

Level density of spin chains of Haldane-Shastry type

Alberto Enciso,^{*} Federico Finkel,[†] and Artemio González-López[‡]

Departamento de Física Teórica II, Universidad Complutense, 28040 Madrid, Spain

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We provide a rigorous proof of the fact that the level density of all known $\text{su}(m)$ spin chains of Haldane-Shastry type associated with the A_{N-1} root system approaches a Gaussian distribution as the number of spins N tends to infinity. Our approach is based on the study of the large- N limit of the characteristic function of the level density, using the description of the spectrum in terms of motifs and the asymptotic behavior of the transfer matrix.

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A well-known conjecture on spin chains of Haldane-Shastry (HS) type [1–10] states that their level density becomes Gaussian as the number of sites tends to infinity. Although this conjecture has been numerically verified for all chains of HS type whose spectrum has been computed in closed form [8,9,11–17], a rigorous proof thereof is lacking, except in a few exceptional cases in which the partition function factorizes [18]. In this paper we settle the conjecture in the affirmative for spin chains of HS type associated with the A_{N-1} root system, which is both the simplest and the most studied case.

Our result has implications in connection with two fundamental conjectures in the theory of quantum chaos that we shall now discuss. The first of these conjectures, due to Berry and Tabor [19], asserts that the probability density of spacings between consecutive levels in the spectrum of a quantum system whose classical analog is integrable follows Poisson's law $p(s) = e^{-s}$. The second conjecture, formulated by Bohigas *et al.* [20], posits that for a fully chaotic quantum system this density is instead given by Wigner's surmise $p(s) = (\pi s/2) \exp(-\pi s^2/4)$, characteristic of the Gaussian orthogonal ensemble in random matrix theory [21]. It is important to bear in mind that in this context the term “spectrum” refers to what is known as the *unfolded* spectrum, which by construction has an approximately uniform level density. The energies of this unfolded spectrum are obtained from the “raw” energies E_i through the mapping $E_i \mapsto \eta_i \equiv \eta(E_i)$, where η is a continuous approximation to the cumulative level density. Thus, the knowledge of this continuous approximation is essential for testing the latter conjectures. It turns out that in all spin chains of HS type studied so far, if one assumes that the continuous part of the cumulative level density is Gaussian, i.e.,

$$\eta(E) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^E e^{-(E' - \mu)^2/(2\sigma^2)} dE' = \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{E - \mu}{\sqrt{2}\sigma} \right) \right]$$

(where μ and σ are the mean and standard deviation of the spectrum), the spacings density follows a characteristic dis-

tribution which is neither of Poisson nor of Wigner type [9,11–17].

Let us briefly recall the definition of the three spin chains we shall deal with in what follows. The Hamiltonian of the original $\text{su}(m)$ Haldane-Shastry chain is defined as

$$H = \frac{1}{2} \sum_{i < j} \frac{1 - \varepsilon S_{ij}}{\sin^2(\xi_i - \xi_j)}, \quad \xi_k = \frac{k\pi}{N}, \quad (1)$$

where (as always hereafter, unless otherwise stated) the sum runs from 1 to the number of spins N , and $\varepsilon = 1$ (respectively, $\varepsilon = -1$) for the ferromagnetic (respectively, antiferromagnetic) chain. The operator S_{ij} permutes the i th and j th spins, i.e., its action on an element $|s_1, \dots, s_i, \dots, s_j, \dots, s_N\rangle$ ($s_k \in \{1, \dots, m\}$) of the canonical spin basis is given by

$$S_{ij} |s_1, \dots, s_i, \dots, s_j, \dots\rangle = |s_1, \dots, s_j, \dots, s_i, \dots\rangle.$$

The permutation operators S_{ij} can be expressed in terms of the generators t_k^a of the fundamental representation of $\text{su}(m)$ at the k th site as

$$S_{ij} = 2 \sum_{a=1}^{m^2-1} t_i^a t_j^a + \frac{1}{m},$$

with the normalization $\operatorname{tr}(t_k^a t_k^b) = \frac{1}{2} \delta^{ab}$. The chain (1) is intimately connected with the Hubbard model. For instance, it can be obtained from the one-dimensional Hubbard model with long-range hopping introduced in Ref. [22] when the on-site interaction tends to infinity and the sites are half-filled. The rational version of the HS chain (1) was subsequently introduced by Polychronakos [3] and Frahm [23]. The Hamiltonian of the Polychronakos-Frahm (PF) chain can be taken as

$$H = \sum_{i < j} \frac{1 - \varepsilon S_{ij}}{(\xi_i - \xi_j)^2}, \quad (2)$$

where the chain sites ξ_k are no longer equidistant, but are given by the zeros of the Hermite polynomial of degree N . Finally, the hyperbolic version of the HS chain, known as the Frahm-Inozemtsev (FI) chain [4], is defined by the Hamiltonian

^{*}aenciso@fis.ucm.es

[†]ffinkel@fis.ucm.es

[‡]Corresponding author; artemio@fis.ucm.es

$$H = \frac{1}{2} \sum_{i < j} \frac{1 - \varepsilon S_{ij}}{\sinh^2(\xi_i - \xi_j)}. \quad (3)$$

The chain sites in this case are given by $\xi_i = \frac{1}{2} \log \zeta_i$, where ζ_i is the i th zero of the Laguerre polynomial $L_N^{\alpha-1}$ with $\alpha > 0$. In particular, unlike the previous two chains, the sites of the FI chain depend on an essential parameter.

Each of chains (1)–(3) can be obtained from a corresponding spin dynamical model of Calogero-Sutherland type [24–26] by applying the so-called freezing trick [3]. As first shown by Polychronakos [27], this connection can be exploited to derive closed-form expressions for the partition functions of the above chains. Remarkably, it turns out that these expressions can be rewritten in a unified way as [11,14,17,28]

$$Z(q) = \sum_{\mathbf{k} \in \mathcal{P}_N} \prod_{i=1}^r d(k_i) q^{\sum_{i=1}^{r-1} \mathcal{F}(K_i)} \prod_{i=1}^{N-r} (1 - q^{\mathcal{F}(K'_i)}), \quad (4)$$

where $q \equiv e^{-1/(k_B T)}$, $\mathbf{k} \equiv (k_1, \dots, k_r)$ is an element of the set \mathcal{P}_N of partitions of N with order taken into account, and the spin degeneracy factor $d(k_i)$ is given by

$$d(k_i) = \begin{cases} \binom{m+k_i-1}{k_i}, & \varepsilon = 1 \\ \binom{m}{k_i}, & \varepsilon = -1. \end{cases}$$

The numbers K_i in Eq. (4) are defined as $K_i = \sum_{j=1}^i k_j \in \{1, \dots, N-1\}$ and $\{K'_1, \dots, K'_{N-r}\} = \{1, \dots, N-1\} \setminus \{K_1, \dots, K_{r-1}\}$. The partition function Z depends on the chain under consideration only through its *dispersion relation* $\mathcal{F}(i)$, given by

$$\mathcal{F}(i) = \begin{cases} i(N-i), & \text{for the HS chain} \\ i, & \text{for the PF chain} \\ i(\alpha+i-1), & \text{for the FI chain.} \end{cases} \quad (5)$$

Starting from Eq. (4), Basu-Mallick *et al.* [28,29] derived a simple set of rules for generating the spectrum of chains (1)–(3) in terms of Young tableaux of certain irreducible representations of the Yangian $Y(\text{gl}(m))$. It can be shown that these rules are equivalent to the explicit formula [30]

$$E_{\mathbf{n}} = \sum_{i=1}^{N-1} \delta(n_i, n_{i+1}) \mathcal{F}(i), \quad \mathbf{n} \equiv (n_1, \dots, n_N), \quad (6)$$

where the quantum numbers n_i independently take the values $1, \dots, m$. As to the function δ , it is given by

$$\delta(j, k) = \begin{cases} 1, & j < k \\ 0, & j \geq k, \end{cases} \quad (7)$$

in the ferromagnetic case, whereas in the antiferromagnetic one it suffices to exchange 0 and 1 in Eq. (7). The vectors $\boldsymbol{\delta}(\mathbf{n}) \in \{0, 1\}^{N-1}$ with components $\delta_i(\mathbf{n}) = \delta(n_i, n_{i+1})$ are in fact the celebrated *motifs* introduced by Haldane *et al.* in Ref. [31]. As pointed out in Ref. [30], Eq. (6) admits an interesting alternative interpretation as the spectrum of a suitable classical one-dimensional vertex model with $N+1$

vertices and N intermediate bonds [30]. Note also that formula (6) for the energies is obtained from the partition function and not vice versa, as is usually the case.

Equation (6) shall be our starting point for establishing the asymptotically Gaussian character of the level density of chains (1)–(3). We shall start by deriving unified formulas for the mean μ and variance σ^2 of the spectrum of the latter chains in terms of the dispersion function \mathcal{F} . Consider first the mean energy, defined by

$$\mu = m^{-N} \sum_{n_1, \dots, n_N=1}^m \sum_{i=1}^{N-1} \delta(n_i, n_{i+1}) \mathcal{F}(i).$$

In the ferromagnetic case ($\varepsilon = 1$), by Eq. (7) the coefficient of $\mathcal{F}(i)$ in the previous expression equals 1 for $n_{i+1} > n_i$, regardless of the values taken by the $N-2$ remaining quantum numbers n_k , and is zero otherwise. Thus,

$$\mu = m^{-N} \sum_{i=1}^{N-1} \binom{m}{2} m^{N-2} \mathcal{F}(i) = \frac{1}{2} \left(1 - \frac{1}{m}\right) \sum_{i=1}^{N-1} \mathcal{F}(i). \quad (8)$$

Using Eq. (5) we easily arrive at the following explicit expression for the mean energy in the ferromagnetic case:

$$\mu = \frac{1}{4} \left(1 - \frac{1}{m}\right) N(N-1) \tilde{\mu}, \quad (9a)$$

where

$$\tilde{\mu} = \begin{cases} \frac{1}{3}(N+1), & \text{for the HS chain} \\ 1, & \text{for the PF chain} \\ \frac{1}{3}(2N+3\alpha-4), & \text{for the FI chain.} \end{cases} \quad (9b)$$

In the antiferromagnetic case ($\varepsilon = -1$), using the analog of Eq. (7) and reasoning as before it is immediate to show that the mean energy is given by

$$\mu = \frac{1}{2} \left(1 + \frac{1}{m}\right) \sum_{i=1}^{N-1} \mathcal{F}(i) = \frac{1}{4} \left(1 + \frac{1}{m}\right) N(N-1) \tilde{\mu}, \quad (10)$$

with $\tilde{\mu}$ as above. It may be easily verified that the latter expression exactly coincides with the formulas for μ previously computed on a case by case basis in Refs. [11,14,17].

The computation of σ^2 , which follows essentially the same lines, is left to Appendix A. The final result (in both the ferromagnetic and the antiferromagnetic cases) reads

$$\sigma^2 = \left(1 - \frac{1}{m^2}\right) \left[\frac{1}{4} \sum_{i=1}^{N-1} \mathcal{F}(i)^2 - \frac{1}{6} \sum_{i=2}^{N-1} \mathcal{F}(i-1) \mathcal{F}(i) \right]. \quad (11)$$

Inserting expressions (5) for the dispersion relation into the latter equation we easily obtain

$$\sigma^2 = \frac{1}{360} \left(1 - \frac{1}{m^2}\right) N(N-1) \tilde{\sigma}^2, \quad (12a)$$

where

$$\tilde{\sigma}^2 = \begin{cases} (N+1)(N^2+11), & \text{for the HS chain} \\ 5(2N+5), & \text{for the PF chain} \\ 6N^3 + 3N^2(5\alpha+2) + N(10\alpha^2 + 25\alpha - 84) + (5\alpha-16)(5\alpha-6), & \text{for the FI chain.} \end{cases} \quad (12b)$$

Again, it is easily checked that Eqs. (12) coincide with the formulas for σ^2 derived in Refs. [11,14,17] by taking traces of appropriate powers of the Hamiltonians (1)–(3).

After these preliminaries, we are now ready to present the main part of our proof. In fact, since the sum of the Hamiltonians of the ferromagnetic and antiferromagnetic chains is a constant, from now on we shall restrict ourselves to the ferromagnetic case unless otherwise stated. As in our previous paper [18], the proof is based on analyzing the limit as N tends to infinity of the (normalized) *characteristic function* of the level density, defined by

$$\begin{aligned} \hat{\phi}(t) &= \langle e^{it(E_{\mathbf{n}} - \mu)/\sigma} \rangle \equiv m^{-N} \sum_{n_1, \dots, n_N=1}^m e^{it(E_{\mathbf{n}} - \mu)/\sigma} \\ &= m^{-N} e^{-i\mu t/\sigma} Z(e^{it/\sigma}). \end{aligned} \quad (13)$$

Note that $\hat{\phi}(t)$ is simply the Fourier transform of the level density, after normalizing the spectrum to zero mean and unit variance. The importance of the characteristic function in the present context lies in the following standard result (see, e.g., Ref. [32]): in the limit $N \rightarrow \infty$, the level density (normalized to unity) approaches a Gaussian with parameters μ and σ provided that

$$\lim_{N \rightarrow \infty} \hat{\phi}(t) = e^{-t^2/2}, \quad (14)$$

for all real t . In order to analyze the asymptotic behavior of the characteristic function (13), we first rewrite the partition function $Z(q)$ of chains (1)–(3) using the explicit formula (6) for the energies, obtaining

$$\begin{aligned} Z(q) &= \sum_{n_1, \dots, n_N=1}^m \prod_{i=1}^{N-1} q^{\delta(n_i, n_{i+1}) \mathcal{F}(i)} \\ &= m^{N-1} \sum_{n, n'=1}^m [T_1(q) T_2(q) \cdots T_{N-1}(q)]_{nn'}, \end{aligned} \quad (15)$$

where the transfer matrix $T_j(q)$ ($j=1, \dots, N-1$) is the $m \times m$ matrix with elements

$$[T_j(q)]_{kl} = \frac{1}{m} q^{\delta(k,l) \mathcal{F}(j)}, \quad 1 \leq k, l \leq m.$$

Note that, since $\delta(k,l)$ depends only on the difference $k-l$ [cf. Eq. (7)], $T_j(q)$ is a Toeplitz matrix. More precisely, using the explicit definition (7) we see that

$$T_j(q) = T(\omega_j(q)), \quad (16)$$

where the matrix $T(\omega)$ is given by

$$T(\omega) = \frac{1}{m} \begin{pmatrix} 1 & \omega^m & \cdots & \omega^m \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \omega^m \\ 1 & \cdots & \cdots & 1 \end{pmatrix}, \quad \omega \in \mathbb{C}, \quad (17)$$

$$\omega_j(q) = q^{\mathcal{F}(j)/m}. \quad (18)$$

The $m \times m$ matrix (17) can be easily diagonalized for arbitrary m and $\omega \neq 0$; see Appendix B for the details. In the particular case in which ω is unimodular, it is shown in the latter appendix that

$$T(\omega) = U(\omega) D(\omega) U^\dagger(\omega), \quad (19)$$

where

$$D(\omega) = \text{diag}[\lambda_1(\omega), \dots, \lambda_m(\omega)], \quad (20)$$

the eigenvalues $\lambda_k(\omega)$ are given by

$$\lambda_k(\omega) = \frac{1}{m} \sum_{l=0}^{m-1} [\omega e^{2\pi i k l / m}]^l, \quad (21)$$

and $U(\omega)$ is the unitary $m \times m$ matrix with entries

$$U_{nn'}(\omega) = \frac{1}{\sqrt{m}} [\omega e^{2\pi i n' / m}]^{m-n}. \quad (22)$$

Let us now go back to the characteristic function (13). Since

$$\omega_j(e^{it/\sigma}) = e^{i\gamma_j t}, \quad \gamma_j \equiv \frac{\mathcal{F}(j)}{m\sigma}, \quad (23)$$

we can apply Eqs. (19)–(22) to the matrices $T_j(e^{it/\sigma}) = T(e^{i\gamma_j t})$ in Eq. (15). We thus readily obtain

$$\hat{\phi}(t) = \frac{e^{-i\mu t/\sigma}}{m} \sum_{n, n'=1}^m M_{nn'}(t), \quad (24)$$

where the $m \times m$ matrix $M(t)$ is given by

$$\begin{aligned} M(t) &= U(e^{i\gamma_1 t}) D(e^{i\gamma_1 t}) B_1(t) \cdots D(e^{i\gamma_{N-2} t}) B_{N-2}(t) \\ &\quad \times D(e^{i\gamma_{N-1} t}) U^\dagger(e^{i\gamma_{N-1} t}), \end{aligned} \quad (25)$$

with

$$B_j(t) = U^\dagger(e^{i\gamma_j t}) U(e^{i\gamma_{j+1} t}). \quad (26)$$

From Eqs. (24)–(26) one can determine the large- N limit of the characteristic function $\hat{\phi}(t)$, as we shall next discuss. To this end, we note first of all that

$$\gamma_j = O(N^{-1/2}), \quad j = 1, \dots, N-1, \quad (27)$$

where the notation $f_j(N) = O(N^a)$ means that when N is large enough $N^{-a}f_j(N)$ is bounded by a constant independent of N and j [note, in particular, that $N^b = O(N^a)$ for $b \leq a$]. Indeed, for the HS and FI chains $\mathcal{F}(j)$ is $O(N^2)$ by Eq. (5), while σ^{-1} is $O(N^{-5/2})$ by Eq. (11). Likewise, for the PF chain $\mathcal{F}(j)$ is $O(N)$, while σ^{-1} is $O(N^{-3/2})$ on account of the latter equation. From Eqs. (22) and (27), it follows that

$$U(e^{i\gamma_j t}) = R + O(N^{-1/2}), \quad (28)$$

where

$$R \equiv U(1) \quad (29)$$

is a constant unitary matrix (independent of N). In order to estimate $B_j(t)$, note first that for all three chains (1)–(3) we have

$$\gamma_j - \gamma_{j+1} = O(N^{-3/2}). \quad (30)$$

Indeed, $\mathcal{F}(j) - \mathcal{F}(j+1)$ is $O(N)$ [respectively, $O(1)$] for the HS and FI (respectively, PF) chains. Taking this into account and using Eq. (22) we immediately obtain

$$\begin{aligned} [B_j(t)]_{nn'} &= \sum_{k=1}^m \overline{U_{kn}(e^{i\gamma_j t})} U_{kn'}(e^{i\gamma_{j+1} t}) \\ &= \frac{1}{m} \sum_{k=1}^m [e^{-i\gamma_j t} e^{-2\pi i n/m}]^{m-k} [e^{i\gamma_{j+1} t} e^{2\pi i n'/m}]^{m-k} \\ &= \frac{1}{m} \sum_{k=1}^m e^{i(\gamma_{j+1} - \gamma_j)(m-k)t} e^{2\pi i(n' - n)(m-k)/m} \\ &= \frac{1}{m} \sum_{k=1}^m e^{2\pi i(n' - n)(m-k)/m} + O(N^{-3/2}) \\ &= \delta_{nn'} + O(N^{-3/2}), \end{aligned}$$

so that

$$B_j(t) = \mathbb{1} + O(N^{-3/2}). \quad (31)$$

On the other hand, from Eq. (21) it easily follows that

$$|\lambda_k(e^{i\gamma_j t})| \leq 1, \quad k = 1, \dots, m. \quad (32)$$

Equation (25) and estimates (28), (31), and (32) immediately yield the asymptotic formula

$$M(t) = R\Lambda(t)R^\dagger + O(N^{-1/2}), \quad (33)$$

where $\Lambda(t)$ is the diagonal matrix with entries

$$\Lambda_k(t) = \prod_{j=1}^{N-1} \lambda_k(e^{i\gamma_j t}), \quad k = 1, \dots, m. \quad (34)$$

Inserting Eqs. (33)–(34) into Eq. (24) and using identity (B1) with $\omega = 1$, we obtain the simple asymptotic estimate

$$\hat{\phi}(t) = e^{-i\mu t/\sigma} \Lambda_m(t) + O(N^{-1/2}). \quad (35)$$

In view of the latter equation, in order to complete our proof of Eq. (14) we just have to determine the asymptotic behavior as $N \rightarrow \infty$ of the eigenvalue $\lambda_m(\omega)$, with $\omega = e^{i\gamma_j t}$

unimodular. By Eq. (21), this eigenvalue is given by

$$\lambda_m(e^{i\gamma_j t}) = \frac{1}{m} \sum_{l=0}^{m-1} e^{il\gamma_j t} = \frac{e^{im\gamma_j t} - 1}{m(e^{i\gamma_j t} - 1)}, \quad (36)$$

provided that $t \notin (2\pi/\gamma_j)\mathbb{Z}$. Note that for any fixed $t \neq 0$ the condition $mt \notin (2\pi/\gamma_j)\mathbb{Z}$ is fulfilled for sufficiently large N on account of Eq. (27), and implies the weaker condition $t \notin (2\pi/\gamma_j)\mathbb{Z}$. Thus, for all $t \neq 0$ Eq. (36) holds if N is large enough, and $\lambda_m(e^{i\gamma_j t}) \neq 0$. The latter equation, together with the elementary Taylor expansion

$$\ln \left[\frac{e^{imx} - 1}{m(e^{ix} - 1)} \right] = \frac{1}{2}(m-1)ix - \frac{1}{24}(m^2-1)x^2 + O(x^4),$$

and identity (8), easily yields the asymptotic formula

$$-\frac{i\mu t}{\sigma} + \ln \Lambda_m(t) = -\frac{t^2}{24}(m^2-1) \sum_{j=1}^{N-1} \gamma_j^2 + O(N^{-1}). \quad (37)$$

In order to estimate the coefficient of t^2 in the previous formula, it suffices to note that Eq. (11) can be equivalently written as

$$\begin{aligned} \frac{1}{12}(m^2-1) \sum_{j=1}^{N-1} \gamma_j^2 &= 1 + \frac{1}{6}(m^2-1) \sum_{j=2}^{N-1} (\gamma_{j-1} - \gamma_j) \gamma_j \\ &= 1 + O(N^{-1}), \end{aligned}$$

where we have used Eqs. (27) and (30) for the last estimate. Hence,

$$-\frac{i\mu t}{\sigma} + \ln \Lambda_m(t) = -\frac{t^2}{2} + O(N^{-1}),$$

which obviously implies Eq. (14) in view of Eq. (35).

We shall conclude by summarizing the main result of this paper and presenting an outline of related future work. We have rigorously shown that for all spin chains of Haldane-Shastry type associated with the A_{N-1} root system the level density (normalized to unity) approaches a Gaussian distribution as the number of sites tends to infinity. Our proof essentially relies on two key properties of these chains, namely, Eq. (6) for the energies in terms of motifs (7), and estimates (27) and (30) involving the large- N behavior of the dispersion relation (5).

Our results admit several natural generalizations. For instance, one could consider the $\mathfrak{su}(n|n')$ supersymmetric extensions of chains (1)–(3), some of which have already been studied in the literature [12,16]. As shown in Ref. [30], in this case Eq. (7) for the motifs should be replaced with

$$\delta(j, k) = \begin{cases} 1, & j > k \text{ or } j = k > n \\ 0, & j < k \text{ or } j = k \leq n, \end{cases}$$

where $j, k = 1, \dots, n+n' \equiv m$. As a consequence, the last n' elements in the main diagonal of the transfer matrix (17) are replaced with ω^m , so that the resulting matrix is no longer Toeplitz. Although this fact certainly complicates the explicit diagonalization of the transfer matrix, we believe that the main ideas behind our proof can still be applied to this case.

It is also natural to consider the generalization of our re-

sult to spin chains of HS type associated with other root systems, such as BC_N or D_N . The main difficulty in this respect is the fact that for these chains no description of the energies in terms of motifs akin to Eq. (6) is known so far. At least for the Sutherland spin chain of BC_N type [5,8], some preliminary results of our group indicate that such a description is possible, and that our proof can be suitably adapted to this case.

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APPENDIX A: COMPUTATION OF σ^2

In this appendix we shall derive Eq. (11) for the variance of the energy of chains (1)–(3). Since, up to an additive constant, the energies of the antiferromagnetic chains differ from those of their ferromagnetic counterparts by a sign change, we need only consider (say) the ferromagnetic case. By definition, $\sigma^2 = \langle E_{\mathbf{n}}^2 \rangle - \mu^2$, where

$$\begin{aligned} \langle E_{\mathbf{n}}^2 \rangle &\equiv m^{-N} \sum_{n_1, \dots, n_N=1}^m E_{\mathbf{n}}^2 \\ &= m^{-N} \sum_{n_1, \dots, n_N=1}^m \sum_{i,j=1}^{N-1} \delta_i(\mathbf{n}) \delta_j(\mathbf{n}) \mathcal{F}(i) \mathcal{F}(j). \end{aligned}$$

Recall that $\delta_i(\mathbf{n}) = \delta(n_i, n_{i+1})$, where $\delta(j, k)$ is given in Eq. (7). Proceeding as in the derivation of Eq. (8), and taking into account that $\delta_i^2 = \delta_i$, we easily obtain

$$\begin{aligned} \langle E_{\mathbf{n}}^2 \rangle &= \frac{1}{2} \left(1 - \frac{1}{m} \right) \sum_{i=1}^{N-1} \mathcal{F}(i)^2 \\ &\quad + 2m^{-N} \sum_{n_1, \dots, n_N=1}^m \sum_{\substack{i,j=1 \\ i < j}}^{N-1} \delta_i(\mathbf{n}) \delta_j(\mathbf{n}) \mathcal{F}(i) \mathcal{F}(j). \end{aligned}$$

If $i < j-1$, the coefficient of $\mathcal{F}(i) \mathcal{F}(j)$ in the last sum equals 1 provided that $n_i < n_{i+1}$ and $n_j < n_{j+1}$, and is zero otherwise. Likewise, the coefficient of $\mathcal{F}(j-1) \mathcal{F}(j)$ ($j=2, \dots, N-1$) is 1 if $n_{j-1} < n_j < n_{j+1}$, and vanishes otherwise. Hence,

$$\begin{aligned} \langle E_{\mathbf{n}}^2 \rangle &= \frac{1}{2} \left(1 - \frac{1}{m} \right) \sum_{i=1}^{N-1} \mathcal{F}(i)^2 + 2m^{-N} \sum_{\substack{i,j=1 \\ i < j-1}}^{N-1} \binom{m}{2} m^{N-4} \mathcal{F}(i) \mathcal{F}(j) \\ &\quad + 2m^{-N} \sum_{j=2}^{N-1} \binom{m}{3} m^{N-3} \mathcal{F}(j-1) \mathcal{F}(j). \end{aligned}$$

Using Eq. (8) for μ and after some straightforward algebra we readily obtain Eq. (11).

APPENDIX B: DIAGONALIZATION OF THE TRANSFER MATRIX

In this appendix we shall show that the transfer matrix $T(\omega)$ in Eq. (17) can be explicitly diagonalized for any non-zero complex number ω . More precisely, we shall derive Eq. (21) for the eigenvalues of $T(\omega)$ and Eq. (22) for the diagonalizing transformation U , proving that U is unitary when ω is unimodular. Consider, to this end, the vectors $v^k(\omega)$ ($k=1, \dots, m$) with components

$$v_j^k(\omega) = [\omega e^{2\pi i k/m}]^{m-j}, \quad j=1, \dots, m.$$

Note, in particular, that $v^m(\omega) = (\omega^{m-1}, \dots, \omega, 1)$. We then have

$$\begin{aligned} m[T(\omega)v^k(\omega)]_j &= \sum_{l=1}^j [\omega e^{2\pi i k/m}]^{m-l} + \omega^m \sum_{l=j+1}^m [\omega e^{2\pi i k/m}]^{m-l} \\ &= \sum_{l=1}^j [\omega e^{2\pi i k/m}]^{m-l} + \sum_{l=j+1}^m [\omega e^{2\pi i k/m}]^{2m-l} \\ &= m\lambda_k(\omega)v_j^k(\omega), \end{aligned}$$

where $\lambda_k(\omega)$ is given by Eq. (21). Thus, $v^k(\omega)$ is an eigenvector of $T(\omega)$ with eigenvalue $\lambda_k(\omega)$. The above calculation shows that for arbitrary $\omega \neq 0$ we have

$$T(\omega) = U(\omega)D(\omega)U^{-1}(\omega),$$

where the matrices $D(\omega)$ and $U(\omega)$ are, respectively, defined by Eqs. (20) and (22). In order to establish Eq. (19), we need only to show that when ω is unimodular the matrix $U(\omega)$ is unitary. In other words, the columns of $U(\omega)$, namely the vectors $v^k(\omega)/\sqrt{m}$ ($k=1, \dots, m$), form an orthonormal basis of \mathbb{C}^m . Indeed, the latter vectors are clearly of unit length, and their scalar product vanishes for $k \neq k'$:

$$\begin{aligned} v^k \cdot v^{k'} &= \sum_{l=1}^m [\bar{\omega} e^{-2\pi i k/m}] \cdot [\omega e^{2\pi i k'/m}]^{m-l} \\ &= \sum_{l=0}^{m-1} e^{2\pi i (k'-k)l/m} = 0, \quad k \neq k'. \end{aligned}$$

For later convenience, we shall also evaluate the sum

$$\begin{aligned} \sum_{n,n'=1}^m U_{nk}(\omega) \overline{U_{n'k}(\omega)} &= \frac{1}{m} \sum_{n,n'=1}^m e^{2\pi i k(m-n)/m} e^{-2\pi i k(m-n')/m} \\ &= \frac{1}{m} \sum_{n,n'=1}^m e^{2\pi i k(n'-n)/m} = m\delta_{km}, \quad |\omega|=1. \end{aligned} \tag{B1}$$

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