*$-action on X induced by Γ. Thus, $\mathbb{C}^*$ acts on the fiber of $L = \mathcal{O}_Z(a_1, a_2)$ over z_0 with weight γ and recall the definition of the numerical function in Theorem 1.1.14, $\mu(\Gamma, x) := \gamma$, the *minimum relevant exponent* of the action of Γ on $x \in Z$, i.e. the minimum exponent of the diagonal of the one parameter subgroup which acts on a non-zero coordinate of the point x.

A weighted filtration $(V_\bullet, n_\bullet)$ of V is a filtration

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_t \subset V_{t+1} = V, \tag{1.2.34}$$

and positive numbers $n_1, n_2, \ldots, n_t > 0$. If $t = 1$ (one-step filtration), then we will take $n_1 = 1$. To a weighted filtration we associate a vector of $\mathbb{C}^p$ defined as $\Gamma = \sum_{i=1}^{t} n_i \Gamma^{(\dim V_i)}$, where

$$\Gamma^{(k)} := \big(\overbrace{k-p, \ldots, k-p}^{k}, \overbrace{k, \ldots, k}^{p-k}\big) \qquad (1 \le k < p)\,. \tag{1.2.35}$$

Hence, the vector is of the form

$$\Gamma = (\overbrace{\Gamma_1, \ldots, \Gamma_1}^{\dim V^1}, \overbrace{\Gamma_2, \ldots, \Gamma_2}^{\dim V^2}, \ldots, \overbrace{\Gamma_{t+1}, \ldots, \Gamma_{t+1}}^{\dim V^{t+1}})\,.$$

Giving the numbers $n_1, \ldots, n_t$ is clearly equivalent to giving the numbers $\Gamma_1, \ldots, \Gamma_{t+1}$, because

$$n_i = \frac{\Gamma_{i+1} - \Gamma_i}{p} \quad \text{and} \quad \sum_{i=1}^{t+1} \Gamma_i \dim V^i = 0\,. \tag{1.2.36}$$

Given a 1-parameter subgroup Γ, we associate a weighted filtration as follows. There is a basis $\{e_1, \ldots, e_p\}$ of V where it has a diagonal form. Let $\Gamma_1 < \cdots < \Gamma_{t+1}$ be the distinct exponents, and let

$$0 \subset V_1 \subset \cdots \subset V_{t+1} = V$$

be the associated filtration, where each V_i is generated by the vectors e_j associated to exponents $\Gamma_j \leq \Gamma_i$. Note that two 1-parameter subgroups define the same filtration if and only if they are conjugate by an element of the parabolic subgroup $P \subset \mathrm{SL}(V)$ defined by the filtration.

Now, let $\mathcal{I} = \{1, ..., t+1\}^{\times s}$ be the set of all multi-indexes $I = (i_1, ..., i_s)$. Define

$$\mu(\Phi, V_\bullet, n_\bullet) = \min_{I \in \mathcal{I}} \{\Gamma_{\dim V_{i_1}} + \cdots + \Gamma_{\dim V_{i_s}} : \Phi|_{(V_{i_1} \otimes \cdots \otimes V_{i_s})^{\oplus c}} \neq 0\}. \tag{1.2.37}$$

If $I_0 = (i_1, \ldots, i_s)$ is the multi-index giving minimum in (1.2.37), we will denote by $\epsilon_i(\overline{\Phi}, V_\bullet, n_\bullet)$ (or just $\epsilon_i(V_\bullet)$ if the rest of the data is clear from the context) the number of elements k of the multi-index I_0 such that $\dim V_k \leq \dim V_i$.

Given a quotient $q : V \otimes \mathcal{O}_X(-m) \twoheadrightarrow E$, for each subspace $V' \subset V$, we define the subsheaf $E_{V'} \subset E$ as the image of the restriction of q to V',

$$\begin{tikzcd} V \otimes \mathcal{O}_X(-m) \arrow[r, two heads] & E \\ V' \otimes \mathcal{O}_X(-m) \arrow[r, two heads] \arrow[u, hook] & E_{V'} \arrow[u, hook] \end{tikzcd} \tag{1.2.38}$$

Note that, in particular, $E_{V'}(m)$ is generated by global sections.

If the quotient $q : V \otimes \mathcal{O}_X(-m) \twoheadrightarrow E$ induces an injection $V \hookrightarrow H^0(E(m))$, and if $E' \subset E$ is a subsheaf, we can define

$$V_{E'} = V \cap H^0(E'(m)) \,. \tag{1.2.39}$$

We will show in Proposition 1.2.30 that all quotients coming from GIT semistable points $(\overline{Q}, [\Phi]) \in Z$ satisfy this injectivity property, then filtrations of subsheaves will define filtrations of vector subspaces and viceversa. Here, there are two Lemmas relating both processes.

Lemma 1.2.27. *Given a point $(\overline{Q}, [\Phi]) \in Z$ such that q induces an injection $V \hookrightarrow H^0(E(m))$, and a weighted filtration $(E_\bullet, n_\bullet)$ of E, we have:*

1. $E_{V_{E_i}} \subset E_i$
2. *If* $\varphi|_{(E_{i_1} \otimes \cdots \otimes E_{i_s})^{\oplus c}} = 0$, *then* $\Phi|_{(V_{E_{i_1}} \otimes \cdots \otimes V_{E_{i_s}})^{\oplus c}} = 0$
3. $\sum_{i=1}^{t} -n_i \epsilon_i(\varphi, E_\bullet, n_\bullet) \leq \sum_{i=1}^{t} -n_i \epsilon_i(\Phi, V_{E_\bullet}, n_\bullet)$

Furthermore, if q induces an isomorphism $V \simeq H^0(E(m))$, all E_i are m-regular and all $E_{i_1} \otimes \cdots \otimes E_{i_s}$ are sm-regular, then 1 becomes an equality, 2 becomes "if and only if" and 3 becomes an equality.

Proof. By definition, $E_{V_{E_i}} = q|_{V_{E_i}} = q(V \cap H^0(E_i(m)) \otimes \mathcal{O}_X(-m)) \subset E_i$ because we are evaluating sections that, in particular, are sections of E_i. And, if E_i is m-regular then it will be generated by their global sections. Hence, every point in the total space of E_i will be the image of any section in $H^0(E_i(m)) \otimes \mathcal{O}_X(-m) = V \cap H^0(E_i(m)) \otimes \mathcal{O}_X(-m)$, provided $V \simeq H^0(E(m))$. This proves 1.

To prove 2, note that Φ, as a morphism of sections, is given at each point by the morphism of sheaves φ. A section of $V_{E_{i_j}}$ is, in particular, a section of $H^0(E_{i_j}(m))$, so if φ vanishes on a factor E_{i_j}, then it vanishes on $E_{i_j}(m)$, and therefore Φ vanishes on $V_{E_{i_j}}$. If $V \simeq H^0(E(m))$ and E_i is m-regular, by 1 we have $E_{V_{E_i}} = E_i$. If $E_{i_1} \otimes \cdots \otimes E_{i_s}$ is sm-regular, then

$$E_{i_1} \otimes \cdots \otimes E_{i_s} \otimes \mathcal{O}_X(sm) = E_{i_1}(m) \otimes \cdots \otimes E_{i_s}(m)$$

is generated by global sections and every element of $E_{i_1}(m) \otimes \cdots \otimes E_{i_s}(m)$ comes from a section of

$$V_{E_{i_1}} \otimes \cdots \otimes V_{E_{i_s}} = H^0(E_{i_1}(m)) \otimes \cdots \otimes H^0(E_{i_s}(m)\ .$$

Therefore, if Φ vanishes, then φ does.

Recall that if I_0 is the multi-index giving minimum in (1.2.37), $\epsilon_i(\overline{\Phi}, V_{E_\bullet}, n_\bullet)$ is the number of elements k of I_0 such that $\dim V_{E_{i_k}} \leq \dim V_{E_i}$, and similarly for $\epsilon_i(\varphi, E_\bullet, n_\bullet)$ with $\operatorname{rk} E_{i_k} \leq \operatorname{rk} E_i$ (c.f. (1.2.4)). Then, if φ (resp. Φ) vanishes on a filter V_i (resp. E_i), the index i cannot be taken into account in the calculation of $\epsilon_i(\overline{\Phi}, V_{E_\bullet}, n_\bullet)$ (resp. $\epsilon_i(\varphi, E_\bullet, n_\bullet)$). Also note that, by 2, if φ vanishes on a filter E_{i_k}, then Φ vanishes on $V_{E_{i_k}}$. Hence, all $V_{E_{i_k}}$ are not counted in $\epsilon_i(\overline{\Phi}, V_{E_\bullet}, n_\bullet)$ if they were not counted in $\epsilon_i(\varphi, E_\bullet, n_\bullet)$. Therefore, $\epsilon_i(\varphi, E_\bullet, n_\bullet) \geq \epsilon_i(\overline{\Phi}, V_{E_\bullet}, n_\bullet)$, and this proves 3. ■

Lemma 1.2.28. *Given a point $(\overline{Q}, [\Phi]) \in Z$ such that q induces an injection $V \hookrightarrow H^0(E(m))$, and a weighted filtration $(V_\bullet, n_\bullet)$ of V, we have:*

1. $V_i \subset V_{E_{V_i}}$
2. $\varphi|_{(E_{V_{i_1}} \otimes \cdots \otimes E_{V_{i_s}})^{\oplus c}} = 0$ *if and only if* $\Phi|_{(V_{i_1} \otimes \cdots \otimes V_{i_s})^{\oplus c}} = 0$
3. $\sum_{i=1}^t -n_i \epsilon_i(\varphi, E_{V_\bullet}, n_\bullet) = \sum_{i=1}^t -n_i \epsilon_i(\Phi, V_\bullet, n_\bullet)$

Proof. Similarly to Lemma 1.2.27, $E_{V_i} = q|_{V_i}$, then $V_i \subset V \cap H^0(E_{V_i}(m)) = V_{E_{V_i}}$, by definition, hence we prove 1.

To prove 2, as Φ is given at each point by the morphism of sheaves φ, and E_{V_i} is generated by the global sections of V_i, if Φ vanishes on sections $V_{i_i} \otimes \cdots \otimes V_{i_s}$, then φ vanishes on the respective sheaves $E_{V_{i_1}} \otimes \cdots \otimes E_{V_{i_s}}$ and viceversa.

Statement 3 follows from 2 and the same argument that in the proof of 3 in Lemma 1.2.27. ∎

Proposition 1.2.29. *The point $(\overline{Q}, [\Phi]) \in Z$ is **GIT semistable** with respect to $\mathcal{O}_Z(a_1, a_2)$ if and only if for every weighted filtration $(V_\bullet, n_\bullet)$ of V*

$$a_1 \sum_{i=1}^{t} n_i(r \dim V_i - \operatorname{rk} E_{V_i} \dim V) + a_2 \sum_{i=1}^{t} n_i(s \dim V_i - \epsilon_i(V_\bullet) \dim V) \leq 0 \qquad (1.2.40)$$

*The point $(\overline{Q}, [\Phi])$ is **GIT stable** if we get a strict inequality for every weighted filtration. In any case, there exists an integer A_2 (depending only on m, P, s, b, c, $\mathcal{D}$) such that it is enough to consider weighted filtrations with $n_i \leq A_2$.*

Proof. By the Hilbert-Mumford criterion, Theorem 1.1.14, a point is GIT semistable (resp. GIT stable) if and only if for all 1-parameter subgroups Γ of $SL(V)$,

$$\mu((\overline{Q}, [\Phi]), \Gamma) = a_1 \mu(\overline{Q}, \Gamma) + a_2 \mu([\Phi], \Gamma) \leq 0$$

(resp. $<$). We have seen that, given a 1-parameter subgroup Γ of $\mathrm{SL}(V)$, we associate a weighted filtration $(V_\bullet, n_\bullet)$ where each exponent Γ_i corresponds to the action of Γ on $V^i = V_i / V_{i-1}$. Denote $\mathcal{I}' = \{1, ..., t+1\}^{\times r}$. Then, the minimum weight of the action of Γ on $\overline{Q} \in \mathbb{P}(\operatorname{Hom}(\wedge^r V, A)$ is (c.f. [Si1] and [HL2]),

$$\mu(\overline{Q}, \Gamma) = \min_{I \in \mathcal{I}'} \{ \Gamma_{i_1} + \cdots + \Gamma_{i_r} : Q|_{V_{i_1} \wedge \cdots \wedge V_{i_r}} \neq 0 \} \ .$$

Note that Γ acts trivially on $A = H^0(\det(E)(rm))$ and observe that the evaluation Q does not vanish on a wedge of r sections, $e_{i_i} \wedge \cdots \wedge e_{i_r}$, whenever $e_{i_1}, ..., e_{i_r}$ span the fiber of E over the generic point $x \in X$. Recall that $\Gamma_1 < \ldots < \Gamma_{t+1}$, then to achieve the minimum we have to take the minimum exponent Γ_1 as many times as possible, then take Γ_2 as many times as possible, and so on, while $Q \neq 0$. Therefore, this occurs when we take Γ_1 a number $\operatorname{rk} E_{V_1}$ of times, then we take Γ_2 a number $(\operatorname{rk} E_{V_2} - \operatorname{rk} E_{V_1})$ of times, etc, and finally we take Γ_{t+1} a number $(\operatorname{rk} E_{V_{t+1}} - \operatorname{rk} E_{V_t})$ of times, hence

$$\mu(\overline{Q}, \Gamma) = \sum_{i=1}^{t+1} \Gamma_i (\operatorname{rk} E_{V_i} - \operatorname{rk} E_{V_{i-1}}) \ .$$

Making calculations

$$\mu(\overline{Q},\Gamma) = r\Gamma_{t+1} - \sum_{i=1}^{t}(\Gamma_{i+1}-\Gamma_i)\operatorname{rk} E_{V_i} = r\Gamma_{t+1} - \sum_{i=1}^{t} n_i \dim V \operatorname{rk} E_{V_i}$$

$$= r(\sum_{i=1}^{t} n_i \dim V_i) - \sum_{i=1}^{t} n_i \dim V \operatorname{rk} E_{V_i} = \sum_{i=1}^{t} n_i(r\dim V_i - \operatorname{rk} E_{V_i}\dim V)\ .$$

Now we calculate the minimum weight of the action of Γ on

$$[\Phi] \in \mathcal{W} = \mathbb{P}\big(((V^{\otimes s})^{\oplus c})^{\vee} \otimes \pi_{J\times R,*}(\pi^*_{X\times J}\mathcal{P}^{\otimes b}\otimes \pi^*_{X\times R}\mathcal{D}(sm))\big)\ ,$$

where note that Γ only acts non trivially on V. Then,

$$\mu([\Phi],\Gamma) = \min_{I\in\mathcal{I}}\{\Gamma_{i_1}+\cdots+\Gamma_{i_s} : \Phi|_{(V_{i_1}\otimes\cdots\otimes V_{i_s})^{\oplus c}} \neq 0\}\ .$$

Similarly, to achieve the minimum, we have to take Γ_1 as many times as V_1 can appear in the multi-index I while the restriction of Φ does not vanish, then take Γ_2 as many times as V_2 can appear minus the number of times V_1 appears, and so on. This can be written in terms of the symbols $\epsilon^i(V_\bullet)$, the number of times that each index i appears on I:

$$\mu([\Phi],\Gamma) = \sum_{i=1}^{t}\Gamma_i\epsilon^i(V_\bullet) = \sum_{i=1}^{t}\Gamma_i(\epsilon_{i+1}(V_\bullet)-\epsilon_i(V_\bullet))$$

$$= s\Gamma_{t+1} - \sum_{i=1}^{t}(\Gamma_{i+1}-\Gamma_i)\epsilon_i(V_\bullet) = s\Gamma_{t+1} - \sum_{i=1}^{t} n_i\dim V\epsilon_i(V_\bullet)$$

$$= s(\sum_{i=1}^{t} n_i\dim V_i) - \sum_{i=1}^{t} n_i\dim V\epsilon_i(V_\bullet) = \sum_{i=1}^{t} n_i(s\dim V_i - \epsilon_i(V_\bullet)\dim V)\ .$$

The last statement follows from an argument similar to the proof of Lemma 1.2.7, with $\mathbb{Z}^r$ replaced by $\mathbb{Z}^p$. ∎

Proposition 1.2.30. *The point $(\overline{Q},[\Phi]) \in Z$ is GIT semistable if and only if for every weighted filtration $(E_\bullet, n_\bullet)$ of E, it is*

$$\sum_{i=1}^{t} n_i\big((r\dim V_{E_i} - \operatorname{rk} E_i\dim V) + \delta(m)(s\operatorname{rk} E_i - \epsilon_i(E_\bullet)r)\big) \leq 0\ . \tag{1.2.41}$$

If $(\overline{Q},[\Phi])$ is GIT stable we get a strict inequality for every weighted filtration. Moreover, if $(\overline{Q},[\Phi])$ is GIT semistable, then the induced map $f_q = H^0(q(m))$, $f_q : V \to H^0(E(m))$ is injective.

Proof. Let us show first that if $(\overline{Q},[\Phi])$ is GIT semistable, then the induced map f_q is injective. Let V' be its kernel and consider the one-step filtration $V' \subset V$. We have $E_{V'}=0$ by definition and, if we calculate $\mu([\Phi],\Gamma)$ for the 1-parameter subgroup Γ associated to the one-step filtration $V' \subset V$, it is

$$\mu([\Phi],\Gamma) = s \dim V' \ ,$$

because $\epsilon_1 = 0$ (V' does not appear in the multi-index giving minimum in (1.2.37) because $E_{V'}=0$). Therefore, applying Proposition 1.2.29,

$$a_1 r \dim V' + a_2 s \dim V' \leq 0 \ ,$$

and $\dim V' = 0$, hence f_q is injective.

Using (1.2.33), the inequality (1.2.40) becomes

$$\sum_{i=1}^{t} n_i \big((r \dim V_i - \operatorname{rk} E_{V_i} \dim V)(P(m) - s\delta(m)) + r\delta(m)(s \dim V_i - \epsilon_i(V_\bullet) \dim V) \big) \leq 0$$

which, setting $P(m) = \dim V$, is equivalent to

$$\sum_{i=1}^{t} n_i \big((r \dim V_i - \operatorname{rk} E_{V_i} \dim V) + \delta(m)(s \operatorname{rk} E_{V_i} - \epsilon_i(V_\bullet) r) \big) \leq 0 \ . \tag{1.2.42}$$

Now let $(\overline{Q},[\Phi])$ be a GIT semistable point. Take a weighted filtration $(E_\bullet, n_\bullet)$ of E. Consider the induced weighted filtration $(V_{E_\bullet}, n_\bullet)$ of V. By Proposition 1.2.29 and using (1.2.33) we have

$$\sum_{i=1}^{t} n_i \big((r \dim V_{E_i} - \operatorname{rk} E_{V_{E_i}} \dim V)(P(m) - s\delta(m)) + r\delta(m)(s \dim V_{E_i} - \epsilon_i(V_{E_\bullet}) \dim V) \big) \leq 0 \ ,$$

which is equivalent to

$$\sum_{i=1}^{t} n_i \big((r \dim V_{E_i} - \operatorname{rk} E_{V_{E_i}} \dim V) + \delta(m)(s \operatorname{rk} E_{V_{E_i}} - \epsilon_i(V_{E_\bullet}) r) \big) \leq 0 \ .$$

Then, by Lemma 1.2.27, using statement 1 we have $E_{V_{E_i}} \subset E_i$, then $\operatorname{rk} E_{V_{E_i}} \leq \operatorname{rk} E_i$, and statement 3 gives $-\epsilon_i(E_\bullet) \leq -\epsilon_i(V_{E_\bullet})$, therefore

$$\sum_{i=1}^{t} n_i \big((r \dim V_{E_i} - \operatorname{rk} E_i \dim V) + \delta(m)(s \operatorname{rk} E_i - \epsilon_i(E_\bullet) r) \big) \leq$$

$$\sum_{i=1}^{t} n_i\big((r\dim V_{E_i} - \operatorname{rk} E_{V_{E_i}} \dim V) + \delta(m)(s \operatorname{rk} E_{V_{E_i}} - \epsilon_i(V_{E_\bullet})r)\big) \leq 0 \ ,$$

hence (1.2.41) holds. Note that, if we start with a GIT stable point, we can substitute the inequalities by strict inequalities.

On the other hand, suppose that (1.2.41) holds. Take a weighted filtration $(V_\bullet, n_\bullet)$ of V. Then we get an induced weighted filtration $(E_{V_\bullet}, n_\bullet)$ of E. For this filtration, (1.2.41) becomes

$$\sum_{i=1}^{t} n_i\big((r\dim V_{E_{V_i}} - \operatorname{rk} E_{V_i} \dim V) + \delta(m)(s \operatorname{rk} E_{V_i} - \epsilon_i(E_{V_\bullet})r)\big) \leq 0 \ .$$

By Lemma 1.2.28, using statement 1 we have $V_i \subset V_{E_{V_i}}$, then $\dim V_i \leq \dim V_{E_{V_i}}$, and statement 2 gives $-\epsilon_i(E_{V_\bullet}) = -\epsilon_i(V_\bullet)$. Hence, applying (1.2.42) (which we have seen that is equivalent to (1.2.40)) for the filtration $(V_\bullet, n_\bullet)$ we get

$$\sum_{i=1}^{t} n_i\big((r\dim V_i - \operatorname{rk} E_{V_i} \dim V) + \delta(m)(s \operatorname{rk} E_{V_i} - \epsilon_i(V_\bullet)r)\big) \leq$$

$$\sum_{i=1}^{t} n_i\big((r\dim V_{E_{V_i}} - \operatorname{rk} E_{V_i} \dim V) + \delta(m)(s \operatorname{rk} E_{V_i} - \epsilon_i(E_{V_\bullet})r)\big) \leq 0 \ ,$$

and therefore the point $(\overline{Q}, [\Phi])$ is GIT semistable by Proposition 1.2.29. If we start with a strict inequality in (1.2.41), we get a GIT stable point. ∎

Theorem 1.2.31. *Assume $m > N$. A point $(\overline{Q}, [\Phi]) \in Z$ is GIT semistable (resp. GIT stable) if and only if the corresponding tensor (E, φ, u) is δ-semistable (resp. δ-stable) and the linear map $f_q : V \simeq H^0(E(m))$ induced by q is an isomorphism.*

Proof. $\Rightarrow$) Suppose $(\overline{Q}, [\Phi]) \in Z$ is GIT semistable and let $(E_\bullet, n_\bullet)$ be a weighted filtration of E. We will use Theorem 1.2.19 to prove that (E, φ, u) is δ-semistable, and similarly for stable.

Recall that, as we have seen in (2. $\Rightarrow$ 3.) in the proof of Theorem 1.2.19, we have this inequality for the dimensions of the vector spaces

$$h^0(E(m)) \leq h^0(E_i(m)) + h^0(E^i(m)) \ .$$

Then, by definition, $V_{E_i} := V \cap H^0(E_i(m))$ and, by dimensions formula,

$$\dim V_{E_i} = \dim V + h^0(E_i(m)) - \dim(V \cup H^0(E_i(m))) \geq$$

$$\dim V + h^0(E_i(m)) - h^0(E(m)) \geq \dim V - h^0(E^i(m)) \ .$$

Recall that $P(m) = \dim V$. Therefore we obtain the inequality of condition 3 in Theorem 1.2.19,

$$\Big(\sum_{i=1}^{t} n_i(\operatorname{rk} E^i P(m) - rh^0(E^i(m)))\Big) + \delta(m)\mu(\varphi, E_\bullet, n_\bullet) =$$

$$\sum_{i=1}^{t} n_i\Big((\operatorname{rk} E^i P(m) - rh^0(E^i(m))) + \delta(m)(s\operatorname{rk} E_i - \epsilon_i(E_\bullet)r)\Big) =$$

$$\sum_{i=1}^{t} n_i\Big((r(\dim V - h^0(E^i(m))) - \operatorname{rk} E_i \dim V) + \delta(m)(s\operatorname{rk} E_i - \epsilon_i(E_\bullet)r)\Big) \leq$$

$$\sum_{i=1}^{t} n_i\Big((r\dim V_{E_i} - \operatorname{rk} E_i \dim V) + \delta(m)(s\operatorname{rk} E_i - \epsilon_i(E_\bullet)r)\Big) \leq 0 \ ,$$

by Proposition 1.2.30. If we start with a GIT stable point we get a strict inequality.

To apply Theorem 1.2.19 we need to show that E is torsion free. By Lemma 1.2.25, there exists a tensor (F, ψ, u) with F torsion free such that $P_E = P_F$ and a exact sequence

$$0 \longrightarrow T(E) \longrightarrow E \xrightarrow{\beta} F$$

where define $E'' := \beta(E)$. Consider a weighted filtration $(F_\bullet, n_\bullet)$ of F. Let $F^i = F/F_i$, and let E^i be the image of E in F^i, $E^i = E''/F_i$. Let E_i be the kernel of $E \longrightarrow E^i$. Then $\operatorname{rk}(F_i) = \operatorname{rk}(E_i) = r_i$, because $\operatorname{rk} E = \operatorname{rk} E^i + \operatorname{rk} E_i$ and $\operatorname{rk} E = \operatorname{rk} E''$. Also, $E^i = E''/F_i \subset F/F_i = F^i$, then $h^0(E^i(m)) \leq h^0(F^i(m))$. Moreover, $\epsilon(\psi, F_\bullet, n_\bullet) = \epsilon(\varphi, E_\bullet, n_\bullet)$, because the difference between filters of F and E occurs in the 0-rank torsion subsheaf. Using this and applying condition 3 in Theorem 1.2.19 to $(F_\bullet, n_\bullet)$, we get

$$\Big(\sum_{i=1}^{t} n_i\big((\operatorname{rk} F^i P(m) - rh^0(F^i(m))\big)\Big) + \delta(m)\mu(\psi, F_\bullet, n_\bullet) =$$

$$\sum_{i=1}^{t} n_i\Big((\operatorname{rk} F^i P(m) - rh^0(F^i(m))) + \delta(m)(s\operatorname{rk} F_i - \epsilon_i(\psi, F_\bullet, n_\bullet)r)\Big) \leq$$

$$\sum_{i=1}^{t} n_i\Big((\operatorname{rk} E^i P(m) - rh^0(E^i(m))) + \delta(m)(s\operatorname{rk} E_i - \epsilon_i(\varphi, E_\bullet, n_\bullet)r)\Big) \leq 0 \ ,$$

and hence Theorem 1.2.19 implies that (F, ψ, u) is δ-semistable.

Apply condition 3 of Theorem 1.2.19 to the one-step filtration $T(E) \subset E$, then $E/T(E) \simeq E''$, and

$$\mathrm{rk}(E'')P(m) - rh^0(E'')(m) + \delta(m)(s\,\mathrm{rk}(T(E)) - \epsilon_1(T(E) \subset E)r) \leq 0$$

$$\Leftrightarrow P(m) - h^0(E'')(m) \leq 0 ,$$

where note that $\mathrm{rk}\, T(E) = 0$ and $\epsilon_1(T(E) \subset E) = 0$. Hence,

$$P(m) \leq h^0(E''(m)) \leq h^0(F(m)) = P(m) ,$$

where the second inequality follows from $E'' \subset F$ and the third equality does from the conclusion of Theorem 1.2.19 about m-regularity of δ-semistable tensors. Hence, equality holds at all places and $h^0(F(m)) = h^0(E''(m))$. Since F is globally generated, $F = E''$ and, therefore, $T(E) = 0$ and E is torsion free. Then, by Theorem 1.2.19, (E, φ, u) is δ-semistable.

Finally, we have seen that if $(\overline{Q}, [\Phi]) \in Z$ is GIT semistable, then the linear map $f_q : V \longrightarrow H^0(E(m))$ is injective by Proposition 1.2.29, and since (E, φ, u) is δ-semistable, then E is m-regular by Proposition 1.2.19. Given that $\dim V = P(m) = h^0(E(m))$, therefore f_q is an isomorphism.

$\Leftarrow$) Suppose (E, φ, u) is δ-semistable, and q induces an isomorphism in the linear map $f_q : V \longrightarrow H^0(E(m))$. Then we have $V_{E'} = H^0(E'(m))$ for any subsheaf $E' \subset E$ and Theorem 1.2.19 condition 2 says that for all weighted filtrations $(E_\bullet, n_\bullet)$ of E,

$$\Big(\sum_{i=1}^{t} n_i(rh^0(E_i) - \mathrm{rk}\, E_i P(m))\Big) + \delta(m)\mu(\varphi, E_\bullet, n_\bullet) =$$

$$\sum_{i=1}^{t} n_i\Big((r \dim V_{E_i} - \mathrm{rk}\, E_i P(m)) + \delta(m)(s\,\mathrm{rk}\, E_i - \epsilon_i(E_\bullet)r)\Big) \leq 0 ,$$

which is exactly (1.2.41) in Proposition 1.2.30, therefore $(\overline{Q}, [\Phi])$ is GIT semistable. Similarly, if (E, φ, u) is δ-stable we obtain a strict inequality and, hence, $(\overline{Q}, [\Phi])$ is GIT stable. ■

Corollary 1.2.32. *Let (E, φ, u) be a δ-semistable tensor and let $(E_\bullet, n_\bullet)$ be a weighted filtration of E. Then the induced morphism $f_q : V \to H^0(E(m))$ is an isomorphism and, therefore, $V = H^0(E(m))$ and $V_{E_i} = H^0(E_i(m))$, for all i.*

Proof. It follows from the proof of Theorem 1.2.31 ■

Now, recall that given V, a vector space of dimension $P(m)$, and a 1-parameter subgroup Γ of $SL(V)$ given in its diagonal form

$$\Gamma = (\overbrace{\Gamma_1, \ldots, \Gamma_1}^{\dim V^1}, \overbrace{\Gamma_2, \ldots, \Gamma_2}^{\dim V^2}, \ldots, \overbrace{\Gamma_{t+1}, \ldots, \Gamma_{t+1}}^{\dim V^{t+1}}) ,$$

we get a weighted filtration $(V_\bullet, n_\bullet)$ of V and a splitting $V = \oplus_i V^i$ of this filtration. Defining $E_{V_i} = q(V_i \otimes \mathcal{O}_X(-m))$ we obtain a weighted filtration $(E_\bullet, n_\bullet)$ of E.

Conversely, let $(E_\bullet, n_\bullet)$ be a weighted filtration of E and $V = \oplus_i V^i$ a splitting of the filtration $V_i = H^0(E_i(m))$. This gives a 1-parameter subgroup Γ of $SL(V)$ defined as $v^i \mapsto t^{\lambda_i} v^i$, for $v^i \in V^i$, with relations (1.2.36).

We will use the following proposition to prove the criterion for S-equivalence.

Proposition 1.2.33. *Suppose that $m > N$. Let (E, φ, u) be a δ-semistable tensor, $f : V \simeq H^0(E(m))$ an isomorphism, and let $(\overline{Q}, [\Phi]) \in Z$ be the corresponding GIT semistable point (c.f. Theorem 1.2.31). The above construction gives a bijection between 1-parameter subgroups of $SL(V)$ with $\mu((\overline{Q}, [\Phi]), \Gamma) = 0$ on the one hand, and weighted filtrations $(E_\bullet, n_\bullet)$ of E with*

$$\sum_{i=1}^{t} n_i \big((r P_{E_i} - \operatorname{rk} E_i P) \big) + \delta \mu(\varphi, E_\bullet, n_\bullet) = 0 \tag{1.2.43}$$

together with a splitting of the filtration $H^0(E_\bullet(m))$ of $V \simeq H^0(E(m))$ on the other hand.

Proof. Let Γ be a 1-parameter subgroup of $SL(V)$ with $\mu((\overline{Q}, [\Phi]), \Gamma) = 0$. Then we get a weighted filtration $(V_\bullet, n_\bullet)$ of V and, by evaluating, a weighted filtration $(E_{V_\bullet}, n_\bullet)$ of E. By hypothesis, the proof of Proposition 1.2.29 gives equality in (1.2.40) applied to $(V_\bullet, n_\bullet)$, and in the proof of Proposition 1.2.30 we have seen that (1.2.40) is equivalent to (1.2.42). Therefore, using Lemma 1.2.28, statement 3, we get

$$\sum_{i=1}^{t} n_i \big((r \dim V_i - \operatorname{rk} E_{V_i} P(m)) + \delta(m)(s \operatorname{rk} E_{V_i} - \epsilon_i(E_{V_\bullet}) r) \big) =$$

$$\sum_{i=1}^{t} n_i \big((r \dim V_i - \operatorname{rk} E_{V_i} P(m)) \big) + \delta(m) \mu(\varphi, E_{V_\bullet}, n_\bullet) = 0 \; ,$$

where we also use that $\dim V = P(m)$ and (1.2.7). Statement 1 of Lemma 1.2.28 gives $V_i \subset V_{E_{V_i}} = H^0(E_{V_i}(m))$, hence

$$\sum_{i=1}^{t} n_i\big((rh^0(E_{V_i}(m)) - \operatorname{rk} E_{V_i} P(m))\big) + \delta(m)\mu(\varphi, E_{V_\bullet}, n_\bullet) \geq 0$$

but, by Theorem 1.2.19, condition 2, this must be non-positive, hence $V_i = H^0(E_{V_i}(m)) = V_{E_{V_i}}$, and last inequality is an equality. By Corollary 1.2.24, $E_{V_i} \in \mathcal{S}_0$, and hence $h^0(E_{V_i}(m)) = P_{E_{V_i}}(m)$ for all i. Therefore, as equality holds for m, Lemma 1.2.23 gives the equality of polynomials

$$\sum_{i=1}^{t} n_i\big((rP_{E_{V_i}} - \operatorname{rk} E_{V_i} P)\big) + \delta\mu(\varphi, E_{V_\bullet}, n_\bullet) = 0 \; .$$

Conversely, let $(E_\bullet, n_\bullet)$ be a weighted filtration of E such that (1.2.43) holds, together with a splitting of the filtration $H^0(E_\bullet(m))$ of $V \simeq H^0(E(m))$, and let Γ be the associated 1-parameter subgroup of $SL(V)$. Evaluating the expression (1.2.43) in m gives, in particular,

$$\sum_{i=1}^{t} n_i\big((rP_{E_i}(m) - \operatorname{rk} E_i P(m))\big) + \delta(m)\mu(\varphi, E_\bullet, n_\bullet) =$$

$$\sum_{i=1}^{t} n_i\big((rP_{E_i}(m) - \operatorname{rk} E_i P(m)) + \delta(m)(s \operatorname{rk} E_i - \epsilon_i(E_\bullet) r)\big) = 0 \; ,$$

using (1.2.7). By the proof of implication (3. $\Rightarrow$ 1.) in Theorem 1.2.19, since we get an equality, it is $E_i \in \mathcal{S}_0$ for all i, hence $\dim V_{E_i} = h^0(E_i(m)) = P_{E_i}(m)$ for all i, and the previous equality becomes

$$\sum_{i=1}^{t} n_i\big((r \dim V_{E_i} - \operatorname{rk} E_i P(m)) + \delta(m)(s \operatorname{rk} E_i - \epsilon_i(E_\bullet) r)\big) = 0 \; .$$

Using the strong version of Lemma 1.2.27, $E_{V_{E_i}} = E_i$ and $\epsilon_i(E_\bullet) = \epsilon_i(V_{E_\bullet})$, then

$$\sum_{i=1}^{t} n_i\big((r \dim V_{E_i} - \operatorname{rk} E_{V_{E_i}} \dim V) + \delta(m)(s \operatorname{rk} E_{V_{E_i}} - \epsilon_i(V_{E_\bullet}) r)\big) = 0$$

which is (1.2.42) applied to the weighted filtration $(V_{E_\bullet}, n_\bullet)$ of V and, by the proof of Proposition 1.2.30, equivalent to equality in (1.2.40), therefore $\mu((\overline{Q}, [\Phi]), \Gamma) = 0$.

We have that V_i generates $E_{V_i}(m)$, then take $H^0(E_{V_i}(m)) = V \cap H^0(E_{V_i}(m)) = V_{E_{V_i}}$, which we have seen that is equal to V_i. Conversely, take E_i, then we have that $H^0(E_i(m)) = V \cap H^0(E_i(m)) = V_{E_i}$ and, evaluating, we obtain $E_{V_{E_i}}$, which we have seen that is equal to E_i. Therefore, this gives the bijection. ∎

1.2.5 Proof of Theorem 1.2.9

Proof of the Theorem 1.2.9. We follow [GS1, Proof of Theorem 1.8] which also follows closely [HL2, Proof of Main Theorem 0.1].

Recall notation from section 1.2.3 and Theorem 1.1.11. We will use Theorem 1.2.31, where we show that GIT semistable points correspond to δ-semistable tensors. Let $\mathfrak{M}_\delta$ (respectively $\mathfrak{M}^s_\delta$) be the GIT quotient of Z (respectively Z^s) by $SL(V)$. Since Z is projective, $\mathfrak{M}_\delta$ is also projective and, by Theorem 1.1.11, $\mathfrak{M}^s_\delta$ is an open subset of the projective scheme $\mathfrak{M}_\delta$. The restriction $Z^s \longrightarrow \mathfrak{M}^s_\delta$ to the stable points is a geometric quotient where the fibers are $SL(V)$-orbits, and hence the points of $\mathfrak{M}^s_\delta$ correspond to isomorphism classes of δ-stable tensors. We have to show that $\mathfrak{M}_\delta$ corepresents the functor $\mathcal{M}_\delta$ (c.f. Definition 1.1.3).

Let (E_T, φ_T, u_T, N) be a family of δ-semistable tensors parametrized by a scheme T, as in (1.2.8). Then, $\mathcal{V} := \pi_{T,*}(E_T \otimes \pi_X^*\mathcal{O}_X(m))$ is locally free on T. The family E_T induces a map $\Delta : T \longrightarrow Pic^d(X)$, sending $t \in T$ to $\det E_t$. We can cover T with small open sets T_i such that for each i we can find an isomorphism

$$\beta_{T_i} : \det E_{T_i} \longrightarrow \overline{\Delta_i}^* \mathcal{P} \ ,$$

where $\mathcal{P}$ is the Poincare bundle defined in (1.2.27), and a trivialization

$$g_{T_i} : V \otimes \mathcal{O}_{T_i} \longrightarrow \mathcal{V}|_{T_i} \ ,$$

where note that $\mathcal{V}|_{t\in T_i} \simeq E_{t\in T_i}(m)$ and $H^0(\mathcal{V}|_{t\in T_i}) \simeq H^0(E|_{t\in T_i}(m)) \simeq V$. Using this trivialization we obtain a family of quotients parametrized by T_i,

$$q_{T_i} : V \otimes \pi_X^*\mathcal{O}_X(-m) \twoheadrightarrow E_{T_i} \ ,$$

giving a map $T_i \longrightarrow \mathcal{H}$. And, using the quotient q_{T_i} and the isomorphism β_{T_i}, we have another family of quotients parametrized by T_i,

$$(V^{\otimes s})^{\oplus c} \otimes \left(\pi_{T_i,*}(\overline{\Delta_i}^* \mathcal{P}^{\otimes b} \otimes \overline{u_{T_i}}^* \mathcal{D} \otimes \pi_X^*\mathcal{O}_X(sm))\right)^\vee \twoheadrightarrow N_{T_i} \ ,$$

giving an element of $\mathcal{W}$ for each $t \in T_i$. Then, using the representability properties of $\mathcal{H}$ and $\mathcal{W}$, we obtain a morphism $T_i \longrightarrow \mathcal{H} \times \mathcal{W}$. By Lemma 1.2.26, this morphism factors through Z' and its image is in Z^{ss}, because a δ-semistable tensor gives a GIT-semistable point (c.f. Theorem 1.2.31). Compose with the geometric quotient to $\mathfrak{M}_\delta$ to obtain maps

$$\hat{f}_i : T_i \xrightarrow{f_i} Z^{ss} \longrightarrow \mathfrak{M}_\delta \ .$$

Note that the morphism f_i is independent of the choice of isomorphism β_{T_i}, because of the universal property of the Poincare bundle $\mathcal{P}$. A different choice of isomorphism g_{T_i} will change f_i to $h_i \cdot f_i$, where $h_i : T_i \longrightarrow GL(V)$, then $\hat{f}_i$ is independent of the choice of g_{T_i}. Glue the morphisms $\hat{f}_i$ to give a morphism

$$\hat{f} : T \longrightarrow \mathfrak{M}_\delta \ ,$$

and hence we have a natural transformation from the moduli functor to the functor of points of $\mathfrak{M}_\delta$,

$$\mathcal{M}_\delta \longrightarrow \underline{\mathfrak{M}_\delta} \ .$$

Recall that there is a tautological family (1.2.31) of tensors parametrized by Z'. By restriction to Z^{ss}, we obtain a tautological family of δ-semistable tensors parametrized by Z^{ss}. If Y is another scheme with a natural transformation $\mathcal{M}_\delta \longrightarrow \underline{Y}$, then the tautological family defines a $SL(V)$-invariant morphism $Z^{ss} \longrightarrow Y$, hence this factors through the quotient $\mathfrak{M}_\delta$. Then, the natural transformation $\mathcal{M}_\delta \longrightarrow \underline{Y}$ factors through $\underline{\mathfrak{M}_\delta}$ and this proves that $\mathfrak{M}_\delta$ corepresents the functor $\mathcal{M}_\delta$. ■

Remark 1.2.34. *Note that this is not a fine moduli space because the analog of the uniqueness result of [HL2, Lemma 1.6] does not hold in general for tensors.*

Now let us give a criterion for S-equivalence. If (E, φ, u) and (F, ψ, u) are two δ-stable tensors then we have seen that they correspond to the same point in the moduli space if and only if they are isomorphic. But if they are strictly δ-semistable (i.e. δ-semistable but not δ-stable), they can be S-equivalent (i.e. they correspond to the same point in the moduli space), but not isomorphic. Hence, given a tensor (E, φ, u), we will show that there exists a canonical representative of its S-equivalence class (E^S, φ^S, u), such that two tensors (E, φ, u) and (F, ψ, u) will be S-equivalent if and only if (E^S, φ^S, u) (F^S, ψ^S, u) are isomorphic.

Let (E, φ, u) be a strictly δ-semistable. Then, by Proposition 1.2.31, the corresponding point $(\overline{Q}, [\Phi])$ is strictly GIT semistable, by Theorem 1.2.29 there exists at least one 1-parameter subgroup Γ of $SL(V)$ with $\mu((\overline{Q}, [\Phi]), \Gamma) = 0$ and, by Proposition 1.2.33, Γ corresponds to a weighted filtration $(E_\bullet, n_\bullet)$ with

$$\sum_{i=1}^{t} n_i\big((rP_{E_i} - \operatorname{rk} E_i P)\big) + \delta\mu(\varphi, E_\bullet, n_\bullet) = 0 \ ,$$

which we will call an **admissible** weighted filtration for a strictly δ-semistable tensor.

Let $\mathcal{I}_0$ be the set of pairs (k, I) where k is an integer with $1 \leq k \leq c$, and $I = (i_1, ..., i_s)$ is a multi-index with $1 \leq i_j \leq t+1$, such that the restriction of φ

$$\varphi_{k,I} : \overbrace{0 \oplus \cdots \oplus 0}^{k-1} \oplus (E_{i_1} \otimes \cdots \otimes E_{i_s}) \oplus \overbrace{0 \oplus \cdots \oplus 0}^{c-k} \longrightarrow (\det E)^{\otimes b} \otimes D_u$$

is nonzero and

$$\gamma_{r_{i_1}} + \cdots + \gamma_{r_{i_s}} = \mu(\varphi, E_\bullet, n_\bullet) \ .$$

Note that, if $(k, I) \in \mathcal{I}_0$ and $I' = (i'_1, ..., i'_s)$ is a multi-index with $I' \neq I$ and $i'_j \leq i_j$ for all j, then $\varphi_{k,I'} = 0$, by definition of $\mu(\varphi, E_\bullet, n_\bullet)$. Hence, if $(k, I) \in \mathcal{I}_0$, the restriction $\varphi_{k,I}$ defines a homomorphism in the quotient

$$\varphi'_{k,I} : \overbrace{0 \oplus \cdots \oplus 0}^{k-1} \oplus (E'_{i_1} \otimes \cdots \otimes E'_{i_s}) \oplus \overbrace{0 \oplus \cdots \oplus 0}^{c-k} \longrightarrow (\det E)^{\otimes b} \otimes D_u \ ,$$

where $E'_i = E_i / E_{i+1}$. If (k, I) is not in $\mathcal{I}_0$, then define $\varphi'_{k,I} := 0$. Therefore, we can define

$$\big(E' = E'_1 \oplus \ldots \oplus E'_{t+1} \ , \ \varphi' = \bigoplus_{(k,I)} \varphi'_{k,I}\big)$$

in which we are using that $\det E \simeq \det E'$, hence (E', φ', u) is well-defined up to isomorphism and we call it the **admissible deformation** associated to the admissible filtration $(E_\bullet, n_\bullet)$ of E. Observe that this notion depends on the weighted filtration chosen.

Proposition 1.2.35. *[GS1, Proposition 4.1] The tensor (E', φ', u) is strictly δ-semistable and it is S-equivalent to (E, φ, u). If we repeat this process, after a finite number of iterations, the process will stop, i.e. we will obtain tensors isomorphic to each other. We call this tensor (E^S, φ^S, u) and it verifies*

1. *The isomorphism class of (E^S, φ^S, u) is independent of the choices made, i.e. the weighted filtrations chosen.*

2. *Two tensors (E, φ, u) and (F, ψ, u) are S-equivalent if and only if (E^S, φ^S, u) is isomorphic to (F^S, ψ^S, u).*

Proof. First, we recall some observations about GIT quotients. Let Z be a projective variety with an action of a group G linearized on an ample line bundle $\mathcal{O}_Z(1)$. Two points in the open subset Z^{ss} of semistable points are GIT equivalent, or they give the same point in the moduli space, if the closures (in Z^{ss}) of their orbits do intersect (c.f. Remark

1.1.16). Let $z \in Z^{ss}$ and let $B(z)$ be the unique closed orbit in the closure $\overline{G\cdot z}$ in Z^{ss} of its orbit $G\cdot z$. If z is not in $B(z)$, there exists a 1-parameter subgroup Γ such that the limit $z_0=\lim_{t\to 0}\Gamma(t)\cdot z$ is in $\overline{G\cdot z}\backslash G\cdot z$ (for instance, we can take the 1-parameter subgroup given by [Si1, Lemma 1.25]). Note that it is $\mu(z,\Gamma)=0$ (by semistability of z, $\mu(z,\Gamma)\le 0$, and if it were negative, then we would have $G\cdot z = B(z)$). Conversely, if Γ is a 1-parameter subgroup with $\mu(z,\Gamma)=0$, then the limit, z_0, is GIT semistable ([GS0, Proposition 2.14]). Observe that $G\cdot z_0\subset \overline{G\cdot z}\backslash G\cdot z$, therefore, $\dim G\cdot z_0 < \dim G\cdot z$. Repeating the process with z_0 instead of z, we get a sequence of points which stops after a finite number of steps, and gives $\tilde{z}\in B(z)$. Two points z_1 and z_2 in Z^{ss} are S-equivalent if and only if $B(z_1)=B(z_2)$.

Let (E,φ,u) be a δ-semistable tensor with an isomorphism $f: V\simeq H^0(E(m))$, and let $z=(\overline{Q},[\Phi])\in Z$ be the corresponding GIT semistable point. Recall from Proposition 1.2.33 the bijection between 1-parameter subgroups Γ of $SL(V)$ with $\mu(z,\Gamma)=0$ on the one hand, and weighted filtrations $(E_\bullet,n_\bullet)$ of E with

$$\Big(\sum_{i=1}^{t} n_i(rP_{E_i}-\operatorname{rk}E_iP)\Big)+\delta\mu(\varphi,E_\bullet,n_\bullet)\Big)=0$$

together with a splitting of the filtration $H^0(E_\bullet(m))$ of $V=H^0(E(m))$ on the other hand. A 1-parameter subgroup Γ acting on z defines a morphism $\mathbb{C}^*\to Z$ which extends to

$$h:\mathbb{C}\longrightarrow Z\ ,$$

with $h(t)=\Gamma(t)\cdot z$ for $t\neq 0$ and whose limit is $h(0)=\lim_{t\to 0}\Gamma(t)\cdot z=z_0$.

If we pull back by h the universal family parametrized by Z, we obtain another family (q_T,E_T,φ_T,u), where

$$E_T=\bigoplus_{i=1}^{t+1}E^i\otimes t^{\gamma_{r_i}}\subset E\otimes_{\mathbb{C}} t^{-N}\mathbb{C}[t]\subset E\otimes_{\mathbb{C}}\mathbb{C}[t,t^{-1}]\ ,$$

where recall that $t^{\Gamma_{r_i}}$ acts on each E^i. We get the morphisms,

$$\begin{array}{ccccc} q_T: V\otimes\mathcal{O}_X(-m)\otimes\mathbb{C}[t] & \xrightarrow{\ \xi\ } & \oplus_i V^i\otimes\mathcal{O}_X(-m)\otimes t^{\Gamma_{r_i}} & \longrightarrow & E_T \\ v^i\otimes 1 & \longmapsto & v^i\otimes t^{\Gamma_{r_i}} & \longmapsto & q(v^i)\otimes t^{\Gamma_{r_i}} \end{array}$$

and

$$\varphi_T : (E_T^{\otimes s})^{\oplus c} \longrightarrow (\det E_T)^{\otimes b} \otimes \overline{u_T}^* \mathcal{D} \otimes \pi_T^* N$$

$$(\underbrace{0,\ldots,0}_{k-1}, w_{i_1} t^{\Gamma_{r_{i_1}}} \cdot \ldots \cdot w_{i_s} t^{\Gamma_{r_{i_s}}}, \underbrace{0,\ldots,0}_{c-k}) \longmapsto \varphi(\underbrace{0,\ldots,0}_{k-1}, w_{i_1} \cdot \ldots \cdot w_{i_s}, \underbrace{0,\ldots,0}_{c-k}) \otimes t^{\Gamma_{r_{i_1}}+\cdots+\Gamma_{r_{i_s}}}$$

Then, (q_t, E_t, φ_t, u) corresponds to $h(t)$ (in particular, if $t \neq 0$, then (E_t, φ_t, u) is canonically isomorphic to (E, φ, u)), and (E_0, φ_0, u) is the admissible deformation associated to $(E_\bullet, n_\bullet)$. Note that 1 follows from the universality of the construction and 2 follows from the previous discussion. ■

1.3 The Harder-Narasimhan filtration

In section 1.2 we have constructed a moduli space for tensors, by restricting the class of objects that we classify, the δ-semistable tensors. This is the usual situation when constructing a moduli space, to restrict the original moduli problem by introducing a stability condition.

In some sense, the construction of a moduli space answers the classification problem for the class of the semistable objects. For the rest, the unstable objects, there is a main tool in algebraic geometry, called the Harder-Narasimhan filtration, to study them.

We will recall the original Harder-Narasimhan filtration for vector bundles and torsion free sheaves in this section. At the end, we will discuss the abstract generalization of this notion for an abelian category.

1.3.1 Harder-Narasimhan filtration for sheaves

We consider, first, the case of vector bundles over curves. Let E be a holomorphic vector bundle over a smooth projective complex curve X. Let

$$\mu(E) := \frac{\deg E}{\operatorname{rk} E}$$

be its **slope**.

Definition 1.3.1. *E is **semistable** if for every proper holomorphic subbundle $F \subset E$, it is $\mu(F) \leq \mu(E)$. If the inequality is strict for every proper subbundle we say that E is*

stable. *A holomorphic vector bundle is* ***unstable*** *if it exists a proper subbundle verifying* $\mu(F) > \mu(E)$.

The holomorphic vector bundles of fixed rank r and degree d over an algebraic curve X of genus g were studied by Grothendieck for $g = 0$ and Atiyah for $g = 1$. With the previous definition of stability, Narasimhan and Seshadri constructed a moduli space for holomorphic bundles over algebraic curves of genus g. They did it, first for bundles of degree 0 (c.f. [NS1]) and later in general (c.f. [NS2]), through representations of the fundamental group, and putting in correspondence the semistable bundles with the semistable points, in the sense of GIT, defined by Mumford. Then, Gieseker (c.f. [Gi1]) gave an algebraic construction for a moduli space of torsion free sheaves over an algebraic surface and Maruyama (c.f. [Ma1]) extended the construction to higher dimensional varieties.

As we announced in Section 1.1, we impose a condition on the objects we are trying to classify, the notion of stability, and restrict our classification problem to the semistable objects. What we have to do then is, in all the moduli problems which arise as GIT quotients of a space by the action of a group, to show that the semistable objects, with respect to the definition of stability we give from the beginning, correspond to the semistable orbits in the sense of Geometric Invariant Theory. Therefore, we obtain a moduli space for the class of semistable objects which is a good quotient (c.f. Definition 1.1.7) where each point corresponds to an S-equivalence class of semistable objects (c.f. Remark 1.1.16). If we restrict the moduli problem to the stable objects, we get a geometrical quotient (c.f. Definition 1.1.8) which is an orbit space where, indeed, each point corresponds to a stable orbit and represents an isomorphism class of stable objects.

Harder and Narasimhan prove the existence of a canonical filtration for a holomorphic vector bundle over a smooth algebraic curve (c.f. [HN]). The construction of the filtration is based on the existence of a unique subbundle which maximally contradicts the stability in Definition 1.3.1 (in [HN, Proposition 1.3.4] this subbundle is called SCSS, a subbundle which *"strongly contradicts the semistability"*), taking the quotient by this subbundle and repeating the process by recursion (c.f. [HN, Lemma 1.3.7]). In that article, Harder and Narasimhan use the filtration to decompose an unstable vector bundle in blocks and calculate some numbers in relation with the cohomology groups of the moduli space. Within the years, the so-called **Harder-Narasimhan filtration** has been proved to be extremely useful in the study of properties of moduli spaces in algebraic geometry.

Let us show how to construct the Harder-Narasimhan filtration in an easy case, where

E is a holomorphic vector bundle of rank r and degree d over a smooth projective complex curve X of genus g.

Suppose that E is unstable and let $\mu(E) = \frac{d}{r}$ be its slope. By definition of stability there are subbundles E' of rank $r' < r$ and degree d, $0 \subsetneqq E' \subsetneqq E$ such that $\mu(E') = \frac{d'}{r'} > \mu(E) = \frac{d}{r}$. We choose E_1 with $\mu(E_1) > \mu(E)$ to be maximal and of maximal rank among those of maximal slope (i.e. if $\exists E_1'$ with $\mu(E_1') = \mu(E_1)$, then $E_1' \subseteq E_1$). We will call E_1 **the maximal destabilizing subbundle of** E. Now we consider the subbundle $F = E/E_1$. If it is semistable we are done, and the Harder-Narasimhan filtration is $0 \subsetneqq E_1 \subsetneqq E$. If not, in analogy with the previous case, there exists $0 \subsetneqq F_1 \subsetneqq F$, of maximal slope and of maximal rank among those of maximal slope, hence we have

$$\begin{array}{ccccccc} 0 & \subsetneqq & E_1 & \subsetneqq & E_2 & \subsetneqq & E \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \subsetneqq & E_2/E_1 = F_1 & \subsetneqq & E/E_1 = F \end{array}$$

Call r_1, r_2, d_1, d_2 the ranks and degrees of E_1 and E_2 respectively. These two properties hold:

- The quotient E_2/E_1 is semistable. Indeed, if $E_2/E_1 = F_1$ were not semistable, there would exists $0 \subsetneqq F_2 \subsetneqq F_1$ with $\mu(F_2) > \mu(F_1)$, contradicting the choice of F_1.

- It is $\mu(E_1/0) = \mu(E_1) > \mu(E_2/E_1)$, because if we had $\mu(E_1) \leq \mu(E_2/E_1) \iff \frac{d_1}{r_1} \leq \frac{d_2-d_1}{r_2-r_1} \iff d_1r_2 - d_1r_1 \leq d_2r_1 - d_1r_1 \iff \frac{d_1}{r_1} \leq \frac{d_2}{r_2} \iff \mu(E_1) \leq \mu(E_2)$, and we have chosen E_1 of maximal slope among the subbundles of E and $E_1 \subsetneqq E_2$.

Repeating the process, if the quotient $G = E/E_2$ is not semistable, we can choose $0 \subsetneqq G_1 \subsetneqq G$ with maximal slope and rank, and we obtain

$$\begin{array}{ccccccccc} 0 & \subsetneqq & E_1 & \subsetneqq & E_2 & \subsetneqq & E_3 & \subsetneqq & E \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \subsetneqq & E_2/E_1 = F_1 & \subsetneqq & E_3/E_1 = F_2 & \subsetneqq & E/E_1 = F \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & 0 = F_1/E_2 & \subsetneqq & E_3/E_2 = G_1 & \subsetneqq & E/E_2 = G \end{array} .$$

By analogy, we get that $F_2/F_1 = \frac{E_3/E_1}{E_2/E_1} \simeq E_3/E_2$ is semistable and

$$\mu(F_1) > \mu(G_1) \iff \mu(E_2/E_1) > \mu(E_3/E_2) .$$

By iterating until we get a semistable quotient E/E_t, we obtain the Harder-Narasimhan filtration:

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E$$

which verifies

- $\mu(E^1) > \mu(E^2) > \mu(E^3) > ... > \mu(E^t) > \mu(E^{t+1}) = \mu$, where $\mu(E^i) = \frac{\deg E^i}{\operatorname{rk} E^i}$
- $E^i := E_i/E_{i-1}$ is semistable, $\forall i \in \{1, ..., t+1\}$ where $E_0 = 0$

And the process has to stop by finiteness of the rank of E.

Therefore, note that we can exhibit unstable vector bundles as extensions of semistable ones in this way. Given an unstable vector bundle we have its Harder-Narasimhan filtration

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E\ .$$

This breaks into short exact sequences

$$\begin{array}{ccccccccc}
0 & \to & \underset{\textit{semistable}}{E_1} & \to & E_2 & \to & \underset{\textit{semistable}}{E_2/E_1} & \to & 0 \\
0 & \to & E_2 & \to & E_3 & \to & \underset{\textit{semistable}}{E_3/E_2} & \to & 0 \\
 & & & & \cdots & & & & \\
0 & \to & E_t & \to & E & \to & \underset{\textit{semistable}}{E/E_t} & \to & 0
\end{array} ,$$

where vector bundles on the right are semistable. Using the Harder-Narasimhan filtration we can think of semistable bundles as building blocks for holomorphic vector bundles.

Now we give the definition and the proof of the existence and uniqueness of the Harder-Narasimhan filtration for torsion free sheaves over smooth projective varieties.

Let X be a smooth projective variety and fix an ample line bundle $\mathcal{O}_X(1)$. For every coherent sheaf over X, E, let P_E its Hilbert polynomial with respect to $\mathcal{O}_X(1)$, i.e $P_E(m) = \chi(E \otimes \mathcal{O}_X(m))$. If P and Q are polynomials, we write $P \leq Q$ if $P(m) \leq Q(m)$ for $m \gg 0$.

Definition 1.3.2. *[Gi1, Definition 0.1] Let E be a torsion free sheaf over X. We say that E is* ***semistable*** *if for all proper subsheaves $F \subset E$, it is*

$$\frac{P_F}{\operatorname{rk} F} \leq \frac{P_E}{\operatorname{rk} E}\ .$$

If the inequality is strict for every proper subsheaf we say that E is ***stable****.*

Note that, if E is a holomorphic vector bundle of rank r and degree d over an algebraic curve X of genus g, the Hilbert polynomial of E is $P_E(m) = rm + d + r(1-g)$ and Definition 1.3.2 is equivalent to Definition 1.3.1. We often refer to Definition 1.3.2 as **Gieseker or Maruyama stability**, whereas Definition 1.3.1 is usually called **Mumford, Takemoto or slope stability**, both definitions coinciding for curves.

Definition 1.3.3. *Let E be a torsion free sheaf of rank r over a smooth projective algebraic variety X. A* ***Harder-Narasimhan filtration*** *for E is a sequence*

$$0 \subset E_1 \subset E_2 \subset \ldots \subset E_t \subset E_{t+1} = E$$

verifying

- *The Hilbert polynomials verify*

$$\frac{P_{E^1}}{\operatorname{rk} E^1} > \frac{P_{E^2}}{\operatorname{rk} E^2} > \ldots > \frac{P_{E^{t+1}}}{\operatorname{rk} E^{t+1}}$$

- *Every E^i is semistable*

where $E^i := E_i/E_{i-1}$.

Remark 1.3.4. *Note that the Harder-Narasimhan filtration with Gieseker stability is a refinement of the one with Mumford stability, with the inequalities holding between the Hilbert polynomials in one case, or their leading coefficients in the other.*

A sheaf E is **pure of dimension** n if its support has dimension n and it has no subsheaves supported on a locus of lower dimension.

Theorem 1.3.5. *[HN, Proposition 1.3.9], [HL3, Theorem 1.3.4] Every pure sheaf E of dimension n over a smooth projective variety X has a unique Harder-Narasimhan filtration.*

Lemma 1.3.6. *Let E be a torsion free sheaf. Then, there exists a subsheaf $F \subset E$ such that for all subsheaves $G \subset E$, one has $\frac{P_F}{\operatorname{rk} F} \geq \frac{P_G}{\operatorname{rk} G}$ and, in case of equality $G \subset F$. Moreover, F is uniquely determined and F is semistable, called the* ***maximal destabilizing subsheaf of*** *E.*

Proof. Note that F to be semistable and uniquely determined follows from the first property.

Define an order relation on the set of subsheaves of E by $F_1 \leq F_2$ if and only if $F_1 \subset F_2$ and $\frac{P_{F_1}}{\operatorname{rk} F_1} \leq \frac{P_{F_2}}{\operatorname{rk} F_2}$. Every ascending chain is bounded by E, then by Zorn's Lemma, for every subsheaf F there exists $F \subset F' \subset E$ such that F' is maximal with respect to $\leq$. Let F be $\leq$-maximal with F of minimal rank among all maximal subsheaves of E. Let us show that F has the required properties.

Suppose there exists $G \subset E$ with $\frac{P_G}{\operatorname{rk} G} \geq \frac{P_F}{\operatorname{rk} F}$. First, note that we can assume $G \subset F$ by replacing G by $F \cap G$. Suppose that $G \nsubseteq F$, then F is a proper subsheaf of $F+G$ and hence $\frac{P_F}{\operatorname{rk} F} > \frac{P_{F+G}}{\operatorname{rk} F+G}$, by definition of F. From the sequence

$$0 \to F \cap G \to F \oplus G \to F + G \to 0$$

we get

$$P_F + P_G = P_{F \oplus G} = P_{F \cap G} + P_{F+G}$$

and

$$\operatorname{rk} F + \operatorname{rk} G = \operatorname{rk}(F \oplus G) = \operatorname{rk}(F \cap G) + \operatorname{rk}(F+G) \ .$$

Calculating we have

$$\operatorname{rk}(F \cap G)\left(\frac{P_G}{\operatorname{rk} G} - \frac{P_{F \cap G}}{\operatorname{rk}(F \cap G)}\right) =$$

$$\operatorname{rk}(F+G)\left(\frac{P_{F+G}}{\operatorname{rk}(F+G)} - \frac{P_F}{\operatorname{rk} F}\right) + (\operatorname{rk} G - \operatorname{rk}(F \cap G))\left(\frac{P_F}{\operatorname{rk} F} - \frac{P_G}{\operatorname{rk} G}\right) \ .$$

Then, together with the two inequalities $\frac{P_F}{\operatorname{rk} F} \leq \frac{P_G}{\operatorname{rk} G}$ and $\frac{P_F}{\operatorname{rk} F} > \frac{P_{F+G}}{\operatorname{rk}(F+G)}$ we obtain

$$\frac{P_G}{\operatorname{rk} G} - \frac{P_{F \cap G}}{\operatorname{rk}(F \cap G)} < 0$$

and hence

$$\frac{P_F}{\operatorname{rk} F} < \frac{P_{F \cap G}}{\operatorname{rk}(F \cap G)} \ ,$$

which proves the assert that we can suppose $G \subset F$.

Now, let $G \subset F$ with $\frac{P_G}{\operatorname{rk} G} > \frac{P_F}{\operatorname{rk} F}$ such that G is $\leq$-maximal in F. Then let $G' \geq G$, $\leq$-maximal in E. We obtain the inequalities $\frac{P_F}{\operatorname{rk} F} < \frac{P_G}{\operatorname{rk} G} \leq \frac{P_{G'}}{\operatorname{rk} G'}$. Because of the maximality of G' and F it is $G' \nsubseteq F$, because otherwise $\operatorname{rk}(G') < \operatorname{rk}(F)$ but $\operatorname{rk}(F)$ is minimal by hypothesis. Therefore, F is a proper subsheaf of $F+G'$ and $\frac{P_F}{\operatorname{rk} F} > \frac{P_{F+G'}}{\operatorname{rk}(F+G')}$. The previous inequalities $\frac{P_F}{\operatorname{rk} F} < \frac{P_{G'}}{\operatorname{rk} G'}$ and $\frac{P_F}{\operatorname{rk} F} > \frac{P_{F+G'}}{\operatorname{rk}(F+G')}$ give

$$\frac{P_{F \cap G'}}{\operatorname{rk}(F \cap G')} > \frac{P_{G'}}{\operatorname{rk} G'} \geq \frac{P_G}{\operatorname{rk} G} \ .$$

Given that $G \subset F \cap G' \subset F$, we get a contradiction with the hypothesis on G. ■

Proof of the Theorem. Lemma 1.3.6 shows the existence of a Harder-Narasimhan filtration for E. Let E_1 the maximal destabilizing subsheaf and suppose that the corresponding quotient E/E_1 has a Harder-Narasimhan filtration,

$$0 \subset G_0 \subset G_1 \subset \ldots \subset G_{t-1} = E/E_1 \ ,$$

by induction hypothesis. We define E_{i+1} as the pre-image of G_i and it is $\frac{P_{E_1}}{\operatorname{rk} E_1} > \frac{P_{E_2/E_1}}{\operatorname{rk} E_2/E_1}$ because, if not, we get $\frac{P_{E_1}}{\operatorname{rk} E_1} \leq \frac{P_{E_2}}{\operatorname{rk} E_2}$, which contradicts the maximality of E_1.

Next we prove the uniqueness. Let $E_\bullet$ and $E'_\bullet$ be two Harder-Narasimhan filtrations of the same sheaf E. We consider, without loss of generality, $\frac{P_{E'_1}}{\operatorname{rk} E'_1} \geq \frac{P_{E_1}}{\operatorname{rk} E_1}$. Let j be the minimal index verifying $E'_1 \subset E_j$. The composition

$$E'_1 \to E_j \to E_j/E_{j-1}$$

is a non-trivial homomorphism of semistable sheaves which implies

$$\frac{P_{E_j/E_{j-1}}}{\operatorname{rk} E_j/E_{j-1}} \geq \frac{P_{E'_1}}{\operatorname{rk} E'_1} \geq \frac{P_{E_1}}{\operatorname{rk} E_1} \geq \frac{P_{E_j/E_{j-1}}}{\operatorname{rk} E_j/E_{j-1}}$$

where first inequality comes from the fact that, if there exists a non-trivial homomorphism between semistable sheaves, then the Hilbert polynomial of the target is greater or equal than the one of the first sheaf. Therefore, equality holds everywhere, and this implies that the index j is equal to 1, so that $E'_1 \subset E_1$. Then, by semistability of E_1, it is $\frac{P_{E'_1}}{\operatorname{rk} E'_1} \leq \frac{P_{E_1}}{\operatorname{rk} E_1}$, and we can repeat the argument interchanging the roles of E_1 and E'_1 to show that $E_1 = E'_1$. By induction we can assume that uniqueness holds for the Harder-Narasimhan filtration of E/E_1 to show that $E'_i/E_1 = E_i/E_1$, which completes the proof. ■

Remark 1.3.7. *If a torsion free sheaf E is already semistable, we can still talk about its Harder-Narasimhan filtration which is the trivial filtration $0 \subset E$.*

Next we show how the Harder-Narasimhan filtration looks like in the easiest case, for an unstable vector bundle over $X = \mathbb{P}^1_{\mathbb{C}}$.

Example 1.3.8. *Let $X = \mathbb{P}^1_{\mathbb{C}}$. We know, by a theorem of Grothendieck, that a vector bundle E over $\mathbb{P}^1_{\mathbb{C}}$ splits on line bundles*

$$E = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_3) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_s)$$

with $a_1 > a_2 > ... > a_s$*, and we call* b_i *the number of times each line bundle* $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_i)$ *appears (c.f. [HL3, Theorem 1.3.1]). Thus, the slope of* E *is the average of the degrees* a_i *of the line bundles appearing in the decomposition of* E*,*

$$\mu(E) = \frac{\deg E}{rk\ E} = \frac{a_1 b_1 + \cdots + a_s b_s}{b_1 + \cdots + b_s} \ .$$

With the notation of Theorem 1.3.5, it is clear that

$$E^1 = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1)$$

with

$$\mu(E^1) = \frac{\overbrace{a_1 + \cdots + a_1}^{b_1\ times}}{b_1} = a_1 > \mu(E)$$

and

$$F = E/E_1 = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_3) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_s) \ .$$

Then it is also

$$E^2 = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_2)$$

which lifts to

$$E_2 = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_2) \ .$$

Repeating the process we obtain a filtration

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_{s-1} \subset E_s = E$$

where

$$E_i = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_i) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(a_i)$$

which is the Harder-Narasimhan filtration. It is clear that each $E^i = E_i/E_{i-1}$ *is semistable and* $\mu(E^1) > \mu(E^2) > \mu(E^3) > ... > \mu(E^{s-1}) > \mu(E^s)$*.*

1.3.2 Harder-Narasimhan filtration in an abelian category

Finally, we would like to close this section with some comments about stability notions and the concept of the Harder-Narasimhan filtration in a more general context. Rudakov defines in [Ru] a notion of stability for objects in an abelian category.

Let $\mathcal{C}$ be an abelian category. We define a **preorder** on the objects making possible to compare two nonzero objects, i.e. if $A \neq 0$, $B \neq 0$ are objects of $\mathcal{C}$ it is one of the following $A \prec B$, $A \succ B$ or $A \asymp B$, being possible to have $A \asymp B$ although $A \neq B$.

Definition 1.3.9. *[Ru, Definition 1.1] A **stability structure** on $\mathcal{C}$ is a preorder on $\mathcal{C}$ such that for every short exact sequence $0 \to A \to B \to C \to 0$ it happens one of the following*

- $A \prec B \Leftrightarrow A \prec C \Leftrightarrow B \prec C$
- $A \succ B \Leftrightarrow A \succ C \Leftrightarrow B \succ C$
- $A \asymp B \Leftrightarrow A \asymp C \Leftrightarrow B \asymp C$

Note that this property is satisfied by the category of holomorphic vector bundles over curves, if we associate to each object E the numerical function given by its slope

$$\mu(E) = \frac{\deg E}{\operatorname{rk} E}$$

and define the preorder as

$$E \prec F \Leftrightarrow \mu(E) < \mu(F) \quad E \asymp F \Leftrightarrow \mu(E) = \mu(F) \ .$$

Also, for torsion free sheaves over projective varieties, if we associate to each sheaf E the polynomial function given by $\frac{P_E}{\operatorname{rk} E}$, where P_E is the Hilbert polynomial of E, we have a stability structure in the corresponding category by defining the preorder with the obvious relations between the polynomial functions.

Definition 1.3.10. *An object $A \in \mathcal{C}$ is **semistable** if it is nonzero and for every nontrivial subobject $B \subset A$, we have $B \preccurlyeq A$. We say that A is **stable** if we have a strict inequality for every nontrivial subobject.*

Let us mention three properties for an abelian category $\mathcal{C}$ to have, in order to assure that a Harder-Narasimhan filtration exists for an unstable object in $\mathcal{C}$ (these properties appear in [Ru]).

Definition 1.3.11. *An object $A \in \mathcal{C}$ is **quasi-noetherian** if a chain verifying*

$$A_1 \subset A_2 \subset \ldots \subset A$$

and

$$A_1 \preccurlyeq A_2 \preccurlyeq \ldots \preccurlyeq A$$

stabilizes. We say that $\mathcal{C}$ is quasi-noetherian if every $A \in \mathcal{C}$ is.

Definition 1.3.12. *An object $A \in \mathcal{C}$ is* ***weakly-noetherian*** *if it is quasi-noetherian and a chain verifying*

$$A_1 \supset A_2 \supset \ldots \supset A$$

and

$$A_1 \succcurlyeq A_2 \succcurlyeq \ldots \succcurlyeq A$$

stabilizes. We say that $\mathcal{C}$ is weakly-noetherian if every $A \in \mathcal{C}$ is.

Definition 1.3.13. *An object $A \in \mathcal{C}$ is* ***weakly-artinian*** *if a chain verifying*

$$A_1 \supset A_2 \supset \ldots \supset A$$

and

$$A_1 \preccurlyeq A_2 \preccurlyeq \ldots \preccurlyeq A$$

stabilizes. We say that $\mathcal{C}$ is weakly-artinian if every $A \in \mathcal{C}$ is.

Theorem 1.3.14. *[Ru, Theorem 2] Let $\mathcal{C}$ be an abelian category with a given stability structure, which is weakly-noetherian and weakly-artinian. For every object $A \in \mathcal{C}$ there exists a filtration*

$$0 \subset A_1 \subset A_2 \subset \ldots \subset A_t \subset A_{t+1} = A$$

such that

- $A^1 \succ A^2 \succ \ldots \succ A^t \succ A^{t+1}$
- *Every A^i is semistable*

where $A^i := A_i/A_{i-1}$.

For an object to be quasi-noetherian it is needed to prove the existence and uniqueness of a maximally destabilizing subobject (c.f. Lemma 1.3.6) and the stronger weakly-noetherian is used when lifting the filtration of the quotient A/A_1, which exists by hypothesis in the recursion (c.f. Proof of Theorem 1.3.5). The condition of being weakly-artinian assures that the recursive process when constructing the Harder-Narasimhan filtration of an object finishes in a finite number of steps, i.e. the Harder-Narasimhan filtration is finite.

An object is called **noetherian** if every ascending chain on it stabilizes. Clearly, being noetherian implies being weakly-noetherian.

Note that the category of coherent sheaves over a projective variety is abelian and noetherian, as well as the category of finite dimensional representations of quivers (which we will see in Chapter 3), hence the existence of a Harder-Narasimhan filtration in these cases can be seen as a particular case of Theorem 1.3.14.

1.4 Kempf theorem

In the previous sections we have studied moduli problems for which we have to impose a stability condition in order to have a moduli space with good properties. By *rigidifying the data*, we add extra data to the objects we are classifying, and this leads us to an action of a group in a space which changes the extra data, for a given object in the moduli problem. Then, we use Geometric Invariant Theory to take the quotient by the group and obtain a moduli space with the desired properties.

In the example of the construction of a moduli space for tensors (c.f. section 1.2), the extra data we add to a tensor is the isomorphism between a vector space and a space of global sections of the (twisted) sheaf of the tensor. Different isomorphisms differ by an element of a general linear group, and this is the group we take the quotient by, using GIT (c.f. subsection 1.2.3).

In this kind of constructions, one of the main points appears to be the correspondence between semistable objects and semistable orbits or GIT semistable points (c.f. Theorem 1.2.31). Recall that GIT stability can be checked by 1-parameter subgroups (c.f. Hilbert-Mumford criterion, Theorem 1.1.14): a point x is unstable if there exists any 1-parameter subgroup Γ which makes some numerical function, the so-called *minimal relevant weight* $\mu(x,\Gamma)$, positive.

The GIT stability criterion exposed in Theorem 1.1.13 asserts that a point x is GIT unstable if there exists a 1-parameter subgroup Γ such that

$$\lim_{t\to 0} \Gamma(t)\cdot \tilde{x} = 0 \ ,$$

where $\tilde{x}$ is a point in the affine cone, lying over x. This is, the Hilbert-Mumford criterion says that the fact of 0 appearing as the limit point in the orbit of the linearized action of the group G can be checked through 1-parameter subgroups. Then, Theorem 1.1.13, the numerical criterion, expresses that fact with the positivity of the numerical function $\mu(x,\Gamma)$.

The function $\mu(x,\Gamma)$ can be thought as a measure of how rapidly we can reach 0 from a point $\tilde{x}$ in the affine cone, lying over x, through different 1-parameter subgroups. Let us see this with an easy example.

Example 1.4.1. *Consider the group $G = SL(3,\mathbb{C})$ and let $\Gamma : \mathbb{C}^* \to SL(3,\mathbb{C})$ be a 1-parameter subgroup. There exists a basis of $\mathbb{C}^3$ where Γ takes the diagonal form*

$$\begin{pmatrix} t^{\Gamma_1} & 0 & 0 \\ 0 & t^{\Gamma_1} & 0 \\ 0 & 0 & t^{\Gamma_3} \end{pmatrix} ,$$

where we order the exponents as $\Gamma_1 < \Gamma_2 < \Gamma_3$ and they verify $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$. Now consider that it acts on $\mathbb{P}^3_{\mathbb{C}}$ and let $x = [0 : x_2 : x_3]$ be a point in homogeneous coordinates, $x_2 \neq 0, x_3 \neq 0$. Let $\tilde{x} = (0, x_2, x_3)$ be a point in the affine cone lying over x. Then, $\lim_{t\to 0}\Gamma(t)\cdot\tilde{x} = \lim_{t\to 0}(0, t^{\Gamma_2}\cdot x_2, t^{\Gamma_3}\cdot x_3)$ and we say that Γ acts on the limit with weight Γ_2, the minimal relevant exponent of the action of Γ over x, i.e. $\mu(x,\Gamma) = \Gamma_2$. A point is GIT unstable if 0 can be reached in the closure of the orbit of the linearized action through 1-parameter subgroups. And $\lim_{t\to 0}\Gamma(t)\cdot\tilde{x} = 0$ if and only if $\Gamma_2 > 0$. Hence, x is GIT unstable if there exists any Γ with $\Gamma_2 > 0$.

Observe that the specific value Γ_2 can be thought as a measure of how rapidly we can move from $\tilde{x} = (0, x_2, 0)$ to 0. The greater Γ_2 is, the faster $\lim_{t\to 0}\Gamma(t)\cdot\tilde{x}$ takes $\tilde{x}$ to 0. Hence, $\mu(x,\Gamma)$ encodes, in this sense, the speed of unstability.

A first question which arises is, could we possibly find a 1-parameter subgroup giving the greatest *speed of unstability* as in Example 1.4.1?

The answer would be: not yet. Note that if we multiply the exponents appearing in the diagonal of Γ by the same integer, we still obtain a 1-parameter subgroup of $SL(3,\mathbb{C})$ giving a positive value for $\mu(x,\Gamma)$, hence it also destabilizes the point x. But the value $\mu(x,\Gamma)$ is multiplied by this integer, hence we cannot yet well define a unique 1-parameter subgroup Γ giving maximum for $\mu(x,\Gamma)$. We have to introduce a notion of *length* in the set of 1-parameter subgroups, to be able to calibrate this kind of features.

Let G be a connected reductive algebraic group over k and let T be a maximal torus. Let N be the normalizer of T and let N/T be the Weyl group. Let $\Gamma(G)$ be the set of 1-parameter subgroups of G. For a k-point $g \in G$ and $\Gamma \in \Gamma(G)$, define $g * \Gamma$ as the 1-parameter subgroup $g*\Gamma(t) = g\cdot\Gamma(t)\cdot g^{-1}$. We define a notion of **length** for $\Gamma \in \Gamma(G)$ (c.f. [Ke, p. 305]).

Definition 1.4.2. *A* ***length*** *is a non-negative real function on $\Gamma(G)$ verifying*

- *If $g \in G$ is k-rational, $\|g * \Gamma\| = \|\Gamma\|$ for any $\Gamma \in \Gamma(G)$.*
- *For any maximal torus T of G, there is a positive definite integral-valued bilinear form $(\ ,\ )$ on $\Gamma(T)$ such that $(\Gamma, \Gamma) = \|\Gamma\|^2$, for any $\Gamma \in \Gamma(T)$.*

As it is pointed out in [Ke], the first property is the invariance of the length by the action of the Weyl group of G with respect to T. And, given a positive definite integral-valued bilinear form $(\ ,\ )$ on $\Gamma(T)$ invariant by the Weyl group, where T is a maximal torus, it corresponds to a unique length $\|\cdot\|$ on $\Gamma(G)$, verifying the property $(\Gamma, \Gamma) = \|\Gamma\|^2$, for any $\Gamma \in \Gamma(G)$.

Remark 1.4.3. *If G is simple in characteristic zero all choices of length will be multiples of the Killing form in the Lie algebra $\mathfrak{g}$ (note that in this case $\Gamma(G) \subset \mathfrak{g}$) and, in general, for an almost simple group in arbitrary characteristic, all lengths differ also by a scalar (c.f. [Ke, p. 305]).*

However, if G has different simple factors, there are more choices of lengths. We can obtain different lengths by choosing a linear combination of the Killing forms in each simple factor with positive coefficients.

Given a choice of length in G, we can define the function appearing on [Ke, Theorem 2.2].

Definition 1.4.4. *Let G be a reductive algebraic group over an algebraically closed field of arbitrary characteristic. Let $G \times X \to X$ be an action of G on a k-scheme X. Consider a length in $\Gamma(G)$, as in Definition 1.4.2. For a point $x \in X$ and a 1-parameter subgroup $\Gamma \in \Gamma(G)$, let $\mu(x, \Gamma)$ be the numerical function of the Hilbert-Mumford criterion as in Theorem 1.1.14. We define the following function*

$$K(x, \Gamma) = \frac{\mu(x, \Gamma)}{\|\Gamma\|}\ .$$

We call this function the ***Kempf function****.*

Remark 1.4.5. *The numerator of the Kempf function is precisely the speed of unstability we discussed about in Example 1.4.1, and the denominator serves for normalizing that quantity with respect to scalar multiples of Γ. Then, we will refer to the 1-parameter subgroup which maximally contradicts the stability condition in the sense of GIT by talking of that Γ which gives maximum for the Kempf function $K(x, \Gamma)$.*

Geometric Invariant Theory contains a study of the dependence of $\mu(x,\Gamma)$ with the 1-parameter subgroup Γ (c.f. [Mu, Section 2.2]). It is based on the previous study of a metric space called the **flag complex**, by Tits, which is the space of 1-parameter subgroups modulo certain equivalence relation for which, the values of $\mu(x,\Gamma)$ that we obtain are multiples. Then, a new function can be defined on this flag complex, whose positivity or negativity coincides with the one of $\mu(x,\Gamma)$, encoding GIT stability but forgetting about rescaling the minimal relevant weight $\mu(x,\Gamma)$ with multiples of the 1-parameter subgroups.

The *conjecture of Mumford-Tits* (as it is stated in the introduction of [Ke], or *Tits' center conjecture* in [MFK, Appendix 2B], see [Mu, p. 64]) says that, if a k-rational point x is unstable with respect to the action of G, we can find a special 1-parameter subgroup giving maximum for the Kempf function $K(x,\Gamma)$. Kempf explores this idea in [Ke] and solves positively the Mumford-Tits conjecture, finding that there exists an special class of 1-parameter subgroups which moves most rapidly toward the origin. Kempf shows it in more generality, using a closed G-invariant set S, instead of just the one point set $\{0\}$, to define a point to be S-unstable if the closure of its orbit intersects S. Kempf uses this to prove that the Hilbert-Mumford criterion (i.e. the checking of the GIT stability of a point by 1-parameter subgroups) holds for actions of algebraic groups over algebraically closed fields of arbitrary characteristic first, and then, the analogous result for perfect fields. For a correspondence between Kempf and Mumford's GIT language, see [MFK, Appendix 2B].

The precise statement of the Kempf's result is the following:

Theorem 1.4.6. *[Ke, Theorem 2.2] Let G be a reductive algebraic group over an algebraically closed field of arbitrary characteristic. Let $G \times X \to X$ be an action of G on a k-scheme X. Let $x \in X$ be a k-point and suppose that x is GIT unstable, i.e. there exists a 1-parameter subgroup Γ such that $\mu(x,\Gamma) > 0$. Define a length in $\Gamma(G)$ as in Definition 1.4.2 and consider the Kempf function $K(x,\Gamma) = \frac{\mu(x,\Gamma)}{\|\Gamma\|}$. Then, the function $K(x,\Gamma)$ achieves a maximum B, taken over all $\Gamma \in \Gamma(G)$ and there exists a parabolic subgroup $P \subset G$ such that in each maximal torus T conjugated by P, there exists a unique 1-parameter subgroup $\Gamma \in \Gamma(T)$ achieving the maximal value $K(x,\Gamma) = B$.*

We can say that the Harder-Narasimhan filtration (c.f. section 1.3) is the best filtration which destabilizes an unstable object, with respect to the given definition of stability, among all possible filtrations by subobjects. Its construction (c.f. Theorem 1.3.5) is based on the existence of a maximally destabilizing subobject (c.f. Lemma 1.3.6 in the case of sheaves), which tells us that there is no better choice for the first element of the filtration. Then, the recursive process of the construction implies that, at every step, we do the best possible, finding maximally destabilizing subobjects for the quotients which successively appear. Given an unstable object, we can obtain a maximally destabilizing subobject, and follow the construction to complete it until the Harder-Narasimhan filtration.

On the other hand, Theorem 1.4.6 implies that, whenever we have a GIT unstable point, we can find a special 1-parameter subgroup giving maximal unstability in the sense of Geometric Invariant Theory, with respect to maximizing the Kempf function in Definition 1.4.4.

Then, consider a notion of stability for a category such that there exists a construction of a moduli space of semistable (or stable) objects. Consider that the construction of the moduli space is given through Geometric Invariant Theory, by means of rigidifying the data and taking the quotient of a space by a group, to get rid of the extra data. In that case, we have a correspondence between unstable objects and GIT unstable objects (as in Theorem 1.2.31 in the construction of the moduli of tensors). In some cases, to give a notion of maximal unstability for an unstable object we have the Harder-Narasimhan filtration. And, to give a notion of GIT maximal unstability we have the 1-parameter subgroup given by Kempf in Theorem 1.4.6.

Therefore, the natural question which arises is,

Is the Harder-Narasimhan filtration related to the 1-parameter subgroup given by Kempf?

The main purpose of this thesis will be to explore this idea by establishing a correspondence between both notions which answers positively the question in different cases.

Chapter 2

Correspondence between Kempf and Harder-Narsimhan filtrations

2.1 Torsion free sheaves over projective varieties

In this first section of the chapter, we describe the main case of the correspondence between the 1-parameter subgroup giving the GIT maximal unstability in the sense of Kempf (c.f. Theorem 1.4.6) and the Harder-Narasimhan filtration (c.f. Theorem 1.3.5). The machinery and the ideas described here will serve, in the remaining sections of the chapter, to prove the analogous result for other other moduli problems.

Let X be a smooth complex projective variety, and let $\mathcal{O}_X(1)$ be an ample line bundle on X. If E is a coherent sheaf on X, let P_E be its Hilbert polynomial with respect to $\mathcal{O}_X(1)$, i.e., $P_E(m) = \chi(E \otimes \mathcal{O}_X(m))$.

We will briefly describe the construction of the moduli space for these objects. This is originally due to Gieseker for surfaces (c.f. [Gi1]), and it was generalized to higher dimension by Maruyama (c.f. [Ma1, Ma2]). First, we give Giesekers's definition of stability for torsion free sheaves. Recall that, if P and Q are polynomials, we write $P \leq Q$ if $P(m) \leq Q(m)$ for $m \gg 0$.

Definition 2.1.1. *[Gi1, Definition 0.1] A torsion free sheaf E on X is called **semistable***

if for all proper subsheaves $F \subset E$, the following inequality between polynomials hold,

$$\frac{P_F}{\operatorname{rk} F} \leq \frac{P_E}{\operatorname{rk} E} \ .$$

If strict inequality holds for every proper subsheaf, we say that E is ***stable****.*

To construct the moduli space of torsion free sheaves with fixed Hilbert polynomial P, we choose a suitably large integer m and consider the Quot scheme parametrizing quotients

$$V \otimes \mathcal{O}_X(-m) \longrightarrow E \tag{2.1.1}$$

where V is a fixed vector space of dimension $P(m)$ and E is a sheaf with $P_E = P$. The Quot scheme has a canonical action by $\mathrm{SL}(V)$. Gieseker (c.f. [Gi1]) gives a linearization of this action on a certain ample line bundle, in order to use Geometric Invariant Theory to take the quotient by the action. The moduli space of semistable sheaves is obtained as the GIT quotient.

As we said, at the beginning of section 1.3, the construction of a moduli space for semistable torsion free sheaves solves the classification problem partially. If a sheaf E is not semistable, it is called **unstable**, and it has a canonical filtration:

Theorem 2.1.2. *[HN, Proposition 1.3.9] Given a torsion free sheaf E, there exists a unique filtration*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E \ ,$$

which satisfies the following properties, where $E^i := E_i/E_{i-1}$:

1. *The Hilbert polynomials verify*

$$\frac{P_{E^1}}{\operatorname{rk} E^1} > \frac{P_{E^2}}{\operatorname{rk} E^2} > \ldots > \frac{P_{E^{t+1}}}{\operatorname{rk} E^{t+1}}$$

2. *Every E^i is semistable*

This filtration is called the ***Harder-Narasimhan filtration*** *of E*

Proof. C.f. Theorem 1.3.5. ∎

In this section we will develop a series of arguments to establish a correspondence between the 1-parameter subgroup given by Kempf in Theorem 1.4.6 and the Harder-Narasimhan filtration in Theorem 2.1.2 to show that both notions do coincide.

2.1.1 Moduli space and Kempf theorem

We will recall Gieseker's construction (c.f. [Gi1]) of the moduli space of semistable torsion free sheaves with fixed Hilbert polynomial P and fixed determinant $\det(E) \cong \Delta$.

Recall that a coherent sheaf is called m-regular if $h^i(E(m-i)) = 0$ for all $i > 0$ (c.f. Definition 1.2.12 and Lemma 1.2.13). Let m be a suitable large integer, so that E is m-regular for all semistable E (c.f. [Ma1, Corollary 3.3.1 and Proposition 3.6]). Let V be a vector space of dimension $p := P(m)$. Given an isomorphism $V \cong H^0(E(m))$ we obtain a quotient

$$q : V \otimes \mathcal{O}_X(-m) \twoheadrightarrow E \ ,$$

hence a homomorphism

$$Q : \wedge^r V \cong \wedge^r H^0(E(m)) \longrightarrow H^0(\wedge^r(E(m))) \cong H^0(\Delta(rm)) =: A$$

and points

$$Q \in \operatorname{Hom}(\wedge^r V, A) \qquad \overline{Q} \in \mathbb{P}(\operatorname{Hom}(\wedge^r V, A)) \ ,$$

where Q is well defined up to a scalar because the isomorphism $\det(E) \cong \Delta$ is well defined up to a scalar, and hence $\overline{Q}$ is a well defined point the the projective space. Two different isomorphisms between V and $H^0(E(m))$ differ by the action of an element of $\mathrm{GL}(V)$, but, since an homothecy does not change the point $\overline{Q}$, to get rid of the choice of isomorphism it is enough to take the quotient by the action of $\mathrm{SL}(V)$.

We recall from section 1.2 the correspondence between weighted filtrations and 1-parameter subgroups. A **weighted filtration** $(V_\bullet, n_\bullet)$ of V is a filtration

$$0 \subset V_1 \subset V_2 \subset \ \cdots \ \subset V_t \subset V_{t+1} = V, \tag{2.1.2}$$

and rational numbers $n_1, n_2, \ldots, n_t > 0$. To a weighted filtration we associate a vector of $\mathbb{C}^p$ defined as $\Gamma = \sum_{i=1}^t n_i \Gamma^{(\dim V_i)}$ where

$$\Gamma^{(k)} := \big(\overbrace{k-p, \ldots, k-p}^{k}, \overbrace{k, \ldots, k}^{p-k}\big) \qquad (1 \le k < p)\,. \tag{2.1.3}$$

Hence, the vector is of the form

$$\Gamma = (\overbrace{\Gamma_1, \ldots, \Gamma_1}^{\dim V^1}, \overbrace{\Gamma_2, \ldots, \Gamma_2}^{\dim V^2}, \ldots, \overbrace{\Gamma_{t+1}, \ldots, \Gamma_{t+1}}^{\dim V^{t+1}}) \ ,$$

where $V^i = V_i/V_{i-1}$. Giving the numbers $n_1,\dots,n_t$ is equivalent to giving the numbers $\Gamma_1,\dots,\Gamma_{t+1}$ by setting

$$n_i = \frac{\Gamma_{i+1}-\Gamma_i}{p} \qquad \text{and} \qquad \sum_{i=1}^{t+1} \Gamma_i \dim V^i = 0 \ .$$

A 1-parameter subgroup of $\mathrm{SL}(V)$ is a non-trivial homomorphism

$$\Gamma : \mathbb{C}^* \to \mathrm{SL}(V) \ .$$

To a 1-parameter subgroup we associate a weighted filtration as follows. There is a basis $\{e_1,\dots,e_p\}$ of V where it has a diagonal form

$$t \mapsto \operatorname{diag}\left(t^{\Gamma_1},\dots,t^{\Gamma_1},t^{\Gamma_2},\dots,t^{\Gamma_2},\dots,t^{\Gamma_{t+1}},\dots,t^{\Gamma_{t+1}}\right)$$

with $\Gamma_1 < \cdots < \Gamma_{t+1}$. Let

$$0 \subset V_1 \subset \cdots \subset V_{t+1} = V$$

be the associated filtration. Finally recall that two 1-parameter subgroups give the same filtration if and only if they are conjugate by an element of the parabolic subgroup of $\mathrm{SL}(V)$ defined by the filtration.

The basis $\{e_1,\dots,e_p\}$, together with a basis $\{w_j\}$ of A, induces a basis of $\mathrm{Hom}(\wedge^r V, A)$ indexed in a natural way by tuples $(i_1,\dots,i_r,j)$ with $i_1 < \cdots < i_r$, and the coordinate corresponding to such an index is acted by the 1-parameter subgroup as

$$Q_{i_1,\cdots,i_r,j} \mapsto t^{\Gamma_{i_1}+\cdots+\Gamma_{i_r}} Q_{i_1,\cdots,i_r,j} \ .$$

The coordinate $(i_1,\dots,i_r,j)$ of the point corresponding to E is non-zero if and only if the evaluations of the sections $e_1,\dots,e_r$ are linearly independent for generic $x \in X$. Therefore, the numerical function (i.e. the *minimal relevant weight*) which has to be calculated to apply Hilbert-Mumford criterion for GIT stability (c.f. Theorem 1.1.14) is

$$\begin{aligned} \mu(\overline{Q},V_\bullet,n_\bullet) &= \min\{\Gamma_{i_1}+\cdots+\Gamma_{i_r} : Q_{i_1,\dots,i_r,j} \neq 0\} \\ &= \min\{\Gamma_{i_1}+\cdots+\Gamma_{i_r} : Q(e_{i_1}\wedge\cdots\wedge e_{i_r}) \neq 0\} \\ &= \min\{\Gamma_{i_1}+\cdots+\Gamma_{i_r} : e_{i_1}(x),\dots,e_{i_r}(x) \\ &\qquad \text{linearly independent for generic } x \in X\} \end{aligned} \tag{2.1.4}$$

After a short calculation (originally due to Gieseker) we obtain

$$\mu(\overline{Q},V_\bullet,n_\bullet) = \sum_{i=1}^{t} n_i(r\dim V_i - r_i \dim V) = \sum_{i=1}^{t+1} \frac{\Gamma_i}{\dim V}(r^i \dim V - r\dim V^i) \tag{2.1.5}$$

(recall $n_i = \frac{\Gamma_{i+1}-\Gamma_i}{p}$), where $r_i = \operatorname{rk} E_i$, E_i is the sheaf generated by evaluation of the sections of V_i and $r^i = \operatorname{rk} E^i$, being $E^i = E_i/E_{i-1}$.

By the Hilbert-Mumford criterion in Theorem 1.1.14, a point

$$\overline{Q} \in \mathbb{P}(\operatorname{Hom}(\wedge^r V, A))$$

is **GIT semistable** if and only if for all weighted filtrations, it is

$$\mu(\overline{Q}, V_\bullet, n_\bullet) \leq 0 \, .$$

A point $\overline{Q}$ is **GIT stable** if we get a strict inequality for all weighted filtrations. Using the previous calculation, this can be stated as follows:

Lemma 2.1.3. *A point $\overline{Q}$ is GIT semistable (resp. GIT stable) if for all weighted filtrations $(V_\bullet, n_\bullet)$*

$$\sum_{i=1}^{t} n_i(r \dim V_i - r_i \dim V) \leq 0$$

(resp. $<$).

A **weighted filtration** $(E_\bullet, n_\bullet)$ of a sheaf E of rank r is a filtration

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E, \tag{2.1.6}$$

and rational numbers $n_1, n_2, \ldots, n_t > 0$. To a weighted filtration we associate a vector of $\mathbb{C}^r$ defined as $\gamma = \sum_{i=1}^{t} n_i \gamma^{(\operatorname{rk} E_i)}$ where

$$\gamma^{(k)} := \big(\overbrace{k-r, \ldots, k-r}^{k}, \overbrace{k, \ldots, k}^{r-k}\big) \qquad (1 \leq k < r) \, .$$

Hence, the vector is of the form

$$\gamma = (\overbrace{\gamma_1, \ldots, \gamma_1}^{\operatorname{rk} E^1}, \overbrace{\gamma_2, \ldots, \gamma_2}^{\operatorname{rk} E^2}, \ldots, \overbrace{\gamma_{t+1}, \ldots, \gamma_{t+1}}^{\operatorname{rk} E^{t+1}}) \, ,$$

where $n_i = \dfrac{\gamma_{i+1} - \gamma_i}{r}$, and $E^i = E_i/E_{i-1}$.

The following theorem follows from [Gi1, Ma1, Ma2].

Theorem 2.1.4. *Let E be a sheaf. There exists an integer $m_0(E)$ such that, for $m > m_0(E)$, the associated point $\overline{Q}$ is GIT semistable if and only if the sheaf is semistable.*

From this property, Gieseker, in the case of algebraic surfaces and, later, Maruyama, for higher dimensional varieties, constructs a moduli space of semistable torsion free sheaves as the GIT quotient, which is a projective scheme (c.f. [Gi1, Theorem 0.3], [Ma2, Theorem 4.11]).

Let E be an unstable torsion free sheaf over X of Hilbert polynomial P. We choose an integer m_0 larger than $m_0(E)$ (c.f. Theorem 2.1.4), also larger than the integer used in Gieseker's construction of the moduli space, and such that E is m-regular. Let V be a vector space of dimension $P(m) = h^0(E(m))$ and fix an isomorphism $V \simeq H^0(E(m))$.

Recall that, through Geometric Invariant Theory, stability of a point in the parameter space can be checked by 1-parameter subgroups (c.f. Hilbert-Mumford criterion, Theorem 1.1.14). In other words, a point is unstable if there exists any 1-parameter subgroup which makes the quantity (2.1.5) positive. It is a natural question to ask if there exists a best way of destabilizing a GIT unstable point in this sense, i.e. a 1-parameter subgroup which gives maximum for (2.1.5).

As we showed in Section 1.4, Kempf explores this idea in [Ke] and answers yes to the question, finding that there exists an special class of 1-parameter subgroups which moves most rapidly toward the origin.

We have seen that giving a weighted filtration, i.e. a filtration of vector subspaces $V_1 \subset \cdots \subset V_t \subset V$ and rational numbers $n_1, \cdots, n_t > 0$, is equivalent to giving a parabolic subgroup with weights, which determines uniquely the vector Γ of a 1-parameter subgroup and two of these 1-parameter subgroup are conjugated by the parabolic and come from the same weighted filtration. Hence, the data of Γ is equivalent to the data of $(V_\bullet, n_\bullet)$.

Define the function in Definition 1.4.4,

$$K(x,\Gamma) = \frac{\mu(x,\Gamma)}{||\Gamma||} \; .$$

as the following function

$$K(x,\Gamma) = \frac{\sum_{i=1}^{t} n_i (r \dim V_i - r_i \dim V)}{\sqrt{\sum_{i=1}^{t+1} \dim V^i \Gamma_i^2}} = \mu(V_\bullet, n_\bullet) \; , \tag{2.1.7}$$

which we call **Kempf function**. The numerator of the function coincides with the calculation of the minimal relevant weight by Hilbert-Mumford criterion for GIT stability (c.f. (2.1.5)), and the denominator is a function $|| \cdot ||$ in the set $\Gamma(SL(V))$ of 1-parameter subgroups of $SL(V)$, which is precisely the norm of the vector

$$\Gamma = (\overbrace{\Gamma_1, \ldots, \Gamma_1}^{\dim V^1}, \overbrace{\Gamma_2, \ldots, \Gamma_2}^{\dim V^2}, \ldots, \overbrace{\Gamma_{t+1}, \ldots, \Gamma_{t+1}}^{\dim V^{t+1}})$$

associated to each 1-parameter subgroup Γ.

To define the Kempf function we need to choose a *length* in $\Gamma(SL(V))$ (c.f. Definition 1.4.2). Recall that for a simple group G (as it is the case of $G = \mathrm{SL}(V)$) every bilinear symmetric invariant form is a multiple of the Killing form (c.f. Remark 1.4.3), and this norm $||\Gamma||$ we choose verifies these properties. Hence, the function we defined in (2.1.7) is a Kempf function as in Definition 1.4.4.

We take the GIT quotient by the group $G = \mathrm{SL}(V)$, for which, Theorem 1.4.6 (c.f. [Ke, Theorem 2.2]) states that whenever there exists any Γ giving a positive value for the numerator of the function (i.e. whenever there exists a 1-parameter subgroup whose numerical function (2.1.5) is positive, which is equivalent to the sheaf E to be unstable), there exists a unique parabolic subgroup containing a unique 1-parameter subgroup in each maximal torus, giving maximum for the Kempf function i.e., there exists a unique weighted filtration for which the Kempf function achieves a maximum.

Note that $\mu(V_\bullet, n_\bullet) = \mu(V_\bullet, \alpha n_\bullet)$, for every $\alpha > 0$, hence by multiplying each n_i by the same scalar α, which we call *rescaling the weights*, we get another 1-parameter subgroup but the same value for the Kempf function. Hence, we divide by the norm in the Kempf function to get a well defined maximal weighted filtration, i.e. defined up to rescaling.

Therefore, Theorem 1.4.6 rewritten in our case asserts the following:

Theorem 2.1.5. *There exists a unique weighted filtration*

$$0 \subset V_1 \subset \cdots \subset V_{t+1} = V$$

*and rational numbers $n_1, \cdots, n_t > 0$, up to multiplication by a scalar, called the **Kempf filtration of V**, such that the Kempf function $\mu(V_\bullet, n_\bullet)$ achieves the maximum among all filtrations and positive weights $n_i > 0$.*

We construct a filtration by subsheaves of E out of the Kempf filtration of V in Theorem 2.1.5. Recall that E is an unstable torsion free sheaf over X of Hilbert polynomial P. Let m be an integer, $m \geq m_0$ and let V be a vector space of dimension $P(m) = h^0(E(m))$ (recall that m_0 was defined before). We fix an isomorphism $V \simeq H^0(E(m))$ and let $V_1 \subset \cdots \subset V_{t+1} = V$ be the filtration of vector spaces given by Theorem 2.1.5, called the

Kempf filtration of V. For each index i, let $E_i^m \subset E$ be the subsheaf generated by V_i under the evaluation map. We call this filtration

$$0 \subseteq E_1^m \subseteq E_2^m \subseteq \cdots \subseteq E_t^m \subseteq E_{t+1}^m = E \ ,$$

the m-**Kempf filtration of E**. Note that it depends on the integer m we choose in the process.

The question we finished Section 1.4 with was

Is the Harder-Narasimhan filtration related to the 1-parameter subgroup given by Kempf?

The maximal unstability with respect to Definition 2.1.1 is given by the Harder-Narasimhan filtration (c.f. Theorem 2.1.2) and the GIT maximal unstability is encoded in the Kempf filtration of V, by Theorem 2.1.5. This filtration of vector subspaces can be evaluated to get a filtration of subsheaves, the m-Kempf filtration of E, depending on an integer m. Therefore, the previous question turns out to be more concrete:

Does the m-Kempf filtration coincide with the Harder-Narasimhan filtration?

The answer will be yes. In the following pages we will develop a technique to prove the following two theorems:

Theorem 2.1.6. *There exists an integer $m' \gg 0$ such that the m-Kempf filtration of E is independent of m, for $m' \geq m$.*

This filtration we obtain, independent of the integer m, will be called the **Kempf filtration** of E.

Theorem 2.1.7. *The Kempf filtration of an unstable torsion free coherent sheaf E coincides with the Harder-Narasimhan filtration of E.*

The method we use to prove Theorem 2.1.6 and Theorem 2.1.7 will be translated to other moduli problems to prove an analogous result in the subsequent sections of this chapter.

2.1.2 Results on convexity

In this subsection we define the machinery which will serve us in the following. We study a function on a convex set, and how to maximize it. It will turn out to be that this function will be in correspondence with the Kempf function and we will use the results of this subsection to figure out properties about the Kempf filtration.

Endow $\mathbb{R}^{t+1}$ with an inner product $(\cdot,\cdot)$ defined by a diagonal matrix

$$\begin{pmatrix} b^1 & & 0 \\ & \ddots & \\ 0 & & b^{t+1} \end{pmatrix}$$

where b^i are positive integers. Let

$$\mathcal{C} = \left\{ x \in \mathbb{R}^{t+1} : x_1 < x_2 < \cdots < x_{t+1} \right\} ,$$

$$\overline{\mathcal{C}} = \left\{ x \in \mathbb{R}^{t+1} : x_1 \leq x_2 \leq \cdots \leq x_{t+1} \right\} ,$$

and let $v = (v_1, \cdots, v_{t+1}) \in \mathbb{R}^{t+1} - \{0\}$ verifying $\sum_{i=0}^{t+1} v_i b^i = 0$. Define the function

$$\begin{array}{rcl} \mu_v : \overline{\mathcal{C}} - \{0\} & \to & \mathbb{R} \\ \Gamma & \mapsto & \mu_v(\Gamma) = \dfrac{(\Gamma, v)}{||\Gamma||} \end{array}$$

and note that $\mu_v(\Gamma) = ||v|| \cdot \cos\beta(\Gamma, v)$, where $\beta(\Gamma, v)$ is the angle between Γ and v. Then, the function $\mu_v(\Gamma)$ does not depend on the norm of Γ and takes the same value on every point of the ray spanned by each Γ.

Assume that there exists $\Gamma \in \overline{\mathcal{C}}$ with $\mu_v(\Gamma) > 0$. In that case, we want to find a vector $\Gamma \in \overline{\mathcal{C}}$ which maximizes the function defined before.

Let $w^i = -b^i v_i$, $w_0 = 0$, $w_i = w^1 + \cdots + w^i$, $b_0 = 0$, and $b_i = b^1 + \cdots + b^i$. Note that $w_{t+1} = 0$, by construction. We draw a graph joining the points with coordinates (b_i, w_i). Note that this graph has $t+1$ segments, each segment has slope $-v_i$ and width b^i. This is the graph drawn with a thin line in the figure. Now draw the convex envelope of this graph (thick line in Figure 2.1), whose coordinates we denote by $(b_i, \widetilde{w_i})$, and let us define $\Gamma_i = -\frac{\widetilde{w_i} - \widetilde{w_{i-1}}}{b^i}$. In other words, the quantities $-\Gamma_i$ are the slopes of the convex envelope graph. We call the vector defined in this way Γ_v. Note that the vector $\Gamma_v = (\Gamma_1, \cdots, \Gamma_{t+1})$ belongs to $\overline{\mathcal{C}}$ by construction and $\Gamma_v \neq 0$.

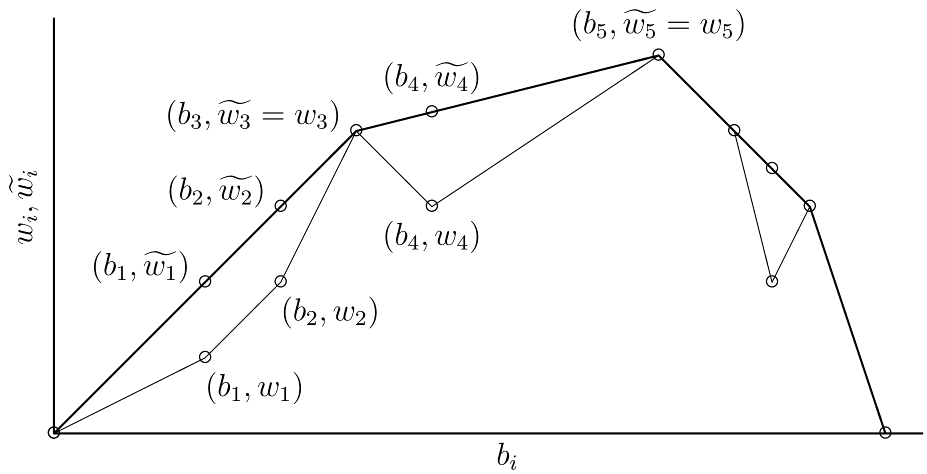


Figure 2.1: Convex envelope Γ_v of v

Remark 2.1.8. *Observe that $\widetilde{w}_i > w_i$, then $\Gamma_i = \Gamma_{i+1}$. Indeed, if $\widetilde{w}_i > w_i$, there will be a segment in the convex envelope joining two vertices such that $w_j = \widetilde{w_j}$ and $w_k = \widetilde{w_k}$, with $j < i$ and $i < k$. Then, it is clear that all segments joining the intermediate heights $\widetilde{w}_l$, $j < l < k$, will have the same slope, in particular $\Gamma_i = \Gamma_{i+1}$.*

Theorem 2.1.9. *The vector Γ_v defined in this way (c.f. Figure 2.1) gives a maximum for the function μ_v on its domain.*

Before proving the theorem we need some lemmas.

Lemma 2.1.10. *Let $v = (v_1, \cdots, v_{t+1}) \in \mathbb{R}^{t+1} - \{0\}$ verifying $\sum_{i=0}^{t+1} v_i b^i = 0$. Let Γ be the point in $\overline{\mathcal{C}}$ which is closest to v. Then Γ achieves the maximum of μ_v.*

Proof. For any $\alpha \in \mathbb{R}^{>0}$, the vector $\alpha\Gamma$ is also in $\overline{\mathcal{C}}$, so in particular Γ is the closest point to v in the line $\alpha\Gamma$. This point is the orthogonal projection of v into the line $\alpha\Gamma$, and the distance is

$$||v|| \sin \beta(v, \Gamma) \ , \tag{2.1.8}$$

where $\beta(\Gamma, v)$ is the angle between Γ and v. But, a vector $\Gamma \in \overline{\mathcal{C}}$ minimizes (2.1.8) if and only if it maximizes

$$||v|| \cos \beta(\Gamma, v) = \frac{(\Gamma, v)}{||\Gamma||} \ ,$$

so the lemma is proved. ■

We say that an affine hyperplane in $\mathbb{R}^{t+1}$ separates a point v from $\mathcal{C}$ if v is on one side of the hyperplane and all the points of $\mathcal{C}$ are on the other side of the hyperplane.

Lemma 2.1.11. *Let $v \notin \overline{\mathcal{C}}$. A point $\Gamma \in \overline{\mathcal{C}} - \{0\}$ gives minimum distance to v if and only if the hyperplane $\Gamma + (v-\Gamma)^{\perp}$ separates v from $\mathcal{C}$.*

Proof. $\Rightarrow$) Let $\Gamma \in \overline{\mathcal{C}}$ and assume that there is a point $w \in \mathcal{C}$ on the same side of the hyperplane as v. The segment going from Γ to w is in $\overline{\mathcal{C}}$ (by convexity of $\overline{\mathcal{C}}$), but there are points in this segment (near Γ), which are closer to v than Γ.

$\Leftarrow$) Let Γ be a point in $\overline{\mathcal{C}}$ such that $\Gamma + (v-\Gamma)^{\perp}$ separates v from $\mathcal{C}$. Let $w \in \overline{\mathcal{C}}$ be another point. Let w' be the intersection of the hyperplane and the segment which goes from w to v. Since the hyperplane separates $\mathcal{C}$ from v, either $w' = w$ or w' is in the interior of the segment. Therefore

$$d(w,v) \geq d(w',v) \geq d(\Gamma, v) \ ,$$

where the last inequality follows from the fact that Γ is the orthogonal projection of v to the hyperplane. ■

Proof of the Theorem 2.1.9. Let $\Gamma_v = (\Gamma_1, ..., \Gamma_{t+1})$ be the vector in the hypothesis of the theorem. If $v \in \overline{\mathcal{C}}$, then $\Gamma_v = v$, and use Lemma 2.1.10 to conclude. If $v \notin \overline{\mathcal{C}}$, by Lemmas 2.1.10 and 2.1.11, it is enough to check that the hyperplane $\Gamma_v + (v - \Gamma_v)^{\perp}$ separates v from $\mathcal{C}$.

Let $\Gamma_v + \epsilon \in \mathcal{C}$, $\epsilon \in \mathbb{R}^{t+1}$. The condition that $\Gamma_v + \epsilon$ belongs to $\mathcal{C}$ means that

$$\epsilon_i - \epsilon_{i+1} < \Gamma_{i+1} - \Gamma_i \tag{2.1.9}$$

The hyperplane separates v from $\mathcal{C}$ if and only if $(v - \Gamma_v, \epsilon) < 0$ for all such ϵ. Therefore we calculate (using the convention $\widetilde{w_0} = 0$, $w_0 = 0$, and $\widetilde{w_{t+1}} = w_{t+1} = 0$)

$$(v - \Gamma_v, \epsilon) = \sum_{i=1}^{t+1} b^i (v_i - \Gamma_i)\epsilon_i = \sum_{i=1}^{t+1} (-w^i + (\widetilde{w}_i - \widetilde{w_{i-1}}))\epsilon_i =$$

$$= \sum_{i=1}^{t+1} \big((\widetilde{w}_i - \widetilde{w_{i-1}}) - (w_i - w_{i-1})\big)\epsilon_i = \sum_{i=1}^{t+1} (\widetilde{w}_i - w_i)(\epsilon_i - \epsilon_{i+1}) \ .$$

If $\widetilde{w}_i = w_i$, then the corresponding summand is zero. On the other hand, if $\widetilde{w}_i > w_i$, then $\Gamma_{i+1} = \Gamma_i$ (c.f. Remark 2.1.8), and (2.1.9) implies $\epsilon_i - \epsilon_{i+1} < 0$. In any case, the summands are always non-positive, and there is at least one which is negative (because $v \notin \overline{\mathcal{C}}$ and then $v \neq \Gamma_v$ and $\widetilde{w}_i > w_i$ for at least one i). Hence

$$(v - \Gamma_v, \epsilon) < 0 \; .$$

■

Therefore, the function $\mu_v(\Gamma)$ achieves its maximum for the value $\Gamma_v \in \overline{\mathcal{C}} - \{0\}$ (or any other point on the ray $\alpha\Gamma_v$) defined as the convex envelope of the graph associated to v.

2.1.3 Graph and identification

In the last section we studied a geometrical function, $\mu_v(\Gamma)$, very similar to the Kempf function. This new function depends on two arguments, one is a vector $\Gamma \in \overline{\mathcal{C}} - \{0\}$, where

$$\overline{\mathcal{C}} = \left\{ x \in \mathbb{R}^{t+1} : x_1 \leq x_2 \leq \cdots \leq x_{t+1} \right\} ,$$

and the other is $v = (v_1, \cdots, v_{t+1}) \in \mathbb{R}^{t+1} - \{0\}$ verifying $\sum_{i=0}^{t+1} v_i b^i = 0$, for certain coefficients b^i of an inner product in an Euclidean space. We will relate both functions where the first argument Γ will be associated to a 1-parameter subgroup (or to a weighted filtration $(V_\bullet, n_\bullet)$ which we recall that is equivalent), and the second one will be associated to the numerical invariants of the Kempf filtration of V,

$$0 \subset V_1 \subset \cdots \subset V_{t+1} = V$$

(c.f. Theorem 2.1.5) and the m-Kempf filtration of E

$$0 \subseteq E_1^m \subseteq E_2^m \subseteq \; \cdots \; \subseteq E_t^m \subseteq E_{t+1}^m = E$$

obtained by evaluating. With this, we will be able to prove properties of the filters appearing on the different m-Kempf filtrations for each m, out from convexity properties of the function μ_v (c.f. Theorem 2.1.9). Both functions have to be maximized by the convex envelope of the graph defined by v, or the Kempf filtration of V, therefore both notions have to correspond to the same filtrations. And to make precise that relation, we have to encode the m-Kempf filtration as a graph.

Definition 2.1.12. *Let $m \geq m_0$. Given $0 \subset V_1 \subset \cdots \subset V_{t+1} = V$, a filtration of vector spaces of V, define*

$$v_{m,i} = m^{n+1} \cdot \frac{1}{\dim V^i \dim V} \left[r^i \dim V - r \dim V^i \right] ,$$

$$b^i_m = \frac{1}{m^n} \dim V^i > 0 \ ,$$

$$w^i_m = -b^i_m \cdot v_{m,i} = m \cdot \frac{1}{\dim V} \big[r \dim V^i - r^i \dim V \big] \ .$$

Also let

$$b_{m,i} = b^1_m + \ldots + b^i_m = \frac{1}{m^n} \dim V_i \ ,$$

$$w_{m,i} = w^1_m + \ldots + w^i_m = m \cdot \frac{1}{\dim V} \big[r \dim V_i - r_i \dim V \big] \ .$$

We call the graph defined by points $(b_{m,i}, w_{m,i})$ the ***graph associated to the filtration*** *$V_\bullet \subset V$.*

Now we can identify the Kempf function (2.1.7) in Theorem 2.1.5

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^t n_i (r \dim V_i - r_i \dim V)}{\sqrt{\sum_{i=1}^{t+1} \dim V^i \Gamma_i^2}} \ ,$$

with the function in Theorem 2.1.9 up to a factor which is a power of m, by defining $v_{m,i}$, the coordinates of vector v_m, and b^i_m, the eigenvalues of the inner product, as in Definition 2.1.12. Note that $-v_{m,i}$ are the slopes of the graph associated to the filtration $V_\bullet \subset V$. To give the weights n_i is the same that to give the coordinates Γ_i (recall the discussion about the correspondence between 1-parameter subgroups of $SL(V)$ and weighted filtrations). Also note that $\sum_{i=1}^{t+1} v_{m,i} b^i_m = 0$. Then, an easy calculation shows that

Proposition 2.1.13. *For every integer m, the following equality holds*

$$\mu(V_\bullet, n_\bullet) = m^{(-\frac{n}{2}-1)} \cdot \mu_{v_m}(\Gamma)$$

between the Kempf function (2.1.7) in Theorem 2.1.5 and the function in Theorem 2.1.9.

In the following, we will omit the subindex m for the numbers $v_{m,i}$, $b_{m,i}$, $w_{m,i}$ in the definition of the graph associated to a filtration of vector spaces, where it is clear from the context. Hence, given $V \simeq H^0(E(m))$ we will refer to a filtration $V_\bullet \subset V$ and a vector $v = (v_1, \ldots, v_{t+1})$ as the vector of the graph associated to the filtration.

Remark 2.1.14. *We introduce the factor m^{n+1} in Definition 2.1.12 for convenience, so that $v_{m,i}$ and b^i_m have order zero on m, because $\dim V = P(m)$ appears in their expressions. Then, the size of the graph does not change when m grows.*

Now, let us prove two lemmas encoding the convexity properties of the graph associated to the Kempf filtration. They will be strongly used in the following, to show properties shared by the possible filters E_i^m appearing in the different m-Kempf filtrations and, finally, to prove Theorems 2.1.6 and Theorem 2.1.7.

Lemma 2.1.15. *Let $0 \subset V_1 \subset \cdots \subset V_{t+1} = V$ be the Kempf filtration of V (cf. Theorem 2.1.5). Let $v = (v_1, ..., v_{t+1})$ be the vector of the graph associated to this filtration by Definition 2.1.12. Then*

$$v_1 < v_2 < \ldots < v_t < v_{t+1} \ ,$$

i.e., ***the graph is convex****.*

Proof. By Theorem 2.1.5 the maximum of $\mu(V_\bullet, n_\bullet)$ among all filtrations $V_\bullet \subset V$ and weights $n_i > 0, \forall i$ is achieved by a unique weighted filtration $(V_\bullet, n_\bullet)$, $n_i > 0, \forall i$, up to rescaling. Let $V_\bullet \subset V$ be this filtration, and allow n_i to vary. By Proposition 2.1.13 $\mu(V_\bullet, n_\bullet)$ is equal to μ_v up to a constant factor. By Theorem 2.1.9, μ_v achieves the maximum on Γ_v. The vector Γ_v corresponds to the weights n_i given by Theorem 2.1.5. Summing up, if $V_\bullet \subset V$ is Kempf filtration of V, then the vector $\Gamma_v = (\Gamma_1, \ldots, \Gamma_{t+1})$ verifies $\Gamma_i < \Gamma_{i+1}, \forall i$.

Assume that, for the Kempf filtration of V, there exists some i such that $v_i \geq v_{i+1}$. Then $v \notin \mathcal{C}$ and, by Lemma 2.1.10, $\Gamma_v \in \overline{\mathcal{C}} \backslash \mathcal{C}$, which means that there exists some j with $\Gamma_j = \Gamma_{j+1}$, but we have just seen that this is impossible. ■

Lemma 2.1.16. *Let $0 \subset V_1 \subset \cdots \subset V_{t+1} = V$ be the Kempf filtration of V (cf. Theorem 2.1.5). Let W be a vector space with $V_i \subset W \subset V_{i+1}$ and consider the new filtration $V'_\bullet \subset V$*

$$\begin{array}{ccccccccccccccc}
0 & \subset & V'_1 & \subset & \cdots & \subset & V'_i & \subset & V'_{i+1} & \subset & V'_{i+2} & \subset & \cdots & \subset & V'_{t+2} & = & V \\
|| & & || & & & & || & & || & & || & & & & & & || \\
0 & \subset & V_1 & & & \subset & V_i & \subset & W & \subset & V_{i+1} & \subset & \cdots & \subset & V_{t+1} & = & V
\end{array} \qquad (2.1.10)$$

Then, $v'_{i+1} \geq v_{i+1}$. We say that ***the Kempf filtration is the convex envelope of every refinement****.*

Proof. The graph associated to $V'_\bullet \subset V$ has one more point than the graph associated to $V_\bullet \subset V$, hence it is a refinement of the graph associated to Kempf filtration of V. Therefore the convex envelope of the graph associated to v' has to be equal to the graph associated to v, and this happens only when the extra point associated to W is not above the graph associated to v, which means that the slope $-v'_{i+1}$ has to be less or equal than $-v_{i+1}$. ■

Remark 2.1.17. *Note that Lemmas 2.1.15 and 2.1.16 assert two properties similar to the ones of the Harder-Narasimhan filtration (c.f. Theorem 2.1.2). This will be the key point in the proof of Theorem 2.1.7.*

Hence, we will prove that, for m large enough, the m-Kempf filtration stabilizes in the sense $E_i^m = E_i^{m+l}, \forall i, \forall l > 0$, in Theorem 2.1.6. The m-Kempf filtration for $m \gg 0$ will be called the Kempf filtration of E, and the goal is to show that it coincides with the Harder-Narasimhan filtration of E in Theorem 2.1.7.

2.1.4 Properties of the m-Kempf filtration

We will show that the filters appearing in the different m-Kempf filtrations form a bounded family.

First recall Lemma 1.2.15 in subsection 1.2.2. Also recall the definition of the Hilbert polynomials of $\mathcal{O}_X$ in (1.2.10) and E in (1.2.11). Then, let us define

$$C = \max\{r|\mu_{\max}(E)| + \frac{d}{r} + r|B| + |A| + 1\ ,\ 1\}, \tag{2.1.11}$$

a positive constant.

Proposition 2.1.18. *Given an integer m and a vector space $V \simeq H^0(E(m))$, we have the Kempf filtration $V_\bullet \subset V \simeq H^0(E(m))$ (c.f. Theorem 2.1.5) and, by evaluation, the m-Kempf filtration $E_\bullet^m \subset E$. There exists an integer m_2 such that for $m \geq m_2$, each filter in the m-Kempf filtration of E has slope $\mu(E_i^m) \geq \dfrac{d}{r} - C$.*

Proof. Choose an $m_1 \geq m_0$ such that for $m \geq m_1$

$$[\mu_{max}(E) + gm + B]_+ = \mu_{max}(E) + gm + B$$

and

$$[\frac{d}{r} - C + gm + B]_+ = \frac{d}{r} - C + gm + B\ .$$

Now, let $m \geq m_1$ and let

$$0 \subseteq E_1^m \subseteq E_2^m \subseteq \ \cdots\ \subseteq E_t^m \subseteq E_{t+1}^m = E$$

be the m-Kempf filtration of E.

Suppose we have a filter $E_i^m \subseteq E$, of rank r_i and degree d_i, such that $\mu(E_i^m) < \frac{d}{r} - C$. The subsheaf $E_i^m(m) \subset E(m)$ satisfies the estimate in Lemma 1.2.15,

$$h^0(E_i^m(m)) \leq \frac{1}{g^{n-1}n!}\big((r_i-1)([\mu_{max}(E_i^m)+gm+B]_+)^n + ([\mu_{min}(E_i^m)+gm+B]_+)^n\big) ,$$

where $\mu_{max}(E_i^m(m)) = \mu_{max}(E_i^m) + gm$ and similarly for μ_{min}.

Note that $\mu_{max}(E_i^m) \leq \mu_{max}(E)$ and $\mu_{min}(E_i^m) \leq \mu(E_i^m) < \frac{d}{r} - C$, so

$$h^0(E_i^m(m)) \leq \frac{1}{g^{n-1}n!}\big((r_i-1)([\mu_{max}(E)+gm+B]_+)^n + ([\frac{d}{r} - C + gm+B]_+)^n\big) ,$$

and, by choice of m,

$$h^0(E_i^m(m)) \leq \frac{1}{g^{n-1}n!}\big((r_i-1)(\mu_{max}(E)+gm+B)^n + (\frac{d}{r} - C + gm+B)^n\big) = G(m) ,$$

where

$$G(m) = \frac{1}{g^{n-1}n!}\big[r_i g^n m^n + ng^{n-1}\big((r_i-1)\mu_{max}(E) + \frac{d}{r} - C + r_i B\big)m^{n-1} + \cdots\big] .$$

Recall that, by Definition 2.1.12, to such filtration we associate a graph with heights, for each j,

$$w_j = w^1 + \ldots + w^j = m \cdot \frac{1}{\dim V}\big[r \dim V_j - r_j \dim V\big] .$$

To reach a contradiction, it is enough to show that $w_i < 0$. In that case, the graph has to be convex by Lemma 2.1.15. If $w_i < 0$ there is a $j < i$ such that $-v_j < 0$, because the graph starts at the origin. Hence, the rest of the slopes of the graph are negative, $-v_k < 0$, $k \geq i$, because the slopes have to be decreasing. Then $w_i > w_{i+1} > \ldots w_{t+1}$, and $w_{t+1} < 0$. But it is

$$w_{t+1} = m \cdot \frac{1}{\dim V}\big[r \dim V_{t+1} - r_{t+1} \dim V\big] = 0 ,$$

because $r_{t+1} = r$ and $V_{t+1} = V$, then the contradiction.

Let us show that $w_i < 0$. Since $E_i^m(m)$ is generated by V_i under the evaluation map, it is $\dim V_i \leq h^0(E_i^m(m))$, hence

$$w_i = \frac{m}{\dim V}\big[r \dim V_i - r_i \dim V\big] \leq$$

$$\leq \frac{m}{P(m)}\big[r h^0(E_i^m(m)) - r_i P(m)\big] \leq \frac{m}{P(m)}\big[rG(m) - r_i P(m)\big] .$$

Hence, $w_i < 0$ is equivalent to

$$\Psi(m) = rG(m) - r_i P_E(m) < 0 \ ,$$

where $\Psi(m) = \xi_n m^n + \xi_{n-1} m^{n-1} + \cdots + \xi_1 m + \xi_0$ is an n^{th}-order polynomial. Let us calculate the n^{th}-coefficient:

$$\xi_n = (rG(m) - r_i P(m))_n = r\frac{r_i g}{n!} - r_i \frac{rg}{n!} = 0 \ .$$

Then, $\Psi(m)$ has no coefficient in order n^{th}. Let us calculate the $(n-1)^{th}$-coefficient:

$$\xi_{n-1} = (rG(m) - r_i P(m))_{n-1} = (rG_{n-1} - r_i \frac{A}{(n-1)!}) \ ,$$

where G_{n-1} is the $(n-1)^{th}$-coefficient of the polynomial $G(m)$,

$$G_{n-1} = \frac{1}{g^{n-1} n!} n g^{n-1} ((r_i - 1)\mu_{max}(E) + \frac{d}{r} - C + r_i B) =$$

$$\frac{1}{(n-1)!}((r_i - 1)\mu_{max}(E) + \frac{d}{r} - C + r_i B) \leq$$

$$\frac{1}{(n-1)!}((r_i - 1)|\mu_{max}(E)| + \frac{d}{r} - C + r_i |B|) \leq$$

$$\frac{1}{(n-1)!}(r|\mu_{max}(E)| + \frac{d}{r} - C + r|B|) < \frac{-|A|}{(n-1)!} \ ,$$

last inequality coming from the definition of C in (2.1.11). Then

$$\xi_{n-1} < r(\frac{-|A|}{(n-1)!}) - r_i \frac{A}{(n-1)!} = \frac{-r|A| - r_i A}{(n-1)!} < 0$$

because $-r|A| - r_i A < 0$.

Therefore $\Psi(m) = \xi_{n-1} m^{n-1} + \cdots + \xi_1 m + \xi_0$ with $\xi_{n-1} < 0$, so there exists $m_2 \geq m_1$ such that for $m \geq m_2$ we will have $\Psi(m) < 0$ and $w_i < 0$, then the contradiction. ■

Proposition 2.1.19. *There exists an integer m_3 such that for $m \geq m_3$ the sheaves E_i^m and $E^{m,i} = E_i^m / E_{i-1}^m$ are m_3-regular. In particular their higher cohomology groups, after twisting with $\mathcal{O}_X(m_3)$, vanish and they are generated by global sections.*

Proof. Note that $\mu(E_i^m) \leq \mu_{\max}(E)$. Then, although E_i^m depends on m, its slope is bounded above and below by numbers which do not depend on m, (cf. Proposition 2.1.18) and furthermore it is a subsheaf of E. Hence, the set of possible isomorphism classes for E_i^m is bounded. Apply Serre Vanishing Theorem choosing $m_3 \geq m_2$. ■

Proposition 2.1.20. *Let $m \geq m_3$. For each filter E_i^m in the m-Kempf filtration, we have* $\dim V_i = h^0(E_i^m(m))$*, therefore* $V_i \cong H^0(E_i^m(m))$.

Proof. Let $V_\bullet \subseteq V$ be the Kempf filtration of V (cf. Theorem 2.1.5) and let $E_\bullet^m \subseteq E$ be the m-Kempf filtration of E. We know that each V_i generates the subsheaf E_i^m, by definition, then we have the following diagram:

$$\begin{array}{ccccccccc}
0 \subset & V_1 & \subset & V_2 & \subset \cdots \subset & V_{t+1} & = & V \\
 & \cap & & \cap & & & & || \\
 & H^0(E_1^m(m)) & \subset & H^0(E_2^m(m)) & \subset \cdots \subset & H^0(E_{t+1}^m(m)) & = & H^0(E(m))
\end{array}$$

Suppose that there exists an index i such that $V_i \neq H^0(E_i^m(m))$. Let i be the index such that $V_i \neq H^0(E_i^m(m))$ and $\forall j > i$ it is $V_j = H^0(E_j^m(m))$. Then we have the diagram:

$$\begin{array}{ccc}
V_i & \subset & V_{i+1} \\
\cap & & || \\
H^0(E_i^m(m)) & \subseteq & H^0(E_{i+1}^m(m))
\end{array} \tag{2.1.12}$$

Therefore $V_i \subsetneq H^0(E_i^m(m)) \subseteq V_{i+1}$ and we can consider a new filtration by adding the filter $H^0(E_i^m(m))$:

$$\begin{array}{ccccc}
V_i & \subset & H^0(E_i^m(m)) & \subset & V_{i+1} \\
|| & & || & & || \\
V'_i & \subset & V'_{i+1} & \subset & V'_{i+2}
\end{array} \tag{2.1.13}$$

Note that we are in situation of Lemma 2.1.16, where $W = H^0(E_i^m(m))$, filtration $V_\bullet$ is (2.1.12) and filtration $V'_\bullet$ is (2.1.13).

The graph associated to filtration $V_\bullet$, by Definition 2.1.12, is given by the points

$$(b_i, w_i) = \Big(\frac{\dim V_i}{m^n}, \frac{m}{\dim V}(r \dim V_i - r_i \dim V)\Big) ,$$

where the slopes of the graph are given by

$$-v_i = \frac{w^i}{b^i} = \frac{w_i - w_{i-1}}{b_i - b_{i-1}} =$$
$$\frac{m^{n+1}}{\dim V}\big(r - r^i \frac{\dim V}{\dim V^i}\big) \leq \frac{m^{n+1}}{\dim V} \cdot r := R$$

and equality holds if and only if $r^i = 0$.

Now, the new point which appears in the graph of the filtration $V'_\bullet$ is

$$Q = \Big(\frac{h^0(E^m_i(m))}{m^n}, \frac{m}{\dim V}(rh^0(E^m_i(m)) - r_i \dim V)\Big) .$$

Point Q joins two new segments appearing in this new graph. The slope of the segment between (b_i, w_i) and Q is, by a similar calculation,

$$-v'_{i+1} = \frac{m^{n+1}}{\dim V} \cdot r = R .$$

By Lemma 2.1.15, the graph is convex, so $v_1 < v_2 < \ldots < v_{t+1}$. As E^m_1 is a non-zero torsion free sheaf, it has positive rank $r_1 = r^1$ and so it follows $v_1 > -R$. On the other hand, by Lemma 2.1.16, $v'_{i+1} \geq v_{i+1}$. Hence

$$-R < v_1 < v_2 < \ldots < v_{i+1} \leq v'_{i+1} = -R ,$$

which is a contradiction.

Therefore, $\dim V_i = h^0(E^m_i(m))$, for every filter in the m-Kempf filtration. ∎

Corollary 2.1.21. *For every filter E^m_i in the m-Kempf filtration, it is $r^i = \operatorname{rk} E^m_i / E^m_{i-1} > 0$.*

Proof. In the proof of Proposition 2.1.20 we have seen that $r^i = 0$ is equivalent to $-v_i = R$. Then, the result follows from that because it is $r^1 = r_1 > 0$ and $-R < v_1 < v_2 < \ldots < v_{t+1}$. ∎

2.1.5 Proof of Theorem 2.1.6: the m-Kempf filtration stabilizes

In Proposition 2.1.19 we have seen that, for any $m \geq m_3$, all the filters E^m_i of the m-Kempf filtration of E are m_3-regular. Hence, $E^m_i(m_3)$ is generated by the subspace $H^0(E^m_i(m_3))$ of $H^0(E(m_3))$, and the filtration of sheaves

$$0 \subset E^m_1 \subset E^m_2 \subset \cdots \subset E^m_{t_m} \subset E^m_{t_m+1} = E$$

is the filtration associated to the filtration of vector spaces

$$0 \subset H^0(E^m_1(m_3)) \subset H^0(E^m_2(m_3)) \subset \cdots \subset H^0(E^m_{t_m}(m_3)) \subset H^0(E^m_{t_m+1}(m_3)) = H^0(E(m_3))$$

by the evaluation map (c.f. Lemma 1.2.13). Note that the dimension of the vector space $H^0(E(m_3))$ does not depend on m and, by Corollary 2.1.21, the length $t_m + 1$ of the m-Kempf filtration of E is, at most, equal to r, the rank of E, a bound which does not also depend on m. Note that, also because of Corollary 2.1.21, each subsheaf in the m-Kempf filtration of E is strictly contained in the following one, for $m \geq m_3$.

Definition 2.1.22. *We call m-**type** to the tuple of different Hilbert polynomials appearing in the m-Kempf filtration of E*

$$(P_1^m, \dots, P_{t_m+1}^m) \ ,$$

where $P_i^m := P_{E_i^m}$.

Note that $P^{i,m} := P_{E_i^m/E_{i-1}^m} = P_{E_i^m} - P_{E_{i-1}^m}$, so they are defined in terms of elements of each m-type.

Proposition 2.1.23. *For all integers $m \geq m_3$, the set of possible m-types*

$$\mathcal{P} = \left\{(P_1^m, \dots, P_{t_m+1}^m)\right\}$$

is finite.

Proof. Once we fix $V \cong H^0(E(m_3))$ of dimension $h^0(E(m_3))$ (which does not depend on m), all possible filtrations by vector subspaces of V are parametrized by a finite-type scheme. Therefore the set of all possible m-Kempf filtrations of E, for $m \geq m_3$, is bounded and $\mathcal{P}$ is finite. ∎

Recall that the vector v can be recovered from the filtration $V_\bullet \subset V$ and the vector Γ from the weights n_i. Then, given m, the m-Kempf filtration achieves the maximum for the Kempf function $\mu(V_\bullet, n_\bullet)$ (c.f. (2.1.7)), which is the same, by Proposition 2.1.13, that achieving the maximum for the function

$$\mu_v(\Gamma) = \frac{(\Gamma, v)}{||\Gamma||} \ ,$$

among all vectors v coming from filtrations $V_\bullet \subset V$ and vectors $\Gamma \in \mathcal{C} - \{0\}$, where

$$\mathcal{C} = \left\{x \in \mathbb{R}^{t+1} : x_1 < x_2 < \cdots < x_{t+1}\right\} .$$

By Definition 2.1.12 we associate a graph to the m-Kempf filtration, given by v_m. Recall that, by Lemma 2.1.15 the graph is convex, meaning $v_m \in \mathcal{C}$, which implies $\Gamma_{v_m} = v_m$ by Lemma 2.1.10. Then, given v_m associated to the m-Kempf filtration

$$\max_{\Gamma \in \overline{\mathcal{C}}} \mu_{v_m}(\Gamma) = \mu_{v_m}(\Gamma_{v_m}) = \frac{(\Gamma_{v_m}, v_m)}{||\Gamma_{v_m}||} = \frac{(v_m, v_m)}{||v_m||} = ||v_m|| \ , \tag{2.1.14}$$

where recall that we defined in Definition 2.1.12

$$v_{m,i} = m^{n+1} \cdot \frac{1}{\dim V^i \dim V}\left[r^i \dim V - r \dim V^i\right] ,$$

$$b^i_m = \frac{1}{m^n} \cdot \dim V^i \ ,$$

and, thanks to Propositions 2.1.19 and 2.1.20, we can rewrite

$$v_{m,i} = m^{n+1} \cdot \frac{1}{P^{i,m}(m)P(m)} \big[r^i P(m) - r P^{i,m}(m) \big] \ ,$$

$$b^i_m = \frac{1}{m^n} \cdot P^{i,m}(m) \ .$$

Let

$$v_{m,i}(l) = m^{n+1} \cdot \frac{1}{P^{i,m}(l)P(l)} \big[r^i P(l) - r P^{i,m}(l) \big]$$

be the coordinates of the graph associated to the m-Kempf filtration but where the polynomials are evaluated at another variable l. Let us define

$$\Theta_m(l) = (\mu_{v_m(l)}(\Gamma_{v_m(l)}))^2 = ||v_m(l)||^2 \ ,$$

where the second equality follows by an argument similar to (2.1.14). Note that $\Theta_m(l)$ is a rational function on l. Let

$$\mathcal{A} = \{\Theta_m : m \geq m_3\}$$

which is a finite set by Proposition 2.1.23. We say that $f_1 \prec f_2$ for two rational functions, if the inequality $f_1(l) < f_2(l)$ holds for $l \gg 0$, and let K be the maximal function in the finite set $\mathcal{A}$, with respect to the defined ordering.

Note that the value $\Theta_m(m)$ is the square of the maximum of the Kempf function $\mu_{v_m}(\Gamma)$, by (2.1.14), achieved for the maximal filtration $V_\bullet \subset V \simeq H^0(E(m))$ of vector spaces which gives the vector v_m. This weighted filtration is the only one which gives the value $\sqrt{\Theta_m(m)}$ for the Kempf function.

Lemma 2.1.24. *There exists an integer $m_4 \geq m_3$ such that $\forall m \geq m_4$, $\Theta_m = K$.*

Proof. Choose m_4 such that $K(l) \geq \Theta_m(l)$, $\forall l \geq m_4$ and every $\Theta_m \in \mathcal{A}$ with equality only when $\Theta_m = K$. Let $m \geq m_4$. Given that the Kempf function achieves the maximum over all possible filtrations and weights (c.f. Theorem 2.1.5), we have $\Theta_m(m) \geq K(m)$, because K is another rational function built with other m'-type, i.e., other values for the polynomials appearing on the rational function (c.f. Definition 2.1.22). Combining both inequalities we obtain $\Theta_m(m) = K(m)$ for all $m \geq m_4$. ∎

Proposition 2.1.25. *Let l_1 and l_2 be integers with $l_1 \geq l_2 \geq m_4$. Then the l_1-Kempf filtration of E is equal to the l_2-Kempf filtration of E.*

Proof. By construction, the filtration

$$H^0(E_1^{l_1}(l_1)) \subset H^0(E_2^{l_1}(l_1)) \subset \cdots \subset H^0(E_{t_1}^{l_1}(l_1)) \subset H^0(E_{t_1+1}^{l_1}(l_1)) = H^0(E(l_1)) \quad (2.1.15)$$

is the l_1-Kempf filtration of $V \simeq H^0(E(l_1))$. Now consider the filtration $V'_\bullet \subset V \simeq H^0(E(l_1))$ defined as follows

$$H^0(E_1^{l_2}(l_1)) \subset H^0(E_2^{l_2}(l_1)) \subset \cdots \subset H^0(E_{t_2}^{l_2}(l_1)) \subset H^0(E_{t_2+1}^{l_2}(l_1)) = H^0(E(l_1))\ . \quad (2.1.16)$$

We have to prove that (2.1.16) is in fact the l_1-Kempf filtration of $V \simeq H^0(E(l_1))$.

Since $l_1, l_2 \geq m_4$, by Lemma 2.1.24 we have $\Theta_{l_1} = \Theta_{l_2} = K$. Then, $\Theta_{l_1}(l_1) = \Theta_{l_2}(l_1)$ and, by uniqueness of the Kempf filtration (c.f. Theorem 2.1.5), the filtrations (2.1.15) and (2.1.16) coincide. Since, in particular $l_1, l_2 \geq m_3$, $E_i^{l_1}$ and $E_i^{l_2}$ are l_1-regular by Proposition 2.1.19. Hence, $E_i^{l_1}(l_1)$ and $E_i^{l_2}(l_1)$ are generated by their global sections (c.f. Lemma 1.2.13) $H^0(E_i^{l_1}(l_1))$ and $H^0(E_i^{l_2}(l_1))$, respectively. By the previous argument, $H^0(E_i^{l_1}(l_1)) = H^0(E_i^{l_2}(l_1))$, therefore $E_i^{l_1}(l_1) = E_i^{l_2}(l_1)$. By tensoring with $\mathcal{O}_X(-l_1)$, this implies that the filtrations $E_\bullet^{l_1} \subset E$ and $E_\bullet^{l_2} \subset E$ coincide. ■

Therefore, Theorem 2.1.6 follows from Proposition 2.1.25 and it is proved that, eventually, the Kempf filtration does not depend on the integer m.

Definition 2.1.26. *If $m \geq m_4$, the m-Kempf filtration of E is called* ***the Kempf filtration of*** *E,*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E\ .$$

2.1.6 Proof of Theorem 2.1.7: Kempf filtration is Harder-Narasimhan filtration

Recall that the Kempf theorem (c.f. Theorem 2.1.5) asserts that given an integer m and $V \simeq H^0(E(m))$, there exists a unique weighted filtration of vector spaces $V_\bullet \subseteq V$ which gives maximum for the Kempf function

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^{t+1} \frac{\Gamma_i}{\dim V}(r^i \dim V - r \dim V^i)}{\sqrt{\sum_{i=1}^{t+1} \dim V^i \Gamma_i^2}}\ .$$

This filtration induces a filtration of sheaves, called the Kempf filtration of E,

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E$$

which is independent of m, for $m \geq m_4$, by Proposition 2.1.25, hence it only depends on E. From now on, we assume $m \geq m_4$.

Based on the fact we can rewrite the Kempf function as a certain scalar product divided by a norm (c.f. Proposition 2.1.13), we shave seen that the Kempf filtration is encoded by a graph with two convexity properties (c.f. Lemmas 2.1.15 and 2.1.16). We can express the data related to the filtration of vector spaces with the data of the filtration of sheaves. Since $m \geq m_3$, the sheaves E_i and E^i are m-regular $\forall i$, and

$$\begin{aligned} \dim V_i &= h^0(E_i(m)) = P_{E_i}(m) =: P_i(m) \\ \dim V^i &= h^0(E^i(m)) = P_{E^i}(m) =: P^i(m) \end{aligned} \tag{2.1.17}$$

(c.f. Proposition 2.1.19 and Proposition 2.1.20). Recall that the Kempf function is a function on m, with order $m^{-\frac{n}{2}-1}$ at zero (c.f. Proposition 2.1.13) then we consider the function K, where

$$K(m) = m^{\frac{n}{2}+1} \cdot \mu(V_\bullet, m_\bullet) = \mu_{v_m}(\Gamma) \ .$$

Making the substitutions (2.1.17), and using the relation $\gamma_i = \frac{r}{P}\Gamma_i$ (c.f. (2.1.2) and (2.1.6)),

$$K(m) = m^{\frac{n}{2}+1} \cdot \frac{\sum_{i=1}^{t+1} \frac{\gamma_i}{r}[(r^i P - rP^i)]}{\sqrt{\sum_{i=1}^{t+1} P^i \frac{P^2}{r^2}\gamma_i^2}} \ ,$$

which is a function on m whose square is a rational function (since P and P^i are polynomials on m). Therefore we get

$$K(m) = m^{\frac{n}{2}+1} \cdot \frac{1}{P} \frac{\sum_{i=1}^{t+1} \gamma_i [r^i P - rP^i]}{\sqrt{\sum_{i=1}^{t+1} P^i \gamma_i^2}} \ .$$

Proposition 2.1.27. *Given a sheaf E, there exists a unique filtration*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E$$

with positive weights $n_1, \ldots, n_t$, $n_i = \frac{\gamma_{i+1}-\gamma_i}{r}$, which gives maximum for the function

$$K(m) = m^{\frac{n}{2}+1} \cdot \frac{\sum_{i=1}^{t+1} P^i \gamma_i [\frac{r^i}{P^i} - \frac{r}{P}]}{\sqrt{\sum_{i=1}^{t+1} P^i \gamma_i^2}} \ .$$

Similarly, we have defined the coordinates v_i (slopes of segments of the graph), as

$$v_i = m^{n+1} \cdot [\frac{r^i}{P^i} - \frac{r}{P}]$$

(c.f. Definition 2.1.12). Therefore we can express the function K as

$$K(m) = m^{-\frac{n}{2}} \cdot \frac{\sum_{i=1}^{t+1} P^i \gamma_i v_i}{\sqrt{\sum_{i=1}^{t+1} P^i \gamma_i^2}} = m^{-\frac{n}{2}} \cdot \frac{(\gamma, v)}{||\gamma||} \ ,$$

where the scalar product is given by the diagonal matrix

$$\begin{pmatrix} P^1 & & & 0 \\ & P^2 & & \\ & & \ddots & \\ 0 & & & P^{t+1} \end{pmatrix}$$

Finally, we use Lemmas 2.1.15 and 2.1.16 to show that the Kempf filtration verifies the two properties of the Harder-Narasimhan filtration for sheaves (c.f. Theorem 1.3.5) hence, by uniqueness, both filtrations have to coincide.

Proposition 2.1.28. *Given the Kempf filtration of a sheaf E,*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E$$

it verifies

$$\frac{P^1}{r^1} > \frac{P^2}{r^2} > \ldots > \frac{P^{t+1}}{r^{t+1}}$$

Proof. The coordinates of the vector v associated to the filtration are, for m large enough, $v_i = m^{n+1} \cdot (\frac{r^i}{P^i} - \frac{r}{P})$. Now apply Lemma 2.1.15 which says that v is convex, i.e. $v_1 < \ldots < v_{t+1}$. ∎

Proposition 2.1.29. *Given the Kempf filtration of a sheaf E,*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E \ ,$$

each one of the blocks $E^i = E_i / E_{i-1}$ is semistable.

Proof. Consider the graph associated to the Kempf filtration of E. Suppose that any of the blocks has a destabilizing subsheaf. Then, it corresponds to a point above of the graph of the filtration. The graph obtained by adding this new point is a refinement of the graph of the Kempf filtration, whose convex envelope is not the original graph, which contradicts Lemma 2.1.16. ∎

Corollary 2.1.30 (c.f. Theorem 2.1.7)**.** *The Kempf filtration of a sheaf E coincides with the Harder-Narasimhan filtration.*

Proof. By Propositions 2.1.28 and 2.1.29 the Kempf filtration verifies the two properties of the Harder-Narasimhan filtration. By uniqueness of the Harder-Narasimhan filtration (c.f. Theorem 1.3.5) both filtrations do coincide. ∎

2.2 Holomorphic pairs

In this section we prove the correspondence between the Kempf filtration and the Harder-Narasimhan filtration for holomorphic pairs. It follows the scheme of the proof given for torsion free coherent sheaves in section 2.1. First, we give some definitions and the notion of stability for the construction of the moduli space of holomorphic pairs. It can be deduced from the construction of the moduli space of tensors in section 1.2.

Let X be a smooth complex projective variety. Let us consider **holomorphic pairs**

$$(E, \varphi : E \to \mathcal{O}_X)$$

given by a coherent torsion free sheaf of rank r with fixed determinant $\det(E) \cong \Delta$ and a morphism to a the structure sheaf $\mathcal{O}_X$. Note that the definition of holomorphic pair coincides with the definition of tensor in Definition 1.2.1, with $s = 1$, $c = 1$, $b = 0$, $R = \operatorname{Spec} \mathbb{C}$ and $\mathcal{D} = \mathcal{O}_X$ is the structure sheaf over $X \times R \simeq X$.

A **weighted filtration** $(E_\bullet, n_\bullet)$ of a sheaf E of rank r is a filtration

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E, \tag{2.2.1}$$

and rational numbers $n_1, n_2, \ldots, n_t > 0$.

Let δ be a polynomial of degree at most $\dim X - 1$ and positive leading coefficient. We rephrase Definition 1.2.5 for the case of holomorphic pairs. See also the calculation made in (1.2.7).

Definition 2.2.1. *A holomorphic pair (E, φ) is δ-**semistable** if for all weighted filtrations $(E_\bullet, n_\bullet)$ (c.f. (2.2.1)),*

$$\sum_{i=1}^{t} n_i (r P_{E_i} - r_i P_E) + \delta \sum_{i=1}^{t} n_i \big(r_i - \epsilon(E_i) r\big) \leq 0 \ ,$$

*where $\epsilon(E_i) = 1$ if $\varphi|_{E_i} \neq 0$ and $\epsilon(E_i) = 0$ otherwise. If the strict inequality holds for every weighted filtration, we say that (E, φ) is δ-**stable**. If (E, φ) is not δ-semistable, we say that it is δ-**unstable**.*

Definition 2.2.2. *Given a holomorphic pair $(E, \varphi : E \to \mathcal{O}_X)$, let $(E', \varphi|_{E'})$ be a **subpair** where $E' \subset E$ is a subsheaf and $\varphi|_{E'}$ is the restriction of the morphism φ. Let $E'' = E/E'$ and define the holomorphic pair $(E'', \varphi|_{E''})$ where, if $\varphi|_{E'} \neq 0$, define $\varphi|_{E''} := 0$, and if $\varphi|_{E'} = 0$, $\varphi|_{E''}$ is the induced morphism in the quotient sheaf. We call $(E'', \varphi|_{E''})$ a **quotient pair** of (E, φ). For every pair $(E, \varphi : E \to \mathcal{O}_X)$, define $\epsilon(E) = 1$ if $\varphi|_E \neq 0$ and $\epsilon(E) = 0$ otherwise. Recall that we define a morphism of pairs $(E, \varphi) \to (F, \psi)$ as a morphism of sheaves $\alpha : E \to F$ such that $\psi \circ \alpha = \varphi$ (c.f. Definition 1.2.1).*

Definition 2.2.3. *Let $(E, \varphi : E \to \mathcal{O}_X)$ be a holomorphic pair. We define the **corrected Hilbert polynomial** of (E, φ) as*

$$\overline{P}_E := P_E - \delta\epsilon(E)$$

Note that the exact sequence of sheaves

$$0 \to E' \to E \to E'' \to 0$$

verify

$$\overline{P}_E = \overline{P}_{E'} + \overline{P}_{E''}$$

for the corrected polynomials.

Remark 2.2.4. *Note that the definition of quotient pair in Definition 2.2.2 does not imply that*

$$0 \to (E', \varphi|_{E'}) \to (E, \varphi) \to (E'', \varphi|_{E''}) \to 0$$

is an exact sequence in the category of tensors, where $E' \subset E$ and $E'' := E/E'$. Nonetheless, we keep that definition for the additivity of the corrected Hilbert polynomials to hold on exact sequences of sheaves.

From Definition 2.2.1 it can be directly deduced the following equivalent definition, which appears on [HL2, Definition 1.1].

Proposition 2.2.5. *A pair (E, φ) is δ-unstable if and only if there exists a subpair $(F, \varphi|_F)$ with $\frac{\overline{P}_F}{\operatorname{rk} F} > \frac{\overline{P}_E}{\operatorname{rk} E}$.*

Proof. If (E,φ) is δ-unstable, there exists a filtration

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E$$

and weights $n_i > 0$ such that

$$\sum_{i=1}^{t} n_i(rP_{E_i} - r_iP_E) + \delta \sum_{i=1}^{t} n_i\big(r_i - \epsilon(E_i)r\big) =$$

$$\sum_{i=1}^{t} n_i\big(r(P_{E_i} - \delta\epsilon(E_i)) - r_i(P_E - \delta)\big) = \sum_{i=1}^{t} n_i(r\overline{P}_{E_i} - r_i\overline{P}_E) > 0 \ .$$

As the weights n_i are positive, there exists any i such that

$$r\overline{P}_{E_i} - r_i\overline{P}_E > 0 \Leftrightarrow$$

$$\frac{\overline{P}_{E_i}}{\operatorname{rk} E_i} > \frac{\overline{P}_E}{\operatorname{rk} E} \ .$$

On the other hand, if there exists $(F,\varphi|_F)$ with $\frac{\overline{P}_F}{\operatorname{rk} F} > \frac{\overline{P}_E}{\operatorname{rk} E}$, the one-step filtration

$$0 \subset F \subset E$$

gives a positive quantity in the expression of Definition 2.2.1. Therefore (E,φ) is δ-unstable. ■

2.2.1 Moduli space of holomorphic pairs

We recall the construction of the moduli space of δ-semistable pairs with fixed polynomial P and fixed determinant $\det(E) \simeq \Delta$. This was done in [HL1] following Gieseker's ideas, and in [HL2] following Simpson's ideas. Here, we use Gieseker's method (although [HL1] assumes that X is a curve or a surface, thanks to Simpson's bound [Si1, Corollary 1.7], we can follow Gieseker's method for any dimension). As we said at the beginning of the section, the construction can be derived from the construction of a moduli space for tensors in section 1.2, where $s = 1$, $c = 1$, $b = 0$, $R = \operatorname{Spec}\mathbb{C}$ and $\mathcal{D} = \mathcal{O}_X$ is the structure sheaf over $X \times R \simeq X$, (c.f. Definition 1.2.1).

Let m be an integer, so that E is m-regular for all semistable E (c.f. [Ma1, Corollary 3.3.1 and Proposition 3.6]). Let V be a vector space of dimension $p := P(m)$. Given an isomorphism $V \cong H^0(E(m))$, we obtain a quotient

$$q : V \otimes \mathcal{O}_X(-m) \twoheadrightarrow E \ ,$$

hence a homomorphism

$$Q : \wedge^r V \cong \wedge^r H^0(E(m)) \longrightarrow H^0(\wedge^r(E(m))) \cong H^0(\Delta(rm)) =: A$$

and points

$$Q \in \mathrm{Hom}(\wedge^r V, A) \qquad \overline{Q} \in \mathbb{P}(\mathrm{Hom}(\wedge^r V, A)) \ .$$

The morphism $\varphi : E \longrightarrow \mathcal{O}_X$ induces a homomorphism

$$\Phi : V = H^0(E(m)) \longrightarrow H^0(\mathcal{O}_X(m)) =: B$$

and hence points

$$\Phi \in \mathrm{Hom}(V, B) \qquad \overline{\Phi} \in \mathbb{P}(\mathrm{Hom}(V, B)) \ .$$

If we change the isomorphism $V \cong H^0(E(m))$ by a homothecy, we obtain another point in the line defined by Q, but the point $\overline{Q}$ does not change, and similarly for $\overline{\Phi}$.

Two different isomorphisms $V \cong H^0(E(m))$ differ by an element of $SL(V)$, hence this group acts on the two projective spaces we have defined. We choose a polarization $\mathcal{O}(a_1, a_2)$ (c.f. (1.2.32)) to give a linearization of the action of $SL(V)$. By the Hilbert-Mumford criterion (c.f. Theorem 1.1.14), a point

$$(\overline{Q}, \overline{\Phi}) \in \mathbb{P}(\mathrm{Hom}(\wedge^r V, A)) \times \mathbb{P}(\mathrm{Hom}(V, B))$$

is **GIT semistable** with respect to the natural linearization on $\mathcal{O}(a_1, a_2)$ if and only if for all weighted filtrations it is

$$\mu(\overline{Q}, V_\bullet, n_\bullet) + \frac{a_2}{a_1}\mu(\overline{\Phi}, V_\bullet, n_\bullet) \leq 0 \ ,$$

where each numerical function μ is the calculation of the minimal relevant weight of the action of a 1-parameter subgroup Γ on each projective space. Recall from section 1.2 the correspondence between 1-parameter subgroups and weighted filtrations $(V_\bullet, n_\bullet)$.

Proposition 2.2.6. *A point $(\overline{Q}, \overline{\Phi})$ is* ***GIT*** *a_2/a_1**-semistable** if for all weighted filtrations $(V_\bullet, n_\bullet)$ we have*

$$\sum_{i=1}^{t} n_i(r \dim V_i - r_i \dim V) + \frac{a_2}{a_1} \sum_{i=1}^{t} n_i\big(\dim V_i - \epsilon_i(\overline{\Phi}) \dim V\big) \leq 0 \ .$$

Proof. C.f. Proposition 1.2.29. ∎

Recall that r_i is the rank of the subsheaf $E_i \subset E$ generated by V_i by the evaluation map. Also recall that, if j is the index giving minimum in (1.2.37), we will define $\epsilon_i(\overline{\Phi}, V_\bullet) = 1$ if $i \geq j$ and $\epsilon_i(\overline{\Phi}, V_\bullet) = 0$ otherwise. We will denote $\epsilon_i(\overline{\Phi}, V_\bullet)$ by just $\epsilon_i(\overline{\Phi})$ if the filtration $V_\bullet$ is clear from the context. Let us call $\epsilon^i(\overline{\Phi}) = \epsilon_i(\overline{\Phi}) - \epsilon_{i-1}(\overline{\Phi})$ and note that $\epsilon_i(\overline{\Phi}, V_\bullet) = \epsilon(E_{V_i})$, in Definition 2.2.2.

Remark 2.2.7. *Note that the definition of $\epsilon_i(\overline{\Phi})$ is independent of the weights $n_\bullet$ or the vector Γ associated to them. Indeed, $\epsilon_i(\overline{\Phi}, V_\bullet) = \epsilon(E_{V_i})$ only depends on the vanishing of the morphism φ on the subsheaves E_{V_i} (c.f. Definition 2.2.2).*

Theorem 2.2.8. *Let (E, φ) be a holomorphic pair. There exists an integer m_0 such that, for $m \geq m_0$, the associated point $(\overline{Q}, \overline{\Phi})$ is GIT a_2/a_1-semistable if and only if the pair is δ-semistable, where*

$$\frac{a_2}{a_1} = \frac{r\delta(m)}{P_E(m) - \delta(m)}$$

Proof. C.f. Theorem 1.2.31. ∎

Let (E, φ) be a δ-unstable holomorphic pair. Let m_0 be the integer in Theorem 2.2.8 (i.e. such that the δ-stability of the tensor coincides with the GIT stability). If necessary, choose another m_0 such that the sheaf E is m_0-regular.

Let $m \geq m_0$ be an integer and let V be a vector space of dimension $P(m) = h^0(E(m))$. Fix an isomorphism $V \simeq H^0(E(m))$. Given a filtration of vector subspaces $0 \subset V_1 \subset \cdots \subset V_{t+1} = V$ and positive numbers $n_1, \cdots, n_t > 0$, i.e., given a weighted filtration, we define now the function

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^t n_i(r \dim V_i - r_i \dim V) + \frac{a_2}{a_1}\sum_{i=1}^t n_i\big(\dim V_i - \epsilon_i(\overline{\Phi}) \dim V\big)(\leq)0}{\sqrt{\sum_{i=1}^{t+1} \dim V^i \Gamma_i^2}} ,$$

which is a **Kempf function** for this problem, as in the case of sheaves (c.f. Definition 1.4.4).

We can apply Theorem 2.1.5 to obtain

$$0 \subset V_1 \subset \cdots \subset V_{t+1} = V , \tag{2.2.2}$$

the **Kempf filtration** of V. Let

$$0 \subseteq (E_1^m, \varphi|_{E_1^m}) \subseteq (E_2^m, \varphi|_{E_2^m}) \subseteq \cdots (E_t^m, \varphi|_{E_t^m}) \subseteq (E_{t+1}^m, \varphi|_{E_{t+1}^m}) \subseteq (E, \varphi) \tag{2.2.3}$$

be the m-**Kempf filtration of the pair** (E,φ), where $E_i^m \subset E$ is the subsheaf generated by V_i under the evaluation map.

We will apply the same techniques as in section 2.1 to prove the following theorem:

Theorem 2.2.9. *There exists an integer $m' \gg 0$ such that the m-Kempf filtration of the holomorphic pair (E,φ) is independent of m, for $m \geq m'$.*

2.2.2 The m-Kempf filtration stabilizes with m

In this subsection we give a proof of Theorem 2.2.9, based on the same arguments as in the case of sheaves. As we did in section 2.1, we associate a graph to the m-Kempf filtration of a δ-unstable pair (E,φ), to relate the Kempf function with the function $\mu_v(\Gamma)$ in Theorem 2.1.9.

Definition 2.2.10. *Let $m \geq m_0$. Given $0 \subset V_1 \subset \cdots \subset V_{t+1} = V$ a filtration of vector spaces of V, let*

$$v_{m,i} = m^{n+1} \cdot \frac{1}{\dim V^i \dim V} \big[r^i \dim V - r \dim V^i + \frac{a_2}{a_1}(\epsilon^i(\overline{\Phi}) \dim V - \dim V^i) \big] \ ,$$

$$b_m^i = \frac{1}{m^n} \dim V^i > 0$$

$$w_m^i = -b_m^i \cdot v_{m,i} = m \cdot \frac{1}{\dim V} \big[r \dim V^i - r^i \dim V + \frac{a_2}{a_1}(\dim V^i - \epsilon^i(\overline{\Phi}) \dim V) \big] \ .$$

Also let

$$b_{m,i} = b_m^1 + \ldots + b_m^i = \frac{1}{m^n} \dim V_i \ ,$$

$$w_{m,i} = w_m^1 + \ldots + w_m^i = m \cdot \frac{1}{\dim V} \big[r \dim V_i - r_i \dim V + \frac{a_2}{a_1}(\dim V_i - \epsilon_i(\overline{\Phi}) \dim V) \big] \ .$$

We call the graph defined by points $(b_{m,i}, w_{m,i})$ the ***graph associated to the filtration*** *$V_\bullet \subset V$.*

Now, applying Proposition 2.1.13, we can identify as well the new Kempf function in Theorem 2.1.5,

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^t n_i (r \dim V_i - r_i \dim V) + \frac{a_2}{a_1} \sum_{i=1}^t n_i \big(\dim V_i - \epsilon_i(\overline{\Phi}) \dim V \big)}{\sqrt{\sum_{i=1}^{t+1} \dim V^i \Gamma_i^2}} \ ,$$

with the function in Theorem 2.1.9, where the coordinates of the graph are now given as in Definition 2.2.10.

We will use Lemmas 2.1.15 and 2.1.16, to give the analogous to Propositions 2.1.18 and 2.1.20 for the case of holomorphic pairs.

Let

$$C = \max\{r|\mu_{\max}(E)| + \frac{d}{r} + r|B| + |A| + \delta_{n-1}(n-1)! + 1\ ,\ 1\} \tag{2.2.4}$$

be positive constant, where δ_{n-1} is the $(n-1)^{th}$-degree coefficient of the polynomial $\delta(m)$ (if $\deg(\delta) < n-1$, then set $\delta_{n-1} = 0$).

Proposition 2.2.11. *Given a sufficiently large m, each filter E_i^m in the m-Kempf filtration of (E,φ) (cf. (2.2.3)) has slope $\mu(E_i^m) \geq \frac{d}{r} - C$.*

Proof. Choose an integer m_1 such that for $m \geq m_1$

$$[\mu_{max}(E) + gm + B]_+ = \mu_{max}(E) + gm + B$$

and

$$[\frac{d}{r} - C + gm + B]_+ = \frac{d}{r} - C + gm + B\ .$$

Let m_2 be such that $P_E(m) - \delta(m) > 0$ for $m \geq m_2$. Now consider $m \geq \max\{m_0, m_1, m_2\}$ and let

$$0 \subseteq (E_1^m, \varphi|_{E_1^m}) \subseteq (E_2^m, \varphi|_{E_2^m}) \subseteq \cdots (E_t^m, \varphi|_{E_t^m}) \subseteq (E_{t+1}^m, \varphi|_{E_{t+1}^m}) \subseteq (E,\varphi)$$

be the m-Kempf filtration of (E,φ).

Suppose we have a filter $E_i^m \subseteq E$, of rank r_i and degree d_i, such that $\mu(E_i^m) < \frac{d}{r} - C$. The subsheaf $E_i^m(m) \subset E(m)$ satisfies the estimate in Lemma 1.2.15,

$$h^0(E_i^m(m)) \leq \frac{1}{g^{n-1}n!}\big((r_i - 1)([\mu_{max}(E_i^m) + gm + B]_+)^n + ([\mu_{min}(E_i^m) + gm + B]_+)^n\big)\ ,$$

where $\mu_{max}(E_i^m(m)) = \mu_{max}(E_i^m) + gm$ and similarly for μ_{min}.

Note that $\mu_{max}(E_i^m) \leq \mu_{max}(E)$ and $\mu_{min}(E_i^m) \leq \mu(E_i^m) < \frac{d}{r} - C$, so

$$h^0(E_i^m(m)) \leq \frac{1}{g^{n-1}n!}\big((r_i - 1)([\mu_{max}(E) + gm + B]_+)^n + ([\frac{d}{r} - C + gm + B]_+)^n\big)\ ,$$

and, by choice of m,

$$h^0(E_i^m(m)) \leq \frac{1}{g^{n-1}n!}\big((r_i - 1)(\mu_{max}(E) + gm + B)^n + (\frac{d}{r} - C + gm + B)^n\big) = G(m)\ ,$$

where

$$G(m) = \frac{1}{g^{n-1}n!}\big[r_i g^n m^n + n g^{n-1}\big((r_i-1)\mu_{max}(E) + \frac{d}{r} - C + r_i B\big)m^{n-1} + \cdots\big] \ .$$

Recall that, by Definition 2.1.12, to the filtration (2.2.2) we associate a graph with heights, for each j

$$w_j = w^1 + \ldots + w^j = m \cdot \frac{1}{\dim V}\big[r \dim V_j - r_j \dim V + \frac{a_2}{a_1}(\dim V_j - \epsilon_j(\overline{\Phi}) \dim V)\big] \ .$$

We will show that $w_i < 0$ and will get a contradiction as in Proposition 2.1.18. Since $E_i^m(m)$ is generated by V_i under the evaluation map, it is $\dim V_i \leq h^0(E_i^m(m))$, hence

$$w_i = \frac{m}{\dim V}\big[r \dim V_i - r_i \dim V + \frac{a_2}{a_1}(\dim V_i - \epsilon_i(\overline{\Phi}) \dim V)\big] \leq$$

$$\frac{m}{P_E(m)}\big[r h^0(E_i^m(m)) - r_i P_E(m) + \frac{r\delta(m)}{P_E(m) - \delta(m)}(h^0(E_i^m(m)) - \epsilon_i(\overline{\Phi})P_E(m))\big] \leq$$

$$\frac{m}{P_E(m)}\big[rG(m) - r_i P_E(m) + \frac{r\delta(m)}{P_E(m) - \delta(m)}(G(m) - \epsilon_i(\overline{\Phi})P_E(m))\big] =$$

$$m \cdot \frac{\big[(P_E(m) - \delta(m))(rG(m) - r_i P_E(m)) + (r\delta(m))(G(m) - \epsilon_i(\overline{\Phi})P_E(m))\big]}{P_E(m)(P_E(m) - \delta(m))} \ .$$

Hence, $w_i < 0$ is equivalent to

$$\Psi(m) = (P_E(m) - \delta(m))(rG(m) - r_i P_E(m)) + (r\delta(m))(G(m) - \epsilon_i(\overline{\Phi})P_E(m)) < 0$$

and $\Psi(m) = \xi_{2n} m^{2n} + \xi_{2n-1} m^{2n-1} + \cdots + \xi_1 m + \xi_0$ is a $(2n)^{th}$-order polynomial. Let us calculate the higher order coefficient:

$$\xi_{2n} = (P_E(m) - \delta(m))_n (rG(m) - r_i P_E(m))_n + (r\delta(m))_n (G(m) - \epsilon_i(\overline{\Phi})P_E(m))_n =$$

$$(P_E(m) - \delta(m))_n (r\frac{r_i g}{n!} - r_i \frac{rg}{n!}) + 0 = 0 \ .$$

Then, $\Psi(m)$ has no coefficient in order $(2n)^{th}$. Let us calculate the $(2n-1)^{th}$-coefficient:

$$\xi_{2n-1} = (P_E(m) - \delta(m))_n (rG(m) - r_i P_E(m))_{n-1} + (r\delta(m))_{n-1} (G(m) - \epsilon_i(\overline{\Phi})P_E(m))_n =$$

$$\frac{rg}{n!}(rG_{n-1} - r_i \frac{A}{(n-1)!}) + r\delta_{n-1}(\frac{r_i g}{n!} - \epsilon_i(\overline{\Phi})\frac{rg}{n!})$$

where G_{n-1} is the $(n-1)^{th}$-coefficient of the polynomial $G(m)$,

$$G_{n-1} = \frac{1}{g^{n-1}n!} n g^{n-1}((r_i - 1)\mu_{max}(E) + \frac{d}{r} - C + r_i B) =$$

$$\frac{1}{(n-1)!}((r_i-1)\mu_{max}(E)+\frac{d}{r}-C+r_iB)\leq$$
$$\frac{1}{(n-1)!}((r_i-1)|\mu_{max}(E)|+\frac{d}{r}-C+r_i|B|)\leq$$
$$\frac{1}{(n-1)!}(r|\mu_{max}(E)|+\frac{d}{r}-C+r|B|)<\frac{-|A|}{(n-1)!}-\delta_{n-1}\ ,$$

last inequality coming from the definition of C in (2.2.4). Then

$$\xi_{2n-1}<\frac{rg}{n!}\big(r(\frac{-|A|}{(n-1)!}-\delta_{n-1})-r_i\frac{A}{(n-1)!}\big)+r\delta_{n-1}\big(\frac{r_ig}{n!}-\epsilon_i(\overline{\Phi})\frac{rg}{n!}\big)=$$
$$\frac{rg}{n!}\big[\big(\frac{-r|A|-r_iA}{(n-1)!}\big)-r\delta_{n-1}+\delta_{n-1}(r_i-\epsilon_i(\overline{\Phi})r)\big]=$$
$$\frac{rg}{n!}\big[\big(\frac{-r|A|-r_iA}{(n-1)!}\big)+\delta_{n-1}(r_i-(1+\epsilon_i(\overline{\Phi}))r)\big]<0$$

because $-r|A|-r_iA<0$, $r_i-(1+\epsilon_i(\overline{\Phi}))r<0$ and $\delta_{n-1}\geq 0$. Note that if $r_i=r$, then $\epsilon_i(\overline{\Phi})=\epsilon_{t+1}(\overline{\Phi})=1$.

Therefore $\Psi(m)=\xi_{2n-1}m^{2n-1}+\cdots+\xi_1m+\xi_0$ with $\xi_{2n-1}<0$, so there exists m_3 such that for $m\geq m_3$ we will have $\Psi(m)<0$ and $w_i<0$, then the contradiction. ∎

Now we can prove the following proposition in a similar way as we proved Proposition 2.1.19.

Proposition 2.2.12. *There exists an integer m_4 such that for $m\geq m_4$ the sheaves E_i^m and $E^{m,i}=E_i^m/E_{i-1}^m$ are m_4-regular. In particular their higher cohomology groups, after twisting with $\mathcal{O}_X(m_4)$, vanish and they are generated by global sections.*

Proposition 2.2.13. *Let $m\geq m_4$. For each filter E_i^m in the m-Kempf filtration of (E,φ) (c.f. (2.2.3)) we have $\dim V_i=h^0(E_i^m(m))$, therefore $V_i\cong H^0(E_i^m(m))$.*

Proof. Let $V_\bullet\subseteq V$ be the Kempf filtration of V (cf. Theorem 2.1.5) and let $(E_\bullet^m,\varphi|_{E_\bullet^m})\subseteq(E,\varphi)$ be the m-Kempf filtration of (E,φ) (cf. (2.2.2) and (2.2.3)). We know that each V_i generates the subsheaf E_i^m, by definition, then we have the following diagram:

$$\begin{array}{ccccccccc} 0\subset & V_1 & \subset & V_2 & \subset\cdots\subset & V_{t+1} & = & V \\ & \cap & & \cap & & & & \| \\ & H^0(E_1^m(m)) & \subset & H^0(E_2^m(m)) & \subset\cdots\subset & H^0(E_{t+1}^m(m)) & = & H^0(E(m)) \end{array}$$

Suppose there exists an index i such that $V_i \neq H^0(E_i^m(m))$. Let i be the index such that $V_i \neq H^0(E_i^m(m))$ and $\forall j > i$ it is $V_j = H^0(E_j^m(m))$. Then we have the diagram:

$$\begin{array}{ccc} V_i & \subset & V_{i+1} \\ \cap & & || \\ H^0(E_i^m(m)) & \subset & H^0(E_{i+1}^m(m)) \end{array} \tag{2.2.5}$$

Therefore $V_i \subsetneq H^0(E_i^m(m)) \subsetneq V_{i+1}$ and we can consider a new filtration by adding the filter $H^0(E_i^m(m))$:

$$\begin{array}{ccccc} V_i & \subset & H^0(E_i^m(m)) & \subset & V_{i+1} \\ || & & || & & || \\ V'_i & & V'_{i+1} & & V'_{i+2} \end{array} \tag{2.2.6}$$

Note that V_i and $H^0(E_i^m)$ generate the same sheaf E_i^m, hence we are in situation of Lemma 2.1.16, where $W = H^0(E_i^m)$, filtration $V_\bullet$ is (2.2.5) and filtration $V'_\bullet$ is (2.2.6).

The graph associated to filtration $V_\bullet$, by Definition 2.1.12, is given by the points

$$(b_i, w_i) = \Big(\frac{\dim V_i}{m^n}, \frac{m}{\dim V}\big(r \dim V_i - r_i \dim V + \frac{a_2}{a_1}(\dim V_i - \epsilon_i(\overline{\Phi}, V_\bullet)\dim V)\big)\Big) ,$$

where the slopes of the graph are given by

$$-v_i = \frac{w^i}{b^i} = \frac{w_i - w_{i-1}}{b_i - b_{i-1}} =$$
$$\frac{m^{n+1}}{\dim V}\big(r - r^i \frac{\dim V}{\dim V^i} + \frac{a_2}{a_1}(1 - \epsilon^i(\overline{\Phi}, V_\bullet)\frac{\dim V}{\dim V^i})\big) \leq$$
$$\frac{m^{n+1}}{\dim V}\big(r + \frac{a_2}{a_1}\big) := R$$

and equality holds if and only if $r^i = 0$. Here note that $r^i = 0$ implies $\epsilon^i(\overline{\Phi}, V_\bullet) = 0$.

Now, the new point which appears in the graph of the filtration $V'_\bullet$ is

$$Q = \Big(\frac{h^0(E_i^m(m))}{m^n}, \frac{m}{\dim V}(rh^0(E_i^m(m)) - r_i \dim V + \frac{a_2}{a_1}(h^0(E_i^m(m)) - \epsilon_i(\overline{\Phi}, V_\bullet)\dim V))\Big) ,$$

where we write $\epsilon_i(\overline{\Phi}, V_\bullet)$ instead of $\epsilon_i(\overline{\Phi}, V'_\bullet)$, because they are equal given that $V_i = V'_i$.

Point Q joins two new segments appearing in this new graph. The slope of the segment between (b_i, w_i) and Q is, by a similar calculation,

$$-v'_{i+1} = \frac{m^{n+1}}{\dim V}(r + \frac{a_2}{a_1}) = R \, .$$

By Lemma 2.1.15, the graph is convex, so $v_1 < v_2 < \ldots < v_{t+1}$. As E_1^m is a non-zero torsion free sheaf, it has positive rank $r_1 = r^1$ and hence it follows $v_1 > -R$.

Recall that, by definition, $\epsilon_i(\overline{\Phi}, V_\bullet)$ is equal to 1 if $\overline{\Phi}|_{V_i} \neq 0$ and 0 otherwise. Then, it is clear that

$$\begin{aligned} \epsilon_j(\overline{\Phi}, V'_\bullet) = \epsilon_j(\overline{\Phi}, V_\bullet) \quad , j \leq i \\ \epsilon_j(\overline{\Phi}, V'_\bullet) = \epsilon_{j-1}(\overline{\Phi}, V_\bullet) \quad , j > i, \end{aligned} \tag{2.2.7}$$

and note that $\epsilon_i(\overline{\Phi}, V_\bullet) = \epsilon_{i+1}(\overline{\Phi}, V'_\bullet)$. Then, the graph associated to $V'_\bullet \subset V$ is a refinement of the graph associated to Kempf filtration $V_\bullet \subset V$, therefore by Lemma 2.1.16, $v'_{i+1} \geq v_{i+1}$. Hence,

$$-R < v_1 < v_2 < \ldots < v_{i+1} \leq v'_{i+1} = -R \ ,$$

which is a contradiction.

Therefore, $\dim V_i = h^0(E_i^m(m))$, for every filter in the m-Kempf filtration. ■

Corollary 2.2.14. *For every filter E_i^m in the m-Kempf filtration of (E, φ) (c.f. (2.2.3)), it is $r^i > 0$.*

Proof. C.f. Corollary 2.1.21. ■

Now let us recall the results on subsection 2.1.5. By Proposition 2.2.12, for any $m \geq m_4$, all the filters E_i^m of the m-Kempf filtration of the pair (E, φ) are m_4-regular and hence, the sheaves of the m-Kempf filtration

$$0 \subset (E_1^m, \varphi|_{E_1^m}) \subset (E_2^m, \varphi|_{E_2^m}) \subset \cdots (E_t^m, \varphi|_{E_t^m}) \subset (E_{t+1}^m, \varphi|_{E_{t+1}^m}) \subset (E, \varphi)$$

are obtained by evaluating the filtration of vector subspaces

$$0 \subset H^0(E_1^m(m_4)) \subset H^0(E_2^m(m_4)) \subset \cdots \subset H^0(E_{t_m}^m(m_4)) \subset H^0(E_{t_m+1}^m(m_4)) = H^0(E(m_4))$$

(c.f. Lemma 1.2.13), of a unique vector space $H^0(E(m_4))$, whose dimension is independent of m. Note that, because of Corollary 2.2.14, each subpair in the m-Kempf filtration of (E, φ) is strictly contained in the following one, for $m \geq m_3$. Let

$$(P_1^m, \ldots, P_{t_m+1}^m)$$

be the m**-type** of the m-Kempf filtration of (E, φ) (c.f. Definition 2.1.22) and let

$$\mathcal{P} = \left\{ (P_1^m, \ldots, P_{t_m+1}^m) \right\}$$

be the set of possible m-types, which is a finite set by the same argument as in Proposition 2.1.23.

By Definition 2.2.10 we associate a graph to the m-Kempf filtration, given by v_m, which, thanks to Propositions 2.2.12 and 2.2.13, can be rewritten as

$$v_{m,i} = m^{n+1} \cdot \frac{1}{P^i_m(m)P(m)} \left[r^i P(m) - rP^i_m(m) + \frac{r\delta(m)}{P(m)-\delta(m)} (\epsilon^i(\overline{\Phi})P(m) - P^i_m(m)) \right] ,$$

$$b^i_m = \frac{1}{m^n} \cdot P^i_m(m) \ .$$

Define

$$v_{m,i}(l) = l^{n+1} \cdot \frac{1}{P^i_m(l)P(l)} \left[r^i P(l) - rP^i_m(l) + \frac{r\delta(l)}{P(l)-\delta(l)} (\epsilon^i(\overline{\Phi})P(l) - P^i_m(l)) \right] ,$$

the coordinates of the graph where the polynomials are evaluated on l and let

$$\Theta_m(l) = (\mu_{v_m(l)}(\Gamma_{v_m(l)}))^2 = ||v_m(l)||^2 \ ,$$

as in (2.1.14). Let $\mathcal{A}$ be the finite set (c.f. Proposition 2.1.23)

$$\mathcal{A} = \{\Theta_m : m \geq m_4\} \ .$$

Let K be a rational function which is maximal in $\mathcal{A}$ and, by a similar argument as in Lemma 2.1.24, there exists an integer m_5 with $\Theta_m = K$, $\forall m \geq m_5$. Finally, we can prove the following

Proposition 2.2.15. *Let l_1 and l_2 be integers with $l_1 \geq l_2 \geq m_5$. Then the l_1-Kempf filtration of E is equal to the l_2-Kempf filtration of the holomorphic pair (E, φ).*

Proof. C.f. Proposition 2.1.25. ■

Therefore, Theorem 2.2.9 follows from Proposition 2.2.15.

Definition 2.2.16. *If $m \geq m_5$, the m-Kempf filtration of (E,φ) is called **the Kempf filtration of** (E,φ),*

$$0 \subset (E_1, \varphi|_{E_1}) \subset (E_2, \varphi|_{E_2}) \subset \cdots (E_t, \varphi|_{E_t}) \subset (E_{t+1}, \varphi|_{E_{t+1}}) \subset (E, \varphi) \ .$$

2.2.3 Harder-Narasimhan filtration for holomorphic pairs

Let $m \geq m_5$. Kempf's theorem (c.f. Theorem 2.1.5) asserts that given $V \simeq H^0(E(m))$, there exists a unique weighted filtration of vector spaces $(V_\bullet, n_\bullet)$ which gives maximum for the Kempf function

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^{t+1} \frac{\Gamma_i}{\dim V}(r^i \dim V - r \dim V^i) + \frac{a_2}{a_1}\sum_{i=1}^{t+1} \frac{\Gamma_i}{\dim V}\left(\epsilon^i(\overline{\Phi}) \dim V - \dim V^i\right)}{\sqrt{\sum_{i=1}^{t+1} \dim V^i \Gamma_i^2}} \ .$$

This filtration induces a filtration of holomorphic subpairs, called the Kempf filtration of (E, φ),

$$0 \subset (E_1, \varphi|_{E_1}) \subset (E_2, \varphi|_{E_2}) \subset \cdots (E_t, \varphi|_{E_t}) \subset (E_{t+1}, \varphi|_{E_{t+1}}) \subset (E, \varphi) \ ,$$

which is independent of m, for $m \geq m_5$, by Theorem 2.2.9, hence it is unique.

We proceed in a similar way to Section 2.1 (c.f. Proof of Theorem 2.1.7), to rewrite the Kempf function for holomorphic pairs in terms of Hilbert polynomials of sheaves. Let $\epsilon^i := \epsilon^i(\overline{\Phi}) = \epsilon^i(\varphi)$ and note that $\epsilon^i = 1$ for the unique index i in the Kempf filtration such that $\varphi|_{E_i} \neq 0$ and $\varphi|_{E_{i-1}} = 0$, and $\epsilon^i = 0$ otherwise. Let us call this index j in the following.

Proposition 2.2.17. *Given a holomorphic pair $(E, \varphi : E \to \mathcal{O}_X)$, there exists a unique filtration*

$$0 \subset (E_1, \varphi|_{E_1}) \subset (E_2, \varphi|_{E_2}) \subset \cdots (E_t, \varphi|_{E_t}) \subset (E_{t+1}, \varphi|_{E_{t+1}}) \subset (E, \varphi)$$

with positive weights $n_1, \ldots, n_t$, which gives maximum for the function

$$K(m) = \frac{m^{\frac{n}{2}+1}}{P} \cdot \frac{\sum_{i=1}^{t+1} \gamma_i [(r^i P - r P^i) + \frac{r\delta}{P-\delta}(\epsilon^i P - P^i)]}{\sqrt{\sum_{i=1}^{t+1} P^i \gamma_i^2}} \ .$$

Similarly, we can express the function K in Proposition 2.2.17 as

$$K(m) = m^{-\frac{n}{2}} \cdot \frac{\sum_{i=1}^{t+1} P^i \gamma_i v_i}{\sqrt{\sum_{i=1}^{t+1} P^i \gamma_i^2}} = m^{-\frac{n}{2}} \cdot \frac{(\gamma, v)}{||\gamma||} \ ,$$

where the coordinates $v_{i,m}$ (slopes of segments of the graph), now are

$$v_i = m^{n+1} \cdot \frac{1}{P^i P}\left[r^i P - r P^i + \frac{r\delta}{P-\delta}(\epsilon^i P - P^i)\right]$$

and the scalar product is, again,

$$\begin{pmatrix} P^1 & & & \\ & P^2 & & \\ & & \ddots & \\ & & & P^{t+1} \end{pmatrix}$$

With Definition 2.2.3, the coordinates of the graph are

$$v_i = m^{n+1} \cdot \frac{r^i r}{P^i(P-\delta)} (\frac{\overline{P}_E}{r} - \frac{\overline{P}_{E^i}}{r^i}) \ ,$$

where $\overline{P}_{E^i} = P^i - \delta\epsilon^i$ is the corrected Hilbert polynomial of the quotient pair $(E^i, \varphi|_{E^i})$ (c.f. Definitions 2.2.2 and 2.2.3).

Now, we define a Harder-Narasimhan filtration for a holomorphic pair, analogously to the notion for torsion free sheaves, substituting the Hilbert polynomials by the corrected Hilbert polynomials.

Definition 2.2.18. *Given a holomorphic pair $(E, \varphi : E \to \mathcal{O}_X)$, a filtration*

$$0 \subset (E_1, \varphi|_{E_1}) \subset (E_2, \varphi|_{E_2}) \subset \cdots (E_t, \varphi|_{E_t}) \subset (E_{t+1}, \varphi|_{E_{t+1}}) \subset (E, \varphi)$$

is called a ***Harder-Narasimhan filtration*** *of (E, φ) if it satisfies these two properties, where $E^i := (E_i/E_{i-1}, \varphi|_{E_i/E_{i-1}})$,*

1. *The corrected Hilbert polynomials verify*

$$\frac{\overline{P}_{E^1}}{\operatorname{rk} E^i} > \frac{\overline{P}_{E^2}}{\operatorname{rk} E^2} > \ldots > \frac{\overline{P}_{E^{t+1}}}{\operatorname{rk} E^{t+1}}$$

2. *Every quotient pair $(E^i, \varphi|_{E^i})$ is δ-semistable as a holomorphic pair.*

Next, we prove the existence and uniqueness for the Harder-Narasimhan filtration of a holomorphic pair. The proof follows similarly to Theorem 1.3.5.

Theorem 2.2.19. *Every pair (E, φ) has a unique Harder-Narasimhan filtration.*

Lemma 2.2.20. *Let (E, φ) be a pair. Then, there exists a subsheaf $F \subseteq E$ such that for all subsheaves $G \subset E$, one has $\frac{\overline{P}_F}{\operatorname{rk} F} \geq \frac{\overline{P}_G}{\operatorname{rk} G}$, and in case of equality $G \subseteq F$. Moreover, F is uniquely determined and $(F, \varphi|_F)$ is δ-semistable, called the* ***maximal destabilizing subpair of*** *(E, φ).*

Proof. The last two assertions follow from the first, where note that being δ-semistable can be checked by subpairs, by Lemma 2.2.5.

Define an order relation on the set of subpairs of (E,φ) by $(F_1,\varphi|_{F_1}) \le (F_2,\varphi|_{F_2})$ if and only if $F_1 \subset F_2$ and $\frac{\overline{P}_{F_1}}{\operatorname{rk} F_1} \le \frac{\overline{P}_{F_2}}{\operatorname{rk} F_2}$. Every ascending chain is bounded by (E,φ), then by Zorn's Lemma, for every subpair $(F,\varphi|_F)$ there exists a $F \subset F' \subset E$ such that $(F',\varphi|_{F'})$ is maximal with respect to $\le$. Let $(F,\varphi|_F)$ be $\le$-maximal with F of minimal rank among all maximal subpairs and let us show that $(F,\varphi|_F)$ is the maximal destabilizing subpair.

Suppose that $\exists\ G \subset E$ with $\frac{\overline{P}_G}{\operatorname{rk} G} \ge \frac{\overline{P}_F}{\operatorname{rk} F}$. First, we show that we can assume $G \subset F$ by replacing G by $G \cap F$. Indeed, if $G \not\subseteq F$, then F is a proper subsheaf of $F+G$ and hence $\frac{\overline{P}_F}{\operatorname{rk} F} > \frac{\overline{P}_{F+G}}{\operatorname{rk} F+G}$, by definition of F. Let the exact sequence

$$0 \to F \cap G \to F \oplus G \to F + G \to 0$$

out of which we get

$$P_F + P_G = P_{F \oplus G} = P_{F \cap G} + P_{F+G}$$

and

$$\operatorname{rk}(F) + \operatorname{rk}(G) = \operatorname{rk}(F \oplus G) = \operatorname{rk}(F \cap G) + \operatorname{rk}(F+G)\ .$$

Calculating we have

$$\operatorname{rk}(F \cap G)\Big(\frac{P_G}{\operatorname{rk} G} - \frac{P_{F\cap G}}{\operatorname{rk}(F \cap G)} =$$

$$\operatorname{rk}(F+G)\Big(\frac{P_{F+G}}{\operatorname{rk}(F+G)} - \frac{P_F}{\operatorname{rk} F}\Big) + (\operatorname{rk}(G) - \operatorname{rk}(F \cap G))\Big(\frac{P_F}{\operatorname{rk} F} - \frac{P_G}{\operatorname{rk} G}\Big)\ .$$

Using

$$\epsilon(F \cap G) + \epsilon(F+G) \le \epsilon(F) + \epsilon(G)$$

we get

$$(P_F - \delta\epsilon(F)) + (P_G) - \delta\epsilon(G)) = (P_{F\cap G} - \delta\epsilon(F \cap G)) + (P_{F+G} - \delta\epsilon(F+G))$$

and, similarly,

$$\operatorname{rk}(F \cap G)\Big(\frac{\overline{P}_G}{\operatorname{rk} G} - \frac{\overline{P}_{F\cap G}}{\operatorname{rk}(F \cap G)}\Big) \le$$

$$\operatorname{rk}(F+G)\Big(\frac{\overline{P}_{F+G}}{\operatorname{rk}(F+G)} - \frac{\overline{P}_F}{\operatorname{rk} F}\Big) + (\operatorname{rk}(G) - \operatorname{rk}(F \cap G))\Big(\frac{\overline{P}_F}{\operatorname{rk} F} - \frac{\overline{P}_G}{\operatorname{rk} G}\Big)\ .$$

Then, using the inequalities $\frac{\overline{P}_F}{\operatorname{rk} F} \le \frac{\overline{P}_G}{\operatorname{rk} G}$ and $\frac{\overline{P}_F}{\operatorname{rk} F} > \frac{\overline{P}_{F+G}}{\operatorname{rk}(F+G)}$, we obtain

$$\frac{\overline{P}_G}{\operatorname{rk} G} - \frac{\overline{P}_{F\cap G}}{\operatorname{rk}(F \cap G)} < 0$$

and hence

$$\frac{\overline{P}_F}{\operatorname{rk} F} < \frac{\overline{P}_{F\cap G}}{\operatorname{rk}(F\cap G)} ,$$

hence we can suppose that $G \subset F$.

Let $G \subset F$ with $\frac{\overline{P}_G}{\operatorname{rk} G} > \frac{\overline{P}_F}{\operatorname{rk} F}$ such that $(G, \varphi|_G)$ is $\leq$-maximal in $(F, \varphi|_F)$. Then let $(G', \varphi|_{G'}) \geq (G, \varphi|_G)$ to be $\leq$-maximal in (E, φ). We obtain, $\frac{\overline{P}_F}{\operatorname{rk} F} < \frac{\overline{P}_G}{\operatorname{rk} G} \leq \frac{\overline{P}_{G'}}{\operatorname{rk} G'}$ and, by maximality of $(G', \varphi|_{G'})$ and $(F, \varphi|_F)$ it is $G' \nsubseteq F$, since otherwise it would be $\operatorname{rk} G < \operatorname{rk} F$ which contradicts the minimality of $\operatorname{rk} F$, therefore F is a proper subsheaf of $F + G'$. Then we obtain $\frac{\overline{P}_F}{\operatorname{rk} F} > \frac{\overline{P}_{F+G'}}{\operatorname{rk}(F+G')}$ and the inequalities $\frac{\overline{P}_F}{\operatorname{rk} F} < \frac{\overline{P}_{G'}}{\operatorname{rk} G'}$ and $\frac{\overline{P}_F}{\operatorname{rk} F} > \frac{\overline{P}_{F+G'}}{\operatorname{rk}(F+G')}$ give

$$\frac{\overline{P}_{F\cap G'}}{\operatorname{rk}(F\cap G')} > \frac{\overline{P}_{G'}}{\operatorname{rk} G'} \geq \frac{\overline{P}_G}{\operatorname{rk} G} .$$

Therefore, as $G \subset F \cap G' \subset F$, we get a contradiction. ■

Proof of the Theorem 2.2.19. With the previous Lemma we are able to show the existence of a Harder-Narasimhan filtration for (E, φ). Let $(E_1, \varphi|_{E_1})$ the maximal destabilizing subpair and suppose that the corresponding quotient $(E/E_1, \varphi|_{E/E_1})$ has a Harder-Narasimhan filtration,

$$0 \subset G_0 \subset G_1 \subset \ldots \subset G_{t-1} = E/E_1 ,$$

by induction. We define E_{i+1} to be the pre-image of G_1 and it is $\frac{\overline{P}_{E_1}}{\operatorname{rk} E_1} > \frac{\overline{P}_{E_2/E_1}}{\operatorname{rk} E_2/E_1}$ because, if not, we would have $\frac{\overline{P}_{E_1}}{\operatorname{rk} E_1} \leq \frac{\overline{P}_{E_2}}{\operatorname{rk} E_2}$, contradicting the maximality of $(E_1, \varphi|_{E_1})$.

To show the uniqueness, suppose that $E_\bullet$ and $E'_\bullet$ are two Harder-Narasimhan filtrations of (E, φ) and consider, without loss of generality, that $\frac{\overline{P}_{E'_1}}{\operatorname{rk} E'_1} \geq \frac{\overline{P}_{E_1}}{\operatorname{rk} E_1}$. Call j an index which is minimal such that $E'_1 \subset E_j$. The composition

$$E'_1 \to E_j \to E_j/E_{j-1}$$

is a non-trivial homomorphism of semistable sheaves which implies

$$\frac{\overline{P}_{E_j/E_{j-1}}}{\operatorname{rk} E_j/E_{j-1}} \geq \frac{\overline{P}_{E'_1}}{\operatorname{rk} E'_1} \geq \frac{\overline{P}_{E_1}}{\operatorname{rk} E_1} \geq \frac{\overline{P}_{E_j/E_{j-1}}}{\operatorname{rk} E_j/E_{j-1}} ,$$

where first inequality comes from the fact that if there exists a non-trivial homomorphism between semistable pairs, then the corrected Hilbert polynomial of the target is greater or equal than the one of the first pair. Hence, equality holds everywhere, implying $j = 1$ so that $E'_1 \subset E_1$. Then, by semistability of the pair $(E_1, \varphi|_{E_1})$, it is $\frac{\overline{P}_{E'_1}}{\operatorname{rk} E'_1} \leq \frac{\overline{P}_{E_1}}{\operatorname{rk} E_1}$, and we can repeat the argument interchanging the roles of E_1 and E'_1 to show that $E_1 = E'_1$. By induction we can assume that uniqueness holds for the Harder-Narasimhan filtration of $(E/E_1, \varphi|_{E/E_1})$. This implies that $E'_i/E_1 = E_i/E_1$ and completes the proof. ∎

Now we will give the analogous to Propositions 2.1.28 and 2.1.29.

Proposition 2.2.21. *Given the Kempf filtration of a holomorphic pair* (E,φ)*,*

$$0 \subset (E_1, \varphi|_{E_1}) \subset (E_2, \varphi|_{E_2}) \subset \cdots (E_t, \varphi|_{E_t}) \subset (E_{t+1}, \varphi|_{E_{t+1}}) \subset (E, \varphi)$$

it verifies

$$\frac{\overline{P}_{E^1}}{\operatorname{rk} E^i} > \frac{\overline{P}_{E^2}}{\operatorname{rk} E^2} > \ldots > \frac{\overline{P}_{E^{t+1}}}{\operatorname{rk} E^{t+1}} .$$

Proof. Let j be the unique index such that $\epsilon^j = 1$. By Lemma 2.1.15 it is

$$v_1 < v_2 < \ldots v_{j-1} < v_j < v_{j+1} < \ldots < v_{t+1} .$$

Note that for $i \neq j$ it is $\overline{P}^i = P^i - \delta\epsilon^i = P^i$, hence $v_{i-1} < v_i$ implies $\frac{\overline{P}_{E^{i-1}}}{\operatorname{rk} E^{i-1}} > \frac{\overline{P}_{E^i}}{\operatorname{rk} E^i}$ for all $i \neq j, j+1$.

Now the inequality $v_{j-1} < v_j$ is

$$\frac{r^{j-1} r}{P^{j-1}(P-\delta)}\Big(\frac{P-\delta}{r} - \frac{P^{j-1}}{r^{j-1}}\Big) < \frac{r^j r}{P^j(P-\delta)}\Big(\frac{P-\delta}{r} - \frac{P^j - \delta}{r^j}\Big)$$

or, equivalently,

$$-\delta \frac{rP^{j-1}}{P-\delta} < P^{j-1}r^j - P^j r^{j-1} .$$

The function $\frac{rP^{j-1}}{P-\delta}$ is a homogeneous rational function whose limit at infinity is r^{j-1}, hence for large values of the variable we obtain this inequality between the polynomials

$$-\delta r^{j-1} < P^{j-1}r^j - P^j r^{j-1} ,$$

which is equivalent to $\frac{\overline{P}_{E^{j-1}}}{\operatorname{rk} E^{j-1}} > \frac{\overline{P}_{E^j}}{\operatorname{rk} E^j}$. A similar argument proves that $\frac{\overline{P}_{E^j}}{\operatorname{rk} E^j} > \frac{\overline{P}_{E^{j+1}}}{\operatorname{rk} E^{j+1}}$. ∎

Proposition 2.2.22. *Given the Kempf filtration of a holomorphic pair* (E,φ)*,*

$$0 \subset (E_1, \varphi|_{E_1}) \subset (E_2, \varphi|_{E_2}) \subset \cdots (E_t, \varphi|_{E_t}) \subset (E_{t+1}, \varphi|_{E_{t+1}}) \subset (E, \varphi)$$

each one of the quotient pairs $(E^i, \varphi|_{E^i})$ *is semistable as a holomorphic pair.*

Proof. Suppose that any of the blocks has a destabilizing subpair and apply a similar argument to the one in Proposition 2.1.29. ■

Hence, having seen the convexity properties of the Kempf filtration in Propositions 2.2.21 and 2.2.22, we get that the Kempf filtration of a holomorphic pair (E, φ) is a Harder-Narasimhan filtration. Given that every holomorphic pair has a unique Harder-Narasimhan filtration by Theorem 2.2.19, therefore it will be the same that the Kempf filtration.

Corollary 2.2.23. *Let (E, φ) be a δ-unstable holomorphic pair. The Kempf filtration is the same that the Harder-Narasimhan filtration.*

Proof. By Propositions 2.2.21 and 2.2.22 the Kempf filtration is a Harder-Narasimhan filtration, which is unique by Theorem 2.2.19, hence both filtrations are the same. ■

2.3 Higgs sheaves

Here we consider the moduli space of Higgs sheaves constructed by Simpson in [Si1, Si2] and use the techniques of the previous sections to prove the analogous result in this case, the correspondence between Kempf and Harder-Narasimhan filtrations.

Let X be a smooth complex projective variety of dimension n. A **Higgs sheaf** is a pair (E, φ) where E is a coherent sheaf over X and $\varphi : E \to E \otimes \Omega^1_X$ verifying $\varphi \wedge \varphi = 0$, a morphism called the **Higgs field**. We call (E, φ) a **Higgs bundle** if E is a locally free sheaf. Recall that $\Omega^1_X = T^*X$, the cotangent bundle.

We say that a Higgs sheaf (E, φ) is **semistable** (in the sense of Gieseker) if for all proper subsheaves $F \subset E$, preserved by φ (i.e. $\varphi\big|_F : F \to F \otimes \Omega^1_X$) we have

$$\frac{P_F}{\operatorname{rk} F} \leq \frac{P_E}{\operatorname{rk} E} \ ,$$

where P_E and P_F are the respective Hilbert polynomials of E and F. We say that (E, φ) is **stable** if we have a strict inequality for every subsheaf preserved by φ.

A Higgs field can be thought as a coherent sheaf $\mathcal{E}$ on the cotangent bundle T^*X, supported on the spectral curve (the eigenvalues of the Higgs field). Note that, to define a sheaf of $\mathcal{O}_{T^*X}$-modules on the total space of T^*X we have to determine how to multiply a section by a function on the vertical variables, which is given by the Higgs field, by definition.

Let Z be a projective completion of T^*X, $Z = \mathbb{P}(T^*X \oplus \mathcal{O})$, and $D = Z - T^*X$, the divisor at infinity.

Lemma 2.3.1. *[Si2, Lemma 6.8] A Higgs sheaf (E, φ) on X is the same thing that a coherent sheaf $\mathcal{E}$ on Z such that $\mathrm{Supp}(E) \cap D = \emptyset$, where $E = \pi_* \mathcal{E}$ and $\pi : T^*X \to X$. This identification is compatible with morphisms, giving an equivalence of categories. The condition that E is torsion free is the same that $\mathcal{E}$ is of pure dimension $n = \dim(X)$.*

Choose k so that

$$\mathcal{O}_Z(1) \stackrel{def}{=} \pi^* \mathcal{O}_X(k) \otimes_{\mathcal{O}_Z} \mathcal{O}_Z(D)$$

is ample on Z (c.f. [Ha, Appendix A, Theorem 5.1]). In particular, $\mathcal{O}_{T^*X}(1) = \pi^* \mathcal{O}_X(k)$. Thus, for any coherent sheaf $\mathcal{E}$ on Z with support not meeting D, the Hilbert polynomials of $\mathcal{E}$ and $\pi_* \mathcal{E}$ differ by rescaling on the variable m

$$P_{\mathcal{E}}(m) = P_{\pi_* \mathcal{E}}(km) \ .$$

Recall that, the condition for E to be torsion free is equivalent to $\mathcal{E}$ being pure of dimension n. To relate the stability of a Higgs sheaf (E, φ) with the stability of the associated sheaf $\mathcal{E}$, we have to modify Definition 2.1.1 as in [Si1, Si2], which was stated only for torsion free sheaves. Recall the expression of the Hilbert polynomial of a sheaf $\mathcal{E}$, in (1.2.11),

$$P_{\mathcal{E}}(m) = \frac{rg}{n!} m^n + \frac{d + r\alpha_{n-1}}{(n-1)!} m^{n-1} + ...$$

We define $r = \mathrm{rk}\, \mathcal{E}$, the rank of $\mathcal{E}$, such that the coefficient of the leading term of the Hilbert polynomial is $\frac{rg}{n!}$. We also define $d = \deg \mathcal{E}$, the degree of $\mathcal{E}$, as the corresponding coefficient appearing in the expression. A coherent sheaf $\mathcal{E}$ is of pure dimension n if it has no subsheaves supported on a lower dimensional locus.

Definition 2.3.2. *A coherent sheaf $\mathcal{E}$ on X is called* ***semistable*** *if it is pure of dimension n, and for all proper subsheaves $\mathcal{F} \subset \mathcal{E}$, it is*

$$\frac{P_{\mathcal{F}}}{\mathrm{rk}\, \mathcal{F}} \leq \frac{P_{\mathcal{E}}}{\mathrm{rk}\, \mathcal{E}} \ .$$

If strict inequality holds for every proper subsheaf, we say that $\mathcal{E}$ is ***stable****. If (E, φ) is not semistable, we say that it is* ***unstable****.*

Given that the Higgs subsheaves of (E,φ) correspond to the coherent subsheaves of $\mathcal{E}$, and since a subsheaf of $\mathcal{E}$ is the same that a subsheaf of $\pi_*\mathcal{E}$ preserved by the action of the symmetric algebra $\mathrm{Sym}^*(T^*X)$, we have the following

Corollary 2.3.3. (E,φ) *is a semistable Higgs sheaf if and only if the corresponding sheaf* $\mathcal{E}$*, by Lemma 2.3.1, is semistable as a coherent sheaf (c.f. Definition 2.3.2).*

2.3.1 Moduli space of Higgs sheaves

Given a polynomial P, we denote by k^*P the polynomial such that $k^*P(m) = P(km)$. Denote by $\mathfrak{M}(\mathcal{O}_{T^*X}, k^*P)$ be the moduli space of coherent sheaves $\mathcal{E}$ over $\mathcal{O}_{T^*X}$ with Hilbert polynomial k^*P. By [Si1, §1], Lemma 2.3.1 and Corollary 2.3.3, the scheme $\mathfrak{M}(\mathcal{O}_{T^*X}, k^*P)$ corepresents the functor $\mathcal{M}_{Higgs}(X,P)$ which associates a scheme S to the set of isomorphism classes of semistable Higgs sheaves (E,φ) on X, over S, with Hilbert polynomial P. Therefore, we put

$$\mathfrak{M}_{Higgs}(X,P) = \mathfrak{M}(\mathcal{O}_{T^*X}, k^*P)$$

and let us construct the scheme $\mathfrak{M}(\mathcal{O}_{T^*X}, k^*P)$ following Simpson's method (c.f. [Si1, §1]).

Let P be a polynomial of degree $n = \dim X$. There exists an integer N, sufficiently large, such that for $m \geq N$, $\mathcal{E}(m)$ is generated by global sections and $h^i(\mathcal{E}(m)) = 0$ for $i > 0$. Then, choose $m \geq N$ and fix an isomorphism

$$\alpha : V \simeq \mathbb{C}^{k^*P(m)} = \mathbb{C}^{P(km)}$$

to obtain a quotient

$$q : V \otimes \mathcal{W} \twoheadrightarrow \mathcal{E} \ ,$$

where $\mathcal{W} = \mathcal{O}_Z(-m)$. Let $\mathcal{H}$ be the Hilbert scheme of quotients

$$\mathcal{H} = \mathrm{Hilb}(V \otimes \mathcal{W}, k^*P) = \{V \otimes \mathcal{W} \to \mathcal{E} \to 0\} \ ,$$

with $P_{\mathcal{E}}(m) = P(km) = k^*P(m)$. We define an embedding of this Hilbert scheme to a projective space. Let $l \gg m$ be an integer such that $H^1(\mathrm{Ker}(V \otimes \mathcal{W} \twoheadrightarrow \mathcal{E})(l)) = 0$. Then, q induces the following homomorphisms

$$q : V \otimes \mathcal{W}(l) \twoheadrightarrow \mathcal{E}(l)$$

$$q' : V \otimes H^0(\mathcal{W}(l)) \twoheadrightarrow H^0(\mathcal{E}(l))$$
$$q'' : \bigwedge^{P(kl)} (V \otimes H^0(\mathcal{W}(l)) \twoheadrightarrow \bigwedge^{P(kl)} H^0(\mathcal{E}(l)) \simeq \mathbb{C}$$

Hence, it defines a Grothendieck's embedding on the Grassmannian manifold

$$\mathcal{H} = \mathrm{Hilb}(V \otimes \mathcal{W}, k^*P(m)) \overset{\mathcal{L}_{l,m}}{\hookrightarrow} \mathbb{P}(\bigwedge^{P(kl)} (V^\vee \otimes H^0(\mathcal{W}(l))^\vee)) ,$$

where $\mathcal{L}_{l,m}$ is the very ample line bundle (depending on l and m) given by the pullback of the canonical line bundle on the Grassmannian by the embedding. Note that, given a point $q \in \mathcal{H}$, if $H^0(q(m)) : V \to H^0(\mathcal{E}(m))$ is an isomorphism, we can recover the sheaf $\mathcal{E}$ together with the isomorphism $\alpha : V \simeq \mathbb{C}^{P(km)}$.

The group $GL(V)$ of changes of isomorphism $V \simeq \mathbb{C}^{P(km)}$, acts on $\mathcal{H}$ and the line bundle $\mathcal{L}_{l,m}$. Note that, if we change the isomorphism by a homothecy we obtain a different point in the line bundle defined by q, hence the point $\overline{q}$ in the projective space is the same. Hence, as in the case of the Gieseker embedding (c.f. subsection 2.1.1), we can get rid of the choice of isomorphism by dividing by the action of $SL(V)$. Let $Q \subset \mathcal{H}$ the $SL(V)$-invariant open subset of quotients where $\mathcal{E}$ is a semistable sheaf of pure dimension $n = \dim X$ and the induced morphism $\alpha : V \simeq \mathbb{C}^{P(km)}$ is an isomorphism. There exists a good quotient (c.f. Definition 1.1.7)

$$\mathfrak{M}(\mathcal{O}_Z, k^*P) = Q/SL(V) ,$$

(c.f. [Si1, Theorem 1.19] and [Mu]). Let $Q' \subset Q$ be the subset of those quotient sheaves $\mathcal{E}$ whose support does not meet D (which is also $SL(V)$-invariant and is, set-theoretically, the inverse image of a subset of $\mathfrak{M}(\mathcal{O}_Z, k^*P)$). Therefore, a good quotient $Q'/SL(V)$ exists and it is equal to an open subset which we denote $\mathfrak{M}(\mathcal{O}_{T^*X}, k^*P) \subset \mathfrak{M}(\mathcal{O}_Z, k^*P)$.

As we said before, $\mathfrak{M}(\mathcal{O}_{T^*X}, k^*P)$ corepresents the functor $\mathcal{M}_{Higgs}(X, P)$, hence,

$$\mathfrak{M}_{Higgs}(X, P) = \mathfrak{M}(\mathcal{O}_{T^*X}, k^*P) .$$

Therefore, to construct a moduli space for Higgs sheaves (E, φ), where P is the fixed Hilbert polynomial of E, we construct a particular moduli space of sheaves. From now on, let us consider semistable coherent sheaves $\mathcal{E}$ over $Z = \mathbb{P}(T^*X \oplus \mathcal{O})$ of pure dimension $n = \dim X$ and fixed Hilbert polynomial P. We consider the construction of the moduli space of sheaves following Simpson's method. Giving a sheaf $\mathcal{E}$ and an isomorphism $V \cong H^0(\mathcal{E}(m))$, we obtain a point $\overline{q}$ in the GIT space of parameters. We have to prove that a point $\overline{q}$ is GIT semistable if and only if it corresponds to a semistable sheaf $\mathcal{E}$, to conclude that a moduli space for semistable sheaves can be obtained as the good quotient of the space of GIT semistable points, by the group $SL(V)$.

Remark 2.3.4. *Note that the sheaves $\mathcal{E}$ are torsion sheaves supported on a subscheme of $Z = \mathbb{P}(T^*X \oplus \mathcal{O})$. In this case we cannot use Gieseker's embedding in the construction of the moduli space, because if we take $\wedge^r \mathcal{E}$ we get the zero sheaf (c.f. subsection 2.1.1). Simpson develops his method based on Grothendieck's ideas which gives a solution to the problem in this case (c.f. [Si1, p. 53]).*

Let us calculate the numerical function on the set of 1-parameter subgroups, to apply the Hilbert-Mumford criterion (c.f. Theorem 1.1.14).

Let l, m be integers as before, and let V be a complex vector space of dimension $P(m)$. Recall that a weighted filtration $(V_\bullet, n_\bullet)$ of V is a filtration

$$0 \subset V_1 \subset V_2 \subset \ \cdots \ \subset V_t \subset V_{t+1} = V, \tag{2.3.1}$$

and positive rational numbers $n_1, n_2, \ldots, n_t > 0$. Let Γ be the 1-parameter subgroup associated to the weighted filtration (c.f. subsection 2.1.1) given by

$$\Gamma = (\overbrace{\Gamma_1, \ldots, \Gamma_1}^{\dim V^1}, \overbrace{\Gamma_2, \ldots, \Gamma_2}^{\dim V^2}, \ldots, \overbrace{\Gamma_{t+1}, \ldots, \Gamma_{t+1}}^{\dim V^{t+1}}) \ ,$$

where $V^i = V_i / V_{i-1}$.

Let $W = H^0(\mathcal{O}(l-m))$ be a vector space where $SL(V)$ acts trivially. The basis $\{e_1, \ldots, e_p\}$, together with a basis $\{w_j\}$ of W, induces a basis of $\bigwedge^{P(l)}(V^\vee \otimes W^\vee)$ indexed in a natural way by tuples $(i_1, \ldots, i_{P(l)}, j)$ (the indexes i_j being skewsymmetric), and the coordinate corresponding to such an index is acted by Γ with exponent

$$\Gamma_{i_1} + \cdots + \Gamma_{i_{P(l)}} \ .$$

The coordinate $(i_1, \ldots, i_{P(l)}, j)$ of the point corresponding to the sheaf $\mathcal{E}$ is non-zero if and only if the evaluations of the sections $e_1, \ldots, e_{P(l)}$ are linearly independent for generic $x \in X$. Therefore, the numerical function in Theorem 1.1.14 is

$$\begin{aligned} \mu(\overline{q}, V_\bullet, n_\bullet) &= \min\{\Gamma_{i_1} + \cdots + \Gamma_{i_{P(l)}} : q''(e_{i_1} \wedge \cdots \wedge e_{i_{P(l)}}) \neq 0\} \\ &= \min\{\Gamma_{i_1} + \cdots + \Gamma_{i_{P(l)}} : e_{i_1}(x), \ldots, e_{i_{P(l)}}(x) \\ &\qquad \text{linearly independent for generic } x \in X\} \end{aligned} \tag{2.3.2}$$

Let $\mathcal{E}_{V_i}$ be the subsheaf generated by V_i and let $\mathcal{E}_{V^i}$ be the subsheaf generated by $V^i = V_i/V_{i-1}$. Let $P_{\mathcal{E}_{V_i}}$ and $P_{\mathcal{E}_{V^i}}$ be the corresponding Hilbert polynomials. Note that $P_{\mathcal{E}_{V_i}} - P_{\mathcal{E}_{V_{i-1}}} = P_{\mathcal{E}_{V^i}}$. Given a 1-parameter subgroup Γ, we get

$$\mu(\overline{q}, V_\bullet, n_\bullet) = \sum_{i=1}^{t} n_i (P(l) \dim V_i - P_{\mathcal{E}_{V_i}}(l) \dim V) =$$

$$\sum_{i=1}^{t+1} \frac{\Gamma_i}{\dim V} (P_{\mathcal{E}_{V^i}}(l) \dim V - P(l) \dim V^i) \ ,$$

where recall that $n_i = \frac{\Gamma_{i+1} - \Gamma_i}{p}$.

By the Hilbert-Mumford criterion, Theorem 1.1.14, a point

$$\overline{q} \in \mathbb{P}(\bigwedge^{P(l)} (V^\vee \otimes W^\vee))$$

is **GIT semistable** if and only if for all weighted filtrations it is

$$\mu(\overline{q}, V_\bullet, n_\bullet) \leq 0 \ .$$

Using the previous calculations, this can be stated as follows:

Proposition 2.3.5. *A point $\overline{q}$ is **GIT semistable** if for all weighted filtrations $(V_\bullet, n_\bullet)$*

$$\sum_{i=1}^{t} n_i (P(l) \dim V_i - P_{\mathcal{E}_{V_i}}(l) \dim V) \leq 0 \ .$$

*A point $\overline{q}$ is **GIT stable** if we get a strict inequality for every weighted filtration.*

Then, the next result completes the sketch of the construction of a moduli space for Higgs sheaves.

Theorem 2.3.6. *[Si1, Theorem 1.19] Fix a polynomial P of degree $n = \dim X$. There exist integers m_0 and l_0 (l_0 depending on m_0) such that for $m \geq m_0$ and $l \geq l_0$, a point q in $\mathrm{Hilb}(V \otimes \mathcal{O}_Z(-m), P)$ is GIT semistable (for the action of $SL(V)$ with respect to the embedding into a projective space), if and only if the quotient $\mathcal{E}$ is a semistable coherent sheaf of pure dimension n and the map $V \to H^0(E(m))$ is an isomorphism.*

Let (E, φ) be an unstable Higgs sheaf and $\mathcal{E}$ its associated coherent sheaf by Lemma 2.3.1. Let m_0, l_0 be integers as in Theorem 2.3.6. Choose $m_1 \geq m_0$, $l_1 \geq l_0$ such that $\mathcal{E}$

is also m_1-regular. Now choose $m \geq m_1$ and fix an isomorphism $V \simeq H^0(\mathcal{E}(m))$. Given a weighted filtration $(V_\bullet, n_\bullet)$, define for each $l \geq l_1$ the function

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^{t} n_i(P(l)\dim V_i - P_{\mathcal{E}_{V_i}}(l)\dim V)}{\sqrt{\sum_{i=1}^{t+1}\dim V^i \Gamma_i^2}} ,$$

which is a **Kempf function** for this problem (c.f. Definition 1.4.4). Note that, for each integer m, this is a polynomial function on l, whose coefficients depend on m.

Let

$$0 \subset V_1 \subset \cdots \subset V_{t+1} = V \tag{2.3.3}$$

be the **Kempf filtration** of vector spaces given by By Theorem 2.1.5, and let

$$0 \subseteq \mathcal{E}_1^m \subseteq \mathcal{E}_2^m \subseteq \ \cdots \ \subseteq \mathcal{E}_t^m \subseteq \mathcal{E}_{t+1}^m = \mathcal{E} \ , \tag{2.3.4}$$

be the m-**Kempf filtration of** $\mathcal{E}$, where $\mathcal{E}_i^m \subset \mathcal{E}$ is the subsheaf generated by V_i under the evaluation map. Making the correspondence of Lemma 2.3.1, we call the m-**Kempf filtration of the Higgs sheaf** (E, φ) to

$$0 \subseteq (E_1^m, \varphi|_{E_1^m}) \subseteq (E_2^m, \varphi|_{E_2^m}) \subseteq \ \cdots \ \subseteq (E_t^m, \varphi|_{E_t^m}) \subseteq (E_{t+1}^m, \varphi|_{E_{t+1}^m}) = (E, \varphi) \ , \tag{2.3.5}$$

where $E_i^m = \pi_* \mathcal{E}_i^m$.

Remark 2.3.7. *Recall that for two rational (in particular polynomial) functions, we define an ordering by saying that $f_1 \prec f_2$ if $f_1(l) < f_2(l)$, for $l \gg 0$. Then, although in the construction of the moduli space and in the definition of the Kempf function we use the integer l, we view the Kempf function as a polynomial function on l, having fixed previously m. We define the m-Kempf filtration of $\mathcal{E}$ as the one which maximizes the Kempf function having fixed m, seen as a polynomial function on l, by Theorem 2.1.5. Note that we also talk about the m-Kempf filtration of $\mathcal{E}$, without mentioning l.*

We will proceed as in the previous sections of the chapter to prove the following theorem:

Theorem 2.3.8. *There exists an integer $m' \gg 0$ such that the m-Kempf filtration of $\mathcal{E}$ is independent of m, for $m \geq m'$.*

2.3.2 The m-Kempf filtration stabilizes with m

We will prove Theorem 2.3.8 in an analogous way to the cases of torsion free sheaves and holomorphic pairs in sections 2.1 and 2.2. First, we associate a graph to the m-Kempf filtration of $\mathcal{E}$.

Definition 2.3.9. *Let $m \geq m_1$ and $l \geq l_1$. Given $0 \subset V_1 \subset \cdots \subset V_{t+1} = V$, a filtration of vector spaces of V, define*

$$v_{m,i}(l) = m^{n+1} \cdot \frac{1}{\dim V^i \dim V} \left[P_{\mathcal{E}_{V^i}}(l) \dim V - P(l) \dim V^i \right] ,$$

$$b_m^i = \frac{1}{m^n} \dim V^i > 0 ,$$

$$w_m^i(l) = -b_m^i \cdot v_{m,i}(l) = m \cdot \frac{1}{\dim V} \left[P(l) \dim V^i - P_{\mathcal{E}_{V^i}}(l) \dim V \right] .$$

Also let

$$b_{m,i} = b_m^1 + \ldots + b_m^i = \frac{1}{m^n} \dim V_i ,$$

$$w_{m,i}(l) = w_m^1(l) + \ldots + w_m^i(l) = m \cdot \frac{1}{\dim V} \left[P(l) \dim V_i - P_{\mathcal{E}_{V_i}}(l) \dim V \right] .$$

We call the graph defined by points $(b_{m,i}, w_{m,i}(l))$ the ***graph associated to the filtration*** *$V_\bullet \subset V$. Note that, having fixed m, the coordinates of the graph are polynomials on l.*

We use a similar argument that in Proposition 2.1.13 to identify the Kempf function in Theorem 2.1.5,

$$\mu(V_\bullet, n_\bullet) = \frac{1}{l^n} \frac{\sum_{i=1}^{t} n_i (P(l) \dim V_i - P_{\mathcal{E}_{V_i}}(l) \dim V)}{\sqrt{\sum_{i=1}^{t+1} \dim V^i \Gamma_i^2}} ,$$

where $n_i = \frac{\Gamma_i - \Gamma_{i-1}}{\dim V}$, with the function in Theorem 2.1.9, where the coordinates of the graph are as in Definition 2.3.9.

Proposition 2.3.10. *For every integer m, the following equality holds*

$$\mu(V_\bullet, n_\bullet) = \frac{1}{m^{(\frac{n}{2}+1)}} \cdot \mu_{v_m(l)}(\Gamma) = \frac{1}{m^{(\frac{n}{2}+1)}} \cdot \frac{(\Gamma, v_m(l))}{\|\Gamma\|} ,$$

between the Kempf function on Theorem 2.1.5 and the function in Theorem 2.1.9.

Remark 2.3.11. *Again, we introduce factors m^{n+1} in Definition 2.3.9 for $v_{m,i}(l)$ and b^i_m to have order zero on m (c.f. Remark 2.1.14). As a result, the graph keeps its dimensions when m grows, the coordinates being polynomials on the variable l.*

In the following, we will omit the integers m and l for the quantities $v_{m,i}(l)$, $b_{m,i}$, $w_{m,i}(l)$ in the definition of the graph associated to a filtration of vector spaces, where it is clear from the context.

We give the analogous to Propositions 2.1.18 and 2.1.20 for the case of Higgs sheaves, using Lemmas 2.1.15 and 2.1.16.

Define

$$C = \max\{r|\mu_{\max}(E)| + \frac{d}{r} + r|B| + |A| + 1 \ , \ 1\}, \tag{2.3.6}$$

a positive constant, where recall that $r = \operatorname{rk} \mathcal{E} = \operatorname{rk} E$ and $d = \deg \mathcal{E} = \deg E$, $E = \pi_* \mathcal{E}$, are the rank and the degree of $\mathcal{E}$ as the corresponding coefficients of the Hilbert polynomial for a pure sheaf.

Proposition 2.3.12. *Given sufficiently large integers m and l, each filter in the m-Kempf filtration of $\mathcal{E}$ has slope $\mu(\mathcal{E}^m_i) \geq \dfrac{d}{r} - C$.*

Proof. Choose an integer $m_2 \geq m_1$ such that for $m \geq m_2$

$$[\mu_{max}(\mathcal{E}) + gm + B]_+ = \mu_{max}(\mathcal{E}) + gm + B \ ,$$

and

$$[\frac{d}{r} - C + gm + B]_+ = \frac{d}{r} - C + gm + B \ .$$

Now let $m \geq m_2$. Let

$$0 \subseteq \mathcal{E}^m_1 \subseteq \mathcal{E}^m_2 \subseteq \ \cdots \ \subseteq \mathcal{E}^m_t \subseteq \mathcal{E}^m_{t+1} = \mathcal{E}$$

be the m-Kempf filtration of $\mathcal{E}$.

Let $\mathcal{E}^m_i \subseteq \mathcal{E}$ a subsheaf of rank r_i and degree d_i, such that $\mu(\mathcal{E}^m_i) < \frac{d}{r} - C$, and suppose that $\mathcal{E}^m_i(m) \subset \mathcal{E}(m)$ satisfies the estimate in Lemma 1.2.15. Analogously to Proposition 2.1.18,

$$h^0(\mathcal{E}^m_i(m)) \leq \frac{1}{g^{n-1}n!}\big((r_i-1)(\mu_{max}(\mathcal{E}) + gm + B)^n + (\frac{d}{r} - C + gm + B)^n\big) = G(m) \ ,$$

where

$$G(m) = \frac{1}{g^{n-1}n!}\big[r_i g^n m^n + n g^{n-1}\big((r_i-1)\mu_{max}(\mathcal{E}) + \frac{d}{r} - C + r_i B\big)m^{n-1} + \cdots\big] \ .$$

By Definition 2.3.9, to such filtration we associate a graph with heights, for each j

$$w_{m,j}(l) = w_m(l)^1 + \ldots + w_m(l)^j = m \cdot \frac{1}{\dim V}\big[P(l)\dim V_j - P_{\mathcal{E}_i^m}(l)\dim V\big] \ ,$$

and we will show that $w_{m,i}(l) < 0$ for m and l large enough, to get a contradiction as in Proposition 2.1.18. Note that the coordinates $w_{m,i}(l)$ are polynomials on l, so $w_{m,i}(l) < 0$ for $l \gg 0$, contradicts Lemma 2.1.15 for the Kempf function constructed with the integer m.

Given that $\mathcal{E}_i^m(m)$ is generated by V_i under the evaluation map, it is $\dim V_i \leq h^0(\mathcal{E}_i^m(m))$, hence

$$w_{m,i}(l) = \frac{m}{\dim V}\big[P(l)\dim V_i - P_{\mathcal{E}_i^m}(l)\dim V\big] \leq$$

$$\frac{m}{P_{\mathcal{E}}(m)}\big[P(l)h^0(\mathcal{E}_i^m(m)) - P_{\mathcal{E}_i^m}(l)P_{\mathcal{E}}(m)\big] \leq \frac{m}{P_{\mathcal{E}}(m)}\big[P(l)G(m) - P_{\mathcal{E}_i^m}(l)P_{\mathcal{E}}(m)\big] \ .$$

Hence, $w_{m,i}(l) < 0$ is equivalent to

$$\Phi_m(l) = P(l)G(m) - P_{\mathcal{E}_i^m}(l)P(m) < 0 \ ,$$

where $\Phi_m(l)$ can be seen as an n^{th}-order polynomial on l,

$$\Phi_m(l) = \alpha_n(m)l^n + \alpha_{n-1}(m)l^{n-1} + \cdots + \alpha_1(m)l + \alpha_0(m) \ .$$

Hence, it is sufficient to show that $\alpha_n(m) < 0$ for an integer m sufficiently large.

Note that

$$\alpha_n(m) = rG(m) - r_iP(m) < 0 \ ,$$

is the same polynomial as $\Psi(m)$ in the proof of Proposition 2.1.18. Hence, by the same argument

$$\alpha_n(m) = \xi_{n-1}m^{n-1} + \cdots + \xi_1 m + \xi_0$$

with $\xi_{n-1} < 0$, so there exists $m_3 \geq m_2$ such that for $m \geq m_3$ we will have $\alpha_n(m) < 0$. Hence, there exists an integer $l \gg 0$, depending on m_3, such that for $m \geq m_3$ we have $\Phi_m(l) \prec 0$ as a polynomial (c.f. Remark 2.3.7), hence $w_{m,i}(l) < 0$, which is a contradiction. ■

Now we can assure the m-regularity of the family of the subsheaves appearing in the different m-Kempf filtrations of $\mathcal{E}$, similarly to Proposition 2.1.19.

Proposition 2.3.13. *There exists an integer m_4 such that for $m \geq m_4$ the sheaves $\mathcal{E}_i^m$ and $\mathcal{E}^{m,i} = \mathcal{E}_i^m / \mathcal{E}_{i-1}^m$ are m_4-regular. In particular their higher cohomology groups, after twisting with $\mathcal{O}_X(m_4)$, vanish and they are generated by global sections.*

Proposition 2.3.14. *Let $m \geq m_4$. For each filter $\mathcal{E}_i^m$ in the m-Kempf filtration, we have $\dim V_i = h^0(\mathcal{E}_i^m(m))$, therefore $V_i \cong H^0(\mathcal{E}_i^m(m))$.*

Proof. The proof follows analogously to the proof of Proposition 2.1.20.

Let $m \geq m_4$. Let $V_\bullet \subseteq V$ be the Kempf filtration of V (cf. (2.3.3)) and let $\mathcal{E}_\bullet^m \subseteq \mathcal{E}$ be the m-Kempf filtration of $\mathcal{E}$ (c.f. (2.3.4)). We know that each V_i generates the subsheaf $\mathcal{E}_i^m$, by definition, then we have the diagram:

$$\begin{array}{ccccccccc}
0 \subset & V_1 & \subset & V_2 & \subset \cdots \subset & V_{t+1} & = & V \\
& \cap & & \cap & & & & || \\
& H^0(\mathcal{E}_1^m(m)) & \subset & H^0(\mathcal{E}_2^m(m)) & \subset \cdots \subset & H^0(\mathcal{E}_{t+1}^m(m)) & = & H^0(\mathcal{E}(m))
\end{array}$$

Let i be the first index such that $V_i \neq H^0(\mathcal{E}_i^m(m))$, then we have the diagram:

$$\begin{array}{ccc}
V_i & \subset & V_{i+1} \\
\cap & & || \\
H^0(\mathcal{E}_i^m(m)) & \subset & H^0(\mathcal{E}_{i+1}^m(m))
\end{array} \tag{2.3.7}$$

Therefore we consider a new filtration by adding the filter $H^0(\mathcal{E}_i^m(m))$

$$\begin{array}{ccccc}
V_i & \subset & H^0(\mathcal{E}_i^m(m)) & \subset & V_{i+1} \\
|| & & || & & || \\
V'_i & & V'_{i+1} & & V'_{i+2}
\end{array} \tag{2.3.8}$$

Then, V_i and $H^0(\mathcal{E}_i^m(m))$ generate the same sheaf $\mathcal{E}_i^m$, hence we are in situation of Lemma 2.1.16, where $W = H^0(\mathcal{E}_i^m(m))$, filtration $V_\bullet$ is (2.3.7) and filtration $V'_\bullet$ is (2.3.8).

Now the graph associated to filtration $V_\bullet$ is, by Definition 2.3.9, given by the points

$$(b_{m,i}, w_{m,i}(l)) = \Big(\frac{\dim V_i}{m^n}, \frac{m}{\dim V}(P(l)\dim V_i - P_{\mathcal{E}_i^m}(l)\dim V)\Big) ,$$

and the slopes $-v_{m,i}(l)$ of the graph are given by

$$-v_{m,i}(l) = \frac{w_m^i(l)}{b_m^i} = \frac{w_{m,i}(l) - w_{m,i-1}(l)}{b_{m,i} - b_{m,i-1}} =$$

$$\frac{m^{n+1}}{\dim V}\big(P(l)-P_{\mathcal{E}^{i,m}}(l)\frac{\dim V}{\dim V^i}\big)\ ,$$

which is an n^{th}-order polynomial on l whose leading coefficient is

$$\alpha^i(m)=\frac{m^{n+1}}{\dim V}\big(r-r^i\frac{\dim V}{\dim V^i}\big)\leq\frac{m^{n+1}}{\dim V}\cdot r:=R\ .$$

Equality holds if and only if $r^i=0$.

The new point which appears in graph of the filtration $V'_\bullet$ is

$$Q=\big(\frac{h^0(\mathcal{E}_i^m(m))}{m^n},\frac{m}{\dim V}(P(l)h^0(\mathcal{E}_i^m(m))-P_{\mathcal{E}_i^m}(l)\dim V)\big)\ ,$$

joining two new segments appearing in this new graph. The slope of the segment between $(b_{m,i},w_{m,i}(l))$ and Q is, similarly,

$$-v'_{m,i}(l)=\frac{m^{n+1}}{\dim V}\cdot P(l)\ ,$$

again an n^{th}-order polynomial on l whose leading coefficient is

$$\alpha'^i(m)=\frac{m^{n+1}}{\dim V}\cdot r=R\ .$$

By Lemma 2.1.15, the graph is convex, so $v_{m,1}(l)<v_{m,2}(l)<\ldots<v_{m,t+1}(l)$. On the other hand, by Lemma 2.1.16, $v'_{m,i}(l)\geq v_{m,i}(l)$. Therefore for a sufficiently large l we have the following inequalities between the leading coefficients of the $-v'_{m,i}(l)$,

$$\alpha^1(m)\geq\alpha^2(m)\geq\ldots\geq\alpha^{t+1}(m)\ ,$$

and

$$\alpha'^i(m)\leq\alpha^i(m)\ .$$

Besides, $r^1=r_1>0$, then $R>\alpha^1(m)$. Indeed, $\mathcal{E}$ is pure, then it has no torsion elements on its support, hence also the subsheaf E_1^m, and a rank 0 pure sheaf should be the zero sheaf. Hence

$$R>\alpha^1(m)\geq\alpha^2(m)\geq\ldots\geq\alpha^i(m)\geq\alpha'^i(m)=R\ ,$$

which is a contradiction.

Therefore, for $m\geq m_4$, every filter in the m-Kempf filtration of $\mathcal{E}$ verifies $\dim V_i=h^0(\mathcal{E}_i^m(m))$. ∎

Corollary 2.3.15. *Given $m\geq m_4$, for every filter $\mathcal{E}_i^m$ in the m-Kempf filtration, it is $r^i>0$.*

Proof. It follows from Proposition 2.3.14 the same way as in Corollary 2.1.21. Indeed, given that $r^i = 0$ is equivalent to $\alpha^i(m) = R$, note that it is $r^1 = r_1 > 0$ and $R > \alpha^1(m) \geq \alpha^2(m) \geq \ldots \geq \alpha^{t+1}(m)$. ■

Next, we again recall the results on subsection 2.1.5.

By Proposition 2.3.13, for any $m \geq m_4$, all the filters $\mathcal{E}_i^m$ of the m-Kempf filtration of $\mathcal{E}$ are m_4-regular and hence, the filtration of sheaves

$$0 \subset \mathcal{E}_1^m \subset \mathcal{E}_2^m \subset \cdots \subset \mathcal{E}_{t_m}^m \subset \mathcal{E}_{t_m+1}^m = \mathcal{E}$$

is the filtration associated to the filtration of vector subspaces

$$0 \subset H^0(\mathcal{E}_1^m(m_4)) \subset H^0(\mathcal{E}_2^m(m_4)) \subset \cdots \subset H^0(\mathcal{E}_{t_m}^m(m_4)) \subset H^0(\mathcal{E}_{t_m+1}^m(m_4)) = H^0(\mathcal{E}(m_4))$$

by the evaluation map (c.f. Lemma 1.2.13), of a unique vector space $H^0(\mathcal{E}(m_4))$, whose dimension does not depend on m. Let $P_i^m := P_{\mathcal{E}_i^m}$ and $P^{i,m} := P_{\mathcal{E}^{i,m}}$. Let

$$(P_1^m, \ldots, P_{t_m+1}^m)$$

be the m**-type** of the m-Kempf filtration of $\mathcal{E}$ (c.f. Definition 2.1.22) and let

$$\mathcal{P} = \big\{(P_1^m, \ldots, P_{t_m+1}^m)\big\}$$

be the set of possible m-types, which is a finite set (c.f. Proposition 2.1.23).

By Definition 2.3.9 we associate a graph to the m-Kempf filtration of $\mathcal{E}$, given by $v_m(l)$. By Propositions 2.3.13 and 2.3.14 it can be rewritten as

$$v_{m,i}(l) = m^{n+1} \cdot \frac{1}{P^{i,m}(m)P(m)} \big[P^{i,m}(l)P(m) - P(l)P^{i,m}(m)\big] \ ,$$

$$b_m^i = \frac{1}{m^n} \cdot P^{i,m}(m) \ .$$

Note that, given the m-Kempf filtration, its m-type is fixed. Hence, the coordinates of the graph, $v_{m,i}(l)$ are polynomials on l, whose coefficients are fixed (c.f. Definition 2.3.9). Then, fixing the m-type, $v_m(l)$ defines a different graph for each l.

We define a functional on $\mathcal{P}$ which assigns to each m-type (to each m-Kempf filtration) the function

$$\Theta_m(l) = (\mu_{v_m(l)}(\Gamma_{v_m(l)}))^2 = ||v_m(l)||^2 \ ,$$

which is, given m, a polynomial function on l (c.f. (2.1.14)). By finiteness of $\mathcal{P}$ there is a finite list of such possible functions

$$\mathcal{A} = \{\Theta_m : m \geq m_4\} \ ,$$

so, analogously to Lemma 2.1.24, we can choose K to be a polynomial function such that there exists an integer m_5 with $\Theta_m = K$, for all $m \geq m_5$, meaning that two polynomial functions do coincide if they do for large values of the variable.

Proposition 2.3.16. *Let a_1 and a_2 be integers with $a_1 \geq a_2 \geq m_5$. The a_1-Kempf filtration of $\mathcal{E}$ is equal to the a_2-Kempf filtration of $\mathcal{E}$.*

Proof. The proof follows analogously to the proof of Proposition 2.1.25. ■

Definition 2.3.17. *If $m \geq m_5$, and for $l \geq l_5$, the m-Kempf filtration of $\mathcal{E}$ is called* ***the Kempf filtration of*** *$\mathcal{E}$,*

$$0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_t \subset \mathcal{E}_{t+1} = \mathcal{E} \; .$$

By applying π_* to the Kempf filtration we obtain the following definition.

Definition 2.3.18. *The following filtration*

$$0 \subset (E_1, \varphi|_{E_1}) \subset (E_2, \varphi|_{E_2}) \subset \;\cdots\; \subset (E_t, \varphi|_{E_t}) \subset (E_{t+1}, \varphi|_{E_{t+1}}) = (E, \varphi) \; ,$$

where $E_i = \pi_* \mathcal{E}_i$ is called ***the Kempf filtration of the Higgs sheaf*** *(E, φ).*

2.3.3 Harder-Narasimhan filtration for Higgs sheaves

Recall that, by the Kempf theorem (c.f. Theorem 2.1.5), given an integer m and $V \simeq H^0(\mathcal{E}(m))$, there exists a unique weighted filtration of vector spaces $V_\bullet \subseteq V$ which gives maximum for the Kempf function, which in this case is

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^{t+1} \frac{\Gamma_i}{\dim V} (P_{\mathcal{E}^i}(l) \dim V - P(l) \dim V^i)}{\sqrt{\sum_{i=1}^{t+1} \dim V^i \Gamma_i^2}} \; .$$

This filtration induces the Kempf filtration of $\mathcal{E}$,

$$0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_t \subset \mathcal{E}_{t+1} = \mathcal{E}$$

which is independent of m, for $m \geq m_5$, by Proposition 2.3.16, hence it only depends on $\mathcal{E}$.

We proceed again as in Section 2.1 (c.f. Proof of Theorem 2.1.7), to rewrite the Kempf function in terms of Hilbert polynomials of sheaves. We set $P = P_{\mathcal{E}}$, $P^i = P_{\mathcal{E}^i}$, and recall the relation

$$\gamma_i = \frac{r}{P(m)} \Gamma_i \; .$$

Proposition 2.3.19. *Given $\mathcal{E}$, a pure sheaf of dimension n, there exists a unique filtration*

$$0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_t \subset \mathcal{E}_{t+1} = \mathcal{E}$$

with positive weights $n_1, \ldots, n_t$, which gives maximum for the function

$$K_m(l) := m^{\frac{n}{2}+1} \cdot \frac{1}{P(m)} \frac{\sum_{i=1}^{t+1} \gamma_i [P^i(l)P(m) - P^i(l)P(m)]}{\sqrt{\sum_{i=1}^{t+1} P^i(m)\gamma_i^2}} \ .$$

The coordinates of the graph $v_{m,i}(l)$ are given by

$$v_{m,i}(l) = m^{n+1} \cdot \frac{1}{P^i(m)P(m)} \big[P^i(l)P(m) - P(l)P^i(m) \big] \ .$$

hence, the function K is

$$K_m(l) = m^{-\frac{n}{2}} \cdot l^n \cdot \frac{\sum_{i=1}^{t+1} P^i(m)\gamma_i v_i}{\sqrt{\sum_{i=1}^{t+1} P^i(m)\gamma_i^2}} = m^{-\frac{n}{2}} \cdot l^n \cdot \frac{(\gamma, v)}{||\gamma||} \ ,$$

where the scalar product is given by

$$\begin{pmatrix} P^1(m) & & & \\ & P^2(m) & & \\ & & \ddots & \\ & & & P^{t+1}(m) \end{pmatrix}$$

Proposition 2.3.20. *Given the Kempf filtration of a sheaf $\mathcal{E}$,*

$$0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_t \subset \mathcal{E}_{t+1} = \mathcal{E}$$

it verifies

$$\frac{P_{\mathcal{E}^1}}{\operatorname{rk} \mathcal{E}^1} > \frac{P_{\mathcal{E}^2}}{\operatorname{rk} \mathcal{E}^2} > \ldots > \frac{P_{\mathcal{E}^{t+1}}}{\operatorname{rk} \mathcal{E}^{t+1}} \ .$$

Proof. By Lemma 2.1.15, the vector $v_m(l)$ is convex for $m \geq m_4$ and $l \gg 0$. Therefore, seeing $v_{m,i}(l)$ as polynomials on l,

$$v_{m,i}(l) < v_{m,i+1}(l) \Leftrightarrow \frac{P_{\mathcal{E}^i}(l)}{P_{\mathcal{E}^i}(m)} - \frac{P(l)}{P(m)} < \frac{P_{\mathcal{E}^{i+1}}(l)}{P_{\mathcal{E}^{i+1}}(m)} - \frac{P(l)}{P(m)} \Leftrightarrow$$

$$\frac{\operatorname{rk}(\mathcal{E}^i)}{P_{\mathcal{E}^i}(m)} < \frac{\operatorname{rk}(\mathcal{E}^{i+1})}{P_{\mathcal{E}^{i+1}}(m)} \Leftrightarrow \frac{P_{\mathcal{E}^i}(m)}{\operatorname{rk}(\mathcal{E}^i)} > \frac{P_{\mathcal{E}^{i+1}}(m)}{\operatorname{rk}(\mathcal{E}^{i+1})} \ ,$$

where the second equivalence holds for $l \gg 0$. ∎

Proposition 2.3.21. *Given the Kempf filtration of $\mathcal{E}$,*

$$0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_t \subset \mathcal{E}_{t+1} = \mathcal{E}$$

each one of the blocks $\mathcal{E}^i = \mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable.

Proof. C.f. Proposition 2.1.29. ■

Theorem 1.3.5, which provides the construction of the Harder-Narasimhan filtration, holds for pure sheaves (c.f. [HL3, Theorem 1.3.4]), as it is the present case of the sheaf $\mathcal{E}$ supported on the cotangent bundle associated to a Higgs sheaf (E,φ).

Proposition 2.3.22. *[HL3, Theorem 1.3.4] Given a pure sheaf $\mathcal{E}$, there exists a unique filtration*

$$0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_t \subset \mathcal{E}_{t+1} = \mathcal{E}\ ,$$

which satisfies these two properties

1. *The Hilbert polynomials verify*

$$\frac{P_{\mathcal{E}^1}}{\operatorname{rk}\mathcal{E}^1} > \frac{P_{\mathcal{E}^2}}{\operatorname{rk}\mathcal{E}^2} > \ldots > \frac{P_{\mathcal{E}^{t+1}}}{\operatorname{rk}\mathcal{E}^{t+1}}$$

2. *Every block $\mathcal{E}^i = \mathcal{E}^i/\mathcal{E}_i$ is semistable.*

*This filtration is called the **Harder-Narasimhan filtration** of $\mathcal{E}$.*

Corollary 2.3.23. *The Kempf filtration of a sheaf $\mathcal{E}$ is the Harder-Narasimhan filtration.*

Proof. See Propositions 2.3.20 and 2.3.21 and use uniqueness of the Harder-Narasimhan filtration of $\mathcal{E}$ in Theorem 2.3.22. ■

Making the correspondence between Higgs sheaves (E,φ) over X and sheaves $\mathcal{E}$ of pure dimension $n = \dim X$ over T^*X we can construct a filtration of Higgs subsheaves

$$0 \subset \pi_*\mathcal{E}_1 \subset \pi_*\mathcal{E}_2 \subset \cdots \subset \pi_*\mathcal{E}_t \subset \pi_*\mathcal{E}_{t+1} = E$$

i.e.

$$0 \subset (E_1, \varphi|_{E_1}) \subset (E_2, \varphi|_{E_2}) \subset \ \cdots\ \subset (E_t, \varphi|_{E_t}) \subset (E_{t+1}, \varphi|_{E_{t+1}}) = (E,\varphi)\ ,$$

where $E_i = \pi_*\mathcal{E}_i$, which coincides with the Harder-Narasimhan filtration of a Higgs sheaf (E,φ), which appears in the literature (c.f. [AB]).

2.4 Rank 2 tensors

In this section we study the case of tensors where the sheaf has rank 2. The moduli space of tensors has been studied in section 1.2. Here we prove the analogous correspondence between Kempf and Harder-Narasimhan filtrations for rank 2 tensors, similarly to the cases of torsion free sheaves, holomorphic pairs and Higgs sheaves.

Let X be a smooth complex projective variety of dimension n. Let E be a coherent torsion free sheaf over X, of rank 2. We call a **rank 2 tensor** the pair consisting of

$$(E, \varphi : \overbrace{E \otimes \cdots \otimes E}^{\text{s times}} \longrightarrow \mathcal{O}_X) \ .$$

These objects are particular cases of the ones studied in section 1.2 for arbitrary s, $c = 1$, $b = 0$, $R = \operatorname{Spec} \mathbb{C}$ and $\mathcal{D} = \mathcal{O}_X$, meaning the structure sheaf over $X \times R \simeq X$, in Definition 1.2.1.

Let δ be a polynomial of degree at most $\dim X - 1 = n - 1$ and positive leading coefficient. Recall the definition of δ-stability for tensors. Recall Definition 1.2.3 and calculation made in (1.2.7). Recall Remark 1.2.4 which says that, in Definition 1.2.3 it suffices to check the condition on filtrations with $\operatorname{rk} E_i < \operatorname{rk} E_{i+1}$. Hence, as the rank of E is 2, the only filtrations we have to check are one-step filtrations, i.e. subsheaves of rank 1, and we can rewrite the stability condition as follows:

Definition 2.4.1. *A rank 2 tensor (E, φ) is δ-**semistable** if for every rank 1 subsheaf $L \subset E$*

$$(2P_L - P_E) + \delta(s - 2\epsilon(L)) \leq 0, \tag{2.4.1}$$

*where $\epsilon(L)$ is the number of times that L appears in the multi-index $(i_1, \ldots, i_s)$ giving the minimum in (1.2.4) and P_E, P_L are the Hilbert polynomials of E and L respectively. If the inequality is strict for every L, we say that (E, φ) is δ-**stable**. If (E, φ) is not δ-semistable, we say that it is δ-**unstable**.*

2.4.1 Moduli space of rk 2 tensors

We recall the main points of the construction of the moduli space for tensors with fixed determinant $\det(E) \cong \Delta$ of degree d and $\operatorname{rk}(E) = 2$. The general construction was explained in section 1.2, following Gieseker's method. The present case can be obtained by setting $c=1$, $b=0$, arbitrary s, $R=\operatorname{Spec}\mathbb{C}$ and $\mathcal{D}=\mathcal{O}_X$, the structure sheaf over $X\times R\simeq X$, in Definition 1.2.1.

Let V be a vector space of dimension $p:=h^0(E(m))$, where m is a suitable large integer (in particular, $E(m)$ generated by global sections and $h^i(E(m))=0$ for $i>0$). Given an isomorphism $V\cong H^0(E(m))$ we obtain a point

$$(\overline{Q},\overline{\Phi})\in\mathbb{P}(\operatorname{Hom}(\wedge^r V,A))\times\mathbb{P}(\operatorname{Hom}(V^{\otimes s},B))\ .$$

If we change the isomorphism $\det(E)\cong\Delta$, we obtain a different point in the line defined by Q. Likewise, if we change the isomorphism $V\cong H^0(E(m))$ by a homothecy, we obtain a different point in the line defined by Q. In both cases, the point $\overline{Q}$ in the projective space is the same. The same applies for $\overline{\Phi}$. If we fix once and for all a basis of V, then giving an isomorphism between V and $H^0(E(m))$ is equivalent to giving a basis of $H^0(E(m))$. A change of basis is given by an element of $\mathrm{GL}(V)$, but, since an homothecy does not change the point $(\overline{Q},\overline{\Phi})$, when we want to get rid of this choice it is enough to divide by the action of $\mathrm{SL}(V)$.

A weighted filtration $(V_\bullet,n_\bullet)$ of V is a filtration

$$0\subset V_1\subset V_2\subset\ \cdots\ \subset V_t\subset V_{t+1}=V, \tag{2.4.2}$$

and rational numbers $n_1, n_2,\dots,n_t>0$, and recall that this is equivalent to giving a 1-parameter subgroup $\Gamma:\mathbb{C}^*\to\mathrm{SL}(V)$ (c.f. subsection 2.1.1) represented by the vector

$$\Gamma=(\overbrace{\Gamma_1,\dots,\Gamma_1}^{\dim V^1},\overbrace{\Gamma_2,\dots,\Gamma_2}^{\dim V^2},\dots,\overbrace{\Gamma_{t+1},\dots,\Gamma_{t+1}}^{\dim V^{t+1}})\ .$$

By the Hilbert-Mumford criterion (c.f. Theorem 1.1.14), a point

$$(\overline{Q},\overline{\Phi})\in\mathbb{P}(\operatorname{Hom}(\wedge^r V,A))\times\mathbb{P}(\operatorname{Hom}(V^{\otimes s},B))$$

is **GIT semistable** with respect to the natural linearization on $\mathcal{O}(a_1,a_2)$ if and only if for all weighted filtrations

$$\mu(\overline{Q},V_\bullet,n_\bullet)+\frac{a_2}{a_1}\mu(\overline{\Phi},V_\bullet,n_\bullet)\le 0\ ,$$

and recall the numerical function which has to be calculated to apply Mumford criterion for GIT stability (c.f. Proposition 1.2.29).

Proposition 2.4.2. *A point $(\overline{Q},\overline{\Phi})$ is* ***GIT*** *a_2/a_1*-***semistable*** *if for all weighted filtrations $(V_\bullet,n_\bullet)$,*

$$\sum_{i=1}^{t} n_i(r\dim V_i - r_i \dim V) + \frac{a_2}{a_1}\sum_{i=1}^{t} n_i\big(s\dim V_i - \epsilon_i(\overline{\Phi})\dim V\big) \leq 0 \ .$$

Here, E_{V_i} is the subsheaf of E generated by V_i and $r_i = \operatorname{rk} E_{V_i}$. If $I=(i_1,\dots,i_s)$ is the multi-index giving minimum in (1.2.37) (c.f. Section 1.2), we will denote by $\epsilon_i(\overline{\Phi},V_\bullet,n_\bullet)$ (or just $\epsilon_i(\overline{\Phi})$ if the rest of the data is clear from the context) the number of elements k of the multi-index I such that $\dim V_k \leq \dim V_i$. Let $\epsilon^i(\overline{\Phi}) = \epsilon_i(\overline{\Phi}) - \epsilon_{i-1}(\overline{\Phi})$.

Then, recall Theorem 1.2.31:

Theorem 2.4.3. *Let (E,φ) be a tensor. There exists an m_0 such that, for $m\geq m_0$ the associated point $(\overline{Q},\overline{\Phi})$ is GIT a_2/a_1-semistable if and only if the tensor is δ-semistable, where*

$$\frac{a_2}{a_1} = \frac{r\delta(m)}{P_E(m) - s\delta(m)} \ .$$

Let X be a smooth complex projective variety of dimension n. Let us consider rank 2 tensors

$$(E,\varphi : \overbrace{E\otimes\cdots\otimes E}^{\text{s times}} \longrightarrow \mathcal{O}_X)$$

given by a coherent torsion free sheaf E of rank 2 over X with fixed determinant $\det(E)\cong\Delta$ and a morphism φ from a tensor product of s copies of E to the trivial line bundle $\mathcal{O}_X$. Let δ be a polynomial of degree at most $\dim X - 1 = n-1$ and positive leading coefficient.

Let (E,φ) be a δ-unstable rank 2 tensor. Let m_0 be an integer as in Theorem 2.4.3 (i.e. such that the δ-stability and the GIT stability coincide) and also such that E is m_0 regular (choosing a larger integer, if necessary). Choose an integer $m\geq m_0$ and let V be a vector space of dimension $P_E(m) = h^0(E(m))$.

Given a filtration of vector subspaces $0\subset V_1\subset\cdots\subset V_{t+1}=V$ and positive numbers $n_1,\cdots,n_t>0$, i.e., given a weighted filtration, we define the following function

$$\mu(V_\bullet,n_\bullet) = \frac{\sum_{i=1}^{t} n_i(r\dim V_i - r_i\dim V) + \frac{a_2}{a_1}\sum_{i=1}^{t} n_i\big(s\dim V_i - \epsilon_i(\overline{\Phi})\dim V\big)}{\sqrt{\sum_{i=1}^{t+1}\dim V^i\Gamma_i^2}} \ ,$$

which is a **Kempf function** for this problem (c.f. Definition 1.4.4), where the numerator of the function coincides with the numerical function in Proposition 2.4.2 and the denominator is a length $||\Gamma||$ in the space of 1-parameter subgroups (c.f. Definition 1.4.2).

Let

$$0 \subset V_1 \subset \cdots \subset V_{t+1} = V \tag{2.4.3}$$

be the **Kempf filtration** of V (c.f. Theorem 2.1.5), and let

$$0 \subseteq (E_1^m, \varphi|_{E_1^m}) \subseteq (E_2^m, \varphi|_{E_2^m}) \subseteq \cdots (E_t^m, \varphi|_{E_t^m}) \subseteq (E_{t+1}^m, \varphi|_{E_{t+1}^m}) \subseteq (E, \varphi) \tag{2.4.4}$$

be the m-**Kempf filtration** of the rank 2 tensor (E,φ), where $E_i^m \subset E$ is the subsheaf generated by V_i under the evaluation map.

We will prove the following

Theorem 2.4.4. *There exists an integer $m' \gg 0$ such that the m-Kempf filtration of the* rk 2 *tensor (E,φ) is independent of m, for $m \geq m'$.*

2.4.2 The m-Kempf filtration stabilizes with m

Let us define the graph associated to the m-Kempf filtration of (E,φ).

Definition 2.4.5. *Let $m \geq m_0$. Given $0 \subset V_1 \subset \cdots \subset V_{t+1} = V$ a filtration of vector spaces of V, let*

$$v_{m,i} = m^{n+1} \cdot \frac{1}{\dim V^i \dim V}\left[r^i \dim V - r \dim V^i + \frac{a_2}{a_1}(\epsilon^i(\overline{\Phi}) \dim V - s \dim V^i)\right] ,$$

$$b_m^i = \frac{1}{m^n} \dim V^i > 0 ,$$

$$w_m^i = -b_m^i \cdot v_{m,i} = m \cdot \frac{1}{\dim V}\left[r \dim V^i - r^i \dim V + \frac{a_2}{a_1}(s \dim V^i - \epsilon^i(\overline{\Phi}) \dim V)\right] .$$

Also let

$$b_{m,i} = b_m^1 + \ldots + b_m^i = \frac{1}{m} \dim V_i ,$$

$$w_{m,i} = w_m^1 + \ldots + w_m^i = m \cdot \frac{1}{\dim V}\left[r \dim V_i - r_i \dim V + \frac{a_2}{a_1}(s \dim V_i - \epsilon_i(\overline{\Phi}) \dim V)\right] .$$

*We call the graph defined by points $(b_{m,i}, w_{m,i})$ the **graph associated to the filtration** $V_\bullet \subset V$.*

Now we prove a crucial Lemma which will let us prove Theorem 2.4.4 using the same method than in previous sections.

Lemma 2.4.6. *The symbols $\epsilon_i(\overline{\Phi}) = \epsilon_i(\overline{\Phi}, V_\bullet, n_\bullet)$ do not depend on the weights $n_\bullet$. Therefore, the graph associated to the filtration only depends on the data $V_\bullet \subset V$, not the weights $n_\bullet$.*

Proof. Note that $\operatorname{rk} E_1 \geq 1$ because it is generated by, at least, a non zero global section. Suppose that $\operatorname{rk} E_1^m = \operatorname{rk} E_2^m = \ldots = \operatorname{rk} E_k^m = 1$ and $\operatorname{rk} E_{k+1}^m = \ldots = \operatorname{rk} E_t^m = \operatorname{rk} E = 2$. Then, for example, E_1^m coincide with E_2^m on an open set and, generically, the behavior with respect to φ is the same, i.e.

$$\overline{\Phi}|_{V_1\otimes\cdots\otimes V_1} = 0 \Leftrightarrow \varphi|_{E_1^m\otimes\cdots\otimes E_1^m} = 0 \Leftrightarrow \varphi|_{E_2^m\otimes E_1^m\cdots\otimes E_1^m} = 0 \ .$$

Therefore, the values $\epsilon_i(\overline{\Phi}, V_\bullet, n_\bullet)$ only depend on the filters E_i^m but not on the specific values of the Γ_i. In fact, they will only depend on Γ_1 and Γ_{k+1}, because they are the minimal ones among the filters of the same rank (c.f. (1.2.4) and (1.2.37)). In this case we will just write $\epsilon_i(\overline{\Phi}, V_\bullet)$, or $\epsilon_i(\overline{\Phi})$, when the filtration is clear from the context. ■

Next, we can identify the Kempf function in Theorem 2.1.5

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^t n_i(r\dim V_i - r_i \dim V) + \frac{a_2}{a_1}\sum_{i=1}^t n_i\big(s\dim V_i - \epsilon_i(\overline{\Phi})\dim V\big)}{\sqrt{\sum_{i=1}^{t+1}\dim V^i \Gamma_i^2}} =$$

$$= \frac{\sum_{i=1}^{t+1}\frac{\Gamma_i}{\dim V}(r^i\dim V - r\dim V^i) + \frac{a_2}{a_1}\sum_{i=1}^{t+1}\frac{\Gamma_i}{\dim V}\big(\epsilon^i(\overline{\Phi})\dim V - s\dim V^i\big)}{\sqrt{\sum_{i=1}^{t+1}\dim V^i\Gamma_i^2}} \ ,$$

where $n_i = \frac{\Gamma_i - \Gamma_{i-1}}{\dim V}$, with the function in Theorem 2.1.9 (c.f. Proposition 2.1.13). Precisely, we use Lemma 2.4.6 to assure that the data of the filters $V_\bullet \subset V$, and the data of the weights $n_\bullet$ are independent, so we can maximize the Kempf function with respect to each of them, independently, as in Theorem 2.1.9.

Proposition 2.4.7. *For every integer m, the following equality holds*

$$\mu(V_\bullet, n_\bullet) = m^{(-\frac{n}{2}-1)} \cdot \mu_{v_m}(\Gamma)$$

between the Kempf function on Theorem 2.1.5 and the function in Theorem 2.1.9.

Proof. By Lemma 2.4.6, we can fix a vector v_m and look for the maximum of the function μ_{v_m} among the corresponding convex cone. ■

In the following, we will omit the subindex m for the numbers $v_{m,i}$, $b_{m,i}$, $w_{m,i}$ in the definition of the graph associated to the filtration of vector spaces, where it is clear from the context. Recall Remark 2.1.14 to understand the meaning of the factors in m in Definition 2.4.5.

Now we use Lemmas 2.1.15 and 2.1.16 to give the analogous to Propositions 2.1.18 and 2.1.20 in this case.

Let us define

$$C = \max\{r|\mu_{\max}(E)| + \frac{d}{r} + r|B| + |A| + s\delta_{n-1}(n-1)! + 1\ ,\ 1\}, \tag{2.4.5}$$

a positive constant, where δ_{n-1} is the leading coefficient of the polynomial $\delta(m)$, of degree $\leq n-1$ (if $\deg\delta < n-1$, set $\delta_{n-1} = 0$).

Proposition 2.4.8. *Given a sufficiently large m, each filter in the m-Kempf filtration of the* rk 2 *tensor (E,φ) has slope $\mu(E_i^m) \geq \frac{d}{r} - C$.*

Proof. Choose an m_1 such that for $m \geq m_1$

$$[\mu_{max}(E) + gm + B]_+ = \mu_{max}(E) + gm + B$$

and

$$[\frac{d}{r} - C + gm + B]_+ = \frac{d}{r} - C + gm + B\ .$$

Let m_2 be such that $P_E(m) - s\delta(m) > 0$ for $m \geq m_2$. Now consider $m \geq \max\{m_0, m_1, m_2\}$ and let

$$0 \subseteq (E_1^m, \varphi|_{E_1^m}) \subseteq (E_2^m, \varphi|_{E_2^m}) \subseteq \cdots (E_t^m, \varphi|_{E_t^m}) \subseteq (E_{t+1}^m, \varphi|_{E_{t+1}^m}) \subseteq (E,\varphi)$$

be the m-Kempf filtration.

Suppose that we have a filter $E_i^m \subseteq E$, of rank r_i and degree d_i, such that $\mu(E_i^m) < \frac{d}{r} - C$. Again, the subsheaf $E_i^m(m) \subset E(m)$ satisfies the estimate in Lemma 1.2.15,

$$h^0(E_i^m(m)) \leq \frac{1}{g^{n-1}n!}\big((r_i - 1)([\mu_{max}(E_i^m) + gm + B]_+)^n + ([\mu_{min}(E_i^m) + gm + B]_+)^n\big)\ .$$

Hence, analogously to Proposition 2.1.18,

$$h^0(E_i^m(m)) \leq \frac{1}{g^{n-1}n!}\big((r_i - 1)(\mu_{max}(E) + gm + B)^n + (\frac{d}{r} - C + gm + B)^n\big) = G(m)\ ,$$

where

$$G(m) = \frac{1}{g^{n-1}n!}\big[r_i g^n m^n + n g^{n-1}\big((r_i - 1)\mu_{max}(E) + \frac{d}{r} - C + r_i B\big)m^{n-1} + \cdots\big]\ .$$

By Definition 2.4.5, to the m-Kempf filtration we associate a graph with heights, for each j

$$w_j = w^1 + \ldots + w^j = m \cdot \frac{1}{\dim V}\big[r\dim V_j - r_j\dim V + \frac{a_2}{a_1}(s\dim V_j - \epsilon_j(\overline{\Phi})\dim V)\big]\ .$$

We will get a contradiction by showing that $w_i < 0$ (c.f. Proposition 2.1.18).

Since $E_i^m(m)$ is generated by V_i under the evaluation map, it is $\dim V_i \leq H^0(E_i^m(m))$, hence

$$w_i = \frac{m}{\dim V}\big[r\dim V_i - r_i \dim V + \frac{a_2}{a_1}(s\dim V_i - \epsilon_i(\overline{\Phi})\dim V)\big] \leq$$

$$\frac{m}{P_E(m)}\big[rh^0(E_i^m(m)) - r_iP_E(m) + \frac{r\delta(m)}{P_E(m)-s\delta(m)}(sh^0(E_i^m(m)) - \epsilon_i(\overline{\Phi})P_E(m))\big] \leq$$

$$\frac{m}{P_E(m)}\big[rG(m) - r_iP_E(m) + \frac{r\delta(m)}{P_E(m)-s\delta(m)}(sG(m) - \epsilon_i(\overline{\Phi})P_E(m))\big] =$$

$$m\cdot\frac{\big[(P_E(m)-s\delta(m))(rG(m)-r_iP_E(m)) + (r\delta(m))(sG(m)-\epsilon_i(\overline{\Phi})P_E(m))\big]}{P_E(m)(P_E(m)-s\delta(m))} .$$

Then, $w_i < 0$ is equivalent to

$$\Psi(m) = (P_E(m)-s\delta(m))(rG(m)-r_iP_E(m)) + (r\delta(m))(sG(m)-\epsilon_i(\overline{\Phi})P_E(m)) < 0 ,$$

and $\Psi(m) = \xi_{2n}m^{2n} + \xi_{2n-1}m^{2n-1} + \cdots + \xi_1 m + \xi_0$ is a $(2n)^{th}$-order polynomial, whose higher order coefficient is

$$\xi_{2n} = (P_E(m)-s\delta(m))_n(rG(m)-r_iP_E(m))_n + (r\delta(m))_n(sG(m)-\epsilon_i(\overline{\Phi})P_E(m))_n =$$

$$(P_E(m)-s\delta(m))_n(r\frac{r_ig}{n!} - r_i\frac{rg}{n!}) + 0 = 0 .$$

The $(2n-1)^{th}$-order coefficient is

$$\xi_{2n-1} = (P_E(m)-s\delta(m))_n(rG(m)-r_iP_E(m))_{n-1} + (r\delta(m))_{n-1}(sG(m)-\epsilon_i(\overline{\Phi})P_E(m))_n =$$

$$\frac{rg}{n!}(rG_{n-1} - r_i\frac{A}{(n-1)!}) + r\delta_{n-1}(s\frac{r_ig}{n!} - \epsilon_i(\overline{\Phi})\frac{rg}{n!})$$

where G_{n-1} is the $(n-1)^{th}$-coefficient of the polynomial $G(m)$,

$$G_{n-1} = \frac{1}{g^{n-1}n!}ng^{n-1}((r_i-1)\mu_{max}(E) + \frac{d}{r} - C + r_iB) =$$

$$\frac{1}{(n-1)!}((r_i-1)\mu_{max}(E) + \frac{d}{r} - C + r_iB) \leq$$

$$\frac{1}{(n-1)!}((r_i-1)|\mu_{max}(E)| + \frac{d}{r} - C + r_i|B|) \leq$$

$$\frac{1}{(n-1)!}(r|\mu_{max}(E)| + \frac{d}{r} - C + r|B|) < \frac{-|A|}{(n-1)!} - s\delta_{n-1} ,$$

last inequality coming from the definition of C in (2.4.5). Then

$$\xi_{2n-1} < \frac{rg}{n!}\big(r(\frac{-|A|}{(n-1)!} - s\delta_{n-1}) - r_i\frac{A}{(n-1)!}\big) + r\delta_{n-1}\big(\frac{r_i g}{n!} - \epsilon_i(\overline{\Phi})\frac{rg}{n!}\big) =$$

$$\frac{rg}{n!}\big[\big(\frac{-r|A| - r_i A}{(n-1)!}\big) - rs\delta_{n-1} + \delta_{n-1}(r_i - \epsilon_i(\overline{\Phi})r)\big] =$$

$$\frac{rg}{n!}\big[\big(\frac{-r|A| - r_i A}{(n-1)!}\big) + \delta_{n-1}(-rs + r_i s - \epsilon_i(\overline{\Phi})r)\big] <$$

$$\frac{rg}{n!}\delta_{n-1}(-rs + r_i s - \epsilon_i(\overline{\Phi})r) \ ,$$

because $-r|A| - r_i A < 0$. Last expression is either zero if $r_i = \operatorname{rk} E = 2$ (because in that case it is $\epsilon_i(\overline{\Phi}) = \epsilon_{t+1}(\overline{\Phi}) = s$), or negative if $r_i = 1$. Hence, $\xi_{2n-1} < 0$.

Therefore $\Psi(m) = \xi_{2n-1}m^{2n-1} + \cdots + \xi_1 m + \xi_0$ with $\xi_{2n-1} < 0$, so there exists an integer m_3 such that for $m \geq \{m_0, m_1, m_2, m_3\}$ we have $\Psi(m) < 0$ and $w_i < 0$, then the contradiction. ■

Similarly to Proposition 2.1.19, we prove

Proposition 2.4.9. *There exists an integer m_4 such that for $m \geq m_4$ the sheaves E_i^m and $E^{m,i} = E_i^m/E_{i-1}^m$ are m_4-regular. In particular their higher cohomology groups, after twisting with $\mathcal{O}_X(m_4)$, vanish and they are generated by global sections.*

Proposition 2.4.10. *Let $m \geq m_4$. For each filter E_i^m in the m-Kempf filtration of the* rk 2 *tensor (E, φ), we have* $\dim V_i = h^0(E_i^m(m))$, *therefore* $V_i \cong H^0(E_i^m(m))$.

Proof. Let $V_\bullet \subseteq V$ be the Kempf filtration of V (cf. Theorem 2.1.5) and let $(E_\bullet^m, \varphi|_{E_\bullet^m}) \subseteq (E, \varphi)$ be the m-Kempf filtration of (E, φ). Analogously to Proposition 2.1.20 we can construct two filtrations

$$\begin{array}{ccccccccccccc}
0 & \subset & \cdots & \subset & V_i & \subset & V_{i+1} & \subset & V_{i+2} & \subset & \cdots & \subset & V \\
 & & & & \cap & & \| & & \| & & & & \\
 & & & & H^0(E_i^m(m)) & \subset & H^0(E_{i+1}^m(m)) & \subset & H^0(E_{i+2}^m(m)) & & & &
\end{array} \tag{2.4.6}$$

and

$$\begin{array}{ccccccccccccc}
0 & \subset & \cdots & \subset & V_i & \subset & H^0(E_i^m(m)) & \subset & V_{i+1} & \subset & \cdots & \subset & V \\
 & & & & \| & & \| & & \| & & & & \\
 & & & & V'_i & & V'_{i+1} & & V'_{i+2} & & & &
\end{array} \tag{2.4.7}$$

to be in situation of Lemma 2.1.16, where $W = H^0(E_i^m(m))$, filtration $V_\bullet$ is (2.4.6) and filtration $V'_\bullet$ is (2.4.7).

Now, the graph associated to filtration $V_\bullet$ is given, by Definition 2.4.5, by the points

$$(b_i, w_i) = \Big(\frac{\dim V_i}{m^n}, \frac{m}{\dim V}\big(r\dim V_i - r_i \dim V + \frac{a_2}{a_1}(s\dim V_i - \epsilon_i(\overline{\Phi}, V_\bullet)\dim V)\big)\Big)\ ,$$

the slopes $-v_i$ of the graph given by

$$-v_i = \frac{w^i}{b^i} = \frac{w_i - w_{i-1}}{b_i - b_{i-1}} =$$
$$\frac{m^{n+1}}{\dim V}\big(r - r^i\frac{\dim V}{\dim V^i} + \frac{a_2}{a_1}(s - \epsilon^i(\overline{\Phi}, V_\bullet)\frac{\dim V}{\dim V^i})\big) \leq$$
$$\frac{m^{n+1}}{\dim V}\big(r + s\frac{a_2}{a_1}\big) := R$$

and equality holds if and only if $r^i = 0$ (note that $r^i = 0$ implies $\epsilon^i(\overline{\Phi}, V_\bullet) = 0$).

The new point which appears in graph of the filtration $V'_\bullet$ is

$$Q = \big(\frac{h^0(E_i^m(m))}{m^n}, \frac{m}{\dim V}(rh^0(E_i^m(m)) - r_i\dim V + \frac{a_2}{a_1}(sh^0(E_i^m(m)) - \epsilon_i(\overline{\Phi}, V_\bullet)\dim V))\big)\ ,$$

where we write $\epsilon_i(\overline{\Phi}, V_\bullet)$ instead of $\epsilon_i(\overline{\Phi}, V'_\bullet)$, by the same argument used in proof of Proposition 2.2.13 (c.f. (2.2.7)).

The slope of the segment between (b_i, w_i) and Q is, similarly,

$$-v'_i = \frac{m^{n+1}}{\dim V}(r + s\frac{a_2}{a_1}) = R\ .$$

By Lemma 2.1.15, the graph is convex, so $v_1 < v_2 < \ldots < v_{t+1}$. Besides, $r^1 = r_1 > 0$, then $-R < v_1$, because E is torsion free, hence also the subsheaf E_1^m, and a rank 0 torsion free sheaf is the zero sheaf. On the other hand, by Lemma 2.1.16, $v'_i \geq v_i$. Hence,

$$-R < v_1 < v_2 < \ldots < v_i \leq v'_i = -R\ ,$$

which is a contradiction.

Therefore, $\dim V_i = h^0(E_i^m(m))$, for every filter in the m-Kempf filtration. ∎

Corollary 2.4.11. *Let $m \geq m_4$. For every filter E_i^m in the m-Kempf filtration of the* rk 2 *tensor (E, φ), it is $r^i > 0$. Therefore, the m-Kempf filtration consists on a rank* 1 *subsheaf, $0 \subset (L^m, \varphi|_{L^m}) \subset (E, \varphi)$.*

Proof. We have seen that $r^i = 0$ is equivalent to $-v_i = R$. Then the result follows from Proposition 2.4.10 because it is $r^1 = r_1 > 0$ and $-R < v_1 < v_2 < \ldots < v_{t+1}$. ■

For any $m \geq m_4$, by Corollary 2.1.21 there is only one filter $(L^m, \varphi|_{L^m})$ in the m-Kempf filtration and, by Proposition 2.1.19, L^m is m_4-regular. Hence, $L^m(m_4)$ is generated by the subspace $H^0(L^m(m_4)) \subset H^0(E(m_4))$ by the evaluation map (c.f. Lemma 1.2.13). Note that the dimension of the vector space $H^0(E(m_4))$ does not depend on m.

The m**-type** of the m-Kempf filtration $0 \subset (L^m, \varphi|_{L^m}) \subset (E, \varphi)$ is the Hilbert polynomial P_{L^m} (c.f. Definition 2.1.22). The set of possible m-types

$$\mathcal{P} = \{P_{L^m}\}$$

is finite, for all integers $m \geq m_3$ (c.f. Proposition 2.1.23).

Rewrite the graph associated to the m-Kempf filtration (c.f. Definition 2.4.5)

$$v_{m,i} = \frac{m^{n+1}}{\dim V^i \dim V}\left[r^i \dim V - r \dim V^i + \frac{a_2}{a_1}(\epsilon^i(\overline{\Phi}) \dim V - s \dim V^i)\right] ,$$

$$b^i_m = \frac{1}{m^n} \cdot \dim V^i ,$$

as

$$v_{m,i} = \frac{m^{n+1}}{P^i_m(m)P(m)}\left[r^i P(m) - rP^i_m(m) + \frac{r\delta(m)}{P(m) - s\delta(m)}(\epsilon^i(\overline{\Phi})P(m) - sP^i_m(m))\right] ,$$

$$b^i_m = \frac{1}{m^n} \cdot P^i_m(m) ,$$

by Propositions 2.4.9 and 2.4.10.

Note that, by Corollary 2.1.21, the graph has only two slopes given by

$$v_{m,1} = \frac{m^{n+1}}{P_{L^m}(m)P(m)}\left[P(m) - 2P_{L^m}(m) + \frac{2\delta(m)}{P(m) - s\delta(m)}(\epsilon_{L^m}P(m) - sP_{L^m}(m))\right] ,$$

$$v_{m,2} = \frac{m^{n+1}}{P_{E/L^m}(m)P(m)}\left[P(m) - 2P_{E/L^m}(m) + \frac{2\delta(m)}{P(m) - s\delta(m)}((s - \epsilon_{L^m})P(m) - sP_{E/L^m}(m))\right] ,$$

where $\epsilon(L^m)$ is the number of times that the subsheaf L^m appears on the minimal multi-index (c.f. (1.2.37) in section 1.2).

The set

$$\mathcal{A} = \{\Theta_m : m \geq m_4\}$$

is finite (c.f. Proposition 2.1.23), where

$$\Theta_m(l) = (\mu_{v_m(l)}(\Gamma_{v_m(l)}))^2 = ||v_m(l)||^2$$

(c.f. (2.1.14)). Let K be the maximal function in $\mathcal{A}$ as in Lemma 2.1.24) for which $\exists\, m_5$ such that for all $m \geq m_5$ it is $\Theta_m = K$.

Proposition 2.4.12. *Let l_1 and l_2 be integers with $l_1 \geq l_2 \geq m_5$. Then the l_1-Kempf filtration of E is equal to the l_2-Kempf filtration of E.*

Proof. C.f. Proposition 2.1.25. ■

Therefore, Theorem 2.4.4 follows from Proposition 2.4.12. Hence, eventually, the Kempf filtration of the $\operatorname{rk} 2$ tensor (E, φ) does not depend on the integer m.

Definition 2.4.13. *If $m \geq m_5$, the m-Kempf filtration of the* $\operatorname{rk} 2$ *tensor* (E, φ)

$$0 \subset (L, \varphi|_L) \subset (E, \varphi)$$

is called the ***Kempf filtration*** *or the* ***Kempf subsheaf*** *of* (E, φ).

2.4.3 Harder-Narasimhan filtration for $\operatorname{rk} 2$ tensors

Kempf theorem (c.f. Theorem 2.1.5) says that, given an integer m and $V \simeq H^0(E(m))$, there exists a unique weighted filtration of vector spaces $V_\bullet \subseteq V$ which gives maximum for the Kempf function

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^{t+1} \frac{\Gamma_i}{\dim V}(r^i \dim V - r \dim V^i) + \frac{a_2}{a_1} \sum_{i=1}^{t+1} \frac{\Gamma_i}{\dim V}\left(\epsilon^i(\overline{\Phi}) \dim V - s \dim V^i\right)}{\sqrt{\sum_{i=1}^{t+1} \dim V^i \Gamma_i^2}} .$$

This filtration induces a unique rank 1 subsheaf $L \subset E$ called the Kempf subsheaf of the $\operatorname{rk} 2$ tensor (E, φ). By Proposition 2.4.12, the subsheaf L does not depend on m, for $m \geq m_5$.

The Kempf function is a function on m (c.f. Proposition 2.4.7). Consider the function

$$K(m) = m^{\frac{n}{2}+1} \cdot \mu(V_\bullet, m_\bullet) = \mu_{v_m}(\Gamma)$$

and, making the substitutions for m sufficiently large

$$\dim V_1 = \dim V^1 = h^0(L(m)) = P_L(m) \ ,$$

$$\dim V^2 = \dim V - \dim V_1 = h^0(E/L(m)) = P_{E/L}(m)$$

we get

$$K(m) = m^{\frac{n}{2}+1} \cdot \frac{\sum_{i=1}^{2} \frac{\gamma_i}{r}[(r^i P - r P^i) + \frac{r\delta}{P - s\delta}(\epsilon^i P - s P^i)]}{\sqrt{\sum_{i=1}^{2} P^i \frac{P^2}{r^2} \gamma_i^2}} ,$$

where we put $P = P_E(m)$, $P^1 = P_L(m)$, $P^2 = P_{E/L}(m)$, $\epsilon^1 = \epsilon(L)$, $\epsilon^2 = s - \epsilon(L)$. Note that $\epsilon^i = \epsilon^i(\overline{\Phi}) = \epsilon^i(\varphi)$ and recall the relations

$$\gamma_i = \frac{r}{P}\Gamma_i \ ,$$

$$\frac{a_2}{a_1} = \frac{r\delta}{P - s\delta} \ .$$

Also recall

$$\frac{\gamma_{i+1} - \gamma_i}{r} = n_i \ ,$$

$$\sum r^i \gamma_i = \gamma_1 + \gamma_2 = 0 \ ,$$

which gives in our case $\gamma_1 = -n_1$, $\gamma_2 = n_1$. Substituting we get

$$K(m) = m^{\frac{n}{2}+1} \cdot \frac{1}{P - s\delta} \frac{-n_1[2(\delta\epsilon^1 - P^1) + (P - \delta s)] + n_1[2(\delta\epsilon^2 - P^2) + (P - \delta s)]}{\sqrt{P^1 n_1^2 + P^2 n_1^2}} =$$

$$m^{\frac{n}{2}+1} \cdot \frac{r}{\sqrt{P}(P - s\delta)}[2P_L - P_E + \delta(s - 2\epsilon(L))] \ .$$

Note that the unique weight n_1 does not appear in the function later from the substitutions, as it was expected from a one-step filtration. Also note that the denominator of the function K is positive (c.f. choice of m_2 in proof of Proposition 2.4.8). Hence, we can state the following theorem.

Theorem 2.4.14. *Given a δ-unstable* rk 2 *tensor* $(E, \varphi : \overbrace{E \otimes \cdots \otimes E}^{s\ times} \longrightarrow \mathcal{O}_X)$, *there exists a unique line subsheaf $L \subset E$ which gives maximum for the polynomial function*

$$K(m) = 2P_L(m) - P_E(m) + \delta(m)(s - 2\epsilon(L)) \ .$$

If X is a one dimensional complex projective variety, i.e. a smooth projective complex curve, we can simplify the function μ. Recall that, by Riemann-Roch, the Hilbert polynomial of a sheaf E of rank r and degree d over a curve of genus g is

$$P_E(m) = rm + d + r(1 - g) \ ,$$

and the polynomial $\delta(m)$ becomes a positive constant τ. In this case, a coherent torsion free sheaf of rank 2 is a vector bundle of rank 2 over X, and the Kempf subsheaf will be a line subbundle.

Theorem 2.4.15. *Given a τ-unstable* rk 2 *tensor* $(E,\varphi:\overbrace{E\otimes\cdots\otimes E}^{s\ times}\longrightarrow\mathcal{O}_X)$ *over a smooth projective complex curve, there exists a unique line subbundle $L\subset E$ which maximizes the quantity*

$$2\deg L-\deg E+\tau(s-2\epsilon(L)) .$$

Note that, if the tensor is unstable, such quantity will be positive, and the graph corresponding to the filtration will be *a cusp* which is a convex graph.

If we define the **corrected Hilbert polynomials** of (E,φ) and $(L,\varphi|_L)$ (c.f. Definition 2.2.3) as

$$\overline{P}_E=P_E-\delta s ,$$

$$\overline{P}_L=P_L-\delta\epsilon(L) ,$$

we recover the notion of stability for rk 2 tensors (c.f. Definition 2.4.1). A rk 2 tensor (E,φ) is **δ-unstable** if there exists a line subsheaf $L\subset E$ such that

$$\frac{\overline{P}_L}{\operatorname{rk}L}>\frac{\overline{P}_E}{\operatorname{rk}E}\Leftrightarrow\overline{P}_L>\frac{\overline{P}_E}{2} .$$

Hence, this procedure allows us define a notion of a Harder-Narasimhan filtration for δ-unstable rk 2 tensors.

Definition 2.4.16. *If (E,φ) is a δ-unstable* rk 2 *tensor, there exists a unique line subsheaf maximizing*

$$2\cdot\overline{P}_L-\overline{P}_E>0 .$$

We call

$$0\subset(L,\varphi|_L)\subset(E,\varphi)$$

*the **Harder-Narasimhan filtration** of (E,φ), and we call L the **Harder-Narasimhan subsheaf** of (E,φ).*

Remark 2.4.17. *We do not know, in principle, how to define a quotient tensor $(E/L,\overline{\varphi}|_{E/L})$, because we do not know, a priori, how to define $\overline{\varphi}|_{E/L}$. This is why we cannot talk about quotient tensors, as in Definition 2.2.2.*

Given the exact sequence of sheaves, $0\to L\to E\to E/L\to 0$, we define the corrected Hilbert polynomial of the quotient as $\overline{P}_{E/L}=\overline{P}_E-\overline{P}_L$, and we have, trivially, the additivity of the corrected polynomials on exact sequences of sheaves. This way we can *consider that Definition 2.4.16 contains the analogous to conditions of Definition 1.3.3 for* rk 2 *tensors. Indeed,*

$$2 \cdot \overline{P}_L - \overline{P}_E > 0 \Leftrightarrow \overline{P}_L > \overline{P}_{E/L} \ ,$$

and the semistability of $(L, \varphi|_L)$ *and* $(E/L, \overline{\varphi}|_{E/L})$ *(whichever definition of* $\overline{\varphi}|_{E/L}$ *we impose), would follow trivially from the fact of they are rank* 1 *tensors.*

Therefore, Definition 2.4.16 gives a notion of Harder-Narasimhan filtration for these objects.

2.4.4 Stable coverings of a projective curve

In this section we use the previous notions for rk 2 tensors over curves where the morphism is symmetric, and the Definition 2.4.16 of the Harder-Narasimhan subsheaf, to define stable coverings of a projective curve and, for the unstable ones, a maximally destabilizing object, in geometrical terms.

In the following, we shall consider rank 2 tensors (E, φ) where E is a rk 2 vector bundle over a smooth projective complex curve X, and

$$\varphi : \overbrace{E \otimes \cdots \otimes E}^{\text{s times}} \longrightarrow \mathcal{O}_X$$

is a symmetric non degenerate morphism. We call it a **symmetric non degenerate rank 2 tensor**. Let τ be a positive real number. Let $\mathbb{P}(E)$ be the projective space bundle of the vector bundle E, which is a ruled algebraic surface (c.f. [Ha, Section V.2]).

The morphism φ is, fiberwise, a symmetric multilinear map

$$\varphi_x : \overbrace{V \otimes \cdots \otimes V}^{\text{s times}} \longrightarrow \mathbb{C} \ ,$$

where $V \simeq \mathbb{C}^2$. Then, φ_x factors through $\mathrm{Sym}^s(V)$, isomorphic to the $(s+1)$-dimensional vector space of homogeneous polynomials of degree s in two variables. Hence, fiberwise, φ can be represented by a polynomial

$$\varphi_x \equiv \sum_{i=0}^{s} a_i(x) X_0^i X_1^{s-i} \tag{2.4.8}$$

which vanishes on s points in $\mathbb{P}(V) \simeq \mathbb{P}^1_{\mathbb{C}}$. Therefore, as φ varies on X, it defines a degree s covering

$$\mathbb{P}(E) \supset X' \to X \ .$$

Suppose that (E,φ) is a τ-unstable rk 2 tensor. Then, by Theorem 2.4.15, there exists a line subbundle $L\subset E$, the **Harder-Narasimhan subbundle**, giving maximum for the quantity

$$2\deg(L)-\deg(E)+\tau(s-2\epsilon(L))\ . \tag{2.4.9}$$

The subbundle L can be seen as a section of $\mathbb{P}(E)$, each fiber L_x corresponding to a point $P=\{L_x\}\in\mathbb{P}^1_{\mathbb{C}}$. Recall from Definition 2.4.1 that $\epsilon(L)=k$ if $\varphi|_{L^{\otimes(k+1)}\otimes E^{\otimes(s-k-1)}}=0$ and $\varphi|_{L^{\otimes k}\otimes E^{\otimes(s-k)}}\neq 0$. Note that here we use the symmetry of the morphism φ. Therefore, $\epsilon(L)=k$ means that, generically, $P=\{L_x\}$ is a zero of multiplicity $s-k$ and, by definition of the covering $X'\to X$, $s-\epsilon(L)$ is exactly the number of branches of X' which generically do coincide with the section defined by L, counted with multiplicity.

Recall Examples 1.1.5 and 1.1.15 in Section 1.1. There, a homogeneous polynomial of degree N, $P=\sum\limits_i a_iX_0^iX_1^{N-i}$, was unstable if it contained a linear factor of degree greater that $\frac{N}{2}$. Now, observe that the restriction of a rank 2 tensor to a point $x\in X$ in (2.4.8), passing to the projectivization $\mathbb{P}(E)$ hence fibers are isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, is precisely one of the homogeneous polynomials in Examples 1.1.5 and 1.1.15. Fiberwise, the morphism φ defines a set of s points in $\mathbb{P}^1_{\mathbb{C}}$. See that, from the point of view of Examples 1.1.5 and 1.1.15, letting $s=N$, the set of points is unstable if there exists a point with multiplicity greater that $\frac{s}{2}$.

Then, as $s-\epsilon(L)$ is the multiplicity of the point defined by the line L_x (the fiber of the Harder-Narasimhan subbundle over x), in the set of s points defined by the morphism φ, following the previous argument, this point $\{L_x\}$ will destabilize the set if

$$s-\epsilon(L)>\frac{s}{2}\Leftrightarrow s-2\epsilon(L)>0\ ,$$

which is the second summand in (2.4.9). Hence, the positivity of $s-2\epsilon(L)$ is equivalent to find a line subbundle L defining a point in the fiber $\mathbb{P}^1_{\mathbb{C}}$, which coincides with one of the zeroes of φ in the fiber, and such that it has multiplicity greater that $\frac{s}{2}$.

To conclude, we can say that the expression (2.4.9) consists of two summands weighted by the parameter τ. First one, $2\deg(L)-\deg(E)$, is measuring the stability of the vector bundle E. Second one, $s-2\epsilon(L)$, is measuring the stability of the morphism or, with the previous observations, the generic stability of the set of points defined in $\mathbb{P}^1_{\mathbb{C}}$, fiberwise, as in Examples 1.1.5 and 1.1.15, when varying along the covering. Therefore, an object destabilizing a rank 2 tensor is an object which contradicts these two stabilities, weighted by τ, and the Harder-Narasimhan subbundle is the unique one which maximally does, for a τ-unstable tensor.

The sets of points in each fiber defined by φ give a covering of degree s,

$$\mathbb{P}(E) \supset X' \to X \ .$$

In the following, we rewrite the stability of the sets of points, fiberwise, as stability for the covering, using intersection theory for ruled surfaces.

Proposition 2.4.18. *[Ha, Proposition V.2.8] Given a ruled surface $\mathbb{P}(E)$, there exists $E' \simeq E \otimes N$, with N line bundle, such that $H^0(E') \neq 0$ but for all line bundles N' with negative degree we have $H^0(E' \otimes N') = 0$. Therefore, $\mathbb{P}(E) = \mathbb{P}(E')$ and the integer $e = -\deg E'$ is an invariant of the ruled surface. Furthermore, in this case, there exists a section $\sigma_0 : X \to \mathbb{P}(E')$ with image C_0, such that $\mathcal{L}(C_0) \simeq \mathcal{O}_X(1)$.*

For a ruled surface $\mathbb{P}(E')$ we say that E' is **normalized** if it satisfies the conditions of the Proposition 2.4.18.

Let $\mathbb{P}(E')$ be a ruled surface with E' normalized. Let $\sigma : X \to \mathbb{P}(E)$ be a section, and let $D = \operatorname{im} \sigma$ a divisor on $\mathbb{P}(E)$. It can be proved that $\deg(L) = -e - C_0 \cdot D$, with these conventions (c.f. [Ha, Proposition V.2.9]). Let us define, by analogy, $\epsilon(\sigma) = \epsilon(D)$ as the number of branches of X' which generically do coincide with D, the section defined by σ, counted with multiplicity.

Definition 2.4.19. *Let $(E, \varphi : \overbrace{E \otimes \cdots \otimes E}^{s \text{ times}} \longrightarrow \mathcal{O}_X))$ be a symmetric non degenerate rank 2 tensor over X. Let $f : X' \to X$ be the covering defined by (E, φ), $X' \subset \mathbb{P}(E)$. Let τ be a positive number. We say that f is **τ-unstable** if there exists a section $\sigma : X \to \mathbb{P}(E)$ with image D, i.e. there exists a line subbundle $L \subset E$, such that the following holds*

$$-2C_0 \cdot D - e + \tau(s - 2\epsilon(D)) > 0$$

Proposition 2.4.20. *Let τ be a positive number. A symmetric non degenerate $\operatorname{rk} 2$ tensor (E, φ) is τ-unstable if and only if the associated covering $f : X' \to X$ is τ-unstable.*

Proof. It is only needed to check that we can assume $X' \subset \mathbb{P}(E')$ with E' normalized (c.f. Proposition 2.4.18), in the definition of stability of f. Let N be a line bundle over X. If we change E by $E' = E \otimes N$, then we have the line subbundle $L \otimes N \subset E'$ (by exactness of the tensor product with locally free sheaves), and

$$\deg(E') = \deg(E \otimes N) = \deg(E) + 2\deg(N) \ ,$$

$$\deg(L \otimes N) = \deg(L) + \deg(N) \ ,$$

so the quantity $2\deg(L) - \deg(E)$ is invariant by tensoring E with a line bundle.

For the invariance of the rest of the formula, also note that we can trivially extend the definition of the morphism φ,

$$\varphi' : (E')^{\otimes s} = E^{\otimes s} \otimes N^{\otimes s} \to \mathcal{O}_X$$

and, then, it is $\epsilon'(L \otimes N) = \epsilon(L)$. ■

Theorem 2.4.21. *If $f : X' \to X$ is a degree s covering coming from a symmetric non degenerate rank 2 tensor (E, φ) which is τ-unstable, then there exists a unique section $\sigma : X \to \mathbb{P}(E)$ with image D, giving maximum for*

$$-2C_0 \cdot D - e + \tau(s - 2\epsilon(D)) > 0 \, .$$

We call σ the ***Harder-Narasimhan section*** *of the covering.*

2.5 Rank 3 tensors and beyond

This final section of chapter 2 contains some observations about the rank 3 tensors case, which is the first one we cannot apply directly the techniques used in the previous sections. The crucial point will be the impossibility of rewriting the Kempf function (c.f. Definition 1.4.4) in this case as a geometrical function as in Proposition 2.1.13 because, as we will see, the argument Γ in that geometrical function (which represents the weights $n_\bullet$ in Definition 1.2.3) depends on the vector v (which represents the filters $E_\bullet$ in Definition 1.2.3) which does not allow us to apply results of subsection 2.1.2.

2.5.1 Independence between multi-indexes and weights

In the previous sections we have been able to carry out the program designed for torsion free coherent sheaves in different cases of tensors: holomorphic pairs in section 2.2 and rk 2 tensors in section 2.4. The proof of the correspondence between the 1-parameter subgroup of Kempf and the Harder-Narasimhan filtration in that case is based on proving properties related with the convexity for an arbitrary filtration of subobjects, to show that the candidates to be the Kempf filtration are very particular, so this filtration is unique. For holomorphic pairs we previously know about the Harder-Narasimhan filtration so, by uniqueness, it coincides with the Kempf filtration. For rank 2 tensors, we define the Harder-Narasimhan filtration as the unique Kempf filtration.

We know that the Kempf filtration gives maximum for some function, the Kempf function, which depends on the data of the filters (ranks, Hilbert polynomials, etc.) and the data of the weights or the exponents of the 1-parameter subgroup. And the key point is to rewrite the Kempf function as a geometrical function in an Euclidean space (c.f. definition of μ_v in (2.1.8)). There we prove that, if we think of the filtration (referring just to the flag $V_\bullet \subset V$ without the weights $n_\bullet$) as a graph, the weights giving the maximum for the function are given by the convex envelope of the graph (c.f. Theorem 2.1.9).

Thanks to the independence between the vector v giving the graph of the m-Kempf filtration $E^m_\bullet \subset E$ and the weights Γ_i, we are able to rewrite the Kempf function as a geometrical function, then it is possible to interpret this geometrical function as depending on two values, v and Γ, independently. For holomorphic pairs this is easily proved by checking that $\epsilon_i(\overline{\Phi}, V_\bullet) = \epsilon(E_i)$ is independent of Γ, because the symbol ϵ_i only depends on the vanishing of the restriction of the morphism φ to E_i (c.f. Lemma 2.2.7). For rk 2 tensors, this is proven in Lemma 2.4.6. Nevertheless, for tensors in general, this is not possible as we are going to show.

Recall Definition 1.2.1 and expression (1.2.4) in the stability condition for tensors,

$$\mu(\varphi, E_\bullet, n_\bullet) = \min_{I\in\mathcal{I}}\{\gamma_{r_{i_1}} + \cdots + \gamma_{r_{i_s}} : \varphi|_{(E_{i_1}\otimes\cdots\otimes E_{i_s})^{\oplus c}} \neq 0\} \ , \tag{2.5.1}$$

where $\mathcal{I} = \{1, ..., t+1\}^{\times s}$ is the set of all multi-indexes $I = (i_i, ..., i_s)$ and $(E_\bullet, n_\bullet)$ is a weighted filtration of E. Recall that the previous quantity was expressed in another way in (1.2.7),

$$\mu(\varphi, E_\bullet, n_\bullet) = \sum_{i=1}^{t} n_i(sr_i - \epsilon_i(E_\bullet)r) \ .$$

Also recall that the data of the multi-index giving minimum in (2.5.1) is equivalent to the data of the $\epsilon_i(E_\bullet)$ in the second expression.

For holomorphic pairs (i.e. $s = 1$ in Definition 1.2.1), Lemma 2.2.7 guarantees that the multi-index does not depend on the weights $\gamma_{r_{i_j}}$ of the 1-parameter subgroup. The multi-index is i if $i = \min\{i : \varphi_{E_i} \neq 0\}$, i.e., $\varphi_{E_{i-1}} = 0$ and $\varphi_{E_i} \neq 0$. Lemma 2.4.6 does the analogous for rk 2 tensors, being the multi-index (in the symmetric case) $(i_1, \dots, i_s) = (1, \dots, 1, 2, \dots, 2)$ where the number of 1's is $\epsilon(L) = k$ if $\varphi|_{L^{\otimes(k+1)}\otimes E^{\otimes(s-k-1)}} = 0$ and $\varphi|_{L^{\otimes k}\otimes E^{\otimes(s-k)}} \neq 0$. However, beyond this cases, we cannot assure the independence of the multi-index with the weights.

Suppose the case $s = 2$, $r = 3$ in Definition 1.2.1, and suppose, for simplicity, that the morphism φ is symmetric. The multi-index in this case will be (i_1, i_2), where $0 \leq i_j \leq 3$ because recall that, in (2.5.1), the vanishing of the morphism is checked generically. Then, the multi-index is checking whether $\varphi|_{E_{i_1} \otimes E_{i_2}} = 0$ or not, hence by the symmetry of φ, the only multi-indexes which can appear are $(1,1)$, $(1,2)$, $(1,3)$, $(2,2)$, $(2,3)$. Therefore, for a given filtration $0 \subset L \subset F \subset E$ with weights $\gamma_1, \gamma_2, \gamma_3$, the following six situations can occur:

1. $\varphi|_{L\otimes L} \neq 0$
2. $\varphi|_{F\otimes L} \neq 0$ and $\varphi|_{L\otimes L} = 0$
3. $\varphi|_{E\otimes L} \neq 0$ and $\varphi|_{F\otimes F} = 0$
4. $\varphi|_{F\otimes F} \neq 0$ and $\varphi|_{E\otimes L} = 0$
5. $\varphi|_{E\otimes F} \neq 0$ and $\varphi|_{E\otimes L} = 0$
6. $\varphi|_{F\otimes L} = 0$, $\varphi|_{F\otimes F} \neq 0$, $\varphi|_{E\otimes L} \neq 0$

Cases $1 - 5$ give a fixed multi-index, $(1,1)$, $(1,2)$, $(1,3)$, $(2,2)$ and $(2,3)$ respectively. However, in case number 6, the multi-index will be $(1,3)$ if $\gamma_1 + \gamma_3 \leq 2\gamma_2$ or $(2,2)$ if $\gamma_1 + \gamma_3 \geq 2\gamma_2$. Hence, this is the simplest case where the multi-index actually depends on the weights γ_i. Therefore, setting $s = 2$ and $r = 3$ in Definition 1.2.1 we get the first case for which these features can occur.

If this happens, we are not able to rewrite the Kempf function as a geometrical function (c.f. (2.1.8)) and prove an analogous to Proposition 2.1.13 to apply the argument of the convex envelope to look for the vector γ giving maximum in Theorem 2.1.9. This is the reason why the general method described in this thesis breaks down and does not apply beyond rank 2 tensors.

2.5.2 Considerations for $\operatorname{rk} 3$ tensors

Let us consider **symmetric rank 3 tensors of two arguments** over a smooth projective curve, i.e. the morphism $\varphi : E \otimes E \to \mathcal{O}_X$ being symmetric. This case is obtained from section 1.2 by setting $s = 2$ $c = 1$, $b = 0$, $R = \operatorname{Spec} \mathbb{C}$ and $\mathcal{D} = \mathcal{O}_X$, the structure sheaf over $X \times R \simeq X$, in Definition 1.2.1.

Let τ be a positive constant and consider a tensor (E,φ) which is τ-unstable (c.f. Definition 1.2.5). Let m_0 be an integer as in Theorem 1.2.31 (such that δ-stability and GIT stability coincide) and such that E is m_0-regular (picking a larger integer, if necessary). Let $m \geq m_0$ and let $V \simeq H^0(E(m))$. In this case, the **Kempf function** (c.f. Definition 1.4.4) is

$$\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^{t} n_i(r\dim V_i - r_i \dim V) + \frac{a_2}{a_1}\sum_{i=1}^{t} n_i\big(2\dim V_i - \epsilon_i(\overline{\Phi})\dim V\big)}{\sqrt{\sum_{i=1}^{t+1}\dim V^i\Gamma_i^2}} ,$$

where

$$\frac{a_1}{a_2} = \frac{r\tau}{P_E(m)-s\tau} = \frac{3\tau}{P_E(m)-2\tau} .$$

Let

$$0 \subset V_1 \subset \cdots \subset V_{t+1} = V$$

be the **Kempf filtration of** V (c.f. Theorem 2.1.5) and let

$$0 \subseteq (E_1^m, \varphi|_{E_1^m}) \subseteq (E_2^m, \varphi|_{E_2^m}) \subseteq \cdots (E_t^m, \varphi|_{E_t^m}) \subseteq (E_{t+1}^m, \varphi|_{E_{t+1}^m}) \subseteq (E,\varphi) .$$

be the m**-Kempf filtration of** (E,φ), by evaluating the V_i. Suppose that we are able to prove properties satisfied by the filters of the m-Kempf filtration (i.e. analogous to Propositions 2.1.18 and 2.1.20) to rewrite the Kempf function as

$$K = \frac{\sum_{i=1}^{t+1}\Gamma_i[(r^i d - r d^i) + \frac{r\tau}{P-2\tau}(\epsilon^i P - 2P^i)]}{\sqrt{\sum_{i=1}^{t+1} P^i\Gamma_i^2}} =$$

$$\frac{\sum_{i=1}^{t+1}\Gamma_i[(r^i d - r d^i) + \tau(r\epsilon^i - 2r^i)]}{\sqrt{\sum_{i=1}^{t+1} r^i\Gamma_i^2}} = \frac{\sum_{i=1}^{t} n_i[(r d_i - r_i d) + \tau(2r_i - \epsilon_i r)]}{\sqrt{\sum_{i=1}^{t+1} r^i\Gamma_i^2}}$$

(c.f. Proposition 2.2.17). Recall that we are considering the case $s = 2$ and $\dim X = 1$ in Definition 1.2.1.

Observe that, in order to achieve a maximum of the function, it is enough to consider saturated filtrations. Indeed, note that $n_i > 0$, $\deg E_i \leq \deg \overline{E_i}$ and $\operatorname{rk} E_i = \operatorname{rk} \overline{E_i}$, hence the value of the function is greater on saturated filtrations. Also, by similar reasons, it is enough to consider filtrations with increasing ranks (c.f. Remark 1.2.4). Therefore, in order to look for the Kempf filtration, i.e., the filtration which maximizes the previous function, we can restrict our attention to filtrations of the form

$$0 \subset (L, \varphi|_L) \subset (F, \varphi|_F) \subset (E,\varphi) , \qquad (2.5.2)$$

for which the Kempf function is

$$\frac{\sum_{i=1}^{3}\Gamma_i[(r^i d - r d^i) + \tau(r\epsilon^i - r^i s)]}{\sqrt{\sum_{i=1}^{3} r^i \Gamma_i^2}} \ .$$

Let (i_1, i_2) be the multi-index in the definition of (2.5.1). Consider a filtration as in (2.5.2) which is τ-destabilizing, i.e. contradicting Definition 1.2.3. We want to check if this filtration is the Kempf filtration, and for that we would like to ask ourselves for the best 1-parameter subgroup giving maximum for the Kempf function. Note that, in the definition of stability for rk 3 tensors, it is not enough to consider one-step filtrations, i.e. subobjects, hence asking for the weights of the filtration is a meaningful question.

The crucial fact is that the coefficients of the Kempf function (understood as the function in Theorem 2.1.9, a function on the exponents Γ_1, Γ_2 and Γ_3) vary with the multi-index (i_1, i_2). The filtration (2.5.2) will give a multi-index in (2.5.1). If the multi-index we obtain falls into one of the cases $1-5$ in the list of the previous subsection, then the multi-index does not depend on the weights. However, if

$$\varphi|_{F\otimes L} = 0 \ , \varphi|_{F\otimes F} \neq 0 \ , \varphi|_{E\otimes L} \neq 0 \ ,$$

we are in case 6, and the multi-index will be $(1,3)$ if $\Gamma_1 + \Gamma_3 \leq 2\Gamma_2$, or $(2,2)$ otherwise. In this case, the vector v (the vector of the graph associated to the filtration) will depend on the multi-index, so we can have two possible vectors associated to the filtration (2.5.2). Call the vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, then the coordinates of these two vectors are

$$\begin{aligned} x_1 &= r^1 d - d^1 r + \tau(r\epsilon_1^{(1,3)} - 2r^1) = v_1 + \tau \\ x_2 &= r^2 d - d^2 r + \tau(r\epsilon_2^{(1,3)} - 2r^2) = v_2 - 2\tau \\ x_3 &= r^3 d - d^3 r + \tau(r\epsilon_3^{(1,3)} - 2r^3) = v_3 + \tau \ , \end{aligned}$$

$$\begin{aligned} y_1 &= r^1 d - d^1 r + \tau(r\epsilon_1^{(2,2)} - 2r^1) = v_1 - 2\tau \\ y_2 &= r^2 d - d^2 r + \tau(r\epsilon_2^{(2,2)} - 2r^2) = v_2 + 4\tau \\ y_3 &= r^3 d - d^3 r + \tau(r\epsilon_3^{(2,2)} - 2r^3) = v_3 - 2\tau \ . \end{aligned}$$

Note that $r^1 = r^2 = r^3 = 1$, $r = 3$, and note that we denote with the upper index the different symbols ϵ_i for each multi-index. Also we call $v_i = r^i d - d^i r$, for each i.

Note that, in both cases, the following holds

$$\sum_{i=1}^{3} x_i = \sum_{i=1}^{3} y_i = \sum_{i=1}^{3} v_i = 0 \ .$$

Suppose that $(\Gamma_1, \Gamma_2, \Gamma_3)$ is the vector giving maximum in the Kempf function. And suppose that it verifies $\Gamma_1 + \Gamma_3 \leq 2\Gamma_2$, hence the multi-index is $(1,3)$, and the vector of the graph associated to the filtration is x. Taking into account that $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$, because Γ is a 1-parameter subgroup of $SL(N)$, the Kempf function will be

$$K = \frac{\Gamma_1 x_1 + \Gamma_2 x_2 + \Gamma_3 x_3}{\sqrt{\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2}} = \frac{\Gamma_1 v_1 + \Gamma_2 v_2 + \Gamma_3 v_3 + \tau(\Gamma_1 + \Gamma_3 - 2\Gamma_2)}{\sqrt{\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2}} =$$

$$\frac{\Gamma_1(v_1 - v_2 + 3\tau) + \Gamma_3(v_3 - v_2 + 3\tau)}{\sqrt{2(\Gamma_1^2 + \Gamma_1\Gamma_2 + \Gamma_3^2)}} \ ,$$

which is a function on two arguments. To maximize the function with respect to Γ_1 and Γ_3 we set the gradient of K equal to 0 which gives

$$\Gamma_1 = \Gamma_3\left(\frac{v_1 + \tau}{v_3 + \tau}\right) \ .$$

If we suppose that the vector giving maximum in the Kempf function verifies $\Gamma_1 + \Gamma_3 \geq 2\Gamma_2$, we obtain the Kempf function

$$K = \frac{\Gamma_1 y_1 + \Gamma_2 y_2 + \Gamma_3 y_3}{\sqrt{\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2}} \ ,$$

which we maximize with respect to Γ_1 and Γ_3 as well, obtaining

$$\Gamma_1 = \Gamma_3\left(\frac{v_1 - 2\tau}{v_3 - 2\tau}\right) \ .$$

Observe that, in both cases, the vector $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$ which maximize the Kempf function is exactly given by the vectors x and y, because the Γ_1 and Γ_3 obtained are precisely multiples of their coordinates.

Note that, $\Gamma_1 + \Gamma_3 \leq 2\Gamma_2$ implies $\Gamma_1 \leq -\Gamma_3$, so to be congruent, in the first case, it has to be $v_1 + v_3 + 2\tau \leq 0$ (here we use that $v_3 > 0$, which has to hold by convexity in Lemma 2.1.15). Similarly, in the second case, it has to be $v_1 + v_3 - 4\tau \geq 0$ (whenever $v_3 - 2\tau \geq 0$). Observe that, as $\tau > 0$, both conditions cannot hold at the same time, then necessarily the multi-index is one of two, either $(1,3)$ or $(2,2)$.

Finally, consider the following example. Let (E,φ) be a rank 3 tensor over $X=\mathbb{P}^1_{\mathbb{C}}$, where $E=\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2)\oplus\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}\oplus\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)$. Consider the filtration $0\subset(L,\varphi|_L)\subset(F,\varphi|_F)\subset(E,\varphi)$ where $L=\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2)$, $F=\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2)\oplus\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$, and suppose that the matrix of the morphism φ, adapted to the filtration $0\subset L\subset F\subset E$, is

$$\begin{pmatrix} 0 & 0 & X \\ 0 & X & X \\ X & X & X \end{pmatrix}$$

where X represents a non zero element. Let $\tau=\frac{1}{3}$ and observe that (E,φ) is τ-unstable, because we can find weights Γ_1, Γ_2 and Γ_3 such that the filtration $0\subset L\subset F\subset E$ contradicts Definition 1.2.3. We consider such filtration and look for the best weights in order to maximize the Kempf function. Because of how φ looks like for this filtration, i.e. not knowing if the multi-index is $(1,3)$ or $(2,2)$, we have to apply the previous analysis.

See that

$$v_1=\operatorname{rk}L\cdot\deg E-\deg L\cdot\operatorname{rk}E=-5$$

$$v_2=\operatorname{rk}F/L\cdot\deg E-\deg F/L\cdot\operatorname{rk}E=1$$

$$v_3=\operatorname{rk}E/F\cdot\deg E-\deg E/F\cdot\operatorname{rk}E=4\ ,$$

hence we can only be in the first case, where the multi-index is given by $(1,3)$. Substituting, we get that the best 1-parameter subgroup is given by the vector,

$$\Gamma=(\frac{-5+\tau}{4+\tau},\frac{1-2\tau}{4+\tau},1)=(\frac{-14}{13},\frac{1}{13},1)\ .$$

Note that we set $\Gamma_3=1$, and recall that the Kempf function is invariant by rescaling the Γ_i (c.f. subsection 2.1.2).

To know if the filtration $0\subset(L,\varphi|_L)\subset(F,\varphi|_F)\subset(E,\varphi)$ is the Kempf filtration, i.e. to know if it is the filtration giving maximum for the Kempf function, we would have to check all possible destabilizing filtrations and the values the Kempf function achieves for them, to choose the greatest value which has to correspond to the Kempf filtration, the candidate to be defined as the Harder-Narasimhan filtration.

In view of this, we can define a class of tensors for which, the ambiguity of two or more multi-indexes which can give the minimum in (1.2.4) or (1.2.37), cannot appear.

Definition 2.5.1. *Let δ be a polynomial with positive leading coefficient of degree at most $n-1$. Let (E,φ) be a δ-unstable tensor over an n-dimensional projective variety. Suppose* *that there does not exist a destabilizing weighted filtration* $(E_\bullet, n_\bullet)$ *(i.e. contradicting the expression of Definition 1.2.3) such that for it, the symbols* $\epsilon_i(\varphi)$ *do depend on the weights* Γ_i *(because, for example, the expression of the morphism* φ *adapted to the filtration is particularly easy). We call these tensors,* ***determined multi-index*** *tensors.*

It is clear that we can develop the same techniques of this chapter to show that the m-Kempf filtration stabilizes with the integer m, and to construct a Harder-Narasimhan filtration for determined multi-index tensors as in Definition 2.5.1.

Chapter 3

Correspondence for representations of quivers

3.1 Representations of quivers on vector spaces

Let Q be a finite quiver, given by a finite set of vertices and arrows between them, and a representation of Q on finite dimensional k-vector spaces, where k is an algebraically closed field of arbitrary characteristic. There exists a notion of stability for such representations given by King (c.f. [Ki]) and, more generally by Reineke (c.f. [Re]) (both particular cases of the abstract notion of stability for an abelian category that we can find in [Ru]), and a notion of the existence of a unique Harder-Narasimhan filtration with respect to that stability condition.

We consider the construction of a moduli space for these objects by King (c.f. [Ki]) and associate to an unstable representation an unstable point, in the sense of Geometric Invariant Theory, in a parameter space where a group acts. Then, the 1-parameter subgroup given by Kempf (c.f. Theorem 1.4.6), which is maximally destabilizing in the GIT sense, gives a filtration of subrepresentations and we prove that it coincides with the Harder-Narasimhan filtration for that representation.

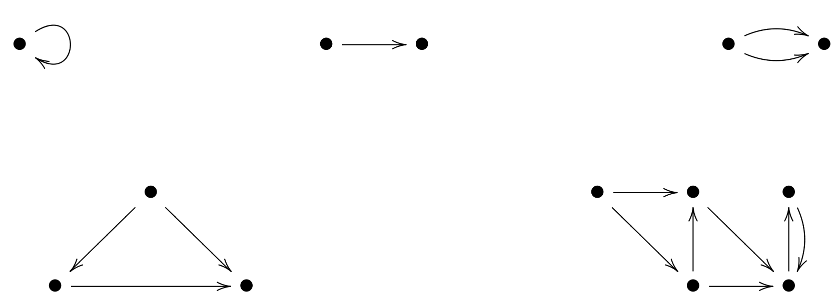
The proof follows the argument given in chapter 2 to establish the correspondence between the 1-parameter subgroup of Kempf and the Harder-Narasimhan filtration for the different cases studied there. However, for representations of quivers on the category of vector spaces, the proof is much simpler, as there is no need of proving that there exists an integer m, sufficiently large, such that the m-Kempf filtration does not depend on m (c.f. subsection 2.1.6), because in this construction of the moduli space there is no such integer m involved.

The definition of stability for a representation of a quiver (c.f. Definition 3.1.1) contains two sets of parameters, the coefficients of the linear functions Θ and σ. In [Ke], the 1-parameter subgroup is taken to maximize certain function which depends on the choice of a linearization of the action of the group we are taking the quotient by, and a *length* in the set of 1-parameter subgroups (c.f. Definition 1.4.2). In the case of sheaves the group is $SL(N)$, which is simple, so any such length is unique up to multiplication by a scalar, whereas for finite dimensional representations of quivers we quotient by a product of general linear groups, so we have to choose a scalar for each factor in the choice of a length. Hence, we set the positive coefficients of σ precisely as these scalars and consider a particular linearization depending on σ and Θ, in order to relate the Harder-Narasimhan filtration of a representation with the 1-parameter subgroup given by Kempf in [Ke] (c.f. Theorem 3.1.15).

3.1.1 Harder-Narasimhan filtration for representations of quivers

A **finite quiver** Q is given by a finite set of vertices Q_0 and a finite set of arrows Q_1. The arrows will be denoted by $(\alpha : v_i \to v_j) \in Q_1$. We denote by $\mathbb{Z}Q_0$ the free abelian group generated by Q_0.

The following figures show different examples of finite quivers:



Fix k, an algebraically closed field of arbitrary characteristic. Let $\operatorname{mod} kQ$ be the category of finite dimensional representations of Q over k. Such category is an abelian category and its objects are given by tuples

$$M = ((M_v)_{v \in Q_0}, (M_\alpha : M_{v_i} \to M_{v_j})_{\alpha : v_i \to v_j})$$

of finite dimensional k-vector spaces and k-linear maps between them. The **dimension vector** of a representation is given by $\underline{\dim}M = \sum_{v\in Q_0} \dim_k M_v \cdot v \in \mathbb{N}Q_0$.

For example, in the previous figures, a representation of the first quiver on the top left on finite dimensional vector spaces is an endomorphism of a vector space, and a representation of the one in the top center is a homomorphism between two vector spaces.

Let Θ be a set of numbers Θ_v for each $v \in Q_0$ and define a linear function $\Theta : \mathbb{Z}Q_0 \to \mathbb{Z}$, by

$$\Theta(M) := \Theta(\underline{\dim}M) = \sum_{v\in Q_0} \Theta_v \dim_k M_v \ .$$

Let σ be a set of strictly positive numbers σ_v for each $v \in Q_0$, and define a (strictly positive) linear function $\sigma : \mathbb{Z}Q_0 \to \mathbb{Z}$, by

$$\sigma(M) := \sigma(\underline{\dim}M) = \sum_{v\in Q_0} \sigma_v \dim_k M_v \ .$$

We call $\sigma(M)$ **the total dimension of** M. we will refer to Θ and σ indistinctly meaning the sets of numbers Θ_v and σ_v or the linear functions.

For a non-zero representation M of Q over k, define its **slope** by

$$\mu_{(\Theta,\sigma)}(M) := \frac{\Theta(M)}{\sigma(M)} \ .$$

Definition 3.1.1. *A representation M of Q over k is (Θ,σ)**-semistable** if for all non-zero subrepresentations M' of M, we have*

$$\mu_{(\Theta,\sigma)}(M') \le \mu_{(\Theta,\sigma)}(M) \ .$$

*If the inequality is strict for every non-zero subrepresentation, we say that M is (Θ,σ)**-stable**. If M is not (Θ,σ)-semistable we say that it is (Θ,σ)**-unstable**.*

Lemma 3.1.2. *If we multiply the linear function Θ by a non-negative integer, or if we add an integer multiple of the strictly positive linear function σ to Θ, the semistable (resp. stable) representations remain semistable (resp. stable).*

Proof. Let $\Theta' = a\cdot\Theta + b\cdot\sigma,\ a,b\in\mathbb{Z},\ a>0$, be another linear function and note that

$$\frac{\Theta'(M')}{\sigma(M')} \le \frac{\Theta'(M)}{\sigma(M)} \Leftrightarrow \frac{a\cdot\Theta(M')+b\cdot\sigma(M)}{\sigma(M')} \le \frac{a\cdot\Theta(M)+b\cdot\sigma(M)}{\sigma(M)}$$

$$\Leftrightarrow \frac{\Theta(M')}{\sigma(M')} \le \frac{\Theta(M)}{\sigma(M)} \ .$$

∎

Remark 3.1.3. *In [Ki], the stability condition (c.f [Ki, Definition 1.1]) is formulated by not considering representations with different dimension vectors. This leads to the construction of a moduli space and S-filtrations (or Jordan-Hölder filtrations) but not to define a Harder-Narasimhan filtration, for which is needed a slope condition as in Definition 3.1.1.*

This slope stability condition, the (Θ,σ)-stability (c.f. Definition 3.1.1), can be turned out into a stability condition as in [Ki], by clearing denominators

$$\theta(M') = \Theta(M)\sigma(M') - \sigma(M)\Theta(M') \ ,$$

where θ is the function in [Ki, Definition 1.1] (observe that $\theta(M)=0$), Θ and σ are as in Definition 3.1.1, and $M' \subset M$ is a subrepresentation.

We will apply this in Proposition 3.1.8, to relate (Θ,σ)-stability with GIT stability.

Remark 3.1.4. *The definition of stability which appears in [Re] considers $\sigma_v = 1$ for each $v \in Q_0$, although we consider a strictly positive linear function σ in general. The notation of σ agrees with [AC], [ACGP], [Sch], while Θ agrees with [Re] but in the other references it is substituted by different notations closer to classical moduli problems where the stability notion depends on parameters (τ-stability or ρ-stability).*

Lemma 3.1.5. *[Ru, Definition 1], [Re, Lemma 4.1] Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of non-zero representations of Q over k. Then $\mu_{(\Theta,\sigma)}(X) < \mu_{(\Theta,\sigma)}(Y)$ if and only if $\mu_{(\Theta,\sigma)}(X) < \mu_{(\Theta,\sigma)}(Z)$ if and only if $\mu_{(\Theta,\sigma)}(Y) < \mu_{(\Theta,\sigma)}(Z)$.*

Proof. Note that $\sigma(Y) = \sigma(X) + \sigma(Z)$ and, therefore

$$\mu_{(\Theta,\sigma)}(Y) = \frac{\Theta(Y)}{\sigma(Y)} = \frac{\Theta(X)+\Theta(Z)}{\sigma(X)+\sigma(Z)} \ ,$$

from which the statement follows. ∎

Theorem 3.1.6. *[Ru, Theorem 2], [Re, Lemma 4.7] Given linear functions Θ and σ, (being σ strictly positive), every representation M of Q over k has a unique filtration*

$$0 \subset M_1 \subset M_2 \subset \ldots \subset M_t \subset M_{t+1} = M$$

verifying the following properties, where $M^i := M_i/M_{i-1}$,

1. $\mu_{(\Theta,\sigma)}(M^1) > \mu_{(\Theta,\sigma)}(M^2) > \ldots > \mu_{(\Theta,\sigma)}(M^t) > \mu_{(\Theta,\sigma)}(M^{t+1})$

2. *The quotients M^i are (Θ,σ)-semistable*

This filtration is called the ***Harder-Narasimhan filtration*** *of M (with respect to Θ and σ).*

Proof. The proof follows the usual argument to show the existence and uniqueness of the Harder-Narasimhan filtration.

Using Lemma 3.1.5 we can prove the existence of a unique subrepresentation M_1, whose slope is maximal among all the subrepresentations of M, and of maximal total dimension $\sigma(M_1)$ among those of maximal slope (c.f. [Ru, Proposition 1.9], [Re, Lemma 4.4]). Then, proceed by recursion on the quotient M/M_1. ∎

3.1.2 Moduli space of representations of quivers

Fix k an algebraically closed field of arbitrary characteristic. Fix a dimension vector $d \in \mathbb{Z}Q_0$ and fix k-vector spaces M_v of dimension d_v for all $v \in Q_0$. Fix linear functions $\Theta, \sigma : \mathbb{Z}Q_0 \to \mathbb{Z}$, being σ strictly positive. We recall the construction by King (c.f. [Ki]) of a moduli space for representations of Q over k with dimension vector d.

Consider the affine k-space

$$\mathcal{R}_d(Q) = \bigoplus_{\alpha: v_i \to v_j} \mathrm{Hom}_k(M_{v_i}, M_{v_j}) \ ,$$

whose points parametrize representations of Q on the k-vector spaces M_v. The reductive linear algebraic group

$$G_d = \prod_{v \in Q_0} GL(M_v)$$

acts on $\mathcal{R}_d(Q)$ by

$$(g_{v_i})_{v_i} \cdot (M_\alpha)_\alpha = (g_{v_j} M_\alpha g_{v_i}^{-1})_{\alpha: v_i \to v_j} \ ,$$

and the G_d-orbits of M in $\mathcal{R}_d(Q)$ correspond bijectively to the isomorphism classes $[M]$ of k-representations of Q with dimension vector d. We will use Geometric Invariant Theory to take the quotient of $\mathcal{R}_d(Q)$ by G_d and construct a moduli space of representations of the quiver Q on the k-vector spaces M_v.

The action of G_d on the affine space $\mathcal{R}_d(Q)$ can be lifted by a character χ to the (necessarily trivial) line bundle L required by the Geometric Invariant Theory. Note that the subgroup of the diagonal scalar matrices in G_d,

$$\Delta = \{(t1, \ldots, t1) : t \in k^*\} \ ,$$

acts trivially on $\mathcal{R}_d(Q)$. Then, we have to choose χ in such a way that Δ acts trivially on the fiber, in other words, $\chi(\Delta) = 1$.

Then, using the linear functions Θ and σ, consider the character

$$\chi_{(\Theta,\sigma)}((g_v)_{v\in Q_0}) := \prod_{v\in Q_0} \det(g_v)^{(\Theta(d)\sigma_v - \sigma(d)\Theta_v)}$$

of G_d, and note that $\chi_{(\Theta,\sigma)}(\Delta) = 1$, because $\sum_{v\in Q_0}(\Theta(d)\sigma_v - \sigma(d)\Theta_v)\cdot d_v = 0$.

Given a linearization of an action by a character χ, we say that f is a **relative invariant of weight** χ^n if $f(g\cdot x) = \chi^n(g)\cdot f(x)\ \forall x$.

Definition 3.1.7. *[Ki, Definition 2.1] A point $x \in \mathcal{R}_d(Q)$ is $\chi_{(\Theta,\sigma)}$-**semistable** if there is a relative invariant of weight $\chi^n_{(\Theta,\sigma)}$, $f \in k[\mathcal{R}_d(Q)]^{G_d,\chi^n_{(\Theta,\sigma)}}$ with $n \geq 1$, such that $f(x) \neq 0$.*

The algebraic quotient will be given by

$$\mathcal{R}_d(Q)/\!\!/(G_d, \chi_{(\Theta,\sigma)}) = \operatorname{Proj}\Big(\bigoplus_{n\geq 0} k[\mathcal{R}_d(Q)]^{G_d,\chi^n_{(\Theta,\sigma)}}\Big)\ .$$

Proposition 3.1.8. *A point $x_M \in \mathcal{R}_d(Q)$ corresponding to a representation $M \in \operatorname{mod} kQ$ is $\chi_{(\Theta,\sigma)}$-semistable (resp. $\chi_{(\Theta,\sigma)}$-stable) for the action of G_d if and only if M is (Θ,σ)-semistable (resp. (Θ,σ)-stable).*

Proof. It follows easily from [Ki, Proposition 3.1] and the observation in Remark 3.1.3. In [Ki], given a linear function θ, a representation M is θ-semistable if $\theta(M) = 0$ and, for every subrepresentation $M' \subset M$, we have $\theta(M') \geq 0$ (c.f. [Ki, Definition 1.1]). Then, [Ki, Proposition 3.1] relates the θ-stability with the χ_θ-stability, where the character is

$$\chi_\theta((g_v)_v) := \prod_{v\in Q_0} \det(g_v)^{\theta_v}\ .$$

Hence, the $\chi_{(\Theta,\sigma)}$-stability with the character given by

$$\chi_{(\Theta,\sigma)}((g_v)_v) := \prod_{v\in Q_0} \det(g_v)^{(\Theta(d)\sigma_v - \sigma(d)\Theta_v)}\ ,$$

is equivalent to the (Θ,σ)-stability in Definition 3.1.1 because, given a subrepresentation $M' \subset M$, the expression

$$\sum_{v\in Q_0}(\Theta(M)\sigma_v - \sigma(M)\Theta_v)\cdot \dim M'_v = \Theta(M)\sigma(M') - \sigma(M)\Theta(M') \geq 0$$

is equivalent to

$$\frac{\Theta(M')}{\sigma(M')} \le \frac{\Theta(M)}{\sigma(M)} .$$

■

Now denote by $\mathcal{R}_d^{(\Theta,\sigma)-ss}(Q)$ the set of $\chi_{(\Theta,\sigma)}$-semistable points.

Theorem 3.1.9. *[Ki, Proposition 4.3], [Re, Corollary 3.7] The moduli space* $\mathfrak{M}_d^{(\Theta,\sigma)}(Q) = \mathcal{R}_d^{(\Theta,\sigma)-ss}(Q)/\!\!/G_d$ *is a projective variety which parametrizes S-equivalence classes of (Θ,σ)-semistable representations of Q of dimension vector d.*

By the Hilbert-Mumford criterion we can characterize $\chi_{(\Theta,\sigma)}$-semistable points by its behavior under the action of 1-parameter subgroups. A 1-parameter subgroup of $G_d = \prod_{v\in Q_0} GL(M_v)$ is a non-trivial homomorphism $\Gamma : k^* \to G_d$. There exist bases of the vector spaces M_v such that Γ takes the diagonal form

$$\begin{pmatrix} t^{\Gamma_{v_1,1}} & & \\ & \ddots & \\ & & t^{\Gamma_{v_1,t_1+1}} \end{pmatrix} \times \cdots \times \begin{pmatrix} t^{\Gamma_{v_s,1}} & & \\ & \ddots & \\ & & t^{\Gamma_{v_s,t_s+1}} \end{pmatrix}$$

where $v_1,\dots,v_s \in Q_0$ are the vertices of the quiver.

Let $x \in \mathcal{R}_d(Q)$ and suppose that $\lim_{t\to 0}\Gamma\cdot x$ exists and is equal to x_0. Then x_0 is a fixed point for the action of Γ, and Γ acts on the fiber of the trivial line bundle over x_0 as multiplication by t^a. Define the following numerical function,

$$\mu_{\chi_{(\Theta,\sigma)}}(x,\Gamma) = -a .$$

The next proposition establishes a variant of the Hilbert-Mumford criterion given in Theorem 1.1.14.

Proposition 3.1.10. *[Ki, Proposition 2.5] A point $x_M \in \mathcal{R}_d(Q)$ corresponding to a representation M is $\chi_{(\Theta,\sigma)}$-semistable if and only if every 1-parameter subgroup Γ of G_d, for which* $\lim_{t\to 0}\Gamma(t)\cdot x_M$ *exists, satisfies* $\mu_{\chi_{(\Theta,\sigma)}}(x_M,\Gamma) \le 0$.

Remark 3.1.11. *Note that in Proposition 3.1.10 we change the sign of the numerical function $\mu_{\chi_{(\Theta,\sigma)}}(x_M,\Gamma)$ with respect to [Ki] (as we did when changing the character in the proof of Proposition 3.1.8), in congruence with [Ke] and the numerical function in Theorem 1.1.14.*

The action of a 1-parameter subgroup Γ of G_d provides a decomposition of each vector space M_v, associated to each vertex $v \in Q_0$, in weight spaces

$$M_v = \bigoplus_{n\in\mathbb{Z}} M_v^n \ ,$$

where $\Gamma(t)$ acts on the weight space M_v^n as multiplication by t^n. Every 1-parameter subgroup, for which $\lim_{t\to 0}\Gamma(t)\cdot x$ exists, determines a weighted filtration $M_\bullet \subset M$ of subrepresentations (c.f. [Ki]),

$$0 \subset M_1 \subset M_2 \subset \ldots \subset M_t \subset M_{t+1} = M \ ,$$

where M_i is the subrepresentation with vector spaces $M_{v,i} := \bigoplus_{n\leq i} M_v^n$ for each vertex $v \in Q_0$, and the weight corresponding to each quotient $M^i = M_i/M_{i-1}$ is Γ_i. Note that two 1-parameter subgroups giving the same filtration are conjugated by an element of the parabolic subgroup of G_d defined by the filtration. Therefore, the numerical function $\mu_{\chi_{(\Theta,\sigma)}}(x_M,\Gamma)$, has a simple expression in terms of the filtration $M_\bullet \subset M$ (c.f. calculation in [Ki]):

$$\mu_{\chi_{(\Theta,\sigma)}}(x_M,\Gamma) = \sum_{v\in Q_0} \Big[\big(\Theta(M)\sigma_v - \sigma(M)\Theta_v\big)\cdot \sum_{i=1}^{t_v+1}\Gamma_{v,i}\dim M_v^i\Big] \ . \tag{3.1.1}$$

Let d_i, d^i be the dimension vectors of the subrepresentation M_i and the quotient $M^i = M_i/M_{i-1}$, respectively. The action of Γ on the point corresponding to a representation M has different weights for each vertex $v \in Q_0$, but collect all different weights Γ_i corresponding to any vertex and form the vector

$$\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_t, \Gamma_{t+1})$$

verifying $\Gamma_1 < \Gamma_2 < \ldots < \Gamma_t < \Gamma_{t+1}$. Hence, (3.1.1) turns out to be

$$\mu_{\chi_{(\Theta,\sigma)}}(x_M,\Gamma) = \sum_{i=1}^{t+1}\Gamma_i\cdot[\Theta(M)\cdot\sigma(M^i) - \sigma(M)\cdot\Theta(M^i)] \ , \tag{3.1.2}$$

and Proposition 3.1.10 can be rewritten in terms of filtrations of M.

Proposition 3.1.12. *A point $x_M \in \mathcal{R}_d(Q)$ corresponding to a representation M of Q over k, is $\chi_{(\Theta,\sigma)}$-semistable if and only if every 1-parameter subgroup Γ of G_d, defining a filtration of subrepresentations of M*

$$0 \subset M_1 \subset M_2 \subset \ldots \subset M_t \subset M_{t+1} = M \ ,$$

satisfies that

$$\mu_{\chi_{(\Theta,\sigma)}}(x_M,\Gamma)=\sum_{i=1}^{t+1}\Gamma_i\cdot[\Theta(M)\cdot\sigma(M^i)-\sigma(M)\cdot\Theta(M^i)]\leq 0\ .$$

3.1.3 Kempf theorem

Given a weighted filtration of M,

$$0\subset M_1\subset M_2\subset\ldots\subset M_t\subset M_{t+1}=M\ ,$$

and $\Gamma_1<\Gamma_2<\ldots<\Gamma_t<\Gamma_{t+1}$, define the following function which is a **Kempf function** (c.f. Definition 1.4.4) for this problem,

$$K(M_\bullet,\Gamma)=\frac{\sum_{i=1}^{t+1}\Gamma_i\cdot[\Theta(M)\cdot\sigma(M^i)-\sigma(M)\cdot\Theta(M^i)]}{\sqrt{\sum_{i=1}^{t+1}\sigma(M^i)\cdot\Gamma_i^2}} \tag{3.1.3}$$

It is a function whose numerator is equal to the numerical function $\mu_{\chi_{(\Theta,\sigma)}}(x_M,\Gamma)$ and the denominator is a *length* of the 1-parameter subgroup Γ. Given a reductive linear algebraic group G, recall the notion of **length** in $\Gamma(G)$, the set of all 1-parameter subgroups (c.f. Definition 1.4.2).

If G is simple, in characteristic zero all choices of length will be multiples of the Killing form in the Lie algebra $\mathfrak{g}$ (note that in this case $\Gamma(G)\subset\mathfrak{g}$). For an almost simple group in arbitrary characteristic (a group G whose center Z is finite and G/Z is simple, e.g. $SL(N)$ in positive characteristic), all lengths differ also by a scalar.

However, in this case, the group is a product of general linear groups, which is not simple. Then, there are several simple factors in the group and we can take a different multiple of the Killing form for each factor. Hence, observe that in the Kempf function (3.1.3), the denominator of the expression is a function verifying the properties of the definition of a length (c.f. Definition 1.4.2). The different multiples we take for each factor appear to be related to the choice of the strictly positive linear function σ.

Therefore, we can rewrite Theorem 2.1.5 in our case as follows:

Theorem 3.1.13. *Given a $\chi_{(\Theta,\sigma)}$-unstable point $x_M\in\mathcal{R}_d(Q)$ corresponding to a representation M, there exists a unique weighted filtration, i.e. $0\subset M_1\subset\cdots\subset M_{t+1}=M$ and real numbers $\Gamma_1<\Gamma_2<\ldots<\Gamma_t<\Gamma_{t+1}$, called the **Kempf filtration of M**, such that the Kempf function $K(M_\bullet,\Gamma)$ achieves the maximum among all filtrations and weights verifying $\Gamma_1<\Gamma_2<\ldots<\Gamma_t<\Gamma_{t+1}$.*

Note that the length we are considering depends on the choice of σ and the Kempf function depends both on the length and the linearization of the group action, hence depends both on Θ and σ. In order to relate the Kempf filtration of M with the Harder-Narasimhan filtration, which also depends on Θ and σ, we have set the parameters conveniently in the expression of the stability condition (c.f. Proposition 3.1.8).

3.1.4 Kempf filtration is Harder-Narasimhan filtration

Finally, we close the section by relating the Kempf filtration in Theorem 3.1.13 and the Harder-Narasimhan filtration in Theorem 3.1.6. We study the geometrical properties of the Kempf filtration by associating to it a graph which encodes the two properties satisfied by the Harder-Narasimhan filtration. We will use the results of subsection 2.1.2.

Let $\Theta : \mathbb{Z}Q_0 \to \mathbb{Z}$ be a linear function and let $\sigma : \mathbb{Z}Q_0 \to \mathbb{Z}$ be a strictly positive linear function. Let M be a representation of Q over an algebraically closed field k of arbitrary characteristic, which is (Θ, σ)-unstable. Consider the $\chi_{(\Theta,\sigma)}$-unstable point $x_M \in \mathcal{R}_d(Q)$ associated to M, by Proposition 3.1.8. Let $0 \subset M_1 \subset \cdots \subset M_{t+1} = M$ and $\Gamma_1 < \Gamma_2 < \ldots < \Gamma_t < \Gamma_{t+1}$ be the Kempf filtration of M, by Theorem 3.1.13.

Let $M^i = M_i/M_{i-1}$ be the quotients of the filtration. Consider the inner product in $\mathbb{R}^{t+1}$ given by the matrix

$$\begin{pmatrix} \sigma(M^1) & & 0 \\ & \ddots & \\ 0 & & \sigma(M^{t+1}) \end{pmatrix}$$

where $\sigma(M^i) > 0$.

Definition 3.1.14. *Given a filtration $0 \subset M_1 \subset \cdots \subset M_{t+1} = M$ of subrepresentations of M, define $v = (v_1, ..., v_{t+1})$, where*

$$v_i = \Theta(M) - \frac{\sigma(M)}{\sigma(M^i)}\Theta(M^i) \ ,$$

*the **graph** or the **vector associated to the filtration**.*

Now we can identify the Kempf function (c.f. (3.1.3)) with the function in Theorem 2.1.9,

$$K(M_\bullet, \Gamma) = \frac{\sum_{i=1}^{t+1} \Gamma_i \cdot [\Theta(M)\sigma(M^i) - \sigma(M)\Theta(M^i)]}{\sqrt{\sum_{i=1}^{t+1} \sigma(M^i) \cdot \Gamma_i^2}} =$$

$$= \frac{\sum_{i=1}^{t+1} \sigma(M^i)\Gamma_i \cdot [\Theta(M) - \frac{\sigma(M)}{\sigma(M^i)}\Theta(M^i)]}{\sqrt{\sum_{i=1}^{t+1} \sigma(M^i) \cdot \Gamma_i^2}} = \frac{(\Gamma, v)}{\|\Gamma\|} = \mu_v(\Gamma) \ .$$

Note that $\sum_{i=1}^{t+1} b^i v_i = 0$.

Theorem 3.1.15. *The Kempf filtration of M is the Harder-Narasimhan filtration of M.*

Proof. The vector v associated to the Kempf filtration of M in Definition 3.1.14 verifies properties in Lemmas 2.1.15 and 2.1.16, which are precisely properties 1 and 2 in Theorem 3.1.6, respectively. Lemma 2.1.15 implies that $v_i < v_{i+1}$, for each i, hence

$$\Theta(M) - \frac{\sigma(M)}{\sigma(M^i)}\Theta(M^i) < \Theta(M) - \frac{\sigma(M)}{\sigma(M^{i+1})}\Theta(M^{i+1}) \Leftrightarrow \frac{\Theta(M^i)}{\sigma(M^i)} > \frac{\Theta(M^{i+1})}{\sigma(M^{i+1})} \ ,$$

and Lemma 2.1.16 implies the (Θ, σ)-semistability of each quotient $M^i = M_i/M_{i-1}$. By uniqueness of the Harder-Narasimhan filtration of M, both filtrations do coincide. ■

3.2 Representations of quivers on coherent sheaves

In this final section we will show the correspondence between the 1-parameter subgroup of Kempf in Theorem 1.4.6 and the filtration of Harder-Narasimhan in Theorem 1.3.5 through the language of representations of quivers. A coherent sheaf will be a representation of a one vertex quiver on the category of coherent sheaves. It will have a H-Kronecker associated, and we will associate to it a representation of another quiver on vector spaces, to use the results of section 3.1.

We first present the Q-sheaves which are representations of a quiver on the category of coherent sheaves, and its relation with the Kronecker modules.

3.2.1 Quiver sheaves and Kronecker modules

Let Q be a quiver and let X be a projective variety. A Q**-sheaf** over X is a representation E of Q in the category of coherent sheaves over X, given by the data of a coherent sheaf E_v for all $v \in Q_0$ and a morphism of sheaves $\phi_\alpha : E_{v_i} \to E_{v_j}$ for all $\alpha \in Q_1$. Let E be a Q-sheaf over X. Let P be a set of polynomials $P_v \in \mathbb{Q}[m]$, indexed by the vertices $v \in Q_0$. A Q-sheaf E has Hilbert polynomial vector P if P_v is the Hilbert polynomial of each sheaf E_v.

Let κ be a set of polynomials $\kappa_v \in \mathbb{Q}[m]$, for $v \in Q_0$, such that $\kappa_v(m) > 0$ for $m \gg 0$ and $\deg \kappa_v = t$ for all $v \in Q_0$, for a fixed integer $t \geq 0$ independent of v. Let us call σ, τ the sets of rational numbers $\sigma_v > 0$, τ_v, indexed by the vertices $v \in Q_0$, such that

$$\kappa_v(m) = \sigma_v m^t + \tau_v m^{t-1} + \dots \ .$$

The **κ-Hilbert polynomial** of a Q-sheaf E is the polynomial $P_\kappa(E) \in \mathbb{Q}[m]$ given by

$$P_\kappa(E, m) := \sum_{v \in Q_0} \kappa_v(m) P_{E_v}(m) \ ,$$

where each $P_{E_v}(m)$ is the Hilbert polynomial of the coherent sheaf E_v.

A Q-sheaf E is called **pure of dimension** e, if the sheaf E_v is pure of dimension e (independent of v), for all $v \in Q_0$. A **Q-subsheaf** of a Q-sheaf E is given by a subsheaf $E'_v \subset E_v$ for each vertex such that the restrictions of the morphisms are compatible.

Definition 3.2.1. *A Q-sheaf E over X is **Gieseker κ-semistable** if it is pure (of any dimension e) and*

$$\frac{P_\kappa(E', m)}{\sum_{v \in_0} \sigma_v \operatorname{rk} E'_v} \leq \frac{P_\kappa(E, m)}{\sum_{v \in_0} \sigma_v \operatorname{rk} E_v} \ \ \text{for } m \gg 0 \ ,$$

*for each non-zero Q-subsheaf $E' \subset E$, and **Gieseker κ-stable** if, furthermore, the inequality is strict for all proper Q-subsheaves $E' \subset E$. If E is not Gieseker κ-semistable we say that it is **Gieseker κ-unstable**.*

By κ-semistable, κ-stable and κ-unstable we mean, in the following, Gieseker κ-semistable, Gieseker κ-stable and Gieseker κ-unstable. We can rewrite Definition 3.2.1 as it appears in [AC].

Lemma 3.2.2. *[AC, Lemma 7] A Q-sheaf E over X is κ-semistable if and only if for all non-zero $E' \subset E$,*

$$\frac{P_\kappa(E', m)}{P_\kappa(E', l)} \leq \frac{P_\kappa(E, m)}{P_\kappa(E, l)} \quad \text{for } l \gg m \gg 0 \ ,$$

for each non-zero Q-subsheaf $E' \subset E$, and Gieseker κ-stable if, furthermore, the inequality is strict for all proper $E' \subset E$.

Álvarez-Cónsul and King in [ACK] give a functorial construction of the moduli space of coherent sheaves over a projective variety by associating to a sheaf a **Kronecker module** which is a representation of a particular quiver on vector spaces.

Let $l > m$ be integers and consider the sheaf $T = \mathcal{O}(-l) \oplus \mathcal{O}(-m)$ together with a finite dimensional k-algebra

$$\begin{pmatrix} k & H \\ 0 & k \end{pmatrix}$$

of operators on T, where $A = L \oplus H \subset \mathrm{End}_X(T)$, $L = k \cdot e_0 \oplus k \cdot e_1$ is the semisimple algebra generated by the two projection operators onto the summands of T, and $H = H^0(\mathcal{O}(l-m)) = \mathrm{Hom}(\mathcal{O}_X(m), \mathcal{O}_X(l))$, acting on T in the off-diagonal way.

We can give a right A-module structure on M by giving a right L-module structure and a right L-module map $M \otimes_L H \to V$. The first one is equivalent to a direct sum decomposition $M = V \oplus W$, being $V = M \cdot e_0$ and $W = M \cdot e_1$, and the second one given by the map

$$\alpha : V \otimes H \to W \ .$$

This structure given on V is called a H**-Kronecker module**. We can also say that A is the path algebra of the quiver with two vertices and, after choosing a basis for H, a number of $\dim H$ arrows between them. A representation of this quiver is also a H-**Kronecker module**, equivalent to the previous definition by the standard equivalence between representations of quivers and modules for their path algebras.

Given a sheaf E, $\mathrm{Hom}_X(T, E)$ can be given a structure of H-Kronecker module. Indeed, it has a natural right module structure over $A \subset \mathrm{Hom}_X(T, T)$, given by composition of maps, and we have the decomposition $\mathrm{Hom}_X(T, E) = H^0(E(m)) \oplus H^0(E(l))$ together with the multiplication map $\alpha_E : H^0(E(m)) \otimes H \to H^0(E(l))$.

Given an A-module $M = V \oplus W$, an A-**submodule** M' is given by $V' \subset V$ and $W' \subset W$ such that $\alpha(V' \otimes H) \subset W'$.

Definition 3.2.3. *[ACK, Definition 2.3] An A module $M = V \oplus W$ is **semistable** if*

$$\frac{\dim V'}{\dim W'} \leq \frac{\dim V}{\dim W}$$

*for every submodule $M' = V' \oplus W' \subset M$. If the previous inequality is strict for every submodule, we say that M is **stable**. If M is not semistable, we say that it is **unstable**.*

We associate to a sheaf E a Kronecker module in this way. An observation in [ACK] points out that the GIT semistability of the orbit of E is equivalent to the natural semistability of the Kronecker module associated (c.f. [ACK, Remark 2.4]). In the following theorem we also relate the stability of the A-module M with the stability of a representation of the quiver $\tilde{Q}$

$$\bullet \longrightarrow \bullet$$

on vector spaces, to use the results of section 3.1.

Theorem 3.2.4. *Let E be a coherent sheaf over X, pure of dimension e, with Hilbert polynomial P. There exists $l \gg m \gg 0$, such that the following are equivalent:*

1. *E is semistable as in Definition 2.1.1.*
2. *E is m-regular and the A-module $M = H^0(E(m)) \oplus H^0(E(l))$ is semistable as in Definition 3.2.3.*
3. *The representation M of the two vertex quiver $\tilde{Q} = \{v_0, v_1\}$ and one arrow between them on k-vector spaces, where $M_{v_0} = H^0(E(m))$, $M_{v_1} = H^0(E(l))$, is (Θ, σ)-semistable as in Definition 3.1.1, where the linear functions Θ and σ are defined as $\Theta(M) = \dim M_{v_0}$, $\sigma(M) = \dim M_{v_0} + \dim M_{v_1}$.*
4. *The point $x_M \in \mathcal{R}_d^{(\Theta,\sigma)}(\tilde{Q})$ is $\chi_{(\Theta,\sigma)}$-semistable, where d is the dimension vector of the representation M, $d_{v_i} = \dim M_{v_i}$.*

Proof. The equivalence between 1 and 2 follows from [ACK, Theorem 5.10]. For the equivalence between 2 and 3 note that, defining the linear functions Θ and σ as in the statement 3, it is

$$\frac{\dim M'_{v_0}}{\dim M'_{v_1}} \leq \frac{\dim M_{v_0}}{\dim M_{v_1}} \Leftrightarrow$$

$$\frac{\dim M'_{v_0}}{\dim M'_{v_0} + \dim M'_{v_1}} \leq \frac{\dim M_{v_0}}{\dim M_{v_0} + \dim M_{v_1}} \Leftrightarrow \frac{\Theta(M')}{\sigma(M')} \leq \frac{\Theta(M)}{\sigma(M)} .$$

The equivalence between 3 and 4 follows from Proposition 3.1.8. ∎

3.2.2 Kempf filtration for Q-sheaves

Here we use Theorem 3.2.4 to show the correspondence between the Kempf Theorem and the Harder-Narasimhan filtration for coherent sheaves (c.f. Theorem 3.2.11), passing through stability for Kronecker modules and stability for representations of quivers on vector spaces. In this way, all ideas involved in this thesis, all different notions of stability, all correspondences of maximal unstability, appear together. Theorem 3.2.11 gives another proof of Theorem 2.1.7.

Let $Q=\{v\}$ be a one vertex quiver without arrows. Then, a Q-sheaf E is a coherent sheaf E and note that the stability condition in Definition 3.2.1 is independent of the choice of a polynomial κ. Let E be an unstable Q-sheaf with Hilbert polynomial P, i.e. an unstable coherent sheaf. Choose $l\gg m\gg 0$ such that Theorem 3.2.4 holds, and such that E is m-regular. By Theorem 3.2.4, the representation $M=H^0(E(m))\oplus H^0(E(l))$ of $\tilde{Q}=\{v_0,v_1\}$ in k-vector spaces is (Θ,σ)-unstable, and the corresponding point $x_M\in \mathcal{R}_d^{(\Theta,\sigma)}(\tilde{Q})$ is $\chi_{(\Theta,\sigma)}$-unstable. By Theorem 3.1.13, let $0\subset M_1\subset\cdots\subset M_{t+1}=M$ and $\Gamma_1<\Gamma_2<\ldots<\Gamma_t<\Gamma_{t+1}$ be the Kempf filtration of M (depending on m and l) which, by Theorem 3.1.15, is the Harder-Narasimhan filtration of M, defined in Theorem 3.1.6. Recall that we denote $M^i=M_i/M_{i-1}$ for each i.

The Kempf function in this case is

$$K(M_\bullet,\Gamma)=\frac{\sum_{i=1}^{t+1}\Gamma_i[\Theta(M)\sigma(M^i)-\sigma(M)\Theta(M^i)]}{\sqrt{\sum_{i=1}^{t+1}\Gamma_i^2\sigma(M^i)}}=\frac{(\Gamma,v)}{\|\Gamma\|}\ ,$$

(c.f. Theorem 3.1.3) where the coordinates of the vector $v=(v_1,...,v_{t+1})$ are given by $v_i=\Theta(M)-\frac{\sigma(M)}{\sigma(M^i)}\Theta(M^i)$, and the scalar product in $\mathbb{R}^{t+1}$ is given by the diagonal matrix with elements $\sigma(M^i)$ (c.f. Definition 3.1.14).

Definition 3.2.5. *[ACK, Definition 5.3] Let $M'=V_0'\oplus V_1'$ and $M''=V_0''\oplus V_1''$ be submodules of an A-module M. We say that M' is **subordinate** to M'' if*

$$V_0'\subset V_0''\ \text{and}\ V_1''\subset V_1'\ .$$

*We say that M' is **tight** if it is subordinate to no submodule other than itself.*

Proposition 3.2.6. *The submodules M_i appearing on the Kempf filtration of M are tight submodules of M.*

Proof. Given m and l, the Kempf filtration of M is, by Theorem 3.1.15, the Harder-Narasimhan filtration of M. Note that, whenever a module M' is subordinate to M'', the slopes verify $\mu_{(\Theta,\sigma)}(M'')\geq\mu_{(\Theta,\sigma)}(M')$. In the construction of the Harder-Narasimhan filtration in Theorem 3.1.6 we look for the unique representation M_1 of maximal slope and of maximal total dimension $\sigma(M_1)$ among those of maximal slope and, then, proceed by recursion. Hence, by that construction, all the submodules have to be tight. ∎

Fix m, l and consider the Kempf filtration of M. Using Proposition 3.2.6 and [ACK, Lemma 5.5], all submodules appearing in the Kempf filtration of M are of the form $M_i = \mathrm{Hom}_X(T, E_i) = H^0(E_i(m)) \oplus H^0(E_i(l))$ for some subsheaves $E_i \subset E$, where $E_i(m)$ is globally generated for each i. Then, define the filtration

$$0 \subset E_1 \subset \cdots \subset E_{t+1} = E \ , \tag{3.2.1}$$

which depends on m and l. We call it the m**-Kempf filtration of the** Q**-sheaf** E.

Now we give the analogous to Proposition 2.1.18 for this case. Fix the positive constant

$$C = \max\{r|\mu_{\max}(E)| + \frac{d}{r} + r|B| + |A| + 1 \ , \ 1\}. \tag{3.2.2}$$

Proposition 3.2.7. *Given integers m, l, let $E_\bullet \subset E$ be the m-Kempf filtration of the Q-sheaf E as in (3.2.1). There exists integers m_2, l_2 such that for $m \geq m_2, l \geq l_2$, each subsheaf $E_i \subset E$ in the m-Kempf filtration has slope $\mu(E_i) \geq \dfrac{d}{r} - C$.*

Proof. The proof follows similarly to Proposition 2.1.18. Choose an integer $m_1 \geq m_0$ such that for every $m \geq m_1$, if we have a filter $E_i^m \subseteq E$ verifying $\mu(E_i^m) < \frac{d}{r} - C$ (hence it satisfies the estimate in Lemma 1.2.15), it is

$$h^0(E_i^m(m)) \leq \frac{1}{g^{n-1}n!}\big((r_i-1)(\mu_{max}(E) + gm + B)^n + (\frac{d}{r} - C + gm + B)^n\big) = G(m) \ ,$$

where

$$G(m) = \frac{1}{g^{n-1}n!}\big[r_i g^n m^n + ng^{n-1}\big((r_i-1)\mu_{max}(E) + \frac{d}{r} - C + r_i B\big)m^{n-1} + \cdots\big] \ .$$

Recall that, by Definition 3.1.14, to any such filtration we associate a graph where $w^i = -b^i \cdot v_i = -\sigma(M^i) \cdot (\Theta(M) - \frac{\sigma(M)}{\sigma(M^i)}\Theta(M^i))$. Then, the heights of the graph, for each i, are

$$w_i = w^1 + \ldots + w^i = \Theta(M_i)\sigma(M) - \Theta(M)\sigma(M_i) =$$

$$\dim M_{v_0,i}(\dim M_{v_0} + \dim M_{v_1}) - \dim M_{v_0}(\dim M_{v_1,i} + \dim M_{v_1,i}) =$$

$$\dim M_{v_0,i} \dim M_{v_1} - \dim M_{v_0} \dim M_{v_1,i} \ .$$

Again, to reach a contradiction, it is enough to show that $w_i < 0$ because, in that case, we get $w_{t+1} < 0$. But it is

$$w_{t+1} = \dim M_{v_0,t+1} \dim M_{v_1} - \dim M_{v_0} \dim M_{v_1,t+1} = 0 \ ,$$

because $M_{v_0,t+1} = M_{v_0}$ and $M_{v_1,t+1} = M_{v_1}$, then the contradiction.

Using Proposition 3.2.5, [ACK, Lemma 5.5], and the m-regularity of E, we get

$$w_i = h^0(E_i(m))P_E(l) - P_E(m)h^0(E_i(l)) \ .$$

Given m and l, and using [ACK, Lemma 5.4 b)], the negativity of the numerical expression given by w_i for each l is equivalent to the negativity of the polynomial expression

$$h^0(E_i(m))P_E - P_E(m)P_{E_i} \ .$$

Let us show that $w_i(m,l) < 0$, for sufficiently larges m and l. By the previous calculations

$$w_i(m,l) = h^0(E_i(m))P_E(l) - P_E(m)P_{E_i}(l) \leq$$

$$G(m)P_E(l) - P_E(m)P_{E_i}(l)) =: \Psi(m,l) \ .$$

where $\Psi(m,l)$ can be seen as an n^{th}-order polynomial on l, whose coefficients are polynomials in m,

$$\Psi(m,l) = \psi_n(m)l^n + \psi_{n-1}(m)l^{n-1} + \cdots + \psi_1(m)l + \psi_0(m) \ .$$

Hence, it is sufficient to show that $\psi_n(m) = rG(m) - r_iP_E(m) < 0$ for sufficiently large m.

Note that $\psi_n(m) = \xi_n m^n + \xi_{n-1}m^{n-1} + \cdots + \xi_1 m + \xi_0$ is an n^{th}-order polynomial. The coefficient in order n^{th} vanishes,

$$\xi_n = (rG(m) - r_iP(m))_n = r\frac{r_i g}{n!} - r_i\frac{rg}{n!} = 0 \ .$$

Let us calculate the $(n-1)^{th}$-coefficient:

$$\xi_{n-1} = (rG(m) - r_iP(m))_{n-1} = (rG_{n-1} - r_i\frac{A}{(n-1)!})$$

where G_{n-1} is the $(n-1)^{th}$-coefficient of the polynomial $G(m)$,

$$G_{n-1} = \frac{1}{g^{n-1}n!}ng^{n-1}((r_i-1)\mu_{max}(E) + \frac{d}{r} - C + r_iB) =$$

$$\frac{1}{(n-1)!}((r_i-1)\mu_{max}(E) + \frac{d}{r} - C + r_iB) \leq$$

$$\frac{1}{(n-1)!}((r_i-1)|\mu_{max}(E)| + \frac{d}{r} - C + r_i|B|) \leq$$

$$\frac{1}{(n-1)!}(r|\mu_{max}(E)|+\frac{d}{r}-C+r|B|) < \frac{-|A|}{(n-1)!}\ ,$$

last inequality coming from the definition of C in (3.2.2). Then

$$\xi_{n-1} < r\big(\frac{-|A|}{(n-1)!}\big) - r_i\frac{A}{(n-1)!} = \frac{-r|A|-r_iA}{(n-1)!} < 0\ ,$$

because $-r|A|-r_iA<0$.

Therefore $\psi_n(m)=\xi_{n-1}m^{n-1}+\cdots+\xi_1 m+\xi_0$ with $\xi_{n-1}<0$, so there exists $m_2\geq m_1$ such that for every $m\geq m_2$ we have $\psi_n(m)<0$ and $w_i(m,l)<0$, for $l\gg 0$, hence the contradiction. ■

Proposition 3.2.8. *There exists an integer m_3 such that for $m\geq m_3$ the sheaves E_i^m and $E^{m,i}=E_i^m/E_{i-1}^m$ are m_3-regular. In particular their higher cohomology groups, after twisting with $\mathcal{O}_X(m_3)$, vanish and they are generated by global sections.*

Proof. The argument follows analogously to Proposition 2.1.19. ■

By Proposition 3.2.8, for any $m\geq m_3$, all the filters E_i of the m-Kempf filtration of E are m_3-regular and hence, the m-Kempf filtration of sheaves

$$0\subset E_1\subset E_2\subset\cdots\subset E_{t_m}\subset E_{t+1}=E\ ,$$

is obtained from the filtration of vector subspaces

$$0\subset H^0(E_1(m_3))\subset H^0(E_2(m_3))\subset\cdots\subset H^0(E_t(m_3))\subset H^0(E_{t+1}(m_3))=H^0(E(m_3))$$

by the evaluation map, of a unique vector space $H^0(E(m_3))$, whose dimension is independent of m.

Let $m\geq m_3$ and let

$$(P_{E_1},\ldots,P_{E_{t+1}})$$

be the m**-type** of the m-Kempf filtration of E (c.f. Definition 2.1.22) and let

$$\mathcal{P}=\big\{(P_{E_1},\ldots,P_{E_{t+1}})\big\}$$

be the finite set of possible vectors for $m\geq m_3$ (c.f. Proposition 2.1.23).

By Definition 3.1.14 we associate a graph to the m-Kempf filtration, which can be rewritten, by Proposition 3.2.8, as

$$v_{m,i}(l)=\Theta(M)-\frac{\sigma(M)}{\sigma(M^i)}\cdot\Theta(M^i)=\dim M_{v_0}-\frac{\dim M_{v_0}+\dim M_{v_1}}{\dim M^i_{v_0}+\dim M^i_{v_1}}\cdot\dim M^i_{v_0}=$$

$$P_E(m) - \frac{P_E(m)+P_E(l)}{P_{E^i}(m)+P_{E^i}(l)} \cdot P_{E^i}(m) \ ,$$

and

$$b^i_m(l) = \dim M^i_{v_0} + \dim M^i_{v_1} = P_{E^i}(m) + P_{E^i}(l) \ .$$

We use notations $v_{m,i}(l)$ and $b^i_m(l)$ because, given the m-Kempf filtration, its m-type is fixed, for $m \geq m_3$, hence the coordinates of the graph can be seen as rational functions in l, whose coefficients are fixed functions in m.

Now we follow the argument in subsection 2.1.5 with the particularities of section 2.3. Define the functional in $\mathcal{P}$,

$$\Phi_m(l) = (\mu_{v_m}(\Gamma_{v_m(l)}))^2 = \|v_m(l)\|^2 \ ,$$

which is a rational function on l (c.f. (2.1.14)). By finiteness of $\mathcal{P}$ there is a finite list of such possible functions

$$\mathcal{A} = \{\Phi_m : m \geq m_3\}$$

and we can choose K among them, such that there exist integers m_4 and l_4 with $\Phi_m(l) = K(l)$, for all $m \geq m_4$ and $l \geq l_4$ (c.f. Lemma 2.1.24).

Proposition 3.2.9. *Let a_1, a_2 be integers with $a_1 \geq a_2 \geq m_4$. The a_1-Kempf filtration of E is equal to the a_2-Kempf filtration of E.*

Proof. C.f. proof of Proposition 2.1.25. ■

Definition 3.2.10. *If $m \geq m_4$, the m-Kempf filtration of E is called **the Kempf filtration of** E,*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E \ .$$

Note that it does not depend on m by Proposition 3.2.9.

Theorem 3.2.11. *Given a one vertex quiver Q, every Q-sheaf E over X pure of dimension e, (i.e. a coherent sheaf pure of dimension e) has a unique filtration*

$$0 \subset E_1 \subset E_2 \subset \ldots \subset E_t \subset E_{t+1} = E$$

verifying the following properties, where $E^i := E_i/E_{i-1}$,

- $\frac{P_{E^1}(m)}{\operatorname{rk} E^1} > \frac{P_{E^2}(m)}{\operatorname{rk} E^2} > \ldots > \frac{P_{E^t}(m)}{\operatorname{rk} E^t} > \frac{P_{E^{t+1}}(m)}{\operatorname{rk} E^{t+1}}$
- *The quotients E^i are semistable*

This filtration is the ***Harder-Narasimhan filtration*** *of E defined in Theorem 1.3.5.*

Proof. Let $\tilde{Q}=\{v_0,v_1\}$. By Theorem 3.2.4, choosing $m\geq m_4$, we associate to an unstable Q-sheaf E a point in the parameter space, $x_M\in\mathcal{R}_d^{(\Theta,\sigma)}(\tilde{Q})$ which is $\chi_{(\Theta,\sigma)}$-unstable. By uniqueness of Theorem 3.1.13, there exists a unique filtration of M

$$0\subset M_1\subset\cdots\subset M_{t+1}=M$$

verifying the two conditions of Theorem 3.1.6, this is $\mu(M^1)>\mu(M^2)>\ldots>\mu(M^t)>\mu(M^{t+1})$ and that the quotients M^i are (Θ,σ)-semistable. Consider the Kempf filtration

$$0\subset E_1\subset E_2\subset\ldots\subset E_t\subset E_{t+1}=E\ ,$$

which does not depend on m, by Proposition 3.2.9.

To the Kempf filtration we associate a graph v, by Definition 3.1.14, and by Lemma 2.1.15, the coordinates v_i are in increasing order, hence

$$v_i<v_{i+1}\Leftrightarrow\Theta(M)-\frac{\sigma(M)}{\sigma(M^i)}\cdot\Theta(M^i)<\Theta(M)-\frac{\sigma(M)}{\sigma(M^{i+1})}\cdot\Theta(M^{i+1})$$

$$\Leftrightarrow\dim M_{v_0}-\frac{\dim M_{v_0}+\dim M_{v_1}}{\dim M^i_{v_0}+\dim M^i_{v_1}}\cdot\dim M^i_{v_0}<\dim M_{v_0}-\frac{\dim M_{v_0}+\dim M_{v_1}}{\dim M^i_{v_0}+\dim M^{i+1}_{v_1}}\cdot\dim M^{i+1}_{v_0}$$

$$\Leftrightarrow\frac{\dim M^i_{v_0}}{\dim M^i_{v_1}}>\frac{\dim M^{i+1}_{v_0}}{\dim M^{i+1}_{v_1}}\ .$$

Using Theorem 3.2.4, the last expression is equivalent to

$$\frac{P_{E^i}(m)}{P_{E^i}(l)}>\frac{P_{E^{i+1}}(m)}{P_{E^{i+1}}(l)}\Leftrightarrow\frac{P_{E^i}(m)}{\operatorname{rk}E^i}>\frac{P_{E^{i+1}}(m)}{\operatorname{rk}E^{i+1}}$$

where the last equivalence follows from Lemma 3.2.2. Using Lemma 2.1.16 as in Proposition 2.1.29, we can see that the Kempf filtration of E verifies the second property of the Harder-Narasimhan filtration as well. ∎

Bibliography

[AB] M. F. Atiyah and R. Bott *The Yang Mills equations over Riemann surfaces,* Phil. Trans. R. Soc. Lond. Ser. A **308** (1982), no. 1505, 523-615.

[AC] L. Álvarez-Cónsul, *Some results on the moduli spaces of quiver bundles,* Geom. Dedicata **139** (2009), 99-120.

[ACGP] L. Álvarez-Cónsul, O. García-Prada, *Hitchin-Kobayashi correspondence, quivers, and vortices,* Comm. Math. Phys. **238** (2003), 1-33.

[ACK] L. Álvarez-Cónsul, A. King, *A functiorial construction of moduli of sheaves,* Invent. Math. **168** (2007), 613-666.

[BGP] S.B. Bradlow, O. García-Prada, *Stable triples, equivariant bundles and dimensional reduction,* Math. Ann. **304** (1996), 225-252.

[Br] L. Bruasse, *Optimal destabilizing vectors in some gauge theoretical moduli problems.* Ann. Inst. Fourier (Grenoble) **56** (2006), no. 6, 1805-1826.

[BT] L. Bruasse and A. Teleman *Harder-Narasimhan filtrations and optimal destabilizing vectors in complex geometry.* Ann. Inst. Fourier (Grenoble) **55** (2005), no. 3, 1017-1053.

[DM] P. Deligne and D. Mumford *The irreducibility of the space of curves of given genus,* Publ. Math. I.H.E.S. **36** (1969), 75-109.

[Gi1] D. Gieseker, *On the moduli of vector bundles on an algebraic surface,* Ann. Math., **106** (1977), 45–60.

[Gi2] D. Gieseker, *Geometric invariant theory and the moduli of bundles,* Lecture Publication Series, IAS/Park City Mathematics Series v.00, (1994).

[Gr] A. Grothendieck, *Techniques de construction et théorèmes d'existence en géométrie algébrique IV: Les schémas de Hilbert,* Séminarie Bourbaki, 1960/1961 **221**.

[GS0] T. Gómez and I. Sols, *Stability of conic bundles,* Internat. J. Math., **11** (2000), 1027-1055.

[GS1] T. Gómez and I. Sols, *Stable tensors and moduli space of orthogonal sheaves,* Preprint 2001, math.AG/0103150.

[GS2] T. Gómez and I. Sols, *Moduli space of principal sheaves over projective varieties,* Ann. of Math. (2) **161** no. 2 (2005), 1037-1092.

[GSZ] T. Gómez, I. Sols, A. Zamora, *A GIT characterization of the Harder-Narasimhan filtration,* arxiv:1112.1886v2, (Preprint 2011).

[Ha] R. Hartshorne, Algebraic Geometry, Grad. Texts in Math. 52, Springer Verlag, 1977.

[He] W. H. Hesselink, *Uniform instability in reductive groups,* J. Reine Angew. Math. **304** (1978), 74-96.

[Hi] D. Hilbert, *Über die vollen Invariantensysteme,* Math. Annalen. **42** 313-373 (1893).

[HK] V. Hoskins and F. Kirwan, *Quotients of unstable subvarieties and moduli spaces of sheaves of Fixed Harder-Narasimhan type,* `arXiv:1103.4731` (preprint 2011). To appear in the Proceedings of the London Mathematical Society.

[HL1] D. Huybrechts and M. Lehn, *Stable pairs on curves and surfaces,* J. Alg. Geom., **4** no. 1 (1995), 67–104.

[HL2] D. Huybrechts and M. Lehn, *Framed modules and their moduli,* Intern. J. Math., **6** no. 2 (1995), 297-324.

[HL3] D. Huybrechts and M. Lehn, The geometry of moduli spaces of sheaves, Aspects of Mathematics E31, Vieweg, Braunschweig/Wiesbaden 1997.

[HN] G. Harder and M. S. Narasimhan, *On the Cohomology Groups of Moduli Spaces of Vector Bundles on Curves,* Math. Ann., **212** (1975), 215-248.

[Ke] G. Kempf, *Instability in invariant theory,* Ann. of Math. (2) **108** no. 1 (1978), 299-316.

[Ki] A. King, *Moduli of representations of finite dimensional algebras,* Quart. J. Math. Oxford Ser. (2), **45** (1994), 180, 515-530.

[LeP] J. Le Potier, *L'espace de modules de Simpson,* Séminarie de géométrie algébrique, Jussieu, fév. (1992).

[Ma1] M. Maruyama, *Moduli of stable sheaves, I.* J. Math. Kyoto Univ. **17** (1977), 91-126.

[Ma2] M. Maruyama, *Moduli of stable sheaves, II.* J. Math. Kyoto Univ. **18** (1978), 557-614.

[Ma3] M. Maruyama, *On boundedness of families of torsion free sheaves,* J. Math. Kyoto Univ. **21** (1981), 673-701.

[Mu] D. Mumford, Geometric invariant theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34. Springer-Verlag, Berlin-New York, 1965.

[MF] D. Mumford, J. Fogarty, Geometric invariant theory. Second Edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34. Springer-Verlag, Berlin-New York, 1982.

[MFK] D. Mumford, J. Fogarty and F. Kirwan, Geometric invariant theory. Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2), 34. Springer-Verlag, Berlin, 1994.

[NS1] M.S. Narasimhan and C.S. Seshadri, *Holomorphic vector bundles on a compact Riemann surface,* Math. Ann. **155** (1964), 69-80.

[NS2] M.S. Narasimhan and C.S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface,* Ann. of Math. **82** (1965), 540-567.

[Ne] P.E. Newstead, Lectures on Introduction to Moduli Problems and Orbit Spaces, Published for the Tata Institute of Fundamental Research, Bombay. Springer-Verlag, Berlin (1978).

[Ra] A. Ramanathan, *Moduli for principal bundles over algebraic curves: I and II,* Proc. Indian Acad. Sci. (Math. Sci.), **106** (1996), 301–328, and 421-449.

[Re] M. Reineke, *Moduli of representations of quivers,* arxiv:0802.2147v1, (Preprint 2008).

[Ri] B. Riemann, *Theorie der Abel'shen Functionen,* J. reine Angew. Math (Crelle's journal), **54** (1857), 115-155.

[Ru] A. Rudakov, *Stability for an abelian category,* J. Algebra, **197** (1997), 231-245.

[Sa] F. Sala, *Restriction theorems for μ-(semi)stable framed sheaves,* `arXiv:1010.5096` (preprint 2010).

[Sch] A. Schmitt, *A universal construction for moduli spaces of decorated vector bundles over curves,* Transformation Groups **161** no. 2 (2004), 167-209.

[Se] C.S. Seshadri, *Space of unitary vector bundles on a compact Riemann surface.* Ann. of Math. (2) **85** (1967), 303-336.

[Si1] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety I,* Publ. Math. I.H.E.S. **79** (1994), 47-129.

[Si2] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety II,* Publ. Math. I.H.E.S. **80** (1994), 5-79.

[We] R.O. Wells, Jr., Differential Analysis on Complex Manifolds, Grad. Texts in Math. 65, Springer Verlag, 1972.

[Za] A. Zamora, *On the Harder-Narasimhan filtration of finite dimensional representations of quivers,* arxiv:1210.5475, (Preprint 2012).