Quasi-exactly solvable spin 1/2 Schrödinger operators

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(Received 31 July 1996; accepted for publication 22 October 1996)

The algebraic structures underlying quasi-exact solvability for spin 1/2 Hamiltonians in one dimension are studied in detail. Necessary and sufficient conditions for a matrix second-order differential operator preserving a space of wave functions with polynomial components to be equivalent to a Schrödinger operator are found. Systematic simplifications of these conditions are analyzed, and are then applied to the construction of new examples of multi-parameter QES spin 1/2 Hamiltonians in one dimension. © *1997 American Institute of Physics.* [S0022-2488(97)03905-4]

I. INTRODUCTION

Symmetries have traditionally played an essential role in quantum mechanics. For a few remarkable Hamiltonians, the knowledge of enough symmetries leads to a complete characterization of the spectrum by algebraic methods.[1] In general, however, the spectrum of an arbitrary Hamiltonian cannot be calculated analytically. During the last decade, a remarkable intermediate class of *quasi-exactly solvable* (QES) spectral problems was introduced, for which a finite part of the spectrum can be computed by purely algebraic methods.[2–4] The key feature in the latter class of spectral problems is that the Hamiltonian H is expressible as a quadratic combination of the generators of a finite-dimensional Lie algebra $\mathfrak{g}$ of first order differential operators preserving a finite-dimensional module of smooth functions $\mathscr{N}$. Thus, H restricts to a linear transformation in the finite-dimensional vector space $\mathscr{N}$, and therefore part of its spectrum can be computed by matrix eigenvalue methods. Appropriate boundary conditions must be imposed so that the eigenfunctions thus obtained qualify as physical wave functions, as, e.g., square integrability if they represent bound states of the system.[5]

These ideas, originally introduced for scalar Hamiltonians describing spinless particles, can be generalized to include particles with spin. The first step in this direction was taken by Shifman and Turbiner,[6] using the fact that a Hamiltonian for a spin 1/2 particle in d spatial dimensions can be constructed from a Lie superalgebra of first order differential operators in d ordinary (commuting) variables and one Grassmann (anticommuting) variable. Alternatively,[7] 2×2 matrices (or $N\times N$ matrices for particles of arbitrary spin[8]) can be used to represent the Grassmann variable. However, in stark contrast with the scalar case, very few examples of matrix QES Schrödinger operators have been found thus far.[6] There are two important conceptual reasons for this fact. First, the algebraic structures underlying partial integrability in the matrix case are richer and less understood than in the scalar case. For one thing, as mentioned before, for matrix Hamiltonians Lie superalgebras of matrix differential operators naturally come into play, whereas in the scalar case only Lie algebras need be considered. Moreover, as we shall explain in Section III, one even has to go beyond Lie superalgebras of matrix differential operators in order to explain quasi-exact solvability in the matrix case.[7,8] Second,[9,5] every scalar second order differential operator in one dimension can be transformed into a Schrödinger operator of the form $-\partial_x^2+V(x)$ by a suitable change of the independent variable x and a local rescaling of the wave function. For matrix differential operators, the analogue of this result—$V(x)$ being now a Hermitian matrix of smooth functions—is no longer true unless the operator satisfies quite stringent conditions, as we shall see in detail in Section IV.

The aim of this paper is to achieve a better theoretical understanding of quasi-exact solvability in the matrix case, which will enable us to construct new examples of matrix QES Schrödinger

0022-2488/97/38(6)/2795/17/$10.00

operators. To this end, in Sections II and III we study the algebraic properties of certain algebras of matrix QES operators, reviewing the literature on the subject and obtaining new results as well. In particular, we give a complete characterization of the form of a QES matrix differential operator preserving a finite-dimensional space of wave functions with polynomial components. For the important particular case of spin 1/2 particles, we derive in Section IV necessary and sufficient conditions for a QES operator to be equivalent to a non-trivial Schrödinger operator. These conditions turn out to be too complicated to be solved in full generality, and so in Sections V and VI we introduce some key simplifications that will prove very useful in the task of finding explicit examples. Finally, the previous results are applied in Section VII to the construction of new examples of multi-parameter QES spin 1/2 Hamiltonians in one dimension.

II. SCALAR QES OPERATORS

We start with the scalar case, introducing the basic concepts and definitions and stating two theorems for the one-dimensional case which will play an important role in what follows. Since the results of this section are fairly standard, we will skip many details and all the proofs, referring the reader to the review articles (Ref. 4) and (Ref. 9) for an in-depth study.

Let M denote an open subset of $\mathbb{R}^d$, and let $\mathscr{D}^1(M)$ be the Lie algebra of first order differential operators

$$X=\sum_{i=1}^{d} \xi^i(z)\,\frac{\partial}{\partial z^i}+\eta(z),\qquad z=(z^1,\ldots,z^d)\in M,$$

acting on $C^\infty(M)$, the Lie bracket being defined as the usual commutator between operators:

$$[X,Y]=XY-YX,\quad X,Y\in\mathscr{D}^1(M).$$

Definition 2.1: A finite-dimensional Lie subalgebra $\mathfrak{g}$ *of* $\mathscr{D}^1(M)$ *is called* quasi-exactly solvable (QES) *if it preserves a finite-dimensional module* $\mathscr{N}\subset C^\infty(M)$. *A differential operator* T *is QES if it lies in the universal enveloping algebra* $\mathscr{U}(\mathfrak{g})$ *of a QES Lie algebra* $\mathfrak{g}$.

In general, quasi-exact solvability of a given differential operator T cannot be ascertained *a priori*. Therefore, the procedure usually followed consists in classifying QES Lie algebras modulo a suitable equivalence relation, and then using the canonical forms in the classification thus obtained to construct QES operators.

Definition 2.2: Two differential operators $T(z)$ *and* $\bar{T}(\bar{z})$ *are* equivalent *if they are related by a change of the independent variables*

$$\bar{z}=\varphi(z) \tag{1}$$

and a local scale transformation by a non-vanishing function $U(z)$, *i.e.*

$$\bar{T}(\bar{z})=U(z)T(z)U^{-1}(z). \tag{2}$$

The corresponding notion of equivalence for QES algebras follows directly, i.e. two QES Lie algebras $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ are *equivalent* if their elements can be mapped into each other by a *fixed* transformation (1)-(2). Their associated finite-dimensional modules $\mathscr{N}$ and $\bar{\mathscr{N}}$ are then related by

$$\bar{\mathscr{N}}=U\cdot\mathscr{N}, \tag{3}$$

the functions being expressed in the appropriate coordinates. The local classification of finite-dimensional QES Lie algebras under the above notion of equivalence has already been completed for the case of one and two (real or complex) variables. Here we shall need only the one-dimensional case.[10–12,2]

Theorem 2.3: *Every (non-singular) QES Lie algebra in one (real or complex) variable is locally equivalent to a subalgebra of one of the Lie algebras*

$$\mathfrak{g}_n = \mathrm{Span}\{\partial_z, z\partial_z, z^2\partial_z - nz, 1\}, \tag{4}$$

where $n \in \mathbb{N}$. The associated $\mathfrak{g}_n$-module is $\mathcal{N}_n = \mathcal{P}_n$, the space of polynomials of degree at most n.

(The two-dimensional case, which is considerably more complicated but is not needed for the sequel, is discussed in Refs. 13, 12 and 14.)

According to the previous theorem, every one-dimensional (scalar) QES differential operator T is locally equivalent to an operator $T \in \mathscr{U}(\mathfrak{g}_n)$ preserving $\mathcal{P}_n$ for a suitable n. A partial converse of the latter result follows from the following remarkable theorem due to Turbiner.[15]

Theorem 2.4: *Let $T^{(k)}$ be a k-th order linear differential operator preserving $\mathcal{P}_n$. We then have:*

(i) *If $n \geqslant k$, then $T^{(k)}$ may be represented by a k-th degree polynomial in the operators*

$$J_n^+ = z^2\partial_z - nz, \quad J_n^0 = z\partial_z - \frac{n}{2}, \quad J_n^- = J_- = \partial_z. \tag{5}$$

(i) *If $k > n$, then $T^{(k)} = T\partial_z^{n+1} + \tilde{T}$, where T is a linear differential operator of order $k-n-1$, and $\tilde{T}$ is a linear differential operator of order at most n satisfying (i).*

The operators $\{J_n^+, J_n^0, J^-\}$ defined above span a QES Lie algebra $\hat{\mathfrak{g}}_n$ isomorphic to $\mathfrak{sl}(2)$, and the Lie algebras $\mathfrak{g}_n$ in Theorem 2.3 are simply a central extension by the constant functions of the corresponding $\hat{\mathfrak{g}}_n$.

III. ALGEBRAIC PROPERTIES OF PVSP OPERATORS

In the last section we have seen that every scalar QES scalar differential operator in one variable is essentially (up to equivalence) a polynomial in the generators of a Lie algebra $\hat{g}_n$ preserving $\mathcal{P}_n$ (for suitable n). When working with vector-valued wave functions, the natural generalization of $\mathcal{P}_n$ is the *polynomial vector space* $\mathcal{P}_{n_1,\ldots,n_N} = \mathcal{P}_{n_1} \oplus \cdots \oplus \mathcal{P}_{n_N}$, with elements $\Psi(z) = (\psi_1(z), \ldots, \psi_N(z))^t$ such that each component ψ_i is a polynomial of degree at most n_i with complex coefficients.

Definition 3.1: An $N \times N$ matrix differential operator T is called polynomial vector space preserving (PVSP) *if it preserves $\mathcal{P}_{n_1,\ldots,n_N} = \mathcal{P}_{n_1} \oplus \cdots \oplus \mathcal{P}_{n_N}$ for some non-negative integers n_i, $i = 1,\ldots,N$.*

We will denote by $\mathcal{P}^{(k)}_{n_1,\ldots,n_N}$ the complex vector space of linear PVSP operators of order at most k preserving $\mathcal{P}_{n_1,\ldots,n_N}$. Following Refs. 7 and 8, we will restrict ourselves in this paper to studying matrix PVSP differential operators. As we will be mainly concerned with spin 1/2 particles, the case $N=2$ deserves special attention.

A. Case $N=2$

Let $n \geqslant \Delta$ be non-negative integers, and consider the following set of matrix differential operators:

$$T^+ = \begin{pmatrix} J^+_{n-\Delta} & 0 \\ 0 & J^+_n \end{pmatrix}, \quad T^0 = \begin{pmatrix} J^0_{n-\Delta} & 0 \\ 0 & J^0_n \end{pmatrix}, \quad T^- = \begin{pmatrix} J^- & 0 \\ 0 & J^- \end{pmatrix}, \quad J = \frac{1}{2}\begin{pmatrix} n+\Delta & 0 \\ 0 & n \end{pmatrix},$$

$$Q_\alpha = z^\alpha \sigma^-, \quad \overline{Q_\alpha} = \overline{q_\alpha}(n,\Delta)\sigma^+, \qquad \alpha = 0,\ldots,\Delta, \tag{6}$$

with

$$\overline{q_\alpha}(n,\Delta) = \prod_{k=1}^{\Delta-\alpha} (z\partial_z - n + \Delta - k)\partial_z^\alpha, \tag{7}$$

where we have adopted the convention that a product with its lower limit greater than the upper one is automatically 1, and

$$\sigma^+ = (\sigma^-)^t = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It can be easily checked that the $6+2\Delta$ operators in (6) (and also any polynomial thereof) preserve $\mathscr{P}_{n-\Delta,n}$. We now introduce a $\mathbb{Z}_2$-grading in the set of 2×2 matrix differential operators $\mathscr{D}_{2\times 2}$ as follows: an operator

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a,b,c,d belong to the space $\mathscr{D}$ of scalar differential operators, is said to be even if $b=c=0$, and odd if $a=d=0$. Therefore, the T's and J are even and the Q's and $\overline{Q}$'s odd. This grading, combined with the usual product (composition) of operators, endows $\mathscr{D}_{2\times 2}$ with an associative superalgebra structure. We can also construct a Lie superalgebra in $\mathscr{D}_{2\times 2}$ by defining a generalized Lie product by

$$[A,B]_s = AB - (-1)^{\deg A \deg B} BA. \tag{8}$$

However, this product does not close within the vector space spanned by our operators (6), except for $\Delta = 0,1$. The explicit commutation relations are as follows:[7,8]

$$[T^+,T^-] = -2T^0, \quad [T^\pm,T^0] = \mp T^\pm,$$

$$[J,T^\epsilon] = 0, \quad [J,Q_\alpha] = -\frac{\Delta}{2} Q_\alpha, \quad [J,\overline{Q_\alpha}] = \frac{\Delta}{2} \overline{Q_\alpha},$$

$$[Q_\alpha,T^\epsilon] = \left(-\alpha + \frac{\Delta}{2}(1+\epsilon)\right) Q_{\alpha+\epsilon}, \quad [\overline{Q_\alpha},T^\epsilon] = \left(\alpha - \frac{\Delta}{2}(1-\epsilon)\right)\overline{Q_{\alpha-\epsilon}},$$

$$\{\overline{Q_\alpha},Q_\beta\} = \begin{cases} M_{\alpha\beta}(T^-)^{\alpha-\beta}, & \alpha \geqslant \beta \\ (T^+)^{\beta-\alpha} M_{\beta\alpha}, & \beta \geqslant \alpha, \end{cases}$$

$$\{Q_\alpha,Q_\beta\} = \{\overline{Q_\alpha},\overline{Q_\beta}\} = 0, \tag{9}$$

where $\epsilon = +,0,-$, and $M_{\alpha\beta}$ is given, for $\alpha \geqslant \beta$, by

$$M_{\alpha\beta} = \prod_{j=0}^{\Delta-\alpha-1} (T^0 + J_c - j - \beta P_2) \prod_{k=0}^{\beta-1} (T^0 + J - k - (\Delta-\alpha)P_1)$$

with $J_c = \Delta - 1 - J$, and

$$P_1 = 1 - P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

As shown in Ref. 7, $M_{\alpha\beta}$ can be expressed in terms of T^0, J, the identity, and the Casimir for the even subalgebra

$$C = -\frac{1}{2}(T^+T^- + T^-T^+) + T^0T^0 = \frac{1}{4}\begin{pmatrix} m(m+2) & 0 \\ 0 & n(n+2) \end{pmatrix}, \tag{10}$$

where $m = n - \Delta$, independently from n and the projectors P_1 and P_2. It can be readily verified that $\{Q_\alpha, Q_\beta\}$ gives a Δ-th order even differential operator, so the vector space spanned by the operators in (6) is not closed under the Lie product (8) whenever $\Delta \geqslant 2$. Moreover, it is not difficult to show that the Lie superalgebra $\mathfrak{s}_\Delta$ generated by the operators (6) is in this case infinite dimensional. Indeed, if we commute $\{Q_\Delta, Q_0\} = (T^-)^\Delta$ with $\{\bar{Q}_0, Q_\Delta\} = (T^+)^\Delta$ iteratively we obtain monomials in T^+, T^0, T^- of increasingly higher order. For $\Delta = 1$ the underlying algebraic structure is the classical simple Lie superalgebra $\mathfrak{osp}(2,2)$,[4,16] whereas for $\Delta = 0$ it is $\mathfrak{h}_1 \oplus \mathfrak{sl}(2)$, where $\mathfrak{h}_1$ is the 3-dimensional Heisenberg superalgebra. As remarked in Ref. 7, in this latter case we can leave the grading aside and replace $J = 1$ by $\tilde{J} = \sigma_3$, ending up with the Lie algebra $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$.

We now state the analogue of Theorem 2.4 for PVSP operators preserving $\mathscr{P}_{n-\Delta,n}$ (a version of this theorem was first mentioned without proof by Turbiner for $\Delta = 1$ in Ref. 16, and subsequently by Brihaye and Kosinski for arbitrary Δ, Ref. 7). It turns out that the operators (6) play the same role in the matrix case as the J's in (5) do in the scalar one.

Theorem 3.2: *Let $n \geqslant m$, and $\Delta = n - m$. Let $T^{(k)}$ be a k-th order differential operator in $\mathscr{P}_{m,n}^{(k)}$. We then have*:

(i) *If $m \geqslant k$, then $T^{(k)}$ is a polynomial in the operators (6), with J replaced by $\tilde{J}$ if $\Delta = 0$.*

(ii) *If $n \geqslant k > m$, then $T^{(k)} = T\partial_z^{m+1} + \tilde{T}$, where T and $\tilde{T}$ are matrix linear differential operators of the form*

$$T = \begin{pmatrix} a^{(k-m-1)} & 0 \\ c^{(k-m-1)} & 0 \end{pmatrix}, \quad \tilde{T} = \begin{pmatrix} \tilde{a}^{(m)} & \tilde{b}^{(k)} \\ \tilde{c}^{(m)} & \tilde{d}^{(k)} \end{pmatrix},$$

where the superscripts indicate the highest possible derivative in each entry, and $\tilde{T}$ satisfies (i).

(iii) *If $k > n$, then $T^{(k)} = T\partial_z^{n+1} + \tilde{T}$, where T is a 2×2 matrix linear differential operator of order $k - n - 1$, and $\tilde{T}: \mathscr{P}_{m,n} \to \mathscr{P}_{m,n}$ is a linear PVSP operator of order at most n verifying (i) or (ii).*

The proof of these results is based on a straightforward analysis, using Theorem 2.4, of the action of the components of $T^{(k)}$ on $\mathscr{P}_{m,n}$. A somewhat weaker version of the previous theorem, namely that any differential operator preserving $\mathscr{P}_{m,n}$ can be expressed as the sum of a polynomial in the generators of $\mathfrak{s}_\Delta$ plus a differential operator annihilating $\mathscr{P}_{m,n}$, follows directly from Burnside's theorem,[17,18] applied to the complexification of $\mathfrak{s}_\Delta$.

The next issue to be addressed is to find out the number of parameters determining a generic k-th order linear differential operator preserving $\mathscr{P}_{m,n}$, that is, the dimension of $\mathscr{P}_{m,n}^{(k)}$. In the scalar case, any k-th degree polynomial in J_n^+, J_n^0, J^- may be constructed from the monomials $\{(J_n^\pm)^r (J_n^0)^{s-r}\}_{r=0}^s$, $s = 0,\ldots,k$, and so $\dim \mathscr{P}_n^{(k)} = (k+1)^2$ if $n \geqslant k$.[15] Remarkably, in the matrix case we have $\dim \mathscr{P}_{m,n}^{(k)} = 4(k+1)^2$ independently of m and n, provided $m \geqslant k \geqslant n - m$,[7] as a consequence of the following Lemma:

Lemma 3.3: The following monomials form a basis of the vector space of polynomials in the operators (6) of differential order at most k:

$$\{X(T^{\pm})^r(T^0)^{s-r}\}_{r=0}^{s}, \quad \{Q_\alpha(T^+)^s\}_{\alpha=1}^{\Delta}, \quad s=0,\ldots,k, \tag{11}$$

where $X=1$, J (or $\tilde{J}$, if $\Delta=0$), Q_0, along with, if $k\geqslant\Delta$:

$$\{\overline{Q_0}(T^{\pm})^r(T^0)^{s-r}\}_{r=0}^{s}, \quad \{\overline{Q_\alpha}(T^-)^s\}_{\alpha=1}^{\Delta}, \quad s=0,\ldots,k-\Delta. \tag{12}$$

Proof: Linear independence of the monomials is straightforward from the definition of the operators. Completeness is a consequence of the following facts. In the first place, every J^s is a linear combination of $\{1, J\}$ (and analogously for $\tilde{J}$). Secondly, JQ_α is proportional to Q_α, and $Q_\alpha Q_\beta=0$ (and the same for the $\overline{Q}$'s). Third, any product $Q_\alpha\overline{Q_\beta}$ is a diagonal PVSP operator, and thus expressible through the T's and J (or $\tilde{J}$). Finally, the formulas ($\alpha\geqslant 1$)

$$Q_\alpha T^0=Q_{\alpha-1}T^+ + \frac{n-\Delta}{2}\,Q_\alpha, \quad Q_\alpha T^-=Q_{\alpha-1}T^0+\frac{n-\Delta}{2}\,Q_{\alpha-1},$$

$$\overline{Q_\alpha}T^0=\overline{Q_{\alpha-1}}T^- + \left(\frac{n}{2}+1\right)\overline{Q_\alpha}, \quad \overline{Q_\alpha}T^+=\overline{Q_{\alpha-1}}T^0+\left(\frac{n}{2}+1\right)\overline{Q_{\alpha-1}},$$

allow us to remove every T^0 and T^- (respectively T^+) from the monomials with Q_α (respectively $\overline{Q_\alpha}$), $\alpha=1,\ldots,\Delta$. Q.E.D.

Corollary 3.4: Let $n\geqslant m\geqslant k$, and $\Delta=n-m$. We then have:

$$\dim \mathscr{P}_{m,n}^{(k)}=\begin{cases}4(k+1)^2, & k\geqslant\Delta\\ (k+1)(3k+\Delta+3), & \Delta>k\end{cases}.$$

If $m<k$, $\dim \mathscr{P}_{m,n}^{(k)}$ is no longer finite, as arbitrary differential operators are involved in this case.

B. Case *N*>2

We now examine briefly some aspects of the case $N>2$. Let $n_1\leqslant\cdots\leqslant n_N$ be non-negative integers, and let $\Delta_{ij}=n_j-n_i$, where $j>i$. Consider the following set of $N\times N$ matrix differential operators:[8]

$$T^\epsilon=\operatorname{diag}(J_{n_1}^\epsilon,\ldots,J_{n_N}^\epsilon), \quad \epsilon=+,0,-,$$

$$P_i=\operatorname{diag}(0,\ldots,0,1,0,\ldots,0),$$

$$Q_\alpha(i,j)=z^\alpha\lambda_{ij}, \quad i>j, \quad \alpha=0,\ldots,\Delta_{ji},$$

$$\overline{Q_\alpha}(i,j)=\overline{q_\alpha}(n_j,\Delta_{ij})\lambda_{ij}, \quad j>i, \quad \alpha=0,\ldots,\Delta_{ij}, \tag{13}$$

where $(\lambda_{ij})_{pq}=\delta_{ip}\delta_{jq}$. It can be readily verified that the operators in (13) preserve $\mathscr{P}_{n_1,\ldots,n_N}$. A complication arising when $N>2$ is to define a suitable composition law between the latter operators. In the approach of Brihaye *et al.*,[8] this composition law is defined to be an anticommutator if both operators are off-diagonal and a commutator otherwise, but the algebra thus obtained is no longer a Lie superalgebra, since the anticommutator of two off-diagonal operators is not always a diagonal one. This reflects the fact that the $\mathbb{Z}_2$-grading we introduced for $N=2$ (i.e. classifying the operators in diagonal and off-diagonal) does not define an associative superalgebra in $\mathscr{D}_{N\times N}$ when $N>2$, for the usual product (composition) of two off-diagonal matrix differential operators is not necessarily diagonal. A possible generalization of this $\mathbb{Z}_2$-grading endowing $\mathscr{D}_{N\times N}$ with an associative superalgebra structure can be defined as follows. An operator $T=a\lambda_{ij}$, where $a\in\mathscr{D}$, is said to be even (respectively odd) if $i+j$ is even (respectively odd). Hence, any diagonal operator is even. We can likewise use this grading and the generalized Lie product (8) to construct a Lie superalgebra structure in $\mathscr{D}_{N\times N}$. It is not clear, however, whether this construction is really useful, and so it will not be further discussed.

As remarked by Brihaye *et al.*,[8] Theorem 3.2 can be easily generalized to arbitrary N, the operators (13) playing the same role as those in (6) for $N=2$. Moreover, it is not difficult to show that $\dim \mathscr{P}^{(k)}_{n_1,\ldots,n_N}$ is still independent of the n_i's if they are large enough and their differences are small enough. More precisely, if $n_1\geqslant k$, we have:

$$\dim \mathscr{P}^{(k)}_{n_1,\ldots,n_N}=\begin{cases} N^2(k+1)^2, & k>\Delta_{1N} \\ \dfrac{N(N+1)}{2}\,(k+1)^2+(k+1)\displaystyle\sum_{i<j}\theta_{ij}, & \Delta_{1N}>k, \end{cases}$$

where $\theta_{ij}=\Delta_{ij}$ if $\Delta_{ij}>k$, and $\theta_{ij}=k+1$ if $\Delta_{ij}\leqslant k$. If $k>n_1$, arbitrary differential operators are involved and thus $\mathscr{P}^{(k)}_{n_1,\ldots,n_N}$ is infinite-dimensional.

Although no attempt will be made here to give a formal definition of a QES algebra of matrix differential operators, it is clear that Definition 2.1 of a QES differential operator is too restrictive in the matrix case. Indeed, the results of this section suggest that in the matrix case one should include at least Lie superalgebras of differential operators—not necessarily finite-dimensional nor spanned by first order operators—preserving a finite-dimensional module of functions among the class of matrix QES algebras. In any case, it is intuitively clear that PVSP operators are just a particular class of QES operators.

IV. SPIN 1/2 SCHRÖDINGER OPERATORS

From now on we will deal only with 2×2 matrix second order differential operators ($N=k=2$ in the notation of the previous sections). We start by formally defining the class of matrix Schrödinger operators:

Definition 4.1: A Schrödinger-like operator is a second order differential operator of the form $H=-\partial_x^2+V(x)$, where V is an arbitrary 2×2 (complex) matrix. A Schrödinger operator (or Hamiltonian) is a Hermitian Schrödinger-like operator, i.e. the matrix V is of the form

$$V=\begin{pmatrix} v_1(x) & v^*(x) \\ v(x) & v_2(x)\end{pmatrix}, \tag{14}$$

where v_1 and v_2 are real-valued functions and v is an arbitrary complex-valued function.

The notion of equivalence we shall use for matrix differential operators is the same as in the scalar case (see Definition 2.2), where now the gauge factor $U(z)$ is an invertible complex 2×2 matrix. We will be interested in constructing one-dimensional Schrödinger operators H equivalent to a second order differential operator T in $\mathscr{P}^{(2)}_{m,n}$, with $\Delta=n-m\geqslant0$. This equivalence can then be used to construct $m+n+2$ eigenfunctions of H from the corresponding ones of T obtained by diagonalization of the $(m+n+2)\times(m+n+2)$ Hermitian matrix representing T in $\mathscr{P}_{m,n}$. We will assume that $m\geqslant2$, and thus T is a polynomial in the operators (6), according to Theorem 3.2

Let $T:\mathscr{P}_{m,n}\to\mathscr{P}_{m,n}$ ($n\geqslant m\geqslant2$) be a second-order PVSP operator. From Theorem 3.2 we have

$$-T=A_2(z)\partial_z^2+A_1(z)\partial_z+A_0(z), \tag{15}$$

where the A_i's are 2×2 matrices with polynomial entries (in this section, capital letters will be reserved for matrices). Assume that $T(z)$ is equivalent to a Schrödinger operator $H(x)$ under a gauge transformation $U(z)$ and a local change of variable by a real-valued function $x=\varphi(z)$, i.e.

$$-T(z)=-U^{-1}(z)H(x)U(z)=\partial_x^2+2A\partial_x-B, \tag{16}$$

with

$$A(x)=\widetilde{U}^{-1}\widetilde{U}_x, \quad B(x)=\widetilde{U}^{-1}V\widetilde{U}-A^2-A_x, \quad \widetilde{U}(x)=U(\varphi^{-1}(x)).$$

Here and in what follows, a subscripted x denotes derivation with respect to x, while derivatives with respect to z will be denoted with a prime $'$. Expressing $T(z)$ in the variable x, we obtain the operator $\widetilde{T}(x)$ given by

$$-\widetilde{T}(x)=[A_2\varphi'^2\partial_x^2+(A_1\varphi'+A_2\varphi'')\partial_x+A_0]_{\varphi^{-1}(x)}, \tag{17}$$

and comparing with (16) we conclude that A_2 must be a multiple of the identity. It then follows that the only monomials in Lemma 3.3 which contribute to A_2 are $\{(T^{\pm})^r(T^0)^{2-r}\}_{r=0}^2$, and taking into account the explicit form of the T^ϵ's (see (6)), we conclude that A_2 is a 4-th degree polynomial p_4 times the identity matrix. (Unless otherwise stated, we will denote by $p_n(z)$ an arbitrary polynomial in z of degree at most n with complex coefficients.) We also deduce that $\varphi(z)$ satisfies the equation $p_4\varphi'^2=1$, or

$$x=\int^z \frac{1}{\sqrt{p_4(s)}}\,ds, \tag{18}$$

and thus the coefficients of p_4 must be real. Identifying the corresponding remaining terms in (16) and (17), we then get

$$A(x)=\frac{1}{2\sqrt{p_4}}\left(A_1-\frac{1}{2}\,p_4'\right)\bigg|_{\varphi^{-1}(x)}, \quad B(x)=-A_0|_{\varphi^{-1}(x)}. \tag{19}$$

Thus, we have shown:

Theorem 4.2: *Let T be a PVSP operator in $\mathscr{P}_{m,n}^{(2)}$, with $n\geqslant m\geqslant 2$. Then T is equivalent to a Schrödinger-like operator if and only if it is of the form*

$$-T=p_4\partial_z^2+A_1\partial_z+A_0. \tag{20}$$

The operator T is equivalent to a Schrödinger operator $-\partial_x^2+V(x)$ if and only if (20) holds, and in addition there is an invertible matrix $\widetilde{U}$ satisfying the differential equation

$$\widetilde{U}_x=\widetilde{U}A \tag{21}$$

and such that

$$V=\widetilde{U}\widetilde{W}\widetilde{U}^{-1}, \quad \textit{where} \quad \widetilde{W}=B+A^2+A_x, \tag{22}$$

is Hermitian (with x, A, and B given by (18) and (19)).

The eigenfunctions of the Hamiltonian H are of the form $\psi(x)=\widetilde{U}\widetilde{\Psi}$ with $\widetilde{\Psi}(x)=\Psi(\varphi^{-1}(x))$, where $\Psi(z)$ is an eigenfunction of T and ψ must satisfy suitable boundary conditions to qualify as a physical wave function.

We note that once an invertible solution $\widetilde{U}$ of equation (21) has been found, any other invertible solution is of the form $U_0\widetilde{U}$ for some U_0 in GL(2,$\mathbb{C}$). In fact, multiplying $\widetilde{U}$ by such U_0 is equivalent to performing a further constant scale transformation by U_0. In the scalar case this additional freedom is absent, for differential operators are unaffected by scale transformations by constant functions. Note also that a scale transformation by an arbitrary constant matrix U_0 will not map every Hamiltonian into another Hamiltonian, unless U_0 belongs to $\mathbb{R}^+\times U(2)$.

A matrix differential operator will be called *uncoupled* if it is either upper or lower triangular. Since A_2 in (15) must be a multiple of the identity matrix, it follows that a PVSP operator of the form (20) will be automatically uncoupled whenever $\Delta>1$, as none of the monomials (12) can then be present in T. Moreover, the following result shows that any Hamiltonian we may obtain from T when $\Delta>1$ will be essentially diagonal:

Proposition 4.3: Every Hamiltonian H obtained from an uncoupled PVSP operator T of the form (20) is diagonal, up to equivalence.

Proof: If T is uncoupled, the integration of equation (21) is straightforward. Multiplying $\widetilde{U}$ from the left by an appropriate U_0 in $\mathbb{R}^+\times U(2)$, we construct a new gauge factor uncoupled in the same way as T. Using this new gauge factor, we obtain a Hamiltonian $\hat{H}$ given by

$$\hat{H}=U_0HU_0^{-1}=U_0\widetilde{U}\widetilde{T}(U_0\widetilde{U})^{-1},$$

which is both diagonal and equivalent to the initial one. Q.E.D.

Consequently, the only cases we need to consider are $\Delta=0,1$.

There are two main difficulties associated with the method just outlined for constructing QES spin 1/2 Hamiltonians. In the first place, one needs to invert the elliptic integral (18) in order to compute z as a function of x, which is no easy task. Secondly, the differential equation (21) cannot in general be solved in closed form, thus preventing us from verifying the Hermiticity of V. The former complication can be overcome, as we shall see in the next section. The latter is more difficult to handle, although imposing further constraints on the initial PVSP operator will contribute to simplify the problem, as shown in Section VI.

We shall finish this section with a few remarks on the physical significance of matrix Schrödinger operators. First of all, one-dimensional 2×2 matrix Schrödinger operators can be obtained by separation of variables from the three-dimensional Pauli Hamiltonian describing a spin 1/2 charged particle in non-relativistic quantum mechanics. Consider, indeed, the Pauli Hamiltonian

$$H_{\mathrm{Pauli}}=(i\nabla+e\mathbf{A})^2+e\phi-e\boldsymbol{\sigma}\cdot\mathbf{B},$$

where ϕ and $\mathbf{A}=(A^1,A^2,A^3)$ are respectively the scalar and vector potential of the external electromagnetic field, $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$ is the magnetic field, $\boldsymbol{\sigma}=(\sigma^1,\sigma^2,\sigma^3)$ are the Pauli matrices, e is the electric charge, and physical units have been chosen so that $\hbar=c=2m=1$. If, for example, the vector and scalar potentials depend only on the x coordinate (and we take, without loss of generality, $A^1=0$) then H_{Pauli} obviously commutes with the y and z components of the linear momentum. The eigenfunctions of H_{Pauli} can then be sought in the form

$$e^{i(p_y y+p_z z)}\psi(x);\neg\ \ p_y,p_z\in\mathbb{R},$$

where the two-component spinor $\psi(x)$ is an eigenfunction of the one-dimensional matrix Schrödinger operator with potential (14) given by

$$v_j(x)=e\phi+(eA^2-p_y)^2+(eA^3-p_z)^2+(-1)^j e\,\frac{dA^2}{dx},\neg\ \ j=1,2,$$

$$v(x)=ie\,\frac{dA^3}{dx}.$$

More surprisingly, one-dimensional 2×2 matrix operators are also directly related to Dirac's relativistic equation for a spin 1/2 charged particle in an external electromagnetic field. To see this, let us write the latter equation as

$$(i\partial\!\!\!/ - eA\!\!\!/ - m)\Psi(x)=0, \tag{23}$$

where $a\!\!\!/ = \gamma^\mu a_\mu$, the γ^μ's are 4×4 matrices satisfying

$$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu},$$

the metric tensor $(g_{\mu\nu})=\mathrm{diag}(1,-1,-1,-1)$ is used to raise and lower indices, $\partial_\mu=\partial/\partial x^\mu$, $x^0=t$, $A^0=\phi$ and m is the particle's mass. Multiplying Dirac's equation by the operator $i\partial\!\!\!/ - eA\!\!\!/ + m$ we easily arrive at the second-order equation

$$\left[(i\partial-eA)^2-m^2-\frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}\right]\Psi=0, \tag{24}$$

where $\sigma^{\mu\nu}=i/2[\gamma^\mu,\gamma^\nu]$ and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the electromagnetic field strength tensor. In the chiral representation of the gamma matrices,[19] (24) decouples into two independent equations for the upper and lower components $\Psi_\pm(x)$ of $\Psi(x)$, namely (cf. Ref. 20)

$$[(i\partial-eA)^2-m^2+e\boldsymbol{\sigma}\cdot(\mathbf{B}\mp i\mathbf{E})]\Psi_\pm=0.$$

If the electromagnetic four-potential A^μ is time-independent, and we look for solutions of Dirac's equation with well-defined energy E, i.e. we set $\Psi_\pm=e^{-iEt}\psi_\pm(x,y,z)$, we obtain the following equation for $\psi_\pm$:

$$[(i\nabla+e\mathbf{A})^2-(E-eA^0)^2+m^2-e\boldsymbol{\sigma}\cdot(\mathbf{B}\mp i\mathbf{E})]\psi_\pm=0, \tag{25}$$

which has the same structure as Pauli's non-relativistic equation. Just as was the case with Pauli's equation, separation of variables in (25) often leads to the eigenvalue problem for a one-dimensional matrix Schrödinger operator. For instance, suppose that $A^0=0$ and that $\mathbf{A}$ is of the form

$$\mathbf{A}=A_\varphi(\rho)\mathbf{e}_\varphi+A_z(\rho)\mathbf{e}_z$$

in cylindrical coordinates (ρ,φ,z). The left-hand side of (25) then commutes with the z components of the linear momentum $(-i\partial_z)$ and the total angular momentum $(-i\partial_\varphi+1/2\sigma^3)$, which allows us to look for solutions of (25) of the form

$$\psi(\rho,\varphi,z)=e^{ip_z z}\begin{pmatrix}R_1(\rho)e^{i(j_z-1/2)\varphi}\\ R_2(\rho)e^{i(j_z+1/2)\varphi}\end{pmatrix}, \quad p_z\in\mathbb{R}, \quad j_z\in\mathbb{N}+\frac{1}{2}. \tag{26}$$

Here we have dropped the subscript $\pm$, since ψ_+ and ψ_- satisfy the same equation. Substituting (26) into (25) we obtain that the two-component spinor $(f_1(x)\;\; f_2(x))^t=x^{1/2}(R_1(x)\;\; R_2(x))^t$ is an eigenfunction of the one-dimensional 2×2 matrix Schrödinger operator on $(0,\infty)$ with potential

$$V(x)=(p_z-eA_z(x))^2+e^2A_\varphi^2(x)+e\,\frac{dA_z}{dx}(x)\sigma^2-e\,\frac{A_\varphi(x)}{x}(2j_z-\sigma^3)+\frac{j_z}{x^2}(j_z-\sigma^3),$$

with eigenvalue E^2-m^2 and boundary condition $f_1(0)=f_2(0)=0$. See also Ref. 21 for a different approach.

V. GL(2) ACTION AND CANONICAL FORMS

In this section we will study how the GL(2) action on the projective line $\mathbb{RP}^1$ induces an automorphism in the superalgebras $\mathfrak{s}_\Delta$ generated by the operators (6). This will allow us to reduce the polynomial $p_4(z)$ to some simple canonical forms, facilitating the evaluation of the integral (18). These ideas were first applied in the context of QES systems to analyze the normalizability of the wave functions of scalar QES Hamiltonians.[5] We introduce some definitions and results in the scalar case, and then show how to extend these concepts to the matrix superalgebras $\mathfrak{s}_\Delta$.

The action of $\mathrm{GL}(2)=\mathrm{GL}(2,\mathbb{R})$ on $\mathbb{RP}^1$ via linear fractional or Möbius transformations,

$$z \mapsto w=\frac{\alpha z+\beta}{\gamma z+\delta}, \quad C=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad |C|=\alpha\delta-\beta\gamma\neq 0, \tag{27}$$

induces an action on $\mathscr{P}_n$, mapping a polynomial $p(w)$ to the polynomial $\overline{p(z)}$ given by

$$\overline{p(z)}=(\gamma z+\delta)^n p\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right). \tag{28}$$

This defines an irreducible multiplier representation of GL(2) in $\mathscr{P}_n$,[22] which will be denoted by $\rho_{n,0}$. Since the infinitesimal generators of this multiplier representation coincide with the generators of $\mathfrak{g}_n$, cf. (4), it follows that the representation $\rho_{n,0}$ induces an automorphism of the Lie algebra $\hat{\mathfrak{g}}_n$ spanned by the J_n^ϵ's in (5).[5] Performing the explicit scale transformation and change of variable,

$$J_n^\epsilon(w)\mapsto(\gamma z+\delta)^n J_n^\epsilon\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)(\gamma z+\delta)^{-n}, \tag{29}$$

we obtain:

$$\begin{pmatrix} J_n^+ \\ J_n^0 \\ J^- \end{pmatrix} \mapsto \frac{1}{|C|}\begin{pmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ \alpha\gamma & \alpha\delta+\beta\gamma & \beta\delta \\ \gamma^2 & 2\gamma\delta & \delta^2 \end{pmatrix}\begin{pmatrix} J_n^+ \\ J_n^0 \\ J^- \end{pmatrix}.$$

Therefore, the J_n^ϵ's transform according to the representation $\rho_{2,-1}=\rho_{2,0}\otimes \det^{-1}$ of GL(2), where $\det^{-1}$ is the reciprocal of the representation $\det: C\mapsto|C|$. It is convenient at this stage to introduce a larger class of representations of GL(2):

Definition 5.1: Let $n\geqslant 0$, i be integers. The (irreducible) multiplier representation $\rho_{n,i}$ of GL(2) *on $\mathscr{P}_n$ is defined by*

$$p(w)\mapsto\overline{p(z)}=(\alpha\delta-\beta\gamma)^i(\gamma z+\delta)^n p\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right).$$

We note the isomorphism between $\rho_{n,i}$ and $\rho_{n,0}\otimes\det^i$. As shown in Ref. 5, a second-degree polynomial in the J_n^ϵ's (in fact, any operator in $\mathscr{P}_n^{(2)}$ if $n\geqslant 2$, or any QES operator on the line modulo equivalence, according to Theorems 2.3 and 2.4) may be written as

$$p_2(J_n^\epsilon)=p\,\partial_z^2+\left(q-\frac{n-1}{2}\,p'\right)\partial_z+r-\frac{n}{2}\,q'+\frac{n(n-1)}{12}\,p'', \tag{30}$$

where p, q, and r are polynomials in z of degrees 4, 2, and 0 respectively. The transformation of $p_2(J_n^\epsilon)$ under the action (29) is easily described in terms of the triple (p,q,r).[5]

Lemma 5.2: Let p_2 be a second-degree polynomial in the operators $J_n^\epsilon(w)$, determined by the triple $(p(w),q(w),r)$. Then, the transformed polynomial $\overline{p_2}$ under the GL(2) *action (29) is determined by the triple $(\overline{p(z)},\overline{q(z)},\overline{r})$ given by*

$$\overline{p(z)}=\frac{(\gamma z+\delta)^4}{|C|^2}\,p\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right),\neg\quad \overline{q(z)}=\frac{(\gamma z+\delta)^2}{|C|}\,q\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right),\neg\quad \overline{r}=r.$$

Therefore, a second-degree polynomial p_2 in the J_n^ϵ's transforms according to the direct sum representation $\rho_{4,-2}\oplus\rho_{2,-1}\oplus\rho_{0,0}$ under the GL(2) action (29). One can choose a particularly simple representative of the GL(2) orbit generated by p_2 by placing the polynomial p (assumed to be real) in its associated triple (p,q,r) in canonical form.[23]

Theorem 5.3: *Every non-zero quartic real polynomial $p(z)$ transforming under the representation $\rho_{4,-2}$ of* GL(2) *is equivalent to one of the following canonical forms:*

$$\begin{array}{llll} (1) & \nu(z^4+\tau z^2+1),\ \ \tau\neq\pm 2, & (5) & \nu(z^2-1),\\ (2) & \nu(z^4+\tau z^2-1),\neg & (6) & \nu z^2,\\ (3) & \nu(z^2+1)^2,\neg & (7) & z,\\ (4) & \nu(z^2+1),\neg & (8) & 1, \end{array} \tag{31}$$

where $\nu\neq 0$ and τ are real numbers.

We now generalize these results to the matrix case. The induced action of GL(2) on $\mathscr{P}_{n-\Delta,n}$ analogous to (28) is:

$$\begin{pmatrix} p^1(w)\\ p^2(w)\end{pmatrix}\mapsto\begin{pmatrix}\overline{p^1}(z)\\ \overline{p^2}(z)\end{pmatrix}=\hat{U}(z)\begin{pmatrix}p^1(w(z))\\ p^2(w(z))\end{pmatrix},$$

where

$$\hat{U}(z)=\operatorname{diag}((\gamma z+\delta)^{n-\Delta},(\gamma z+\delta)^n),$$

and $w(z)$ is given by (27). This representation of GL(2) in $\mathscr{P}_{n-\Delta,n}$ is obviously isomorphic to $\rho_{n-\Delta,0}\oplus\rho_{n,0}$. The following Lemma describes the induced action on $\mathfrak{s}_\Delta$:

Lemma 5.4: The action of GL(2) *on $\mathfrak{s}_\Delta$ given by*

$$X(w)\mapsto\overline{X(z)}=\hat{U}(z)X\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)\hat{U}^{-1}(z),\neg\quad X\in\mathfrak{s}_\Delta,\neg \tag{32}$$

defines a Lie superalgebra automorphism. The generators of $\mathfrak{s}_\Delta$, cf. (6), transform according to the following irreducible representations:

$$\{T^\epsilon\}\to\rho_{2,-1},\neg\quad\{J\}\to\rho_{0,0},$$

$$\{Q_\alpha\}\to\rho_{\Delta,0},\neg\quad\{\overline{Q_\alpha}\}\to\rho_{\Delta,-\Delta}.$$

A straightforward generalization of equation (30) and of Lemma 5.2 to a second-degree polynomial in the T^ϵ's shows that the (real) polynomial p_4 in (20) transforms according to the representation $\rho_{4,-2}$ under the GL(2) action (32). We will henceforth assume that p_4 is one of the canonical forms given in Theorem 5.3. The integral (18) and the inverse $z=\varphi^{-1}(x)$ can then be easily computed for each of these canonical forms.[9]

Before finishing this section, let us point out that equations (21) and (22) adopt a simpler form in the variable z, because in that case only rational functions appear. Explicitly, the equation for the gauge factor reads:

$$U' = U\hat{A}, \quad \text{with} \quad \hat{A}(z) = \frac{A|_{\varphi(z)}}{\sqrt{p_4}} = \frac{1}{2p_4}\left(A_1 - \frac{1}{2}\,p_4'\right), \tag{33}$$

while $\widetilde{U}$ and $\widetilde{W}$ in equation (22) are substituted by U and W, where

$$W = -A_0 + p_4\hat{A}' + p_4\hat{A}^2 + \tfrac{1}{2}\,p_4'\hat{A}. \tag{34}$$

We now use Theorem 3.2, Lemma 3.3 and the explicit form of the operators (6) to compute $\hat{A}$ and A_0 for the most general operator in $\mathscr{P}^{(2)}_{n-\Delta,n}$ of the form (20), in the cases $\Delta=0$, 1. We denote by p_n^α the polynomial $\Sigma_{i=0}^n \alpha_i z^i$, where α_i are arbitrary complex numbers. If the polynomial p_4 is *not* one of the first three canonical forms, we obtain:

Case $\Delta=0$:

$$p_4\hat{A} = \begin{pmatrix} p_2^a & p_2^b \\ p_2^c & p_2^d \end{pmatrix}, \quad A_0 = \begin{pmatrix} \hat{a}_0 - 2na_2z & \hat{b}_0 - 2nb_2z \\ \hat{c}_0 - 2nc_2z & \hat{d}_0 - 2nd_2z \end{pmatrix}. \tag{35}$$

Case $\Delta=1$:

$$p_4\hat{A} = \begin{pmatrix} p_2^a & p_1^b \\ p_3^c & p_2^d \end{pmatrix}, \quad A_0 = \begin{pmatrix} \hat{a}_0 - 2(n-1)a_2z & -2nb_1 \\ \hat{c}_0 + \hat{c}_1 z - 2(n-1)c_3z^2 & \hat{d}_0 - 2nd_2z \end{pmatrix}, \tag{36}$$

where $\hat{a}_0$, $\hat{b}_0$, $\hat{c}_0$, $\hat{c}_1$, and $\hat{d}_0$ are arbitrary complex numbers. If p_4 is one of the first three canonical forms, the following extra terms are present in $p_4\hat{A}$ and A_0:

$$p_4\hat{A} \to -\mathrm{diag}((n-\Delta)\nu z^3, n\nu z^3), \tag{37}$$

$$A_0 \to +\mathrm{diag}((n-\Delta)(n-\Delta-1)\nu z^2, n(n-1)\nu z^2). \tag{38}$$

VI. THE GAUGE FACTOR

In this section we will deal with the differential equation (33) for the gauge factor $U(z)$. As remarked in Section IV, this equation cannot be solved in closed form for every $\hat{A}$. Alternatively, (33) can be written as two (uncoupled) identical first-order linear systems; unfortunately, the associated scalar second-order ordinary differential equation is not any of the standard equations of Mathematical Physics.

However, if we restrict ourselves to matrices $\hat{A}$ satisfying the equation

$$\left[\hat{A}(z), \int_{z_0}^{z} \hat{A}(s)\,ds\right] = 0, \tag{39}$$

for some $z_0 \in \mathbb{R}$, we can readily integrate the gauge equation (33). Recall that this condition on $\hat{A}$ was indeed verified by the QES Schrödinger operator found by Shifman and Turbiner.[6] If (39) is satisfied, we shall say that we are in the *commuting case*. In this case, the general solution of the gauge equation is given by:

$$U(z) = U_0 \exp \int_{z_0}^{z} \hat{A}(s)\,ds, \tag{40}$$

TABLE I. Matrix $\check{A}(z)$ for the canonical forms $1-8(\alpha,\beta,\gamma\in\mathbb{C})$.

	p_4 in canonical forms 1-3	p_4 in canonical forms 4-8
$\Delta=0$	$p_2\begin{pmatrix}\alpha & \beta\\ \gamma & -\alpha\end{pmatrix}-n\nu z^3$	$p_2\begin{pmatrix}\alpha & \beta\\ \gamma & -\alpha\end{pmatrix}$
$\Delta=1$	$(p_2-\nu z^3)\begin{pmatrix}n-1 & 0\\ \gamma & n\end{pmatrix}$	$p_1\begin{pmatrix}\alpha & \beta\\ \gamma & -\alpha\end{pmatrix},\quad p_2\begin{pmatrix}\alpha & 0\\ \gamma & -\alpha\end{pmatrix},\quad p_3\begin{pmatrix}0 & 0\\ \gamma & 0\end{pmatrix}$

where U_0 is in $\mathrm{GL}(2,\mathbb{C})$. We will make use of the following elementary Lemma to describe the most general form of $\hat{A}$ in the commuting case:

Lemma 6.1: Let $M(z)$ be a 2×2 matrix satisfying equation (39). Then M is of the form:

$$M=f(z)M_0+g(z), \tag{41}$$

where f and g are scalar functions, and M_0 is a 2×2 constant matrix.

With this Lemma in mind, looking at the expressions for $\hat{A}(\Delta=0,1)$ that we obtained in Section V, cf. (35)–(37), we find that the most general $\hat{A}$ satisfying (39) is of the form

$$p_4\hat{A}=\hat{p}_2(z)+\check{A}(z),$$

where, as usual, $\hat{p}_2$ denotes a second-degree polynomial in z with complex coefficients, and the matrix $\check{A}$ is given in Table I.

Note that if $\Delta=1$ and p_4 is one of the first three canonical forms, every Hamiltonian we can possibly obtain will be diagonal, modulo equivalence.

Finally, let us remark that in the *non*-commuting case (that is, when $[\hat{A},\int^z\hat{A}]\neq 0$), we may still be able to integrate (33) explicitly by imposing other constraints on $\hat{A}$, as e.g. assuming it is uncoupled. Alternatively, if p_4 is not any of the first three canonical forms, and we assume that the columns (or rows) of $\hat{A}$ are proportional to each other (the ratio of the respective entries being a constant), we can also reduce (33) to quadratures. Unfortunately, we have not been able to find any interesting examples of QES Hamiltonians in the non-commuting case.

VII. EXAMPLES

In this final section we exhibit some new examples of spin 1/2 Schrödinger operators equivalent to a PVSP operator of the form (20). In the previous section we have seen how, by restricting ourselves to the commuting case, we were able to integrate equation (33) explicitly. This is not, however, the end of the problem, for we must still check that the matrix

$$V=UWU^{-1}\big|_{\varphi^{-1}(x)}, \tag{42}$$

with W given by (34), is Hermitian. Moreover, one has to make sure that the algebraic eigenfunctions of H (of the form $U\Psi\big|_{z=\varphi(x)}$, with $\Psi(z)\in\mathscr{P}_{n-\Delta,n}$) satisfy appropriate boundary conditions. We shall restrict ourselves in this paper to QES Hamiltonians with square integrable eigenfunctions, corresponding to bound states of the system. We will also require the expected value of the potential V to be bounded from below, i.e.

$$\langle\psi,V\psi\rangle\geqslant c\|\psi\|^2,\quad \text{with}\quad \psi\in L^2(\mathbb{R})\oplus L^2(\mathbb{R}), \tag{43}$$

for some $c\in\mathbb{R}$. Again, if we start with the most general PVSP operator of the form (20) and we limit ourselves to the commuting case, even if no conceptual difficulty is involved, the algebraic constraints imposed by all these conditions turn out to be very complicated to solve in full generality. This situation is analogous to what we find when trying to construct scalar QES Schrödinger operators in more than one spatial dimension.[4,9] In fact, in the commuting case with $\Delta=0$ we were not able to find any non-trivial example satisfying all the required physical conditions. We conjecture that in this case all QES potentials are either non-normalizable, not bounded from below, or diagonalizable by a constant gauge transformation.

We now present some relevant examples of QES spin 1/2 Hamiltonians for the commuting case with $\Delta=1$.

Example 1: Let T be the four-parameter PVSP operator given by

$$-T=(T^0)^2+2a_2T^+ +2(n+1)T^0-2JT^0+2b_1\overline{Q_0}+2b_0\overline{Q_1}-2b_1Q_0T^0$$
$$-2b_0Q_0T^- -(4a_2b_0+(3n+1)b_1)Q_0-4a_2b_1Q_1-\left(2\hat{d}_0+n+\frac{1}{2}\right)J,$$

with all the parameters real. Since $p_4=z^2$ we are in case 6 of Theorem 5.3. Solving equation (18) for z, we obtain $z=e^x$. The gauge factor reads:

$$U(z)=\sqrt{z}e^{a_2z}\begin{pmatrix}\cos u & \sin u\\ -\sin u & \cos u\end{pmatrix}, \quad \text{where} \quad u=-\frac{b_0}{z}+b_1 \log z.$$

Using (42), we obtain a potential $V(x)$ with entries given by (see (14)):

$$v_j=-b_0^2e^{-2x}-2b_0b_1e^{-x}+(2n+1)a_2e^x+a_2^2e^{2x}+(-1)^j(\alpha(x)\cos 2\tilde{u}-\beta(x)\sin 2\tilde{u}), \quad j=1,2$$

$$v=\alpha(x)\sin 2\tilde{u}+\beta(x)\cos 2\tilde{u},$$

where $\tilde{u}=b_1x-b_0e^{-x}$, and

$$\alpha(x)=-\frac{\hat{d}_0}{2}+a_2e^x, \quad \beta(x)=(2n+1)b_1+2a_2(b_0+b_1e^x).$$

We have ignored a constant multiple of the identity in V, which is equivalent to fixing a new origin in the energy scale. It may be easily verified that the expected value of the potential is bounded from below, that is, equation (43) holds, if and only if $b_0=0$. (Note, however, that even in this case the amplitude of the oscillations of $v(x)$ tends to infinity as $x\to+\infty$.) Finally, the condition $a_2<0$ is necessary and sufficient to ensure the square integrability of the eigenfunctions $\psi(x)$.

Example 2: As our second example, we consider:

$$-T=T^-T^0+2a_1T^0+\left(2a_0+n-\tfrac{1}{2}\right)T^- -JT^- +2b_1\overline{Q_0}-2b_1Q_0T^0$$
$$-b_1(4a_0+3n+1)Q_0-4a_1b_1Q_1+2(2\hat{a}_0-a_1)J,$$

where all the coefficients are real, and $b_1\neq 0$. Since $p_4=z$ (case 7), we have $z=x^2/4$. The gauge factor is chosen as follows:

$$U(z)=z^{a_0}e^{a_1z}\begin{pmatrix}\cos b_1z & \sin b_1z\\ -\sin b_1z & \cos b_1z\end{pmatrix}.$$

The entries of the potential $V(x)$ are given by

$$v_j = \frac{2a_0(2a_0-1)}{x^2} + \frac{1}{4}(a_1^2 - b_1^2)x^2 + (-1)^j\left(\hat{a}_0 \cos\frac{b_1 x^2}{2} - \alpha(x)\sin\frac{b_1 x^2}{2}\right),$$

$$v = \hat{a}_0 \sin\frac{b_1 x^2}{2} + \alpha(x)\cos\frac{b_1 x^2}{2},$$

with $j=1$, 2, and $\alpha(x)$ is defined by

$$\alpha(x) = \frac{b_1}{2}(4a_0 + 4n + 1 + a_1 x^2).$$

We first note that the potential is singular at the origin unless $a_0 = 0,1/2$. Let us introduce the parameter $\lambda = 2a_0 - 1$, in terms of which the coefficient of x^{-2} in v_j is $\lambda(\lambda+1)$. If λ is a non-negative integer l, we may regard

$$(-\partial_x^2 + V(x) - E)\psi(x) = 0, \quad 0 < x < \infty, \tag{44}$$

as the radial equation obtained after separating variables in the three-dimensional Schrödinger equation with a spherically symmetric Hamiltonian given by

$$\hat{H} = -\Delta + U(r), \quad \text{with} \quad U(x) = V(x) - \frac{l(l+1)}{x^2},$$

where Δ denotes the usual flat Laplacian. Given a non-negative integer l and a spherical harmonic $Y_{lm}(\theta,\phi)$, if ψ is an eigenfunction for the equation (44) satisfying

$$\lim_{x\to 0^+} \psi(x) = 0, \tag{45}$$

then

$$\hat{\Psi}(r,\theta,\phi) = \frac{\psi(r)}{r} Y_{lm}(\theta,\phi)$$

will be an eigenfunction for $\hat{H}$ with angular momentum l. If λ is not a non-negative integer, we shall consider (44) as the radial equation for the singular potential $U(r) = V(r)$ at zero angular momentum. The potential $U(r)$ is physically meaningful, in the sense that the Hamiltonian $\hat{H}$ admits self-adjoint extensions and its spectrum is bounded from below, whenever $\lambda \neq -1/2$.[24,5] The boundary condition (45) must be satisfied in the singular case for *all* values of λ. This boundary condition is verified if and only if $a_0 > 0$. The expected value of the potential is bounded from below whenever

$$\left|\frac{a_1}{b_1}\right| > 1 + \sqrt{2}.$$

Finally, the conditions

$$a_0 \geqslant 0, \quad a_1 < 0,$$

ensure that ψ lies in $L^2(I) \oplus L^2(I)$, where $I = [0,\infty)$ in the singular case or at zero angular momentum, or $I = \mathbb{R}$ in the non-singular one-dimensional case.

ACKNOWLEDGMENT

The authors would like to acknowledge the partial financial support of the DGICYT under Grant No. PB95-0401.

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