# LINEABILITY, SPACEABILITY, AND ADDITIVITY CARDINALS FOR DARBOUX-LIKE FUNCTIONS

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Abstract. We introduce the concept of *maximal lineability cardinal number*, $\mathrm{m}\mathcal{L}(M)$, of a subset $M$ of a topological vector space and study its relation to the cardinal numbers known as: additivity $A(M)$, homogeneous lineability $\mathcal{HL}(M)$, and lineability $\mathcal{L}(M)$ of $M$. In particular, we will describe, in terms of $\mathcal{L}$, the lineability and spaceability of the families of the following Darboux-like functions on $\mathbb{R}^n$, $n \geq 1$: extendable, Jones, and almost continuous functions.

## 1. Preliminaries and background

The work presented here is a contribution to a recent ongoing research concerning the following general question: *For an arbitrary subset $M$ of a vector space $W$, how big can be a vector subspace $V$ contained in $M \cup \{0\}$?* The current state of knowledge concerning this problem is described in the very recent survey article [4]. So far, the term *big* in the question was understood as a cardinality of a basis of $V$; however, some other measures of bigness (i.e., in a category sense) can also be considered.

Following [1, 23] (see, also, [13]), given a cardinal number $\mu$ we say that $M \subset W$ is $\mu$*-lineable* if $M \cup \{0\}$ contains a vector subspace $V$ of the dimension $\dim(V) = \mu$. Consider the following *lineability* cardinal number (see [2]):

$$\mathcal{L}(M) = \min\{\kappa \colon M \cup \{0\} \text{ contains no vector space of dimension } \kappa\}.$$

Notice that $M \subset W$ is $\mu$-lineable if, and only if, $\mu < \mathcal{L}(M)$. In particular, $\mu$ is the maximal dimension of a subspace of $M \cup \{0\}$ if, and only if, $\mathcal{L}(M) = \mu^+$. The number $\mathcal{L}(M)$ need not be a cardinal successor (see, e.g., [1]); thus, the maximal dimension of a subspace of $M \cup \{0\}$ does not necessarily exist.

If $W$ is a vector space over the field $K$ and $M \subset W$, let

$$\mathrm{st}(M) = \{w \in W \colon (K \setminus \{0\})w \subset M\}.$$

Notice that

$$\text{if } V \text{ is a subspace of } W, \text{ then } V \subset M \cup \{0\} \text{ if, and only if, } V \subset \mathrm{st}(M) \cup \{0\}. \tag{1}$$

In particular,

$$\mathcal{L}(M) = \mathcal{L}(\mathrm{st}(M)). \tag{2}$$

Recall also (see, e.g., [15]) that a family $M \subset W$ is said to be *star-like* provided $\mathrm{st}(M) = M$. Properties (1) and (2) explain why the assumption that $M$ is star-like appears in many results on lineability.

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A simple use of Zorn's lemma shows that any linear subspace $V_0$ of $M \cup \{0\}$ can be extended to a maximal linear subspace $V$ of $M \cup \{0\}$. Therefore, the following concept is well defined.

**Definition 1.1** (maximal lineability cardinal number)**.** Let $M$ be any arbitrary subset of a vector space $W$. We define

$$\mathrm{m}\mathcal{L}(M) = \min\{\dim(V) \colon V \text{ is a maximal linear subspace of } M \cup \{0\}\}.$$

Although this notion might seem similar to that of maximal-lineability and maximal-spaceability (introduced by Bernal-González in [3]) they are, in general, not related.

In any case, (1) implies that $\mathrm{m}\mathcal{L}(M) = \mathrm{m}\mathcal{L}(\mathrm{st}(M))$.

**Remark 1.2.** It is easy to see that $\mathcal{HL}(M) = \mathrm{m}\mathcal{L}(M)^+$, where $\mathcal{HL}(M)$ is a homogeneous lineability number defined in [2]. (This explains why $\mathcal{HL}$ is always a successor cardinal, as shown in [2].) Clearly we have

$$\mathcal{HL}(M) = \mathrm{m}\mathcal{L}(M)^+ \leq \mathcal{L}(M).$$

The inequality may be strict, as shown in [2].

For $M \subset W$ we will also consider the following *additivity* number (compare [2]), which is a generalization of the notion introduced by T. Natkaniec in [20, 21] and thoroughly studied by the first author [7–11] and F.E. Jordan [18] for $V = \mathbb{R}^{\mathbb{R}}$ (see, also, [16]):

$$A(M, W) = \min\big(\{|F| \colon F \subset W \ \&\ (\forall w \in W)(w + F \not\subset M)\} \cup \{|W|^+\}\big),$$

where $|F|$ is the cardinality of $F$ and $w + F = \{w + f \colon f \in F\}$. Most of the times the space $W$, usually $W = \mathbb{R}^{\mathbb{R}}$, will be clear by the context. In such cases we will often write $A(M)$ in place of $A(M, W)$.

We are mostly interested in the topological vector spaces $W$. We say that $M \subset W$ is *$\mu$-spaceable* with respect to a topology $\tau$ on $W$, provided there exists a $\tau$-closed vector space $V \subset M \cup \{0\}$ of dimension $\mu$. In particular, we can consider also the following *spaceability* cardinal number:

$$\mathcal{L}_\tau(M) = \min\{\kappa \colon M \cup \{0\} \text{ contains no } \tau\text{-closed subspace of dimension } \kappa\}.$$

Notice that $\mathcal{L}(M) = \mathcal{L}_\tau(M)$ when $\tau$ is the discrete topology.

In what follows, we shall focus on spaces $W = \mathbb{R}^X$ of all functions from $X = \mathbb{R}^n$ to $\mathbb{R}$ and consider the topologies $\tau_u$ and $\tau_p$ of uniform and pointwise convergence, respectively. In particular, we write $\mathcal{L}_u(M)$ and $\mathcal{L}_p(M)$ for $\mathcal{L}_{\tau_u}(M)$ and $\mathcal{L}_{\tau_p}(M)$, respectively. Clearly

$$\mathcal{L}_p(M) \leq \mathcal{L}_u(M) \leq \mathcal{L}(M).$$

Recall also a series of definitions that shall be needed throughout the paper.

**Definition 1.3.** For $X \subseteq \mathbb{R}^n$ a function $f \colon X \to \mathbb{R}$ is said to be

- *Darboux* if $f[K]$ is a connected subset of $\mathbb{R}$ (i.e., an interval) for every connected subset $K$ of $X$;
- *Darboux* in the sense of Pawlak if $f[L]$ is a connected subset of $\mathbb{R}$ for every arc $L$ of $X$ (i.e., $f$ maps path connected sets into connected sets);
- *almost continuous* (in the sense of Stallings) if each open subset of $X \times \mathbb{R}$ containing the graph of $f$ contains also a continuous function from $X$ to $\mathbb{R}$;

- a *connectivity* function if the graph of $f \restriction Z$ is connected in $Z \times \mathbb{R}$ for any connected subset $Z$ of $X$;
- *extendable* provided that there exists a connectivity function $F\colon X \times [0,1] \to \mathbb{R}$ such that $f(x) = F(x,0)$ for every $x \in X$;
- *peripherally continuous* if for every $x \in X$ and for all pairs of open sets $U$ and $V$ containing $x$ and $f(x)$, respectively, there exists an open subset $W$ of $U$ such that $x \in W$ and $f[\mathrm{bd}(W)] \subset V$.

The above classes of functions are denoted by $\mathrm{D}(X)$, $\mathrm{D_P}(X)$, $\mathrm{AC}(X)$, $\mathrm{Conn}(X)$, $\mathrm{Ext}(X)$, and $\mathrm{PC}(X)$, respectively. The class of continuous functions from $X$ into $\mathbb{R}$ is denoted by $\mathrm{C}(X)$. We will drop the domain $X$ if $X = \mathbb{R}$.

**Definition 1.4.** A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called

- *everywhere surjective* if $f[G] = \mathbb{R}$ for every nonempty open set $G \subset \mathbb{R}^n$;
- *strongly everywhere surjective* if $f^{-1}(y) \cap G$ has cardinality $\mathfrak{c}$ for every $y \in \mathbb{R}$ and every nonempty open set $G \subset \mathbb{R}^n$; this class was also studied in [9], under the name of $\mathfrak{c}$ strongly Darboux functions;
- *perfectly everywhere surjective* if $f[P] = \mathbb{R}$ for every perfect set $P \subset \mathbb{R}^n$ (i.e., when $f^{-1}(r)$ is a Bernstein set for every $r \in \mathbb{R}$ (compare [6, chap. 7]));
- a *Jones function* (see [17]) if $f \cap F \neq \varnothing$ for every closed set $F \subset \mathbb{R}^n \times \mathbb{R}$ whose projection on $\mathbb{R}^n$ is uncountable.

The classes of these functions are written as $\mathrm{ES}(\mathbb{R}^n)$, $\mathrm{SES}(\mathbb{R}^n)$, $\mathrm{PES}(\mathbb{R}^n)$, and $\mathrm{J}(\mathbb{R}^n)$, respectively. We will drop the domain $\mathbb{R}^n$ if $n = 1$.

**Definition 1.5.** A function $f\colon \mathbb{R} \to \mathbb{R}$ has:

- the *Cantor intermediate value property* if for every $x, y \in \mathbb{R}$ and for each perfect set $K$ between $f(x)$ and $f(y)$ there is a perfect set $C$ between $x$ and $y$ such that $f[C] \subset K$;
- the *strong Cantor intermediate value property* if for every $x, y \in \mathbb{R}$ and for each perfect set $K$ between $f(x)$ and $f(y)$ there is a perfect set $C$ between $x$ and $y$ such that $f[C] \subset K$ and $f \restriction C$ is continuous;
- the *weak Cantor intermediate value property* if for every $x, y \in \mathbb{R}$ with $f(x) < f(y)$ there exists a perfect set $C$ between $x$ and $y$ such that $f[C] \subset (f(x), f(y))$;
- *perfect roads* if for every $x \in \mathbb{R}$ there exists a perfect set $P \subset \mathbb{R}$ having $x$ as a bilateral (i.e., two sided) limit point for which $f \restriction P$ is continuous at $x$.

The above classes of functions shall be denoted by CIVP, SCIVP, WCIVP, and PR, respectively.

Notice that all classes defined in the above three definitions are star-like.

The text is organized as follows. In Section 2 we study the relations between additivity and maximal lineability numbers. Sections 3 and 4 focus on the set of extendable functions on $\mathbb{R}$ and $\mathbb{R}^n$, respectively. Surprisingly enough, we shall obtain very different results when moving from $\mathbb{R}$ to $\mathbb{R}^n$. The lineability of some of the above functions have been recently partly studied (see, e.g., [2, 14–16]) but here we shall give definitive answers concerning the lineability and spaceability of several previous studied classes.

## 2. Relation between additivity and lineability numbers

The goal of this section is to examine possible values of numbers $A(M)$, $\mathrm{m}\mathcal{L}(M)$, and $\mathcal{L}(M)$ for a subset $M$ of a linear space $W$ over an arbitrary field $K$. We will concentrate on the cases when $\varnothing \neq M \subsetneq W$, since it is easy for the cases $M \in \{\varnothing, W\}$. Indeed, as it can be easily checked, one has $A(\varnothing) = \mathcal{L}(\varnothing) = 1$ and $\mathrm{m}\mathcal{L}(\varnothing) = 0$; $A(W) = |W|^+$, $\mathcal{L}(W) = \dim(W)^+$, and $\mathrm{m}\mathcal{L}(W) = \dim(W)$.

**Proposition 2.1.** *Let $W$ be a vector space over a field $K$ and let $\varnothing \neq M \subsetneq W$. Then*

(i) $2 \leq A(M) \leq |W|$ *and* $\mathrm{m}\mathcal{L}(M) < \mathcal{L}(M) \leq \dim(W)^+$*;*

(ii) *if* $A(\mathrm{st}(M)) > |K|$*, then* $A(\mathrm{st}(M)) \leq \mathrm{m}\mathcal{L}(M)$*.*

*In particular, if $M$ is star-like, then $A(M) > |K|$ implies that*

(iii) $A(M) \leq \mathrm{m}\mathcal{L}(M) < \mathcal{L}(M) \leq \dim(W)^+$*.*

*Proof.* (i) These inequalities are easy to see.

(ii) This can be proved by an easy transfinite induction. Alternatively, notice that A. Bartoszewicz and S. Głąb proved, in [2, corollary 2.3], that if $M \subset W$ is star-like and $A(M) > |K|$, then $A(M) < \mathcal{HL}(M)$. Hence, $A(\mathrm{st}(M)) > |K|$ implies that $A(\mathrm{st}(M)) < \mathcal{HL}(\mathrm{st}(M)) = \mathrm{m}\mathcal{L}(\mathrm{st}(M))^+ = \mathrm{m}\mathcal{L}(M)^+$. Therefore, $A(\mathrm{st}(M)) \leq \mathrm{m}\mathcal{L}(M)$. □

In what follows, we will restrict our attention to the star-like families, since, by Proposition 2.1, other cases could be reduced to this situation. Our next theorem shows that, for such families and under assumption that $A(M) > |K|$, the inequalities (3) constitute all that can be said on these numbers.

**Theorem 2.2.** *Let $W$ be an infinite dimensional vector space over an infinite field $K$ and let $\alpha$, $\mu$, and $\lambda$ be the cardinal numbers such that $|K| < \alpha \leq \mu < \lambda \leq \dim(W)^+$. Then there exists a star-like $M \subsetneq W$ containing $0$ such that $A(M) = \alpha$, $\mathrm{m}\mathcal{L}(M) = \mu$, and $\mathcal{L}(M) = \lambda$.*

The proof of this theorem will be based on the following two lemmas. The first of them shows that the theorem holds when $\alpha = \mu$, while the second shows how such an example can be modified to the general case.

**Lemma 2.3.** *Let $W$ be an infinite dimensional vector space over an infinite field $K$ and let $\mu$ and $\lambda$ be the cardinal numbers such that $|K| < \mu < \lambda \leq \dim(W)^+$. Then there exists a star-like $M \subsetneq W$ containing $0$ such that $A(M) = \mathrm{m}\mathcal{L}(M) = \mu$ and $\mathcal{L}(M) = \lambda$.*

*Proof.* For $S \subset W$, let $V(S)$ be the vector subspace of $W$ spanned by $S$.

Let $B$ be a basis for $W$. For $w \in W$, let $\mathrm{supp}(w)$ be the smallest subset $S$ of $B$ with $w \in V(S)$ and let $c_w \colon \mathrm{supp}(w) \to K$ be such that $w = \sum_{b \in \mathrm{supp}(w)} c_w(b) b$. Let $E$ be the set of all cardinal numbers less than $\lambda$ and choose a sequence $\langle B_\eta \colon \eta \in E \rangle$ of pairwise disjoint subsets of $B$ such that $|B_0| = \mu$ and $|B_\eta| = \eta$ whenever $0 \neq \eta \in E$. Define

$$M = \mathcal{A} \cup \bigcup_{\eta \in E} V(B_\eta),$$

where

$$\mathcal{A} = \{w \in W \colon c_w(b_0) = c_w(b_1) \text{ for some } b_0 \in \mathrm{supp}(w) \cap B_0,\ b_1 \in \mathrm{supp}(w) \setminus B_0\}.$$

We will show that $M$ is as desired.

Clearly, $M$ is star-like and $0 \in M \subsetneq W$. Also, $\mathcal{L}(M) \geq \lambda$, since for any cardinal $\eta < \lambda$ the set $M$ contains a vector subspace $V(B_\eta)$ with $\dim(V(B_\eta)) \geq \eta$.

To see that $A(M) \geq \mu$, choose an $F \subset W$ with $|F| < \mu$. It is enough to show that $|F| < A(M)$, that is, that there exists a $w \in W$ with $w + F \subset \mathcal{A}$. As $\operatorname{supp}(F) = \bigcup_{v \in F} \operatorname{supp}(v)$ has cardinality at most $|F| + \omega < \mu = |B_0| < \lambda \leq |B \setminus B_0|$, there exist $b_0 \in B_0 \setminus \operatorname{supp}(F)$ and $b_1 \in B \setminus (B_0 \cup \operatorname{supp}(F))$. Let $w = b_0 + b_1$ and notice that $w + F \subset \mathcal{A} \subset M$, since for every $v \in F$ we have $b_0 \in \operatorname{supp}(w + v) \cap B_0$, $b_1 \in \operatorname{supp}(w + v) \setminus B_0$, and $c_{w+v}(b_0) = 1 = c_{w+v}(b_1)$.

Next notice that the inequalities $|K| < \mu \leq A(M)$ and Proposition 2.1 imply that $\mu \leq A(M) \leq \mathrm{m}\mathcal{L}(M)$. Thus, to finish the proof, it is enough to show that $\mathrm{m}\mathcal{L}(M) \leq \mu$ and $\mathcal{L}(M) \leq \lambda$.

To see that $\mathrm{m}\mathcal{L}(M) \leq \mu$, it is enough to show that $V(B_0)$ is a maximal vector subspace of $M$. Indeed, if $V$ is a vector subspace of $W$ properly containing $V(B_0)$, then there exists a non-zero $v \in V \cap V(B \setminus B_0)$. Choose a $b_0 \in B_0$ and a non-zero $c \in K \setminus \{c_v(b) \colon b \in \operatorname{supp}(v)\}$. Then $cb_0 + v \in V \setminus M$. So, $V(B_0)$ is a maximal vector subspace of $M$ and indeed $\mathrm{m}\mathcal{L}(M) \leq \dim(V(B_0)) = \kappa$.

To see that $\mathcal{L}(M) \leq \lambda$, choose a vector subspace $V$ of $W$ of dimension $\lambda$. It is enough to show that $V \setminus M \neq \varnothing$. To see this, for every ordinal $\eta \leq \lambda$ let us define $\hat{B}_\eta = \bigcup \{B_\zeta \colon \zeta \in E \cap \eta\}$. Notice that

$$\text{for every } \eta < \lambda \text{ there is a non-zero } w \in V \text{ with } \operatorname{supp}(w) \cap \hat{B}_\eta = \varnothing.$$

Indeed, if $\pi_\eta \colon W = V(\hat{B}_\eta) \oplus V(B \setminus \hat{B}_\eta) \to V(\hat{B}_\eta)$ is the natural projection, then there exist distinct $w_1, w_2 \in V$ with $\pi_\eta(w_1) = \pi_\eta(w_2)$ (as $|V(\hat{B}_\eta)| < \lambda = \dim(V)$). Then $w = w_1 - w_2$ is as required.

Now, choose a non-zero $w_1 \in V$ with $\operatorname{supp}(w_1) \cap B_0 = \operatorname{supp}(w_1) \cap \hat{B}_1 = \varnothing$. Then, $w_1 \notin \mathcal{A}$ and if $\operatorname{supp}(w_1) \not\subset \hat{B}_\lambda = \bigcup_{\eta \in E} B_\eta$, then also $w_1 \notin \bigcup_{\eta \in E} V(B_\eta)$, and we have $w_1 \in V \setminus M$. Therefore, we can assume that $\operatorname{supp}(w_1) \subset \hat{B}_\lambda = \bigcup_{\eta < \lambda} \hat{B}_\eta$. Let $\eta < \lambda$ be such that $\operatorname{supp}(w_1) \subset \hat{B}_\eta$ and choose a non-zero $w_2 \in V$ with $\operatorname{supp}(w_2) \cap \hat{B}_\eta = \varnothing$. Then $w = w_2 - w_1 \in V \setminus M$ (since $w \notin \mathcal{A}$, being non-zero with $\operatorname{supp}(w) \cap B_0 = \varnothing$, and $w \notin \bigcup_{\eta \in E} V(B_\eta)$, as its support intersects two different $B_\eta$). □

**Lemma 2.4.** *Let $W$, $W_0$, and $W_1$ be the vector spaces over an infinite field $K$ such that $W = W_0 \oplus W_1$. Let $M \subsetneq W_0$ and*

$$\mathcal{F} = M + W_1 = \{m + W \colon m \in M \ \&\ w \in W_1\}.$$

*Then*

(I) *If $M$ is star-like, then $\mathcal{F}$ is also star-like.*
(II) $A(\mathcal{F}, W) = A(M, W_0)$.
(III) *If $0 \in M$, then* $\mathrm{m}\mathcal{L}(\mathcal{F}) = \mathrm{m}\mathcal{L}(M) + \dim(W_1)$.
(IV) *If $0 \in M$ and* $\dim(W_1) < \mathcal{L}(M)$*, then* $\mathcal{L}(\mathcal{F}) = \mathcal{L}(M) + \dim(W_1)$.

*Proof.* In the following, let $\pi_0 \colon W = W_0 \oplus W_1 \to W_0$ be the canonical projection.

(I) Let $x \in \mathcal{F}$ and $\lambda \in K \setminus \{0\}$. Since $M$ is star-like and $\pi_0(x) \in M$, we have that $\pi_0(\lambda x) = \lambda \pi_0(x) \in M$, and hence $\lambda x \in M + W_1 = \mathcal{F}$.

(II) Let us see that $A(M, W_0) \leq A(\mathcal{F}, W)$. To this end, let $\kappa < A(M, W_0)$. We need to prove that $\kappa < A(\mathcal{F}, W)$. Indeed, if $F \subset W$ and $|F| = \kappa$, then $|\pi_0[F]| \leq$

$|F| = \kappa$. So, there exists a $w_0 \in W_0$ such that $\pi_0[w_0 + F] = w_0 + \pi_0[F] \subset M$, that is, $w_0 + F \subset M + W_1 = \mathcal{F}$. Therefore, $\kappa < A(\mathcal{F}, W)$.

To see that $A(\mathcal{F}, W) \leq A(M, W_0)$ let $\kappa < A(\mathcal{F}, W)$. We need to show that $\kappa < A(M, W_0)$. Indeed, let $F \subset W_0$ be such that $|F| = \kappa$. Since $|F| < A(\mathcal{F}, W)$, there is a $w \in W$ with $w + F \subset \mathcal{F}$. Then $\pi_0(w) \in W_0$ and $\pi_0(w) + F = \pi_0[w + F] \subset \pi_0[\mathcal{F}] = M$, so indeed $\kappa < A(M)$.

(iii) First notice that it is enough to show that

$$V \text{ is a maximal vector subspace of } \mathcal{F} \text{ if, and only if, } V = V_0 + W_1, \text{ where } V_0 \text{ is a maximal vector subspace of } M. \tag{3}$$

Indeed, if $V$ is a maximal vector subspace of $\mathcal{F}$ with $\mathrm{m}\mathcal{L}(\mathcal{F}) = \dim(V)$, then, by (3), $\mathrm{m}\mathcal{L}(\mathcal{F}) = \dim(V) = \dim(V_0) + \dim(W_1) \geq \mathrm{m}\mathcal{L}(M) + \dim(W_1)$. Conversely, if $V_0$ is a maximal vector subspace of $M$ with $\mathrm{m}\mathcal{L}(M) = \dim(V_0)$, then we have $\mathrm{m}\mathcal{L}(M) + \dim(W_1) = \dim(V_0) + \dim(W_1) = \dim(V_0 + W_1) \geq \mathrm{m}\mathcal{L}(\mathcal{F})$.

To see (3), take a maximal vector subspace $V$ of $\mathcal{F}$. Notice that $W_1 \subset V$, since $V \subset V + W_1 \subset \mathcal{F} + W_0 = \mathcal{F}$ and so, by the maximality, $V + W_1 = V$. In particular, $V = V_0 + W_1 \subset \mathcal{F} = M + W_1$, where $V_0 = \pi_0[V]$. Thus, $V_0$ is a vector subspace of $M$. It must be maximal, since for any its proper extension $\hat{V}_0 \subset M$, the vector space $\hat{V}_0 + W_1 \subset \mathcal{F}$ would be a proper extension of $V$.

Conversely, if $V_0$ is a maximal vector subspace of $M$, then $V = V_0 + W_1$ is a vector subspace of $\mathcal{F}$. If cannot have a proper extension $\hat{V} \subset \mathcal{F}$, since then the vector space $\pi_0[\hat{V}] \subset M$ would be a proper extension of $V_0$.

(iv) To see that $\mathcal{L}(\mathcal{F}) \leq \dim(W_1) + \mathcal{L}(M)$, choose a vector space $V \subset \mathcal{F}$. We need to show that $\dim(V) < \dim(W_1) + \mathcal{L}(M)$. Indeed, $V_1 = V + W_1$ is a vector subspace of $\mathcal{F} + W_1 = \mathcal{F}$ and $\dim(V) \leq \dim(V_1) = \dim(W_1) + \dim(\pi_0[V_1])$, since $V_1 = W_1 \oplus \pi_0[V_1]$. Therefore, $\dim(V) \leq \dim(W_1) + \dim(\pi_0[V_1]) < \dim(W_1) + \mathcal{L}(M)$, since $\dim(W_1) < \mathcal{L}(M)$ and $\dim(\pi_0[V_1]) < \mathcal{L}(M)$, as $\pi_0[V_1]$ is a vector subspace of $M = \pi_0[\mathcal{F}]$. So, $\mathcal{L}(\mathcal{F}) \leq \dim(W_1) + \mathcal{L}(M)$.

To see that $\dim(W_1) + \mathcal{L}(M) \leq \mathcal{L}(\mathcal{F})$, choose a $\kappa < \dim(W_1) + \mathcal{L}(M)$. We need to show that $\kappa < \mathcal{L}(\mathcal{F})$, that is, that there exists a vector subspace $V$ of $\mathcal{F}$ with $\dim(V) \geq \kappa$. First, notice that $\dim(W_1) < \mathcal{L}(M)$ and $\kappa < \dim(W_1) + \mathcal{L}(M)$ imply that there exists a $\mu < \mathcal{L}(M)$ such that $\kappa \leq \dim(W_1) + \mu < \dim(W_1) + \mathcal{L}(M)$. (For finite value of $\mathcal{L}(M)$, take $\mu = \max\{\kappa - \dim(W_1), 0\}$; for infinite $\mathcal{L}(M)$, the number $\mu = \max\{\kappa, \dim(W_1)\}$ works.) Choose a vector subspace $V_0$ of $M$ with $\dim(V_0) \geq \mu$. Then the vector subspace $V = V_0 + W_1 = V_0 \oplus W_1$ of $\mathcal{F}$ is as desired, since we have $\dim(V) = \dim(W_1) + \dim(V_0) \geq \dim(W_1) + \mu \geq \kappa$. □

*Proof of Theorem 2.2.* Represent $W$ as $W_0 \oplus W_1$, where $\dim(W_0) = \lambda$ and $\dim(W_1) = \mu$. Use Lemma 2.3 to find a star-like $M \subsetneq W_0$ containing 0 such that $A(M, W_0) = \mathrm{m}\mathcal{L}(M) = \alpha$ and $\mathcal{L}(M) = \lambda$. Let $\mathcal{F} = M + W_1 \subsetneq B$. Then, by Lemma 2.4, $\mathcal{F} \ni 0$ is star-like such that $A(\mathcal{F}) = A(M, W_0) = \alpha$, $\mathrm{m}\mathcal{L}(\mathcal{F}) = \mathrm{m}\mathcal{L}(M) + \dim(W_1) = \alpha + \mu = \mu$, and $\mathcal{L}(\mathcal{F}) = \mathcal{L}(M) + \dim(W_2) = \lambda + \alpha = \lambda$, as required. □

A. Bartoszewicz and S. Głąb have asked [2, open question 1] whether the inequality $A(\mathcal{F})^+ \geq \mathcal{HL}(\mathcal{F})$ (which is equivalent to $A(\mathcal{F}) \geq \mathrm{m}\mathcal{L}(\mathcal{F})$) holds for any family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$. Of course, for the star-like families $\mathcal{F}$ with $A(\mathcal{F}) > \mathfrak{c}$, a positive answer to this question would mean that, under these assumptions, we have $A(\mathcal{F}) = \mathrm{m}\mathcal{L}(\mathcal{F})$. Notice that Theorem 2.2 gives, in particular, a negative answer to this question.

We do not have a comprehensive example, similar to that provided by Theorem 2.2, for the case when $A(M) \leq |K|$. However, the machinery built above, together with the results from [2], lead to the following result.

**Theorem 2.5.** *Let $W$ a vector space over an infinite field $K$ with $\dim(W) \geq 2^{|K|}$. If $2 \leq \kappa \leq |W|$, there exists a star-like family $\mathcal{F} \subsetneq W$ containing $0$ such that $A(\mathcal{F}) = \kappa$ and $\mathrm{m}\mathcal{L}(\mathcal{F}) = \dim(W)$ (so that $\mathcal{L}(\mathcal{F}) = \dim(W)^+$).*

*Proof.* Represent $W$ as $W = W_0 \oplus W_1$, where $\dim(W_0) = 2^{|K|}$ and $\dim(W_1) = \dim(W)$. If $2 \leq \kappa \leq |K|$, then, by [2, Theorem 2.5], there exists a star-like family $M \subset W_0$ such that $A(M, W_0) = \kappa$. Notice that the family constructed in that result contains 0. Then, by Lemma 2.4, the family $\mathcal{F} = M + W_1$ satisfies that $A(\mathcal{F}) = A(M, W_0) = \kappa$ and $\mathrm{m}\mathcal{L}(\mathcal{F}) = \mathrm{m}\mathcal{L}(M) + \dim(W_1) = \dim(W)$. □

## 3. Spaceability of Darboux-like functions on $\mathbb{R}$

Recall (see, e.g., [8, chart 1] or [7]) that we have the following strict inclusions, indicated by the arrows, between the Darboux-like functions from $\mathbb{R}$ to $\mathbb{R}$. The next theorem, strengthening the results presented in the table from [4, page 14], determines fully the lineability, $\mathcal{L}$, and spaceability, $\mathcal{L}_p$, numbers for these classes.

$$\begin{tikzcd}
 & & \mathrm{AC} \arrow[r] & \mathrm{Conn} \arrow[r] & \mathrm{D} \arrow[dr] & \\
\mathrm{C} \arrow[r] & \mathrm{Ext} \arrow[ur] \arrow[dr] & & & & \mathrm{PC} \\
 & & \mathrm{SCIVP} \arrow[r] & \mathrm{CIVP} \arrow[r] \arrow[dr] & \mathrm{PR} \arrow[ur] & \\
 & & & & \mathrm{WCIVP} &
\end{tikzcd}$$

Figure 1. Relations between the Darboux-like classes of functions from $\mathbb{R}$ to $\mathbb{R}$. Arrows indicate strict inclusions.

**Theorem 3.1.** $\mathcal{L}_p(\mathrm{Ext}) = (2^{\mathfrak{c}})^+$. *In particular, all Darboux-like classes of functions from Figure 1, except* C, *are $2^{\mathfrak{c}}$-spaceable with respect to the topology of pointwise convergence.*

*Proof.* In [11, corollary 3.4] it is shown that there exists an $f \in \mathrm{Ext}$ and an $F_\sigma$ first category set $M \subset \mathbb{R}$ such that

$$\text{if } g \in \mathbb{R}^{\mathbb{R}} \text{ and } g \restriction M = f \restriction M, \text{ then } g \in \mathrm{Ext}. \tag{4}$$

It is easy to see that for any real number $r \neq 0$ the function $rf$ satisfies the same property.

Notice also that there exists a family $\{ h_\xi \in \mathbb{R}^{\mathbb{R}} \colon \xi < \mathfrak{c} \}$ of increasing homeomorphisms such that the sets $M_\xi = h_\xi[M]$, $\xi < \mathfrak{c}$, are pairwise disjoint. (See, e.g., [11, lemma 3.2].) It is easy to see that each function $f_\xi = f \circ h_\xi^{-1}$ satisfies (4) with the set $M_\xi$. Increasing one of the sets $M_\xi$, if necessary, we can also assume that $\{M_\xi \colon \xi < \mathfrak{c}\}$ is a partition of $\mathbb{R}$. Let $\vec{f} = \langle f_\xi \restriction M_\xi \colon \xi < \mathfrak{c} \rangle$ and define

$$V(\vec{f}) = \left\{ \bigcup_{\xi < \mathfrak{c}} t(\xi)(f_\xi \restriction M_\xi) \colon t \in \mathbb{R}^{\mathfrak{c}} \right\}. \tag{5}$$

It is easy to see that $V(\vec{f})$ is $2^{\mathfrak{c}}$-dimensional $\tau_p$-closed linear subspace of Ext. $\square$

As the cardinality of the family $\mathcal{B}$or of Borel functions from $\mathbb{R}$ to $\mathbb{R}$ is $\mathfrak{c}$, Theorem 3.1 easily implies that $\mathrm{Ext}\setminus\mathcal{B}\mathrm{or}$ is $2^{\mathfrak{c}}$-lineable: $\mathcal{L}(\mathrm{Ext}\setminus\mathcal{B}\mathrm{or}) = (2^{\mathfrak{c}})^+$. Actually, we have an even stronger result:

**Proposition 3.2.** $\mathcal{L}_p(\mathrm{Ext}\cap\mathrm{SES}\setminus\mathcal{B}\mathrm{or}) = (2^{\mathfrak{c}})^+$.

*Proof.* The function $f\restriction M$ satisfying (4) may also have the property that

$$M \text{ is } \mathfrak{c}\text{-dense in } \mathbb{R} \text{ and } f\restriction M \text{ is SES non-Borel.} \tag{6}$$

Indeed, this can be ensured by enlarging $M$ by a $\mathfrak{c}$-dense first category set $N\subset\mathbb{R}\setminus M$ and redefining $f$ on $N$ so that $f\restriction N$ is non-Borel and SES.

Now, if $f$ satisfies both (4) and (6) and $\vec{f} = \langle f_\xi\restriction M_\xi \colon \xi<\mathfrak{c}\rangle$ is defined as in Theorem 3.1, then the space $V(\vec{f})$ given in (5) is as required. $\square$

Notice also that $\mathrm{Ext}\cap\mathrm{PES} = \mathrm{PR}\cap\mathrm{PES} = \varnothing$. In particular, the space $V$ from Proposition 3.2 is disjoint with PES.

**Remark 3.3.** Clearly, Theorem 3.1 implies that Ext is $2^{\mathfrak{c}}$-lineable. This result has been also independently proved by T. Natkaniec. (See preprint [22].) The technique used in [22] is similar, but different from that used in the proof of Theorem 3.1.

Recall, that it is known that $\mathcal{L}(\mathrm{AC}\setminus\mathrm{Ext}) = (2^{\mathfrak{c}})^+$. See [15] or [4, page 14]. However, we do not know what the exact values of the following cardinals are.

**Problem 3.4.** Determine the following numbers:

$$\mathcal{L}_p(\mathcal{F}\setminus\mathcal{G}),\ \mathcal{L}_u(\mathcal{F}\setminus\mathcal{G}),\ \text{and}\ \mathcal{L}_u(\mathcal{F}\setminus\mathcal{G})$$

for $\mathcal{F}\in\{\mathrm{Conn}\setminus\mathrm{AC}, \mathrm{D}\setminus\mathrm{Conn}, \mathrm{PC}\setminus\mathrm{D}\}$ and $\mathcal{G}\in\{\mathrm{SCIVP}, \mathrm{CIVP}, \mathrm{PR}\}$.

**Problem 3.5.** Is it consistent with the axioms of set theory ZFC that either $A(\mathcal{F}) < \mathrm{m}\mathcal{L}(\mathcal{F})$ or $\mathrm{m}\mathcal{L}(\mathcal{F})^+ < \mathcal{L}(\mathcal{F})$ for any of the classes $\mathcal{F}\in\{\mathrm{Ext}, \mathrm{AC}, \mathrm{Conn}, \mathrm{D}, \mathrm{PC}\}$?

Notice, that the generalized continuum hypothesis GCH implies that $A(\mathcal{F}) = \mathrm{m}\mathcal{L}(\mathcal{F})$ and $\mathrm{m}\mathcal{L}(\mathcal{F})^+ = \mathcal{L}(\mathcal{F})$ for every $\mathcal{F}\in\{\mathrm{Ext}, \mathrm{AC}, \mathrm{Conn}, \mathrm{D}, \mathrm{PC}\}$.

## 4. Spaceability of Darboux-like functions on $\mathbb{R}^n$, $n\geq 2$

Recall (see, e.g., [8, chart 2] or [7]) that we have the following strict inclusions, indicated by the arrows, between the Darboux-like functions from $\mathbb{R}^n$ to $\mathbb{R}$ for $n\geq 2$.

$$\begin{tikzcd}
& \mathrm{Conn}(\mathbb{R}^n) \arrow[d, equal] & & \mathrm{AC}(\mathbb{R}^n) \\
\mathrm{C}(\mathbb{R}^n) \arrow[r] & \mathrm{Ext}(\mathbb{R}^n) \arrow[r] & \mathrm{AC}(\mathbb{R}^n)\cap\mathrm{D}(\mathbb{R}^n) \arrow[ur] \arrow[dr] & \\
& \mathrm{PC}(\mathbb{R}^n) \arrow[u, equal] & & \mathrm{D}(\mathbb{R}^n)
\end{tikzcd}$$

Figure 2. Relations between the Darboux-like classes of functions from $\mathbb{R}^n$ to $\mathbb{R}$, $n\geq 2$. Arrows indicate strict inclusions.

The proof of the next theorem will be based on the following result [12, Proposition 2.7]:

**Proposition 4.1.** *Let $n > 0$ and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a peripherally continuous function. Then for any $x_0 \in \mathbb{R}^n$ and any open set $W$ in $\mathbb{R}^n$ containing $x_0$, there exists an open set $U \subseteq W$ such that $x_0 \in U$ and the restriction of $f$ to $\mathrm{bd}(U)$ is continuous. Moreover, given any $\varepsilon > 0$, the set $U$ can be chosen so that $|f(x_0) - f(y)| < \varepsilon$ for every $y \in \mathrm{bd}(U)$.*

**Theorem 4.2.** *For $n \geq 2$, $\mathcal{L}_p(\mathrm{Ext}(\mathbb{R}^n)) = \mathcal{L}_u(\mathrm{Ext}(\mathbb{R}^n)) = \mathcal{L}(\mathrm{Ext}(\mathbb{R}^n)) = \mathfrak{c}^+$. In particular, the classes $\mathrm{C}(\mathbb{R}^n)$ and $\mathrm{Ext}(\mathbb{R}^n)$ are $\mathfrak{c}$-spaceable with respect to the pointwise convergence topology $\tau_p$ but are not $\mathfrak{c}^+$-lineable.*

*Proof.* First, notice that $\mathcal{L}_p(\mathrm{C}(\mathbb{R}^n)) = \mathfrak{c}^+$ is justified by the space $\mathrm{C}_0$ of all continuous functions linear on the interval $[k, k+1]$ for every integer $k \in \mathbb{Z}$. Indeed, $\mathrm{C}_0$ is linearly isomorphic to $\mathbb{R}^{\mathbb{Z}}$.

Now, since $\mathfrak{c}^+ = \mathcal{L}_p(\mathrm{C}(\mathbb{R}^n)) \leq \mathcal{L}_p(\mathrm{Ext}(\mathbb{R}^n)) \leq \mathcal{L}_u(\mathrm{Ext}(\mathbb{R}^n)) \leq \mathcal{L}(\mathrm{Ext}(\mathbb{R}^n))$, it is enough to show that $\mathcal{L}(\mathrm{Ext}(\mathbb{R}^n)) \leq \mathfrak{c}^+$, that is, that $\mathrm{Ext}(\mathbb{R}^n)$ is not $\mathfrak{c}^+$-lineable. To see this, by way of contradiction, assume that there exists a vector space $V \subset \mathrm{Ext}(\mathbb{R}^n)$ of cardinality greater than $\mathfrak{c}$. Fix a countable dense set $D \subset \mathbb{R}^n$ and let $\langle\langle x_k, \varepsilon_k\rangle\colon k < \omega\rangle$ be an enumeration of $D \times \{2^{-m}\colon m < \omega\}$. By Proposition 4.1, for every function $f \in \mathrm{Ext}(\mathbb{R}^n)$ and $k < \omega$ we can choose an open neighborhood $U_k^f$ of $x_k$ of the diameter at most $\varepsilon_k$ such that $f \restriction \mathrm{bd}(U_k^f)$ is continuous. Consider the mapping $V \ni f \mapsto T_f = \langle f \restriction \mathrm{bd}(U_k^f)\colon k < \omega\rangle$. Since its range has cardinality $\mathfrak{c}$, there are distinct $f_1, f_2 \in V$ with $T_{f_1} = T_{f_2}$. In particular, $f = f_1 - f_2 \in V$ is equal zero on the set $M = \bigcup_{k<\omega} \mathrm{bd}(U_k^{f_1})$. Notice that the complement $M^c$ of $M$ is zero-dimensional. We will show that $f$ is not extendable, by showing that it does not satisfy Proposition 4.1.

Indeed, since $f_1 \neq f_2$, there is an $x \in \mathbb{R}^n$ with $f(x) \neq 0$. Let $\varepsilon = |f(x)|$ and let $W$ be any bounded neighborhood of $x$. Then, there is no set $U$ as required by Proposition 4.1.

To see this, notice that for any open set $U \subseteq W$ with $x \in U$, its boundary is of dimension at least 1. In particular, $M \cap \mathrm{bd}(U) \neq \varnothing$ and, for $y \in M \cap \mathrm{bd}(U)$, we have $|f(x) - f(y)| = |f(x)| = \varepsilon$. □

Theorem 4.2 determines the values of the numbers $\mathcal{L}_p(\mathcal{F})$, $\mathcal{L}_u(\mathcal{F})$, and $\mathcal{L}(\mathcal{F})$ for $\mathcal{F} \in \{\mathrm{C}(\mathbb{R}^n), \mathrm{Ext}(\mathbb{R}^n), \mathrm{Conn}(\mathbb{R}^n), \mathrm{PR}(\mathbb{R}^n)\}$ and $n \geq 2$. In the remainder of this section we will examine these cardinal numbers for the remaining classes from the diagram in Figure 2. For this, we will need the following fact, improving a recent result of the second author, see [14, Theorem 2.2].

**Proposition 4.3.** $\mathcal{L}_p(J(\mathbb{R}^n)) = (2^{\mathfrak{c}})^+$ *for every $n \geq 1$. In particular, the families $J(\mathbb{R}^n)$, $\mathrm{PES}(\mathbb{R}^n)$, $\mathrm{SES}(\mathbb{R}^n)$, and $\mathrm{ES}(\mathbb{R}^n)$ are $2^{\mathfrak{c}}$-spaceable with respect to the topology of pointwise convergence.*

*Proof.* Let $\{\mathcal{B}_\xi\colon \xi < \mathfrak{c}\}$ be a decomposition of $\mathbb{R}^n$ into pairwise disjoint Bernstein sets. For every $\xi < \mathfrak{c}$, let $f_\xi\colon B_\xi \to \mathbb{R}$ be such that $f_\xi \cap F \neq \varnothing$ for every closed set $F \subset \mathbb{R}^n \times \mathbb{R}$ whose projection on $\mathbb{R}^n$ is uncountable. (All of this can be easily constructed by transfinite induction. See, e.g., [6].) Notice that

$$\text{if } g \in \mathbb{R}^{\mathbb{R}} \text{ and } g \restriction M_\xi = r\ f_\xi \text{ for some } \xi < \mathfrak{c} \text{ and } r \neq 0, \text{ then } g \in J(\mathbb{R}^n).$$

Now, if $\vec{f} = \langle f_\xi \restriction M_\xi\colon \xi < \mathfrak{c}\rangle$ and $V(\vec{f})$ is given by (5), then $V(\vec{f})$ is $2^{\mathfrak{c}}$-dimensional $\tau_p$-closed linear subspace of $J(\mathbb{R}^n)$. □

Every function in $\mathrm{J}(\mathbb{R}^n)$ is surjective. In particular, the above result implies that the class of surjective functions is $2^{\mathfrak{c}}$-lineable. One could also wonder about the lineability of the family of one-to-one functions from $\mathbb{R}^n$ to $\mathbb{R}$, given below.

**Remark 4.4.** The family of one-to-one functions from $\mathbb{R}^n$ to $\mathbb{R}$ is 1-lineable but not 2-lineable.

*Proof.* Clearly the family is 1-lineable. To see that is not 2-lineable, choose two injective linearly independent functions $f$ and $g$ generating a linear space $Z$. Take arbitrary $x \neq y$ in $\mathbb{R}^n$ and consider the function $h = f + \alpha g \in Z \setminus \{0\}$, where $\alpha = (f(x) - f(y))/(g(y) - g(x)) \in \mathbb{R}$. Then, we have $h(x) = h(y)$, so $Z$ contains a function which is not one-to-one. □

Other examples of 1-lineable but not 2-lineable sets and, in general, not lineable sets can be found in [4, 5].

**Theorem 4.5.** *For $n \geq 2$, $\mathrm{J}(\mathbb{R}^n) \subset \mathrm{AC}(\mathbb{R}^n) \setminus \mathrm{D}(\mathbb{R}^n)$. In particular, the class $\mathrm{AC}(\mathbb{R}^n) \setminus \mathrm{D}(\mathbb{R}^n)$ is $2^{\mathfrak{c}}$-spaceable and $\mathcal{L}_p(\mathrm{AC}(\mathbb{R}^n) \setminus \mathrm{D}(\mathbb{R}^n)) = (2^{\mathfrak{c}})^+$.*

*Proof.* By Proposition 4.3, it is enough to show that $\mathrm{J}(\mathbb{R}^n) \subset \mathrm{AC}(\mathbb{R}^n) \setminus \mathrm{D}(\mathbb{R}^n)$. Clearly, $\mathrm{J}(\mathbb{R}^n) \subset \mathrm{AC}(\mathbb{R}^n) \cap \mathrm{PES}(\mathbb{R}^n)$ for any $n \geq 1$. Thus, it is enough to show that $\mathrm{PES}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n) = \varnothing$ for $n \geq 2$. But this follows immediately from the fact that, under $n \geq 2$, every Bernstein set in $\mathbb{R}^n$ is connected. □

**Remark 4.6.** Notice that, since $\mathrm{AC}(\mathbb{R}^n) \subset \mathrm{D_P}(\mathbb{R}^n)$, then, for $n \geq 2$, we have $\mathcal{L}_p(\mathrm{D_P}(\mathbb{R}^n) \setminus \mathrm{D}(\mathbb{R}^n)) = (2^{\mathfrak{c}})^+$. So, $\mathrm{D_P}(\mathbb{R}^n) \setminus \mathrm{D}(\mathbb{R}^n)$ is also $2^{\mathfrak{c}}$-spaceable.

**Theorem 4.7.** *For $n \geq 2$, $\mathcal{L}_p(\mathrm{D}(\mathbb{R}^n) \setminus \mathrm{AC}(\mathbb{R}^n)) = (2^{\mathfrak{c}})^+$. In particular, the class $\mathrm{D}(\mathbb{R}^n) \setminus \mathrm{AC}(\mathbb{R}^n)$ is $2^{\mathfrak{c}}$-spaceable.*

*Proof.* Let $\pi_1 \colon \mathbb{R}^n \to \mathbb{R}$ the projection of $\mathbb{R}^n$ on its first coordinate. Let $W = V(\vec{f}) \subset \mathrm{J}$ be the vector space of cardinality $2^{\mathfrak{c}}$ build in Proposition 4.3. Then the vector space

$$V = \{\, f \circ \pi_1 : f \in W \,\}$$

is obviously contained in $\mathrm{D}(\mathbb{R}^n)$ and has dimension $2^{\mathfrak{c}}$. On the other side, if $f \in W$ then $f \circ \pi_1$ cannot be in $\mathrm{AC}(\mathbb{R}^n)$, because then $f$ would be continuous. (See [19].) This is not possible, because $\mathrm{J} \cap \mathrm{C} = \varnothing$. Therefore, $V \subset \mathrm{D}(\mathbb{R}^n) \setminus \mathrm{AC}(\mathbb{R}^n)$. To finish, let us remark that the space $V$ is also closed by pointwise convergence. □

**Remark 4.8.** Notice that, in $\mathbb{R}^n$ (for every $n \in \mathbb{N}$), we have that $\mathrm{AC} \setminus \mathrm{Ext}$ is $2^{\mathfrak{c}}$-spaceable (since this class contains the Jones functions). Also, in $\mathbb{R}$, $\mathrm{J} \subset \mathrm{AC} \setminus \mathrm{SCIVP} \subset \mathrm{AC} \setminus \mathrm{Ext}$ and, since $\mathcal{L}_p(\mathrm{J}) = (2^{\mathfrak{c}})^+$, we have (from the previous results) that

$$\mathcal{L}_p(\mathrm{AC} \setminus \mathrm{Ext}) = \mathcal{L}_u(\mathrm{AC} \setminus \mathrm{Ext}) = (2^{\mathfrak{c}})^+.$$

**Problem 4.9.** For $n \geq 2$, determine the values of the numbers $\mathcal{L}_p(\mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n))$, $\mathcal{L}_u(\mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n))$, and $\mathcal{L}_p(\mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n))$.

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