



Monotone continuous dependence of solutions of singular quenching parabolic problems

Jesus Ildefonso Díaz¹ · Jacques Giacomoni²

Received: 15 April 2022 / Accepted: 27 August 2022
© The Author(s) 2022

Abstract

We prove the continuous dependence, with respect to the initial datum of solutions of the “quenching parabolic problem” $\partial_t u - \Delta u + \chi_{\{u>0\}} u^{-\beta} = \lambda u^p$, with zero Dirichlet boundary conditions, when $\beta \in (0, 1)$, $p \in (0, 1]$, $\lambda \geq 0$ and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points (x, t) where $u(x, t) > 0$. Notice that the absorption term $\chi_{\{u>0\}} u^{-\beta}$ is singular and monotone decreasing which does not allow the application of standard monotonicity arguments.

Keywords Quenching type singular parabolic equations · Continuous dependence · Nonnegative solutions · Very weak solutions

Mathematics Subject Classification 35K55 · 35K67 · 35K65

1 Introduction.

It is well-known that monotonicity methods allow to prove the existence and uniqueness of semilinear parabolic problems

$$\partial_t u - \Delta u + f(x, u) = 0$$

in presence of additive monotone non-decreasing or Lipschitz-continuous nonlinear terms (see, e.g., the many references of the survey [16]). The extension to the case in which $f(x, u)$ is singular but monotone increasing as $f(x, u) = -a(x)/u^\beta$, for some $a(x) > 0$ and $\beta > 0$ was treated in numerous previous papers in the literature.

The situation changes radically when the term $f(x, u)$ is neither Lipschitz-continuous nor monotone-nondecreasing, for instance because $f(x, u)$ involves a singular decreasing

✉ Jesus Ildefonso Díaz
ji_diaz@mat.ucm.es

Jacques Giacomoni
jacques.giacomoni@univ-pau.fr

¹ Instituto de Matemática Interdisciplinar, Univ. Complutense de Madrid, Plaza de las Ciencias no 3, 28040 Madrid, Spain

² Université de Pau et des Pays de l'Adour, LMAP (UMR E2S-UPPA CNRS 5142), Bat. IPRA, Avenue de l'Université, F-64013 Pau, France

dependence on u . This is the case of the family of problems which some authors refer as “quenching problems” since their solutions quench in a finite time ($u(x, t) \equiv 0$ for any $t \geq t^*$ and any x , for some $t^* > 0$). This type of problems will be the object of the present paper. More precisely, we consider the problem

$$P_{(\lambda, u_0)} \begin{cases} \partial_t u - \Delta u + \chi_{\{u>0\}} u^{-\beta} = \lambda u^p & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0(\cdot) & \text{on } \Omega, \end{cases} \tag{1}$$

under the main structural assumptions

$$\beta \in (0, 1), p \in (0, 1], \lambda \geq 0,$$

and

$$0 \leq u_0 \in L^1(\Omega),$$

where Ω is a smooth bounded domain in \mathbb{R}^N and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points $(x, t) \in \Omega \times (0, T)$ where $u(x, t) > 0$.

We recall that parabolic equations, involving as zero order term a negative power of the unknown, are quite common in the literature since 1960. The pioneering paper by Fulks and Maybee [25] was motivated by the study of the heat conduction in an electric medium. Perhaps, one of the first papers dealing with the equation (1) was [32] in the study of Electric Current Transient in Polarized Ionic Conductors (in fact for $\beta = 1$). The literature on this type of problems increased then very quickly and models arising in other contexts were mentioned by different authors, specially when regarding the equation of (1) as the limit case of models in chemical catalyst kinetics (Langmuir-Hinshelwood model) or of models in enzyme kinetics (see [14, 19] for the elliptic case and [2, 36, 34] [41] for the parabolic equation). See also many other references in the survey [30] and the monograph [27]) and the extension to the case of quasilinear diffusion operators made in [6, 28] [7, 8] and [9]). We also mention that the equation also arises in the context of the study of space-charge problems ([38]), the Euler-Poisson system in Maxwell-Vlasov problems ([1]) and in hydrodynamic quantum fluids ([26]). What makes equations like (1) specially interesting is the fact that the solutions may raise to a free boundary defined as the boundary of the set $\{(x, t): u(x, t) > 0\}$ (see, e.g. [15]). In many contexts the boundary conditions are not zero but, for instance $u = 1$ and thus the terminology of “quenching problem” was used in the literature to denote the appearance of blow-up result on $\partial_t u$ for the first time in which $u = 0$ (see, e.g. [32, 35, 36]). It can be proved (see, e.g. [12, 14]) that the solution starts to growth near the boundary of its initial support (let us assume, for simplicity, that is is given as the whole domain Ω) as

$$u(t, x) \geq C\delta(x)^{v_0} \text{ with } v_0 = \frac{2}{1 + \beta}$$

where

$$\delta(x) := \text{dist}(x, \partial\Omega)$$

(which we shall denote simply as δ). It is also well-known that the uniqueness of solution may fail (see Winkler [41]) except for the case in which there is not a free boundary (see

[12]). In particular, it is known that the solution is not necessarily continuously dependent on the norm $\|u_0\|_{L^\infty(\Omega)}$.

The main purpose of this work is to obtain a result on the continuous dependence (on the initial datum) of the solutions of $P_{(\lambda, u_0)}$, extending and improving the results of [10] in which the existence of solutions (and some partial uniqueness result) was obtained in the class of solutions

$$\mathcal{M}(v) := \left\{ u \in C([0, T]; L^1(\Omega)) \mid \forall T' \in (0, T), \text{ there exists } C(T') > 0 \text{ such that :} \right. \\ \left. \forall t \in (0, T'), u(t, x) \geq C(T')\delta(x)^v \text{ in } \Omega \right\}, \tag{2}$$

when

$$v \in \left(0, \frac{2}{1 + \beta} \right]. \tag{3}$$

To be more precise we introduce the notion of solution we shall use in this paper:

Definition 1 A function $u \in C([0, T]; L^1(\Omega))$ is called a mild solution of (1) if $\chi_{\{u>0\}} u^{-\beta} \in L^1(\Omega \times (0, T))$ and u fulfills the identity

$$u(\cdot, t) = S(t)u_0(\cdot) - \int_0^t S(t-s)(\chi_{\{u>0\}} u^{-\beta}(\cdot, s) - \lambda u^p) ds, \text{ in } L^1(\Omega), \tag{4}$$

where $S(t)$ is the $L^1(\Omega)$ -semigroup corresponding to the Laplace operator with homogeneous Dirichlet boundary conditions.

We recall some results already proved in the literature (see, e.g. [10]):

Theorem 2 ([10]) *Let $0 \leq u_0 \in L^1(\Omega)$. Then,*

i) there exists the (global) maximal nonnegative mild solution u of (1), i.e. such that for any other mild solution v of (1) we have $0 \leq v \leq u$ in $\Omega \times [0, T]$. Moreover, for any $0 < \tau < T$, $u \in L^2(\tau, T; W_0^{1,2}(\Omega)) \cap L^\infty(\Omega \times (\tau, T))$.

ii) there exists a finite time, $T^ > 0$ such that $u(\cdot, t)$ vanishes in $L^1(\Omega)$ for $t > T^*$. Moreover, T^* only depends on $\|u_0\|_{L^1(\Omega)}$, N and $|\Omega|$.*

iii) u is the unique solution to problem (1) in the class of mild solutions that have the same quenching time T^ .*

We also point out that it is possible to deal with a more general class of initial data $u_0 \in L^1(\Omega; \delta)$ by working in the class of very weak solutions (see, e.g. the general references [3, 18, 37]). Some regularity results involving the distance to the boundary, for a different class of equations including some gradient terms, can be found in [23] and [24].

Our main result gives the continuous dependence of solutions with respect to the initial data (implying, obviously, the uniqueness of solutions) as well as a smoothing effect with respect to the initial datum which improves, for $N \geq 3$, the standard smoothing effect associated to the linear heat equation (see Remark 8 below). We will use strongly some

Hilbertian techniques, so we will consider initial data in $L^1(\Omega) \cap L^2(\Omega; \delta)$ and we will prove that if two solutions are in the class $u, v \in \mathcal{M}(v)$, i.e. with $\delta^{-\nu}u, \delta^{-\nu}v \geq C$ then we can estimate the $L^2(\Omega)$ -norm of $\delta^{-\gamma}[u(t) - v(t)]_+$ for suitable $\gamma \in (0, 1]$ in terms of the $L^2(\Omega; \delta)$ -norm of $[u_0 - v_0]_+$. Notice that this implies, automatically, an estimate on the $L^2(\Omega)$ -norm of $[u(t) - v(t)]_+$ (see Remark 9 below). We conjecture, from the above comments, that the result below holds for initial data in $L^2(\Omega; \delta)$, but we will not develop this point of view in this paper.

Theorem 3 *Let $u_0, v_0 \in L^1(\Omega) \cap L^2(\Omega; \delta)$. Let u, v be weak solutions of $P_{(\lambda, u_0)}$ and $P_{(\lambda, v_0)}$, respectively such that $u, v \in \mathcal{M}(v)$ for some $\nu \in \left(0, \frac{2}{1+\beta}\right]$. Then, for any $T' \in (0, T)$ and for any $t \in (0, T')$, we have*

$$\|\delta^{-\gamma}[u(t) - v(t)]_+\|_{L^2(\Omega)} \leq Ct^{-\frac{2\gamma+1}{4}} \|[u_0 - v_0]_+\|_{L^2(\Omega; \delta)}, \tag{5}$$

with

$$\gamma := \min \{ \nu(1 + \beta), 1 \} \tag{6}$$

and for some constant $C > 0$ independent of T' if constants $C(T')$ given in the condition $u, v \in \mathcal{M}(v)$ are independent of T' . In particular, $u_0 \leq v_0$ implies that for any $t \in [0, T]$,

$$u(t, \cdot) \leq v(t, \cdot) \quad \text{a.e. in } \Omega$$

and

$$\|\delta^{-\gamma}(u(t) - v(t))\|_{L^2(\Omega)} \leq Ct^{-\frac{2\gamma+1}{4}} \|u_0 - v_0\|_{L^2(\Omega; \delta)}. \tag{7}$$

As an application, we will prove the uniqueness of the positive solution for the following stationary problem with a singular absorption:

$$P_{(\lambda, F, \beta, p)} \begin{cases} -\Delta u + \chi_{\{u>0\}} u^{-\beta} = \lambda_1 u^p + \lambda_2 F(x) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases}$$

where $\lambda_1, \lambda_2 \geq 0$ and

$$F \in L^1(\Omega), \quad F \geq 0 \text{ a.e. in } \Omega, \tag{8}$$

improving or completing different results in the literature (see, e.g. [12, 19, 29] and [17]).

2 On the continuous dependence

For the proof of Theorem 3 we shall need some well-known auxiliary results. The first one is a singular version of the Gronwall’s inequality which is specially useful in the study of non-globally Lipschitz perturbations of the heat equation (see [4, p. 288], [5, Lemma 8.1.1, p. 125]).

Lemma 4 *Let $T > 0, A \geq 0, 0 \leq a, b \leq 1$ and let f be a non-negative function with $f \in L^p(0, T)$ for some $p > 1$ such that $p' \cdot \max\{a, b\} < 1$ (where $\frac{1}{p} + \frac{1}{p'} = 1$). Consider a non-negative function $\varphi \in L^\infty(0, T)$ such that, for almost every $t \in (0, T)$,*

$$\varphi(t) \leq At^{-a} + \int_0^t (t-s)^{-b} f(s) \varphi(s) ds. \tag{9}$$

Then, there exists $C > 0$ only depending on T, a, b, p and $\|f\|_{L^p(0,T)}$ such that, for almost every $t \in (0, T)$,

$$\varphi(t) \leq ACt^{-a}. \tag{10}$$

We shall also use some regularizing effects properties satisfied by the semigroup $S(t)$ of the heat equation with zero Dirichlet boundary conditions.

Lemma 5

1. There exists $C > 0$ such that, for any $t > 0$ and any $u_0 \in L^2(\Omega)$,

$$\|\nabla S(t)u_0\|_{L^2(\Omega)} \leq Ct^{-\frac{1}{2}} \|u_0\|_{L^2(\Omega)}. \tag{11}$$

2. There exists $C > 0$ such that, for any $t > 0$ and any $u_0 \in L^1(\Omega)$,

$$\|S(t)u_0\|_{L^2(\Omega)} \leq Ct^{-\frac{N}{4}} \|u_0\|_{L^1(\Omega)}. \tag{12}$$

3. There exists $C > 0$ such that, for any $t > 0$, any $m \in (0, 1]$ and any $u_0 \in L^2(\Omega; \delta^{2m})$,

$$\|S(t)u_0\|_{L^2(\Omega)} \leq Ct^{-\frac{m}{2}} \|u_0\|_{L^2(\Omega; \delta^{2m})}. \tag{13}$$

4. There exists $C > 0$ such that, for any $t > 0$, any $p \in [1, +\infty)$ and any $u_0 \in L^p(\Omega, \delta)$,

$$\|S(t)u_0\|_{L^p(\Omega)} \leq Ct^{-\frac{1}{2p}} \|u_0\|_{L^p(\Omega, \delta)}. \tag{14}$$

Proof Properties 1 and 2 are classical (see e.g. [40, remarque III.5] or [5, Proposition 3.5.7, Proposition 3.5.2]). Property 3 was established in [12, Proposition 1.12] in terms of the function $v_0 := u_0 \delta^m$, i.e.

$$\|S(t)\delta^{-m}v_0\|_{L^2(\Omega)} \leq Ct^{-\frac{m}{2}} \|v_0\|_{L^2(\Omega)}.$$

Property 4 was proved in [39, Remark 2.1-(c), p. 179]. □

Proof (of Theorem 3) By the constant variations formula, we know that for any $t \in [0, T]$,

$$u(t) - v(t) = S(t)(u_0 - v_0) + \int_0^t S(t-s)(h(u(s)) - h(v(s))) ds \quad \text{in } \Omega, \tag{15}$$

where $h(u) := \lambda u^p - \chi_{\{u>0\}} u^{-\beta}$. By the convexity of the function $u \mapsto u^{-\beta}$, the concavity of the function $u \mapsto u^p$ and the assumption that $u, v \in \mathcal{M}(v)$, we deduce that

$$h(u) - h(v) \leq (C_1 \delta^{-(\beta+1)v} + C_2 \delta^{(1-p)v})(u - v)_+ \leq C \delta^{-(\beta+1)v} (u - v)_+ \quad \text{in } \Omega. \tag{16}$$

Thus, if we denote $w := u - v$, we get for any $\tau, t \in [0, T]$ with $\tau \leq t$

$$w_+(t) \leq S(t - \tau)w_+(\tau) + C \int_{\tau}^t S(t - s)\delta^{-(\beta+1)\nu} w_+(s) ds. \tag{17}$$

Now we adapt to our framework some arguments of the proof of [12, Lemma 1.11, p. 1823] concerning other forcing nonlinear terms (see Remark 6). We multiply (17) by the weight $\delta^{-\gamma}$, with $\gamma \in [0, 1]$ to be chosen later, and take the L^2 -norms. Then,

$$\|\delta^{-\gamma} w_+(t)\|_{L^2(\Omega)} \leq \|\delta^{-\gamma} S(t - \tau)w_+(\tau)\|_{L^2(\Omega)} + C \int_{\tau}^t \|S(t - s)\delta^{-[(\beta+1)\nu+\gamma]} w_+(s)\|_{L^2(\Omega)} ds.$$

Let us fix $s, t > 0$ and let us call $\psi := S(t - s)\delta^{-(\beta+1)\nu} w_+(s)$. Then, by Hölder inequality,

$$\|\delta^{-\gamma} \psi\|_{L^2(\Omega)}^2 = \int_{\Omega} \frac{\psi^2}{\delta^{2\gamma}} dx \leq \left(\int_{\Omega} \frac{\psi^2}{\delta^2} dx \right)^{\gamma} \left(\int_{\Omega} \psi^2 dx \right)^{1-\gamma}$$

(note that the limit cases $\psi \equiv 0$ and $\psi \equiv 1$ are allowed). Then, applying Hardy inequality,

$$\|\delta^{-\gamma} \psi\|_{L^2(\Omega)} \leq C \|\nabla \psi\|_{L^2(\Omega)}^{\gamma} \|\psi\|_{L^2(\Omega)}^{1-\gamma}.$$

By property 1 of Lemma 5, to $\frac{t-s}{2}$, we get

$$\|\delta^{-[(\beta+1)\nu+\gamma]} S(t - s)w_+(s)\|_{L^2(\Omega)} \leq C(t - s)^{-\frac{\gamma}{2}} \|S\left(\frac{t - s}{2}\right)\delta^{-(\beta+1)\nu} w_+(s)\|_{L^2(\Omega)}. \tag{18}$$

Analogously, using property 4 of Lemma 5,

$$\|\delta^{-\gamma} S(t)w_+(0)\|_{L^2(\Omega)} \leq Ct^{-\frac{\gamma}{2}} \|S\left(\frac{t}{2}\right)w_+(0)\|_{L^2(\Omega)} \leq Ct^{-\left(\frac{\gamma}{2} + \frac{1}{4}\right)} \|w_+(0)\|_{L^2(\Omega;\delta)}. \tag{19}$$

In order to apply the singular Gronwall’s inequality, we must relate the weights $\delta^{-\gamma}$ and $\delta^{-(\beta+1)\nu}$ keeping in mind that $\gamma \in [0, 1]$. To do that, we apply property 3 of Lemma 5 for some $m \in [0, 1]$. We shall take

$$(\beta + 1)\nu = \gamma + m. \tag{20}$$

Indeed, if $(\beta + 1)\nu \in (1, 2]$, then we take $\gamma = 1, m = (\beta + 1)\nu - 1$ and we apply point 3 of Lemma 5 to the initial datum:

$$\|S\left(\frac{t - s}{2}\right)\delta^{-(\beta+1)\nu} w_+(s)\|_{L^2(\Omega)} = \|S\left(\frac{t - s}{2}\right)\delta^{-(m+1)} w_+(s)\|_{L^2(\Omega)} \leq C(t - s)^{-\frac{m}{2}} \|\delta^{-\gamma} w_+(s)\|_{L^2(\Omega)}. \tag{21}$$

On the other hand, if $(\beta + 1)\nu \in [0, 1]$, we can take $\gamma = (\beta + 1)\nu$ and thus, since $S(t - s)$ is a contraction in $L^2(\Omega)$, we get

$$\|S\left(\frac{t - s}{2}\right)\delta^{-(\beta+1)\nu} w_+(s)\|_{L^2(\Omega)} = \|S\left(\frac{t - s}{2}\right)\delta^{-\gamma} w_+(s)\|_{L^2(\Omega)} \leq \|\delta^{-\gamma} w_+(s)\|_{L^2(\Omega)}, \tag{22}$$

which corresponds to (20) with $m = 0$. In other words,

$$\gamma = \min\{1, (\beta + 1)\nu\}$$

and

$$m = \max\{(\beta + 1)\nu - 1, 0\}.$$

Collecting the previous inequalities, we arrive to

$$\|\delta^{-\gamma} w_+(t)\|_{L^2(\Omega)} \leq C t^{-\frac{2\gamma+1}{4}} \|w_+(0)\|_{L^2(\Omega;\delta)} + C \int_0^t (t-s)^{-\frac{m}{2}} \|\delta^{-\gamma} w_+(s)\|_{L^2(\Omega)} ds.$$

Thus, we can apply Lemma 4 with $a = \frac{2\gamma+1}{4} \in \left[\frac{1}{4}, \frac{3}{4}\right]$, $b = \frac{m}{2}$ and $A = C\|w_+(0)\|_{L^2(\Omega;\delta)}$ to deduce that

$$\|\delta^{-\gamma} w_+(t)\|_{L^2(\Omega)} \leq C t^{-\frac{2\gamma+1}{4}} \|w_+(0)\|_{L^2(\Omega;\delta)}.$$

□

Remark 6 The comparison principle was proved in [12] for a generalized version of equation $P_{(\lambda, u_0)}$ in the sense that the right hand side term was replaced by a function $f : [0, +\infty) \rightarrow [0, +\infty)$ satisfying:

$$0 \leq f \in C^2([0, +\infty)), \tag{23}$$

$$f \text{ is concave, increasing and } \lim_{u \rightarrow +\infty} \frac{f(u)}{u} = 0. \tag{24}$$

Note that $f(u) := u^p$ does not satisfy (23) if $p \in (0, 1)$. In fact, condition (24) is neither satisfied if $p = 1$. It was also shown in [12] that for the mere purpose to get the comparison $u \leq v$, the growth condition near $\partial\Omega$ is only required for the "supersolution" v (in fact, conclusion i) in [12, Lemma 1.11] remains true if u is a subsolution and v a supersolution *i.e.* we can replace the symbol "=" in the equation of $P_{(\lambda, u_0)}$ by " \leq " (in the case of u) and/or by " \geq " (in the case of v).

Remark 7 Sufficient conditions on the initial datum u_0 ensuring that there exists some weak solution of $P_{(\lambda, u_0)}$ belonging to the class $\mathcal{M}(v)$ were given in [12, Theorem 1.10] under conditions (23) and (24). Similar results can be obtained in our framework ($p \in (0, 1)$) by applying the super and subsolutions built in [14, 19] and [18].

Remark 8 The smoothing effect, with respect to initial datum, given by formula (5) is sharper than the one given in (12) for the solutions of the linear heat equation. Indeed, it is easy to check that $2\gamma + 1 \leq N$, for $N \geq 3$. Moreover the exponent in (5) depends on the value of exponent β of the singular term.

Remark 9 It is clear that the estimate of the $L^2(\Omega)$ -norm of $\delta^{-\gamma}[u(t) - v(t)]_+$ given in (5) implies, automatically an estimate on the $L^2(\Omega)$ -norm of $[u(t) - v(t)]_+$. Indeed, it suffices to write

$$\begin{aligned} \|[u(t) - v(t)]_+\|_{L^2(\Omega)} &= \|\delta^\gamma \delta^{-\gamma}[u(t) - v(t)]_+\|_{L^2(\Omega)} \leq \\ &\|\delta^\gamma\| \|\delta^{-\gamma}[u(t) - v(t)]_+\|_{L^2(\Omega)} \leq C(\Omega) \|\delta^{-\gamma}[u(t) - v(t)]_+\|_{L^2(\Omega)}. \end{aligned}$$

As mentioned in the Introduction, we can apply the above result to the case of suitable elliptic problems:

Corollary 10 Assume $\beta \in (0, 1)$, $p \in (0, 1)$ and either

$$\lambda_1 = 0 \text{ and } \lambda_2 > 0 \text{ is large enough} \tag{25}$$

or

$$\lambda_1 > 0 \text{ is large enough (and } \lambda_2 \geq 0). \tag{26}$$

Then, problem $P_{(\lambda,F,\beta,p)}$ has at most a positive solution.

Proof Assume (25). By [19, Corollary 1, p. 1335], there exists $\lambda^* > 0$ such that if $\lambda > \lambda^*$ then, there exists a positive solution u of the problem $P_{(\lambda,F,\beta,p)}$. Precisely, if λ is large enough it was shown that any positive solution must verify

$$u(x) \geq \lambda \varphi_1(x)^{\frac{2}{1+\beta}} \quad \text{in } \Omega,$$

with φ_1 the first eigenfunction of $-\Delta$ with homogeneous Dirichlet conditions. Then, assume that $\lambda > \lambda^*$ is large enough and that there are two different solutions u_∞ and v_∞ of $P_{(\lambda,F,\beta,p)}$. By taking $u_0 = u_\infty$ and $v_0 = v_\infty$ as initial data in $P_{(\lambda,u_0)}$ and $P_{(\lambda,v_0)}$, since u_∞ and v_∞ are obviously respective solutions of the mentioned parabolic problems, we get that $C(T')$ is independent of T' and thus $u_\infty - v_\infty$ satisfies the same estimate as in the case $F(x) \equiv 0$, in particular we get

$$\left\| \delta^{-1} (u_\infty - v_\infty)_+ \right\|_{L^2(\Omega)} \leq Ct^{-\frac{2\gamma+1}{4}} \|(u_\infty - v_\infty)_+\|_{L^2(\Omega,\delta)}.$$

Making $t \nearrow +\infty$ and reversing the role of u_∞ and v_∞ , we get that $u_\infty = v_\infty$.

For the case (26), it was proved in [13, Lemma 3.11, p. 321] that any positive solution must satisfy the growth condition for some $\nu \in (1, \frac{2}{1+\beta})$ and thus, arguing as above (but now with $\gamma \in (0, 1)$), we arrive at the same conclusion. \square

Remark 11 The study of the limit case λ^* (concerning assumption (25)) is quite special since the maximal solution u_∞^* is not always strictly positive. The growth estimate with $\nu = \frac{2}{1+\beta}$ (and thus the strict positivity) was proved in [13, Theorem 2.4, p. 307] under the additional condition

$$\frac{3\beta + 1 + 2\sqrt{\beta^2 + \beta}}{\beta + 1} > \frac{N}{2}.$$

An example showing that this condition is almost optimal was given in [11, Theorem 3.2].

Remark 12 In the one-dimensional case it is possible to give the exact multiplicity of positive solutions (see [31] for the case of (25) and $F(x) \equiv F_0 > 0$; see also [17] for (26) with $F(x) = 0$, $\beta \in (0, \frac{1}{q+1})$ and the q -Laplacian as diffusion operator).

Acknowledgements The research of JID was partially supported by the project ref. PID2020-112517GB-I00 of the AEI (Spain) and the Research Group MOMAT (Ref. 910480) of the UCM. The authors have no competing interests to declare that are relevant to the content of this article.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long

as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Ambroso, A., Méhats, F., Raviart, P.A.: On singular perturbation problems for the nonlinear Poisson equation. *Asympt. Anal.* **25**(1), 39–91 (2001)
2. Bandle, C., Brauner, C.M.: Singular perturbation method in a parabolic problem with free boundary, BAIL IV (Novosibirsk, 1986) singular perturbation method in a parabolic problem with free boundary, *Boole Press Conf. Ser.*, **8**, Boole, Dún Laoghaire, 7–14, 1986
3. Brezis, H., Cazenave, T., Martel, Y., Ramiandrisoa, A.: Blow up for $u_t - \Delta u = g(u)$ revisited. *Adv. Differ. Eq.* **1**(1), 73–90 (1996)
4. Brézis, H., Cazenave, T.: A nonlinear heat equation with singular initial data. *J. Anal. Math.* **68**, 277–304 (1996)
5. Cazenave, T., Haraux, A.: An introduction to semilinear evolution equations. *Oxford Lecture Series in Mathematics and its Applications*, vol **13**. The Clarendon Press, Oxford University Press, New York, 1998
6. Dao, A.N., Díaz, J.I.: A gradient estimate to a degenerate parabolic equation with a singular absorption term: the global quenching phenomena. *J. Math. Anal. Appl.* **437**(1), 445–473 (2016)
7. Dao, A.N., Díaz, J.I.: The extinction versus the blow-up: global and non-global existence of solutions of source types of degenerate parabolic equations with a singular absorption. *J. Differ. Eq.* **263**(10), 6764–6804 (2017)
8. Dao, A.N., Díaz, J.I., Kha, H.V.: Complete quenching phenomenon and instantaneous shrinking of support of solutions of degenerate parabolic equations with nonlinear singular absorption. *Proc. Roy. Soc. Edinb. Sect. A* **149**(5), 1323–1346 (2019)
9. Dao, N.A., Díaz, J.I., Nguyen, Q.B.H.: Pointwise gradient estimates in multi-dimensional slow diffusion equations with a singular quenching term. *Adv. Nonlinear. Stud.* **20**(2), 477–502 (2020)
10. Dao, A.N., Díaz, J.I., Sauvy, P.: (2016) Quenching phenomenon of singular parabolic problems with L^1 initial data. *Electron. J. Differ. Eq.* **136**, 1–16 (2016)
11. Dávila, J., Montenegro, M.: Remarks on positive and free boundary solutions to a singular equation. *Rev. Integr. Temas. Mat.* **28**(2), 85–100 (2010)
12. Dávila, J., Montenegro, M.: Existence and asymptotic behavior for a singular parabolic equation. *Trans. Amer. Math. Soc.* **357**(5), 1801–1828 (2004)
13. Dávila, J., Montenegro, M.: Positive versus free boundary solutions to a singular elliptic equation. *J. Anal. Math.* **90**, 303–335 (2003)
14. Díaz, J.I.: *Nonlinear partial differential equations and free boundaries. Vol. I. Elliptic Equations*, *Research Notes in Mathematics*, 106. Pitman, London, 1985
15. Díaz, J.I.: On the free boundary for quenching type parabolic problems via local energy method. *Comunicacion. Pure. Appl. Anal* **13**(5), 1799–1814 (2014)
16. Díaz, J.I.: New applications of monotonicity methods to a class of non-monotone parabolic quasilinear sub-homogeneous problems. *Pure. Appl. Funct. Anal.* **5**(4), 925–949 (2020)
17. Díaz, J.I., Hernández, J., Mancebo, F.J.: Branches of positive and free boundary solutions for some singular quasilinear elliptic problems. *J. Math. Anal. Appl.* **352**(1), 449–474 (2009)
18. Díaz, J. I., Hernández, J., Rakotoson, J. M.: On very weak non-negative solutions to some second order semilinear elliptic problems with a singular absorption term, In preparation
19. Díaz, J.I., Morel, J.M., Oswald, L.: An elliptic equation with singular nonlinearity. *Comm. Partial. Differ. Eq.* **12**(12), 1333–1344 (1987)
20. Díaz, J.I., Rakotoson, J.M.: On very weak solutions of semilinear elliptic equations with right hand side data integrable with respect to the distance to the boundary. *Discret. Contin. Dyn. Syst.* **27**, 1037–1058 (2010)
21. Fila, M., Kawohl, B.: *Asymptot. Anal. Quenching. Prob.* **22**(2), 563–577 (1992)
22. Fila, M., Levine, A.H., Vázquez, J.L.: Stabilization of solutions of weakly singular quenching problems. *Proc. Amer. Math. Soc.* **119**, 555–559 (1993)

23. Filippucci, R., Lombardi, S.: Fujita type results for parabolic inequalities with gradient terms. *J. Differ. Equ.* **268**, 1873–1910 (2020)
24. Filippucci, R., Pucci, P., Souplet, Ph.: A Liouville-type theorem in a half-space and its applications to the gradient blow-up behavior for superquadratic diffusive Hamilton-Jacobi equations. *Comm. Partial. Differ. Equ.* **45**, 321–349 (2020)
25. Furks, W., Maybee, J.S.: A singular Non-Linear equation. *Osaka. Math. J.* **12**, 1–19 (1960)
26. Gamba, I.M., Jüngel, A.: Positive solutions to singular second and third order differential equations for quantum fluids. *Arch. Rational. Mech. Anal.* **156**(3), 183–203 (2001)
27. Ghergu, M., Radulescu, V.: *Singular Elliptic Problems*. Oxford University Press, Bifurcation and Asymptotic Analysis (2008)
28. Giacomoni, J., Sauvy, P., Shmarev, S.: Complete quenching for a quasilinear parabolic equation. *J. Math. Anal. Appl.* **410**(2), 607–624 (2014)
29. Guo, Z., Wei, J.: On the cauchy problem for a reaction-diffusion equation with a singular nonlinearity. *J. Different. Equ.* **240**(2), 279–323 (2007)
30. Hernández, Jesús and Mancebo, Francisco J: 2006 Singular elliptic and parabolic equations. In *Handbook of Differential equations* (ed. M. Chipot and P. Quittner), **3** Elsevier: 317–400
31. Horváth, T.L., Simon, P.L.: On the exact number of solutions of a singular boundary-value problem. *Different. Integral. Equ.* **22**(7–8), 787–796 (2009)
32. Kawarada, H.: On solutions of initial-boundary problem for $u_t = u_{xx} + 1/(1 - u)$. *Publ. Res. Inst. Math. Sci.* **10**(3), 729–736 (1974)
33. Kawohl, B.: Remarks on quenching. *Doc. Math. J. DMV* **1**(9), 199–208 (1996)
34. Kawohl, B., Kersner, R.: On degenerate diffusion with very strong absorption. *Math. Method. Appl. Sci.* **15**(7), 469–477 (1992)
35. Levine, H.A.: Quenching and beyond: a survey of recent results, in *Nonlinear mathematical problems in industry, II* (Iwaki, 1992), vol. 2 of GAKUTO Internat. Ser. Math. Sci. Appl. GAKUTO. Internat. Ser. Math. Sci. Appl **2**, 501–512 (1993)
36. Phillips, D.: Existence of solutions of quenching problems. *Appl. Anal.* **24**(4), 253–264 (1987)
37. Rakotoson, J.-M.: Regularity of a very weak solution for parabolic equations and applications. *Adv. Different. Equ.* **16**(9–10), 867–894 (2011)
38. Rokhlenko, A., Lebowitz, J.L.: Space charge limited 2-d electron flow between two flat electrodes in a strong magnetic field. *Phys. Rev. Lett.* **91**(085002), 1–4 (2003)
39. Souplet, Ph.: Optimal regularity conditions for elliptic problems via L^p_θ -spaces. *Duke. Math. J.* **127**(1), 175–192 (2005)
40. Véron, L.: Effets régularisants de semi-groupes non-linéaires dans des espaces de Banach. *Ann. Fac. Sci. Toulouse. Math.* **1**(2), 171–200 (1979)
41. Winkler, M.: Nonuniqueness in the quenching problem. *Math. Ann.* **339**(3), 559–597 (2007)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.