Robust estimators for the log-logistic model based on ranked set sampling

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Abstract

In this paper, we introduce a novel family of estimators for the shape and scale parameters of the log-logistic distribution being robust when rank set sample method for data selection is used. Rank set sampling effectively reduces the influence of extreme data points. The log-logistic distribution is a versatile model, suitable in various fields such as Economics, Engineering, and Hydrology. Our proposed family of estimators is based on the density power divergence, chosen for its demonstrated robustness and efficiency. Notably, this family includes the classical maximum likelihood estimator as a special case. Besides explicit forms of the estimators, their asymptotic distribution is derived, proving the consistency of the estimators. Finally, a comprehensive simulation study illustrates the significant robustness of the proposed estimators in the presence of data contamination, while also performing competitively with traditional estimators, including the maximum likelihood estimator in terms of efficiency.

Keywords: Log-logistic distribution, rank set sample, Minimum density power divergence estimator, robustness, efficiency.
AMS 2001 Subject Classification: 62F35, 62J12.

1 Introduction

In this paper we deal with the log-logistic distribution. This distribution arises as the logarithmic transformation of the logistic distribution and it is a special case of the Burr type-II distribution family (Burr, 1942) and the Kappa distribution family (Mielke and Johnson, 1973). The log-logistic distribution was first studied in (Bain, 1974) and it has shown to be a suitable model for many different applications. For example, it has been applied in Hydrology (Shoukri et al., 1988) serving as a reliable option for modeling river flows. In Economics, it is a well-known distribution for studying wealth or income distributions, where it is known as the Fisk distribution (Fisk, 1961). Other fields where log-logistic distribution has been applied include Engineering (Ashkar and Mahdi, 2003) and Medical Sciences for cancer studies (Gupta et al., 1999).

The log-logistic distribution depends on two parameters, here denoted by α and β; the parameter α models the scale of the distribution and coincides with the median of the distribution, while parameter β is a shape parameter controlling the form of the distribution. A common challenge in the practical application of the log-logistic distribution is the accurate estimation of its parameters α β. The classical method is the maximum likelihood estimator (MLE) approach, relying on the likelihood function of the observed data. Some works dealing with MLE can be found in (Balakrishnan and Malik, 1987; Kantam and Srinavasa, 2002; Reath et al., 2018). Although the MLE is a BAN (Best Asymptotically Normal) estimator and hence provides asymptotic efficient estimations, it is also known that the MLE is highly non-robust estimators, meaning that it can be heavily influenced by outlying observations. To deal with this problem, several robust estimators have been proposed in the literature, mainly based on position measures that are more robust to data contamination (see for example (Ashkar and Mahdi, 2003; Gupta et al., 1999; Shoukri et al., 1988; Abbas and Tang, 2016; He et al., 2021)). A survey of these procedures can be found in (Ma et al., 2023), where it is also introduced the repeated median (RM) estimator.

In a recent paper (Felipe et al., 2023), we introduced the minimum density power divergence estimators (MDPDE) based on density power divergence (DPD) as a robust alternative for estimating the parameters of the log-logistic distribution. The DPD family was first presented in (Basu et al., 1998) and it includes the Kullback-Leibler divergence as a particular case. It depends on a tuning parameter τ controlling the trade-off between efficiency and robustness; smaller values of τ yield more efficient but less robust estimators, while large values of τ results in more robust although less efficient estimators. However, for the MDPDE with moderate values of τ the trade-off between robustness and efficiency is well-balanced. Indeed, the MDPDE have been proven to be both efficient and highly robust in various situations, so that they seem to be a logical choice when seeking for robust estimators with a small loss of efficiency. In (Felipe et al., 2023), it has been shown that there exist values of τ providing estimations competitive with MLE in terms of efficiency and robust in the presence of outliers.

In an infinite population following a parametric distribution depending on some unknown parameters, the most commonly method used to infer these unknown parameters is by using a simple random sample from the population under consideration. In this situation it is guaranteed, from a probabilistic point of view, that each observation in the random sample is sufficiently representative of the population under consideration. However, there are other sampling procedures that offer a better representation of the population under consideration. For instance, alternative sampling designs such as stratified sampling and cluster sampling can improve the representation of the sample from the underlying population.. Moreover, another important consideration in collecting a sample is minimizing the costs associated with obtaining it. In this regard, an adequate sampling procedure should minimize the sample size required to achieve a pre-established precision for making pertinent inferences about the parameters of the population under consideration. This is the main purpose of the ranking set sample (RSS).

In this paper, we deal with the case in which data have been obtained via RSS procedure instead of random sampling. This way to build samples was first introduced by McIntyre in (McIntyre, 1952) and it is aimed to obtain samples avoiding extreme values. Hence, it is expected to obtain samples where (possible) sample contamination might be reduced. McIntyre observed that the relative efficiency, i.e. the ratio between the variance of the mean of a simple random sample of size k and the variance of the mean of a ranked set sample of the same size, is not much less than $(k+1)/2$ for symmetric or moderately asymmetric distributions. He also noticed that the relative efficiency diminishes with increasing asymmetry of the underlying distribution but it is always greater than 1. A first study of the theoretical properties of RSS appears in (Takahasi and Wakimoto, 1968). Since then, many papers applying RSS have been published. To mention a few (Dell and Clutter, 1972; Stokes, 1980; Kvam and Samaniego, 1993; Chen et al., 2016; Presnell and Bohn, 1999; Abu-Dayyeh and Muttlak, 1996; Li and Chuiv, 1997; Kaur et al., 1997; Al-Saleh and Al-Hadhrami, 2003) and (Chen et al., 2013) applied RSS.

The MLEs for the parameters of the log-logistic distribution based on RSS were obtained in (He et al., 2020). The main purpose of this paper is to introduce and study the MDPDE for the parameters of the log-logistic distribution and to examine the robustness of this new family of estimators when rank set sampling (RSS) is used.

The rest of the paper goes as follows: Section 2 introduces the MDPDE based on RSS for the parameters of the log-logistic model. The asymptotic distribution of the proposed family of estimators is studied in Section 3. In Section 4, we present a procedure to select the optimal tuning parameter τ under RSS. Section 5 presents an extensive simulation to evaluate the behavior of these new estimators and compare it with other estimators appearing in the literature. In Section 6, we apply this new family of estimators in a real data situation. We finish with some final conclusions and open problems for future research. All proofs of the theoretical results established in the paper are provided in the Supplementary Material, as well as additional numerical results.

2 The minimum density power divergence estimator for the log-logistic model

2.1 Basic concepts

A random variable X is said to have a **log-logistic distribution** if it probability density function is given by

$$f_{\alpha,\beta}(x) = \frac{\beta\alpha^{\beta}x^{\beta-1}}{\left(x^{\beta}+\alpha^{\beta}\right)^{2}}, \ x > 0. \tag{1}$$

where the parameter $\alpha > 0$ is a scale parameter and the parameter $\beta > 0$ is a shape parameter. Moreover, the log-logistic distribution is unimodal when $\beta > 1$ and its dispersion decreases as β increases. As it has been pointed out in the Introduction, the log-logisitc distribution has been applied in many different research fields and it specially useful in Economics.

Let us now discuss the **rank set sample procedure** (RSS) introduced in (McIntyre, 1952). Let $X_1^1, ..., X_n^1$ be a random sample of size n from the random variable X and let us denote by Y_1 the first order statistic (or the smallest order statistic), $Y_1 \equiv X_{(1)}^1 = \min\left(X_1^1, ..., X_n^1\right)$. Consider $X_1^2, ..., X_n^2$ a second random sample of size n from the same variable and denote by $Y_2 \equiv X_{(2)}^2$ the second order statistics. Following the same idea, consider the i-th random sample from X, $X_1^i, ..., X_n^i$ and denote by $Y_i \equiv X_{(i)}^i$ the i-th order statistic with $i = 1, 2, ..., n$. For the n-th random sample from X, $X_1^n, ..., X_n^n$, $Y_n \equiv X_{(n)}^n$ denotes the largest ordered statistic. This process may be repeated several times (cycles) by dividing the original sample in several groups and applying the previous procedure for each of the groups. An example of this approach is explained in Section 6.

Following this approach, $Y_1, ..., Y_n$ are independent observations but they are not identically distributed. Certainly, it can be easily seen that the distribution of Y_i is given by

$$f_{\alpha,\beta}^{(i)}(y) = \frac{c\,(i,n)\ \ \beta\left(\frac{y}{\alpha}\right)^{i\beta-1}}{\alpha\left(1+\left(\frac{y}{\alpha}\right)^{\beta}\right)^{n+1}}\ y > 0, \qquad y > 0, \tag{2}$$

being

$$c\,(i,n) = \frac{n!}{(n-i)!(i-1)!}.$$

This way of selecting individuals in the final sample has been proved to be more stable in the presence of outliers.

Let us finally introduce the family of **density power divergence** (DPD). This family of divergences has been introduced in (Basu et al., 1998) and it has been applied in many different fields. Given two densities f and g, the DPD measure, $d_\tau(f,g)$, between g and f, as the function of a single tuning parameter τ (≥ 0), is defined by

$$d_\tau(g,f) = \int \left\{ f^{1+\tau}(x) - \left(1+\frac{1}{\tau}\right) f^{\tau}(x)g(x) + \frac{1}{\tau}g^{1+\tau}(x) \right\} dx. \tag{3}$$

Taking limits for $\tau \to 0$, it follows that

$$d_0(g,f) = \lim_{\tau\to 0} d_\tau(g,f) = \int g(x)\log f(x)dx - \int g(x)\log f(x)dx,$$

i.e. the classical Kullback-Leibler divergence (see (Pardo, 2006)) for more details about Kullback-Leibler divergence). The parameter τ controls the trade-off between efficiency and robustness of the MDPDE. Hence, it has been shown in many different problems that DPD offers robust estimators in the presence of outliers at a reduced cost in terms of efficiency, while likelihood-based approach usually behaves poorly in the presence of outliers, (see e.g. (Basu et al., 2022a; Balakrishnan et al., 2019b,a; Patra et al., 2013; Balakrishnan et al., 2024)).

2.2 Minimum density power divergence estimators

Let us then turn to the problem of estimating the parameters of the log-logistic distribution when RSS is considered via DPD. As discussed earlier $Y_1, ..., Y_n$ are independent observations but they are not identically distributed. Thus, we shall assume that the probability density function of each ranked sample $Y_i, i = 1, ..., n$ is different from the rest. For making inference, we aim to model the underlying densities g_i $i = 1, ..., n$, by assuming a parametric form $f^{(i)}_{\alpha,\beta}(y), i = 1, ..., n$. In the following, we assume that the ranked samples follow a log-logistic distribution with density functions $f^{(i)}_{\alpha,\beta}(y)$ as defined in Equation (2). Note that the distributions $f^{(i)}_{\alpha,\beta}(y)$ differs for each sample, but the unknown parameters $\boldsymbol{\theta}= (\alpha, \beta)$ remain common for all of them.

Therefore, our interest is to estimate $\boldsymbol{\theta}= (\alpha, \beta)$ by minimizing the DPD between the observed data and the assumed log-logistic model. Because the underlying density is different for each ranked sample Y_i, we need to calculate the divergence between data and model separately for each data point. After considering all data points, the average divergence between the empirical densities obtained from data points and the assumed parametric densities should be minimized.

Because the DPD measures the similarity or dissimilarity between two distributions, the best parameter value $\boldsymbol{\theta}$ such that $f^{(i)}_{\boldsymbol{\theta}}(y)$ approximates the true distribution g_i should minimize the divergence between the these true and assumed densities. Therefore, the minimum density power divergence functional at the distribution g_i, denoted by $\boldsymbol{T}_\tau(g_i)$, is defined as

$$d_\tau(g_i, f^{(i)}_{\boldsymbol{T}_\tau(g_i)}) = \min_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} d_\tau(g_i, f^{(i)}_{\boldsymbol{\theta}}). \tag{4}$$

here $\boldsymbol{T}_\tau(g_i) \in \boldsymbol{\Theta}$ denotes the best parameter so the distributions g_i and $f^{Y_i}_{\boldsymbol{T}_\tau(g_i)}$ are as close as possible. Now, if $d_\tau(g_i, f^{(i)}_{\alpha,\beta})$ denotes the DPD between the density estimate corresponding to the i-th data point and the associated model density, for the complete data we should minimize the averaged divergences given by

$$d_\tau\left((g_1, ..., g_n), (f^{(1)}_{\boldsymbol{\theta}}, ..., f^{(n)}_{\boldsymbol{\theta}})\right) = \frac{1}{n}\sum_{i=1}^{n} d_\tau(g_i, f^{(i)}_{\alpha,\beta}). \tag{5}$$

However, the true distributions $g_i, i = 1, ..., n$ are generally unknown in practice. Hence, these distributions need to be estimated from the observed data. Since we only have one datum for each distribution, the best empirical estimate of the underlying density is given by the degenerate distribution at the observed point. We will denote this degenerate distribution by $\widehat{g}_i$.

We define the **minimum density power divergence estimator** (MDPDE) with tuning parameter τ, $\widehat{\boldsymbol{\theta}} = (\hat{\alpha}_\tau, \hat{\beta}_\tau)$, as the values of α and β minimizing the averaged DPD between the assumed model densities and its empirical estimates, given by

$$d_\tau\left((\widehat{g}_1, ..., \widehat{g}_n), (f^{(1)}_{\boldsymbol{\theta}}, ..., f^{(n)}_{\boldsymbol{\theta}})\right) = \frac{1}{n}\sum_{i=1}^{n} d_\tau(\widehat{g}_i, f^{(i)}_{\alpha,\beta}) \tag{6}$$

is attained. Here, we are denoting $f^{Y_i}_{\alpha,\beta}$ by $f^{(i)}_{\alpha,\beta}$. As Kullback-Leibler leads to MLE, if we consider $\tau \geq 0$, it follows that MDPDE family generalizes the MLE to a broader family with a tuning parameter determining the efficiency and robustness of the estimator. We will discuss this generalization for the log-logistic distribution later.

It can be easily seen (see e.g. (Ghosh and Basu, 2013)) that the MDPDE for $\tau > 0$ can be obtained by minimizing the following surrogate function

$$H_{n,\tau}\left(\alpha,\beta\right) = \frac{1}{n}\sum_{i=1}^{n}\left\{\int_0^\infty f^{(i)}_{\alpha,\beta}(y)^{\tau+1}dy - \left(1+\frac{1}{\tau}\right)f^{(i)}_{\alpha,\beta}(y_i)^{\tau}\right\}. \tag{7}$$

For $\tau = 0$, the objective function yields

$$H_{n,0}(\alpha,\beta) = \lim_{\tau\to 0} H_{n,\tau}(\alpha,\beta) = -\frac{1}{n}\sum_{i=1}^{n}\log f^{(i)}_{\alpha,\beta}(y_i). \tag{8}$$

From the above, the objective function based on the DPD with $\tau = 0$ is equivalent to the likelihood function of the model. Therefore, minimizing the DPD with $\tau = 0$, $H_{n,0}(\alpha,\beta)$, in α and β leads to the MLE. Hence, the proposed MDPDE family extends the MLE of the log-logistic parameters.

The following results states the explicit expression of the first term in the above equation,

$$\int_0^\infty f^{(i)}_{\alpha,\beta}(y)^{\tau+1}dy. \tag{9}$$

Proposition 1 *For a log-logistic density function $f^{(i)}_{\alpha,\beta}(y)$ with scale and shape parameters α and β, respectively, we have that*

$$\int_0^\infty f^{(i)}_{\alpha,\beta}(y)^{\tau+1}dy = \frac{c\left(i,n\right)\beta^{\tau}}{\alpha^{\tau}}B\left(\frac{\left(n+i-1\right)\tau\beta+\tau+\beta\left(n+i-1\right)}{\beta},\frac{i\beta\tau+i\beta-\tau}{\beta}\right),$$

where $B(a,b)$ is the Beta function with arguments a and b, i.e,

$$B\left(a,b\right) = \int_0^1 x^{a-1}\left(1-x\right)^{b-1}dx. \tag{10}$$

Proof. See Supplementary Material. ■

Based on the previous theorem the objective function given in Equation (7) for $\tau > 0$ can be written for the log-logistic model as

$$\begin{aligned}H_{n,\tau}\left(\alpha,\beta\right) =& \frac{1}{n}\sum_{i=1}^{n}\left\{c(i,n)^{\tau+1}\left(\frac{\beta}{\alpha}\right)^{\tau}B\left(\frac{\left(n+i-1\right)\tau\beta+\tau+\beta\left(n+i-1\right)}{\beta},\frac{i\beta\tau+i\beta-\tau}{\beta}\right)\right.\\ &\left.-\left(1+\frac{1}{\tau}\right)\frac{c(i,n)^{\tau}\beta^{\tau}\left(\frac{y_i}{\alpha}\right)^{\tau(i\beta-1)}}{\alpha^{\tau}\left(1+\left(\frac{y_i}{\alpha}\right)^{\beta}\right)^{\tau(n+1)}}\right\}.\end{aligned} \tag{11}$$

In what follows, we will denote by $\Psi(a)$ the digamma function, defined as the logarithmic derivative of the gamma function. Having an explicit expression of $H_{n,\tau}\left(\alpha,\beta\right)$ allows us to compute the corresponding estimating equations. Hence, the following holds.

Theorem 2 *The MDPDE for the parameters α,β of the log-logisitc distribution when RSS is applied are given as the solutions of the system*

$$
\begin{aligned}
0 &= \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{c(i,n)}{\alpha^{\tau}}B\left(\frac{(n+i-1)\,\tau\beta+\tau+\beta\,(n+i-1)}{\beta},\frac{i\beta\tau+i\beta-\tau}{\beta}\right)\frac{(-\tau)}{\alpha}\right.\\
&\quad\left.-\left(1+\frac{1}{\tau}\right)y_i^{\tau(i\beta-1)}\frac{\alpha^{\beta\tau(n-i+1)}}{\left(\alpha^{\beta}+y_i^{\beta}\right)^{\tau(n+1)}}\left[\frac{y_i^{\beta}\beta\tau(n-i+1)-\alpha^{\beta}\beta\tau i}{\alpha\left(\alpha^{\beta}+y_i^{\beta}\right)}\right]\right\}.\\
0 &= \sum_{i=1}^{n}c(i,n)B\left(\frac{(n+i-1)\,\tau\beta+\tau+\beta\,(n+i-1)}{\beta},\frac{i\beta\tau+i\beta-\tau}{\beta}\right)\\
&\quad\left[1+\frac{1}{\beta}\left(\Psi\left(\frac{i\beta\tau+i\beta-\tau}{\beta}\right)-\Psi\left(\frac{(n+i-1)\,\tau\beta+\tau+\beta\,(n+i-1)}{\beta}\right)\right)\right]\\
&\quad-\left(1+\frac{1}{\tau}\right)\beta\left(\frac{y_i}{\alpha}\right)^{\tau(i\beta-1)}\\
&\quad\left[\left(\frac{1}{\beta}+i\ln\left(\frac{y_i}{\alpha}\right)\right)\left(1+\left(\frac{y_i}{\alpha}\right)^{\beta}\right)^{\tau(n-1)}-(n+1)\left(\frac{y_i}{\alpha}\right)^{\beta}\ln\left(\frac{y_i}{\alpha}\right)\left(1+\left(\frac{y_i}{\alpha}\right)^{\beta}\right)^{\tau(n-1)-1}\right].
\end{aligned}
$$

Proof. See Supplementary Material. ■

The estimating equations are highly non-linear and its solution does not have a closed form. Therefore, the solution of these estimating equations need to be obtained numerically via a software. We remark that the same problem arises for MLE.

3 Asymptotic distribution of the MDPDE for the log-logistic model for the RSS

In this section, we derive the asymptotic distribution of MDPDE for the log-logisitc model assuming RSS in three different situations. First, we assume that β is known and only α needs to be estimated. Next, we consider the case in which α is known and it is parameter β that needs to be estimated. We finally treat the case in which both parameters need to be estimated. The asymptotic variance-covariance will be important for constructing asymptotic confidence intervals for α and β as well as asymptotic ellipsoidal confidence regions for α and β jointly.

To obtain the asymptotic distribution, we use the theory developed in (Ghosh and Basu, 2013) for independent but not identically distributed observation. Therein it is proved that given $Y_1, ..., Y_n$ independent but not identically distributed variables, all of them depending on a common parameter $\boldsymbol{\theta}$, then under some mild conditions the MDPDE $\widehat{\boldsymbol{\theta}}_\tau$ of $\boldsymbol{\theta}$ satisfies

$$
\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_\tau-\boldsymbol{\theta}\right)\underset{n\longrightarrow\infty}{\overset{\mathcal{L}}{\longrightarrow}}\mathcal{N}\left(0,J_\tau^{-1}(\boldsymbol{\theta})K_\tau(\boldsymbol{\theta})J_\tau^{-1}(\boldsymbol{\theta})\right), \tag{12}
$$

being

$$
J_\tau(\boldsymbol{\theta})=\frac{1}{n}\sum_{i=1}^{n}J_\tau^{(i)}(\boldsymbol{\theta}),\qquad \boldsymbol{K}_\tau(\boldsymbol{\theta})=\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{K}_\tau^{(i)}(\boldsymbol{\theta}). \tag{13}
$$

and

$$
\xi_\tau(\boldsymbol{\theta})=\frac{1}{n}\sum_{i=1}^{n}\xi_\tau^{(i)}(\boldsymbol{\theta}). \tag{14}
$$

Matrices $J_\tau^{(i)}(\boldsymbol{\theta}), K_\tau^{(i)}(\boldsymbol{\theta})$ and $\xi_\tau^{(i)}(\boldsymbol{\theta})$ are given, for $i = 1, ..., n$ as

$$J_\tau^{(i)}(\boldsymbol{\theta}) = \int_{\mathbb{R}} \left(\frac{\partial log f_{\boldsymbol{\theta}}^{(i)}(x)}{\partial \boldsymbol{\theta}} \right)^2 f_{\boldsymbol{\theta}}^{(i)}(x)^{\tau+1} dx, \tag{15}$$

$$\boldsymbol{K}_\tau^{(i)}(\boldsymbol{\theta}) = \boldsymbol{J}_{2\tau}^{(i)}(\boldsymbol{\theta}) - \boldsymbol{\xi}_\tau^{(i)}(\boldsymbol{\theta}) \boldsymbol{\xi}_\tau^{(i)}(\boldsymbol{\theta})^T, \tag{16}$$

and

$$\xi_\tau^{(i)}(\boldsymbol{\theta}) = \int_{\mathbb{R}} \frac{\partial \log f_{\boldsymbol{\theta}}^{(i)}(x)}{\partial \boldsymbol{\theta}} f_{\boldsymbol{\theta}}^{(i)}(x)^{\tau+1} dx. \tag{17}$$

In general, finding explicit expressions for $\boldsymbol{J}_\tau^{(i)}, \boldsymbol{K}_\tau^{(i)}$ and $\boldsymbol{\xi}_\tau^{(i)}$ is a complex problem. However, in next subsections we will succeed in deriving such expressions for the log-logistic distribution.

3.1 Asymptotic distribution of $\widehat{\alpha}_\tau$

For stating the asymptotic distribution of $\widehat{\alpha}_\tau$, assuming β is known, it suffices to apply the previous result for $\boldsymbol{\theta} = \alpha$ as follows

$$\sqrt{n}\left(\widehat{\alpha}_\tau - \alpha\right) \underset{n\longrightarrow\infty}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}\left(0, J_\tau^{-1}(\alpha) K_\tau(\alpha) J_\tau^{-1}(\alpha)\right),$$

being $J_\tau(\alpha)$ and $\xi_\tau(\alpha)$ the corresponding values given in Equations (13) and (14), respectively. Consequently, we need to compute $J_\tau^{(i)}(\alpha), K_\tau^{(i)}(\alpha)$ and $\xi_\tau^{(i)}(\alpha)$ defined in Equations 15, 16 and 17, respectively, for the particular case of the log-logistic distribution. Here, $J_\tau^{(i)}(\alpha)$ and $\xi_\tau^{(i)}(\alpha)$ refer to the corresponding values for the i-th ordered statistic of the log-logistic distribution given by

$$J_\tau^{(i)}(\alpha) = \int_0^\infty \left(\frac{\partial log f_\alpha^{(i)}(x)}{\partial \alpha} \right)^2 f_\alpha^{(i)}(x)^{\tau+1} dx, \quad \xi_\tau^{(i)}(\alpha) = \int_0^\infty \frac{\partial \log f_\alpha^{(i)}(x)}{\partial \alpha} f_\alpha^{(i)}(x)^{\tau+1} dx.$$

The next theorem provides a explicit expression of $J_\tau^{(i)}(\alpha)$.

Theorem 3 *For the log-logistic distribution with density function given as in Equation (1), we have*

$$J_\tau^{(i)}(\alpha) = A_1 + A_2 + A_3,$$

where A_1, A_2, A_3 are given by

$$\begin{aligned}
A_1 &= \left(\frac{\beta}{\alpha}\right)^{\tau+2} (n-i+1)^2 c(i,n)^{\tau+1} B\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}, \frac{i\tau\beta + i\beta - \tau}{\beta}\right). \\
A_2 &= \left(\frac{\beta}{\alpha}\right)^{\tau+2} (n-i+1)^2 c(i,n)^{\tau+1} \cdot \frac{(n-i+1)\tau\beta + (n-i+2)\beta + \tau}{\beta\left[(n+1)\tau + (n+2)\right]} \\
&\quad \cdot \frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta\left[(n+1)\tau + (n+1)\right]} \\
&\quad \cdot B\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}, \frac{i\tau\beta + i\beta - \tau}{\beta}\right). \\
A_3 &= \left(\frac{\beta}{\alpha}\right)^{\tau+2} 2(n-i+1)(n+1) c(i,n)^{\tau+1} \cdot \frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{(n+1)\tau\beta + (n+1)\beta} \\
&\quad B\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}, \frac{i\tau\beta + i\beta - \tau}{\beta}\right).
\end{aligned}$$

Proof. See Supplementary Material. ■

Corollary 4 *The Fisher information for* α *is given by*

$$I_F^{(i)}(\alpha) = J_{\tau=0}^{(i)}(\alpha) = \left(\frac{\beta}{\alpha}\right)^2 (1 + \frac{4}{3} - 2) = \frac{\beta^2}{3\alpha^2}.$$

Proof. See Supplementary Material. ■

After obtaining the expression of $J_\tau^{(i)}(\alpha)$, we are going to get the expression of $K_\tau^{(i)}(\alpha)$.

Theorem 5 *For the log-logistic distribution with density function given as in Equation (1), it holds*

$$K_\tau^{(i)}(\alpha) = J_{2\tau}^{(i)}(\alpha) - \xi_\tau^{(i)}(\alpha)^2,$$

being

$$\xi_\tau^{(i)}(\alpha) = \left(\frac{\beta}{\alpha}\right)^{\tau+1} B\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}, \frac{i\tau\beta + i\beta - \tau}{\beta}\right)\left(\frac{-\tau}{\beta + \tau\beta}\right).$$

Proof. See Supplementary Material. ■

Corollary 6 *We can observe that for* $\tau = 0$ *we have*

$$\xi_{\tau=0}^{(i)}(\alpha) = 0 \text{ and } K_{\tau=0}^{(i)}(\alpha) = I_F^{(i)}(\alpha).$$

As discussed earlier, the MLE is included on the MDPDE family as the limiting case at $\tau = 0$. Therefore, taking limits for $\tau \to 0$ on the previous expressions we should obtain the asymptotic variance of the MLE, that is, the Fisher information value. Indeed, joining Corollaries 4 and 6, proves so.

An important application of the asymptotic variance of $\widehat{\alpha}_\tau$ obtained in Theorems 3 and 5 is the construction of asymptotic confidence intervals for the parameter α. A $100(1-r)\%$ asymptotic confidence interval for α is given by

$$\left(\widehat{\alpha}_\tau \pm z_{r/2}\sigma(\widehat{\alpha}_\tau)\right),$$

where $z_{r/2}$ is the upper $r/2$ quantile of a standard normal distribution and

$$\sigma^2(\widehat{\alpha}_\tau) = J_\tau^{-1}(\widehat{\alpha}_\tau) K_\tau(\widehat{\alpha}_\tau) J_\tau^{-1}(\widehat{\alpha}_\tau).$$

3.2 Asymptotic distribution of $\widehat{\beta}_\tau$

Regarding the asymptotic distribution of $\widehat{\beta}_\tau$, assuming α is known, we apply Equation (12) to obtain the following result

$$\sqrt{n}\left(\widehat{\beta}_\tau - \beta\right) \underset{n\longrightarrow\infty}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}\left(0, J_\tau^{-1}(\beta) K_\tau(\beta) J_\tau^{-1}(\beta)\right).$$

To obtain an explicit expression of the asymptotic variance of $\widehat{\beta}_\tau$,

$$\sigma^2(\widehat{\beta}_\tau) = J_\tau^{-1}(\widehat{\beta}_\tau) K_\tau(\widehat{\beta}_\tau) J_\tau^{-1}(\widehat{\beta}_\tau),$$

it just suffices to obtain

$$J_\tau^{(i)}(\beta) = \int_0^\infty \left(\frac{\partial log f_\beta^{(i)}(x)}{\partial\beta}\right)^2 f_\beta^{(i)}(x)^{\tau+1} dx, \quad \xi_\tau^{(i)}(\beta) = \int_0^\infty \frac{\partial \log f_\beta^{(i)}(x)}{\partial\beta} f_\beta^{(i)}(x)^{\tau+1} dx,$$

provided in the next theorem.

Theorem 7 *For the log-logistic distribution with density function given as in Equation (1), it holds have*

$$J_{\tau}^{(i)}(\beta) = C_1 + C_2 + C_3 + C_4 - C_5 - C_6,$$

where $C_1, C_2, C_3, C_4, C_5, C_6$ *are given by*

$$\begin{aligned}
C_1 &= \left(\frac{\beta}{\alpha}\right)^{\tau} \frac{c(i,n)^{\tau+1}}{\beta^2} B\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}, \frac{i\tau\beta + i\beta - \tau}{\beta}\right). \\
C_2 &= \left(\frac{\beta}{\alpha}\right)^{\tau} \frac{c(i,n)^{\tau+1}}{\beta^2} \cdot i^2 B\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}, \frac{i\tau\beta + i\beta - \tau}{\beta}\right) \\
&\quad \times \left[\Psi'\left(\frac{i\tau\beta + i\beta - \tau}{\beta}\right) + \Psi'\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}\right)\right. \\
&\quad \left. + \left(\Psi\left(\frac{i\tau\beta + i\beta - \tau}{\beta}\right) - \Psi\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}\right)\right)^2\right]. \\
C_3 &= \left(\frac{\beta}{\alpha}\right)^{\tau} \frac{c(i,n)^{\tau+1}}{\beta^2} \cdot (n+1)^2 B\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}, \frac{i\tau\beta + (i+2)\beta - \tau}{\beta}\right) \\
&\quad \times \left[\Psi'\left(\frac{i\tau\beta + (i+2)\beta - \tau}{\beta}\right) + \Psi'\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}\right)\right. \\
&\quad \left. + \left(\Psi\left(\frac{i\tau\beta + (i+2)\beta - \tau}{\beta}\right) - \Psi\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}\right)\right)^2\right]. \\
C_4 &= \left(\frac{\beta}{\alpha}\right)^{\tau} \frac{c(i,n)^{\tau+1}}{\beta^2} \cdot 2iB\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}, \frac{i\tau\beta + i\beta - \tau}{\beta}\right) \\
&\quad \left[\Psi\left(\frac{i\tau\beta + i\beta - \tau}{\beta}\right) - \Psi\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}\right)\right]. \\
C_5 &= \left(\frac{\beta}{\alpha}\right)^{\tau} \frac{c(i,n)^{\tau+1}}{\beta^2} \cdot 2(n+1)B\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}, \frac{i\tau\beta + (i+1)\beta - \tau}{\beta}\right) \\
&\quad \left[\Psi\left(\frac{i\tau\beta + (i+1)\beta - \tau}{\beta}\right) - \Psi\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}\right)\right]. \\
C_6 &= \left(\frac{\beta}{\alpha}\right)^{\tau} \frac{c(i,n)^{\tau+1}}{\beta^2} \cdot 2i(n+1)B\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}, \frac{i\tau\beta + (i+1)\beta - \tau}{\beta}\right) \\
&\quad \times \left[\Psi'\left(\frac{i\tau\beta + (i+1)\beta - \tau}{\beta}\right) + \Psi'\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}\right)\right. \\
&\quad \left. + \left(\Psi\left(\frac{i\tau\beta + (i+1)\beta - \tau}{\beta}\right) - \Psi\left(\frac{(n-i+1)\tau\beta + (n-i+1)\beta + \tau}{\beta}\right)\right)^2\right].
\end{aligned}$$

Proof. See Supplementary Material. ■

The following theorem presents the expression of $K_{\tau}^{(i)}(\beta)$.

Theorem 8 *For the log-logistic distribution given in (1) we have,*

$$K_{\tau}^{(i)}(\beta) = J_{2\tau}^{(i)}(\beta) - \xi_{\tau}^{(i)}(\beta)^2,$$

being

$$\xi_{\tau}^{(i)}(\beta) = \left(\frac{\beta}{\alpha}\right)^{\tau}\frac{c(i,n)^{\tau+1}}{\beta}B\left(\frac{(n-i+1)\tau\beta+(n-i+1)\beta+\tau}{\beta},\frac{i\tau\beta+i\beta-\tau}{\beta}\right)$$
$$\frac{\tau}{1+\tau}\cdot\left[\frac{1}{\beta}\left(\Psi\left(\frac{i\tau\beta+i\beta-\tau}{\beta}\right)-\Psi\left(\frac{(n-i+1)\tau\beta+(n-i+1)\beta+\tau}{\beta}\right)\right)+1\right].$$

Proof. See Supplementary Material. ■

Based on Theorems 7 and 8, we get the expression of the asymptotic variance $\sigma^2(\beta)$ of $\widehat{\beta}_\tau$, and based on this asymptotic variance, similar to what has been done for α, a $100(1-r)\%$ asymptotic confidence interval for β is given by

$$\left(\widehat{\beta}_\tau \pm z_{r/2}\sigma(\widehat{\alpha}_\tau)\right).$$

3.3 Asymptotic distribution of $\left(\widehat{\alpha}_\tau, \widehat{\beta}\tau\right)$

The previous sections presented the asymptotic distribution of the marginal random variables $\widehat{\alpha}_\tau$ and $\widehat{\beta}_\tau$. The main purpose of this section is to get the joint asymptotic distribution of the bivariate random variable $\left(\widehat{\alpha}_\tau, \widehat{\beta}_\tau\right)$.

The asymptotic distribution of $\left(\widehat{\alpha}_\tau, \widehat{\beta}_\tau\right)$, we have,

$$\sqrt{n}\left(\left(\widehat{\alpha}_\tau, \widehat{\beta}_\tau\right)^T-(\alpha,\beta)^T\right)\underset{n\longrightarrow\infty}{\overset{\mathcal{L}}{\longrightarrow}}\mathcal{N}\left(\mathbf{0},\boldsymbol{J}_\tau^{-1}(\alpha,\beta)\boldsymbol{K}_\tau(\alpha,\beta)\boldsymbol{J}_\tau^{-1}(\alpha,\beta)\right)$$

being

$$\boldsymbol{J}_\tau(\alpha,\beta)=\begin{pmatrix} J_\tau^{11}(\alpha,\beta) & J_\tau^{12}(\alpha,\beta)\\ J_\tau^{12}(\alpha,\beta) & J_\tau^{22}(\alpha,\beta)\end{pmatrix}$$

with

$$J_\tau^{11}(\alpha,\beta)=J_\tau(\alpha),\ J_\tau^{22}(\alpha,\beta)=J_\tau(\beta),\ J_\tau^{12}(\alpha,\beta)=\frac{1}{n}\sum_{i=1}^{n}J_\tau^{(i)12}(\alpha,\beta),$$

$$\boldsymbol{K}_\tau(\alpha,\beta)=\begin{pmatrix} K_\tau^{11}(\alpha,\beta) & K_\tau^{12}(\alpha,\beta)\\ K_\tau^{12}(\alpha,\beta) & K_\tau^{22}(\alpha,\beta)\end{pmatrix}$$

with

$$K_\tau^{11}(\alpha,\beta)=K_\tau(\alpha),\ K_\tau^{22}(\alpha,\beta)=K_\tau(\beta),\ K_\tau^{12}(\alpha,\beta)=\frac{1}{n}\sum_{i=1}^{n}K_\tau^{(i)12}(\alpha,\beta)$$

where

$$K_\tau^{(i)12}(\alpha,\beta)=J_{2\tau}^{(i)12}(\alpha,\beta)-\boldsymbol{\xi}_\tau^{(i)}(\alpha,\beta)^T\boldsymbol{\xi}_\tau^{(i)}(\alpha,\beta),$$

and

$$J_\tau^{(i)12}(\alpha,\beta)=\int_0^\infty\left(\frac{\partial\log f_{\alpha,\beta}(x)^{(i)}}{\partial\alpha}\right)\left(\frac{\partial\log f_{\alpha,\beta}^{(i)}(x)}{\partial\beta}\right)f_{\alpha,\beta}^{(i)}(x)^{\tau+1}dx,$$

and

$$\xi_\tau(\alpha,\beta)=\frac{1}{n}\sum_{i=1}^{n}\xi_\tau^{(i)}(\alpha,\beta)=\left(\xi_\tau^{(i)}(\alpha),\xi_\tau^{(i)}(\beta)\right)^T.$$

As we have already obtained the expressions of $J_\tau(\alpha), J_\tau(\beta), K_\tau(\alpha)$ and $K_\tau(\beta)$, we just need to derive the expression for $J_\tau^{(i)12}(\alpha,\beta)$. This is accomplished in the next theorem.

Theorem 9 *For the log-logistic distribution given in Equation (1), we have*

$$J_\tau^{(i)12}(\alpha,\beta)=E_1+E_2-E_3-E_4-E_5+E_6,$$

where E_1,E_2,E_3,E_4,E_5 and E_6 are given by

$$\begin{aligned}
E_1 &= \frac{\beta^\tau}{\alpha^{\tau+1}}B\left(\frac{\tau\beta+\beta+\tau}{\beta},\frac{\tau\beta+\beta-\tau}{\beta}\right).\\
E_2 &= \frac{\beta^\tau}{\alpha^{\tau+1}}B\left(\frac{\tau\beta+\beta+\tau}{\beta},\frac{\tau\beta+\beta-\tau}{\beta}\right)\left\{\Psi\left(\frac{\tau\beta+\beta-\tau}{\beta}\right)-\Psi\left(\frac{\tau\beta+\beta+\tau}{\beta}\right)\right\}.\\
E_3 &= 2\frac{\beta^\tau}{\alpha^{\tau+1}}B\left(\frac{\tau\beta+\beta+\tau}{\beta},\frac{\tau\beta+2\beta-\tau}{\beta}\right)\left\{\Psi\left(\frac{\tau\beta+2\beta-\tau}{\beta}\right)-\Psi\left(\frac{\tau\beta+\beta+\tau}{\beta}\right)\right\}.\\
E_4 &= 2\frac{\beta^\tau}{\alpha^{\tau+1}}B\left(\frac{\tau\beta+2\beta+\tau}{\beta},\frac{\tau\beta+\beta-\tau}{\beta}\right).\\
E_5 &= 2\frac{\beta^\tau}{\alpha^{\tau+1}}B\left(\frac{\tau\beta+2\beta+\tau}{\beta},\frac{\tau\beta+\beta-\tau}{\beta}\right)\left\{\Psi\left(\frac{\tau\beta+\beta-\tau}{\beta}\right)-\Psi\left(\frac{\tau\beta+2\beta+\tau}{\beta}\right)\right\}.\\
E_6 &= 4\frac{\beta^\tau}{\alpha^{\tau+1}}B\left(\frac{\tau\beta+2\beta+\tau}{\beta},\frac{\tau\beta+2\beta-\tau}{\beta}\right)\left\{\Psi\left(\frac{\tau\beta+2\beta-\tau}{\beta}\right)-\Psi\left(\frac{\tau\beta+2\beta+\tau}{\beta}\right)\right\}.
\end{aligned}$$

Proof. See Supplementary Material. ■

Next corollary shows that our asymptotic results extend the corresponding formulas in (He et al., 2020) related to MLE. For $\tau=0$, we obtain the following result:

Corollary 10 *For the log-logistic distribution given in (1) we have*

$$\boldsymbol{J}_0^{(i)}(\alpha,\beta)=\begin{pmatrix} J_0^{(i)11}(\alpha,\beta) & J_0^{(i)12}(\alpha,\beta) \\ J_0^{(i)12}(\alpha,\beta) & J_0^{(i)22}(\alpha,\beta) \end{pmatrix},$$

where $J_0^{(i)11}(\alpha,\beta)=J_0^{(i)}(\alpha)$ and $J_0^{(i)22}(\alpha,\beta)=J_0^{(i)}(\beta)$ has been obtained in (He et al., 2020) and

$$J_0^{(i)12}(\alpha,\beta)=\frac{1}{\alpha}\frac{1}{n+2}\left[n-2i+1-i(n-i+1)(\Psi(i)-\Psi(n-i+1))\right].$$

Next result shows that $\boldsymbol{J}_0^{(i)}(\alpha,\beta)$ is a diagonal matrix.

Remark 11 *At $\tau=0$ it holds that*

$$\sum_{i=1}^{n} i(n-i+1)(\Psi(i)-\Psi(n-i+1))=0.$$

On the other hand,

$$\sum_{i=1}^{n}(n-2i+1)=n(n+1)-2\frac{n(n+1)}{2}=0.$$

Consequently,

$$\sum_{i=1}^{n}J_0^{(i)12}\left(\alpha,\beta\right)=0.$$

and

$$\boldsymbol{J}_0^{(i)}(\alpha,\beta)=\left(\begin{array}{cc} J_0^{(i)}(\alpha) & 0\\ 0 & J_0^{(i)}(\beta)\end{array}\right).$$

The asymptotic variance-covariance matrix of $\left(\widehat{\alpha}_\tau,\widehat{\beta}_\tau\right)$ is given by

$$\boldsymbol{\Sigma}(\alpha,\beta)=\boldsymbol{J}_\tau^{-1}(\alpha,\beta)\boldsymbol{K}_\tau(\alpha,\beta)\boldsymbol{J}_\tau^{-1}(\alpha,\beta).$$

This can be used to obtain an asymptotical ellipsoidal confidence region for (α,β), as well to obtain a measure of the relative efficiency of $\left(\widehat{\alpha}_\tau,\widehat{\beta}_\tau\right)$. It is clear that

$$n\left(\left(\widehat{\alpha}_\tau,\widehat{\beta}_\tau\right)^T-(\alpha,\beta)^T\right)\left(\boldsymbol{J}_\tau^{-1}(\alpha,\beta)\boldsymbol{K}_\tau(\alpha,\beta)\boldsymbol{J}_\tau^{-1}(\alpha,\beta)\right)^{-1}\left(\left(\widehat{\alpha}_\tau,\widehat{\beta}_\tau\right)-(\alpha,\beta)\right)\underset{n\longrightarrow\infty}{\overset{\mathcal{L}}{\longrightarrow}}\chi_2^2,$$

and therefore an asymptotical ellipsoidal confidence region for (α,β) is given by

$$C_{n,\tau}^r=\left\{(\alpha,\beta)\,|n\left(\left(\widehat{\alpha}_\tau,\widehat{\beta}_\tau\right)^T-(\alpha,\beta)^T\right)\left(\boldsymbol{J}_\tau^{-1}(\alpha,\beta)\boldsymbol{K}_\tau(\alpha,\beta)\boldsymbol{J}_\tau^{-1}(\alpha,\beta)\right)^{-1}\left(\left(\widehat{\alpha}_\tau,\widehat{\beta}_\tau\right)-(\alpha,\beta)\right)\leq c\right\}.$$

Now, if we choose $c=\chi^2_{2,r}$, the $100(1-r)$-quantile of the chi-square distribution with 2 degrees of freedom, we have that $P_{\alpha,\beta}(C_{n,\tau}^r)$ tends to $1-r$ as $n\to\infty$. In other words, $C_{n,\tau}^r$ is an ellipsoidal confidence for (α,β) having limiting confidence coefficient $1-r$ as $n\to\infty$. On the other hand, the volume of the ellipsoidal region $C_{n,\tau}^r$ is given by

$$\chi^2_{2,r}det\left(\left(\boldsymbol{J}_\tau^{-1}(\alpha,\beta)\boldsymbol{K}_\tau(\alpha,\beta)\boldsymbol{J}_\tau^{-1}(\alpha,\beta)\right)^{-1}\right)^{1/2}.$$

Based on it, a measure of the asymptotic efficiency of $\left(\widehat{\alpha}_\tau,\widehat{\beta}_\tau\right)$ for $\tau>0$ with respect to the MLE $\left(\widehat{\alpha},\widehat{\beta}\right)$ is given by

$$\left(\frac{det\left(\boldsymbol{I}_F(\alpha,\beta)\right)}{det\left(\left(\boldsymbol{J}_\tau^{-1}(\alpha,\beta)\boldsymbol{K}_\tau(\alpha,\beta)\boldsymbol{J}_\tau^{-1}(\alpha,\beta)\right)^{-1}\right)}\right)^{1/2}.$$

4 Optimal tuning parameter

As previously discussed the tuning parameter τ of the MDPDE controls the trade-off between efficiency and robustness on the estimation; large values of τ result in more robust although less efficient estimators and conversely, the small the tuning parameter is, the more efficient but less robust the resulting estimator would be. Hence, it could be of interest to determine the optimal value of τ. This optimal value would depend, of course, on the amount of contaminated data. Therefore, it is not possible to find an overall optimal value of τ in practical situations. In general (see for instance (Balakrishnan et al., 2024,b) and references therein), moderate large values of τ provide MDPDE with robust estimations without a too high loss of efficiency in relation to the MLE.

The optimal value of the parameter should minimize the estimation error, as measured by an appropriate accuracy function such as estimation bias, the asymptotic variance or its combination, the mean squared error. These proposal have been discussed in the literature. For example, (Warwick and Jones, 2005) provided an useful data-driven procedure for the choice of the tuning parameter τ by minimizing the (estimated) mean squared error of the MDPDE. Their procedure relies on a pilot estimator, which needed to be selected in advance and so the outcome may be influenced by this choice. In the following we shall denote by $\widehat{\alpha}_p$ (if β is known) the pilot estimator for α, and by $\widehat{\beta}_p$ (if α is known) the pilot estimator for β. For the joint estimators vectors, we denote by $\widehat{\boldsymbol{\theta}}_p = (\widehat{\alpha}_p, \widehat{\beta}_p)$ the pilot estimator for $\boldsymbol{\theta} = (\alpha, \beta)$.

The main idea in (Warwick and Jones, 2005) is to assume that optimal values of τ will produce MDPDE with the smallest estimation error. Based on this concept, it is proposed to minimize an estimate of the Mean Squared Error (MSE) in the estimation given by

$$\widehat{MSE}_\tau(\widehat{\alpha}_\tau) = (\widehat{\alpha}_\tau - \widehat{\alpha}_p)^2 + \frac{1}{n}\left(J_\tau^{-1}(\widehat{\alpha}_\tau)K_\tau(\widehat{\alpha}_\tau)J_\tau^{-1}(\widehat{\alpha}_\tau)\right), \tag{18}$$

$$\widehat{MSE}_\tau(\widehat{\beta}_\tau) = (\widehat{\beta}_\tau - \widehat{\beta}_p)^2 + \frac{1}{n}\left(J_\tau^{-1}(\widehat{\beta}_\tau)K_\tau(\widehat{\beta}_\tau)J_\tau^{-1}(\widehat{\beta}_\tau)\right), \tag{19}$$

and

$$\begin{aligned}\widehat{MSE}_\tau(\widehat{\alpha}_\tau, \widehat{\beta}_\tau) = &\left((\widehat{\alpha}_\tau, \widehat{\beta}_\tau)^T - (\widehat{\alpha}_p, \widehat{\beta}_p)^T\right)^T \left((\widehat{\alpha}_\tau, \widehat{\beta}_\tau) - (\widehat{\alpha}_p, \widehat{\beta}_p)\right) \\ &+ \frac{1}{n} Trace\left(\boldsymbol{J}_\tau^{-1}(\widehat{\alpha}_\tau, \widehat{\beta}_\tau)\boldsymbol{K}_\tau(\widehat{\alpha}_\tau, \widehat{\beta}_\tau)\boldsymbol{J}_\tau^{-1}(\widehat{\alpha}_\tau, \widehat{\beta}_\tau)\right),\end{aligned} \tag{20}$$

respectively, where matrices $\boldsymbol{J}_\tau$ and $\boldsymbol{K}_\tau$ are as defined in Section 3. The previous formulas can be split into two terms that measure different types of estimation error: the bias of the estimator (first term) and the asymptotic variance of the estimator (second term). Each of these terms could be used separately as a criterion function for selecting the optimal τ.

On the other hand, there are several proposals for the choice of the pilot estimator in the literature. Note that this choice of the pilot estimator is crucial on the final choice of the tuning parameter, as it invariantly attracts the final estimator. To avoid this drawback, (Basak et al., 2021) considered an iterative algorithm that replaces at each iteration the value of the pilot estimator with the corresponding MDPDE for the optimal value of τ obtained in (18), (19) or (20) in the previous iteration. Thus, although the algorithm is initialized with a suitable robust estimator, the final choice of τ is more pilot-independent.

It is interesting to note that in the cited paper (Basak et al., 2021), it is empirically shown that in some statistical models, if the pilot estimators are one of the MDPDE, all of them lead to the same iterated optimal choice.

Alternatively, from our numerical analysis in Section 5, minimizing the bias of the estimation leads to a similar choice of the optimal tuning parameter while reducing computational burden, as it avoids the need to estimate the asymptotic variance. Therefore, for the log-logistic distribution, minimizing the bias would be sufficient for a data-driven choice of the optimal DPD tuning parameter. Algorithm 1 summarizes the previous discussions to find the optimal tuning parameter τ.

Algorithm 1 Optimal DPD Tuning Parameter

Step 1: Fix the convergence rate ϵ and maximum number of iterations M. Choose an initial pilot estimator for α, β or the joint vector (α, β), as $\widehat{\alpha}_p, \widehat{\beta}_p$ or $(\widehat{\alpha}_p, \widehat{\beta}_p)$, respectively.
Step 2: Minimize the corresponding MSE (resp. Bias) depending on the unknown parameter to estimate, namely $\widehat{MSE}_\tau(\widehat{\alpha}_\tau), \widehat{MSE}_\tau(\widehat{\beta}_\tau)$ or $\widehat{MSE}_\tau(\widehat{\alpha}_\tau, \widehat{\beta}_\tau)$, (resp. $\widehat{Bias}_\tau(\widehat{\alpha}_\tau), \widehat{Bias}_\tau(\widehat{\beta}_\tau)$ or $\widehat{Bias}_\tau(\widehat{\alpha}_\tau, \widehat{\beta}_\tau)$) given in Equations (18), (19) and (20) respectively (resp. the first terms of the Equations). Denote τ^* the optimal value of the DPD tuning parameter.
Step 3: Fit the MDPDE with parameter τ^*
Step 4: If $|\widehat{\alpha}_{\tau^*} - \widehat{\alpha}_p| < \epsilon, |\widehat{\beta}_{\tau^*} - \widehat{\beta}_p| < \epsilon$ or $||(\widehat{\alpha}_{\tau^*}, \widehat{\beta}_{\tau^*}) - (\widehat{\alpha}_p, \widehat{\beta}_p)|| < \epsilon$, respectively, or the maximum number of iterations is reached, end.
Step 5: Otherwise, set $\widehat{\alpha}_p = \widehat{\alpha}_{\tau^*}, \widehat{\beta}_p = \widehat{\beta}_{\tau^*}$, or $(\widehat{\alpha}_p, \widehat{\beta}_p) = (\widehat{\alpha}_{\tau^*}, \widehat{\beta}_{\tau^*})$ and return to Step 2.

In Section 5 we discuss in depth the practical choice of the tunning parameter via simulation.

5 Numerical analysis

To assess the performance of MDPDE in estimating log-logistic parameters for data coming from a rank sample, we have conducted a simulation study. This simulation set-up is similar to the one proposed in (Ma et al., 2023). Although that paper deals with random sampling instead of RSS, we feel that the simulation study is very complete and treats different types of contamination covering different values of the parameters. In this study, we compare MDPDE coming from RSS for different values of parameters α and β and for different sample sizes. We consider that α is fixed to 1, while β attains values $\beta = 1.5, 2.5, 5.0, 10$. The motivation for fixing α is that, as it is a scale parameter, we can focus on the case $\alpha = 1$. This has been done for the log-logistic distribution e.g. in (Ma et al., 2023). The sample sizes are set as $n = 10, 25, 50, 75, 100$. For a fixed sample size, we generate n different samples of size n. For the i-th sample $(X_1^i, ..., X_n^i)$, we define $Y_i = X_{(i)}^i$ the i-th ordered statistic. Finally, we consider the sample $(Y_1, ..., Y_n)$.

For each combination of model parameters and sample size, we estimate the log-logistic parameters based on the sample $(Y_1, ..., Y_n)$ using the MDPDE approach with different values of the tuning parameter, namely $\tau = 0, 0.1, 0.2, ..., 1.0$. Here the MDPDE with tuning parameter $\tau = 0$ states for the MLE. For the seek of completeness, we have considered three more estimators proposed in the literature; the repeated median estimator (RM), the sample median estimator (SM), and the Hodge-Lehmann and Shamos estimator (HL) (see (Ma et al., 2023)) for samples coming from RSS. A brief discussion about these estimators is presented in the Supplementary Material. See also (Ma et al., 2023) and references therein for more details.

Samples are generated randomly from the log-logistic distribution with true parameters (α_0, β_0). The MLE and the MDPDE estimators are obtained by maximizing the corresponding DPD-based objective functions (8) and (11) respectively, using a quasi-Newton algorithm for optimization with bounds on the variables. The bounds considered are $\left[\frac{1}{3}\alpha_0, 3\alpha_0\right]$ for α and $[1, 3\beta_0]$ for β. The initial point of the algorithm is chosen as a random number from an uniform distribution $U(0.9, 1.1)$ The programming language used is Fortran 95 with double precisiona and the optimization was performed using subroutines from the NAG library with the default tolerance for convergence.

To avoid any possible influence of the sample, we repeat this process for each combination of (α, β) and each estimation procedure $M = 1000$ times. Besides, to measure the performance of the different methods, we compute for the bias and squared error (SE) on the estimation for the i-th sample as follows

$$Bias_i(\alpha) = \hat{\alpha}_i - \alpha, \; Bias_i(\beta) = \hat{\beta}_i - \beta,$$
$$SE_i(\alpha) = (\hat{\alpha}_i - \alpha)^2, \; SE_i(\beta) = (\hat{\beta}_i - \beta)^2,$$

and the average over the $M = 1000$ simulations as mean bias and root mean squared error (RMSE)

$$Bias(\alpha) = \frac{1}{M}\sum_{i=1}^{M} Bias_i(\alpha), \; RMSE(\alpha) = \sqrt{\frac{1}{M}\sum_{i=1}^{M} SE_i(\alpha)},$$

$$Bias(\beta) = \frac{1}{M}\sum_{i=1}^{M} Bias_i(\beta), \; RMSE(\beta) = \sqrt{\frac{1}{M}\sum_{i=1}^{M} SE_i(\beta)}.$$

Further, at observation we compute the joint quantities of absolute error (AE) and SE

$$AE_i = |\hat{\alpha}_i - \alpha| + |\hat{\beta}_i - \beta| \text{ and } SE_i = (\hat{\alpha}_i - \alpha)^2 + (\hat{\beta}_i - \beta)^2,$$

and we compute the mean absolute error (MAE) and RMSE over the $M = 1000$ simulations as follows

$$MAE = \frac{1}{M}\sum_{i=1}^{M} AE_i \text{ and } RMSE = \sqrt{\frac{1}{M}\sum_{i=1}^{M} SE_i}.$$

Finally, as a measure of the relative efficiency of any given estimator T with respect to the MLE, we compute

$$RE(T) = \frac{RMSE(T)}{RMSE(MLE)}.$$

Firstly, we consider an uncontaminated scenario. For the sake of brevity, The empirical error measures for all simulation setups are provided in the Supplementary Material (Tables 1 to 20). Here, we present the results for the case $n = 100, \alpha = 1, \beta = 5$ in Table 1 as an example. The rest of the simulations exhibit similar behavior.

As expected, MLE performs very well in the absence of contamination and it performance improves under increasing sample sizes. For MDPDEs with positive values of the tuning parameter, their performance is quite similar to the MLE, achieving a relative efficiency of almost one for small and moderate values of the tuning parameter. Moreover, the performance approaches that of MLE as the sample size increases, indicating that the efficiency loss is less significant for large samples. Indeed for small values of τ, ($\tau \leq 0.6$) the behavior is quite similar to the MLE. Hence, except for the case of lowest sample size $n = 10$, it can be seen that the MDPDE for $\tau < 0.6$ performs as well as the MLE in the absence of contamination. This behavior was expected by the asymptotic properties of MDPDE obtained in Section. Compared to other estimators proposed in the literature, the only competitor to MDPDE is RM, but MDPDE outperforms RM for small values of τ (≤ 0.4) in all scenarios. Next, note that the relative efficiency of MDPDE tends to increase with the sample size, and then stabilizes. We remark that the point of stabilization depends on the value of the tuning parameter τ, and it increases with τ. For example, the value tends to stabilize for $n = 75$ at $\tau = 0.6$ and $n = 25$ at $\tau = 0.2$. Moreover, the MDPDEs with positive values of the tuning parameter are unbiased when estimating the scale parameter α, even for small sample sizes, while there is a tendency for overestimating β for small sample sizes. This tendency also appears for MLE.

It should be noted, as discussed in the introduction and Section 2, that the estimation accuracy when using RSS is lower than that for a random sample of the same size. To illustrate this improvement for the log-logistic

	Bias(α)	Bias(β)	MAE	RMSE(α)	RMSE(β)	RMSE	RE	$\hat{\alpha}$	$\hat{\beta}$
MLE	-0.00001	0.00754	0.10319	0.00497	0.12436	0.12446	1.00000	0.99999	5.00754
$DPD_{0.1}$	-0.00000	0.00540	0.09279	0.00523	0.11041	0.11053	0.88808	1.00000	5.00540
$DPD_{0.2}$	-0.00000	0.00537	0.09203	0.00565	0.10887	0.10901	0.87586	1.00000	5.00537
$DPD_{0.3}$	-0.00002	0.00622	0.09566	0.00610	0.11298	0.11314	0.90905	0.99998	5.00622
$DPD_{0.4}$	-0.00004	0.00744	0.10117	0.00654	0.11918	0.11936	0.95902	0.99996	5.00744
$DPD_{0.5}$	-0.00006	0.00881	0.10703	0.00697	0.12588	0.12607	1.01294	0.99994	5.00881
$DPD_{0.6}$	-0.00008	0.01023	0.11268	0.00739	0.13240	0.13261	1.06548	0.99992	5.01023
$DPD_{0.7}$	-0.00011	0.01164	0.11787	0.00781	0.13851	0.13873	1.11466	0.99989	5.01164
$DPD_{0.8}$	-0.00013	0.01301	0.12253	0.00822	0.14415	0.14438	1.16005	0.99987	5.01301
$DPD_{0.9}$	-0.00015	0.01435	0.12683	0.00863	0.14932	0.14957	1.20175	0.99985	5.01435
$DPD_{1.0}$	-0.00017	0.01564	0.13092	0.00904	0.15409	0.15435	1.24016	0.99983	5.01564
RM	-0.00027	-0.07905	0.11775	0.00532	0.14008	0.14018	1.12631	0.99973	4.92095
SM	0.00388	-1.96975	1.98093	0.01405	1.97545	1.97550	15.8726	1.00388	3.03025
HL	0.00008	-2.11092	2.11494	0.00506	2.11189	2.11189	16.9684	1.00008	2.88908

Table 1: Estimation errors for α and β for different estimators using RSS. Sample size is set $n = 100$ and the true values of the parameters are $\alpha = 1, \beta = 5.0$. The different columns present the averaged Bias and averaged MSE for each of the parameters, and averaged estimates for α and β over $M = 1000$ repetitions.

distribution, we present in Table 2 the corresponding joint MAE and RMSE when random sampling is used for $n = 100, \alpha = 1, \beta = 5$, as well as the averaged estimated parameter under this set-up. Comparing these results with the ones presented in Table 1, (Third column for each measure) it is noted that the performance of the MDPDEs, including the MLE and RM, clearly worsen for the simple random sample, hence highlighting the convenience of using RSS.

For the rest of the simulations results, we refer to the tables in the Supplementary Material of this paper and the Supplementary Material of (Felipe et al., 2023).

As explained in the Introduction, the main motivation for considering MDPDEs instead of the MLE relies on their robustness properties. Hence, we next examine the performance of these estimators in the presence of contamination for ranked samples. For this purpose, we repeat the previous simulation study with $\alpha = 1$, $\beta = 5$, and $n = 100$ under four different scenarios of contamination:

- **Case 1:** For a fixed percentage of data p, we change the value in the sample for a new value coming from a log-logistic distribution with $\alpha = 1, \beta = 0.2$.
- **Case 2:** For a fixed percentage of data p, we change the value in the sample for a new value coming from a log-logistic distribution with $\alpha = 4, \beta = 10$.
- **Case 3:** For a fixed percentage of data p, we change the value in the sample for a new value coming from a uniform distribution $\mathcal{U}(0, 20)$.
- **Case 4:** For a fixed percentage of data p, we change the value in the sample for a new value of 50.

The percentage of contamination ranges from no contamination ($p = 0\%$) to $p = 40\%$, increasing the contamination by a 5% at each step. We use RMSE to assess the goodness-of-fit for each estimator and we repeat this procedure $M = 1000$ times to avoid the possible influence of a specific sample. Figures 1 and 2

	MAE	RMSE	$\hat{\alpha}$	$\hat{\beta}$
MLE	0.36844	0.43431	1.00030	5.06549
$DPD_{0.1}$	0.37232	0.44030	1.00024	5.06584
$DPD_{0.2}$	0.38300	0.45529	1.00016	5.07067
$DPD_{0.3}$	0.39717	0.47472	1.00007	5.07793
$DPD_{0.4}$	0.41270	0.49589	0.99999	5.08649
$DPD_{0.5}$	0.42850	0.51711	0.99990	5.09562
$DPD_{0.6}$	0.44379	0.53734	0.99982	5.10485
$DPD_{0.7}$	0.45786	0.55603	0.99975	5.11385
$DPD_{0.8}$	0.47062	0.57299	0.99968	5.12247
$DPD_{0.9}$	0.48213	0.58821	0.99962	5.13062
$DPD_{1.0}$	0.49251	0.60182	0.99957	5.13828
RM	0.40334	0.47948	0.99994	5.01636
SM	1.93207	1.93817	1.00447	3.10069
HL	2.09938	2.08758	1.00038	2.92855

Table 2: Estimation errors for α and β for different estimators using random sample. Sample size is set $n = 100$ and the true values of the parameters are $\alpha = 1, \beta = 5.0$. The different columns present the averaged Bias and averaged MSE, and averaged estimates for α and β over $M = 1000$ repetitions.

present the RMSE of the MLE, $MDPDE_{0.3}, MDPDE_{0.5}, MDPDE_{0.8}$ and RM against sample contamination. Tables with the corresponding results are provided in the Supplementary Material (Tables 21 to 14).

The results illustrate that, while the MLE performs very well in the absence of contamination, its performance worsens sharply in the presence of contamination. However, if we turn to MDPDE with positive values of the tuning parameter, we can see that they are competitive (indeed, we could say almost equivalent) in the absence of contamination for small values of τ, while they provide very stable results in the presence of contamination. Indeed, MDPDE with positive values of τ are very little affected even under highly contaminated scenarios. This behavior of the MDPDEs and MLE occurs in all the scenarios considered in the simulations. On the other hand, the RM estimator also performs more robustly than the MLE, although it is outperformed by MDPDE for $\tau \in [0.2, 0.6]$. As expected, the performance of the MDPDE improves with higher values of the tuning parameter τ in the presence of contamination.

In conclusion, it seems at the light of this simulation results that the MDPDEs are an appealing alternative to MLE and other estimation methods proposed in the literature, as they behaves competitively to the MLE in the absence of contamination and exhibits high stability when contamination is introduced.

We finish this section by evaluating Algorithm 1 presented in Section 6 for the choice of the optimal tuning parameter using both proposals, the estimated bias and estimated MSE of the MDPDE, as objective function for the choice of τ. Table 3 presents the optimal value of the DPD tuning parameter τ selected by both bias and MSE optimality criteria, as well as associated estimation errors obtained under increasing sample sizes and two scenarios of contamination; pure data (0% contamination) and 10% of contamination by Case 2. The true value of the parameters is set to $\alpha = 1$ and $\beta = 5.0$ From the previous discussions about the performance of the estimators and compromise between robustness and efficiency, we choose the MDPDE with $\tau = 0.3$ as the pilot estimator.

The presented results indicate that the proposed algorithm (using any of the suggested optimality criteria), performs as expected; it selects low values of the tuning parameter in the absence of contamination (all scenarios yielded values below 0.25) and higher values when contamination is present in the data. This behavior suggests that the algorithm effectively distinguishes between the two scenarios and adjusts the tuning

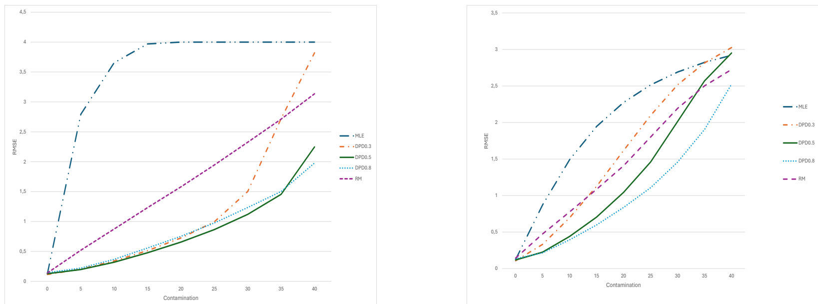

Figure 1: RMSE under increasing contamination (Case 1 on the left and Case 2 on the right). On the x-axis the different degrees of contamination and on y-axis the corresponding RMSE.

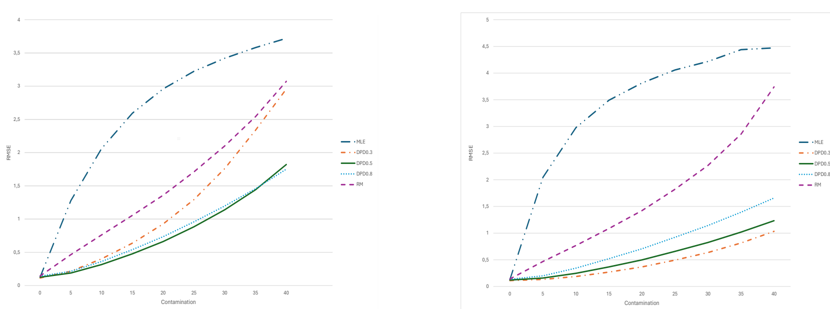

Figure 2: RMSE under increasing contamination (Case 3 on the left and Case 4 on the right). On the x-axis the different degrees of contamination and on y-axis the corresponding RMSE.

Criteria	Cont.	τ^*	Bias(α)	Bias(β)	RMSE(α)	RMSE(β)	$\hat{\alpha}$	$\hat{\beta}$
				$n = 25$				
Bias	0%	0.225	-0.0023	0.1978	0.0217	0.3386	0.9977	5.1978
MSE	0%	0.152	-0.0011	0.1619	0.0208	0.3451	0.9990	5.1619
Bias	10%	0.675	0.0523	-0.0434	0.0608	0.4031	1.0523	4.9566
MSE	10%	0.637	0.0539	-0.1011	0.0620	0.4065	1.0539	4.8989
				$n = 50$				
Bias	0%	0.224	-0.0016	0.0901	0.0106	0.1666	0.9984	5.0901
MSE	0%	0.173	-0.0012	0.0743	0.0103	0.1691	0.9988	5.0743
Bias	10%	0.823	0.0447	-0.1025	0.0470	0.2086	1.0447	4.8975
MSE	10%	0.802	0.0452	-0.1213	0.0474	0.2100	1.0452	4.8787
				$n = 75$				
Bias	0%	0.217	-0.0006	0.0652	0.0067	0.1152	0.9994	5.0652
MSE	0%	0.170	-0.0003	0.0556	0.0066	0.1167	0.9997	5.0556
Bias	10%	0.928	0.0131	0.0944	0.2101	1.6229	1.0131	5.0944
MSE	10%	0.921	0.0143	0.0890	0.2075	1.6228	1.0143	5.0890
				$n = 100$				
Bias	0%	0.2058	-0.0006	0.0729	0.0062	0.4995	0.9994	5.0729
MSE	0%	0.1606	-0.0006	0.0654	0.0052	0.4994	0.9994	5.0654
Bias	10%	0.8058	0.0535	0.5120	0.2038	2.5211	1.0535	5.5120
MSE	10%	0.8052	0.0530	0.5013	0.2031	2.4995	1.0530	5.5013

Table 3: Optimal tuning parameter τ^* obtained using the Bias and MSE selection criteria in Algorithm 1 under different sample sizes and two levels of contamination; 0% (pure data) and 10% of contamination (case 2)

parameter accordingly. It is also worth noting that the MSE optimality criterion tends to select slightly lower values of τ, although both criteria exhibit a similar trend. Given that the Bias criterion offers significantly lower computational times, we recommend its use.

6 Application to real data

Let us now consider the data studied in (Wolfe, 2012) and previously treated in (Nussbaum and Sinha, 1997). This example also provides a situation where RSS is considered as a better option than random sampling. In this problem, we are concerned with air pollution caused by unburned hydrocarbons emitted by automobiles. To reduce this pollution, a reformulated gasoline reducing the volability is used. In order to check that gasoline stations comply with air regulations, samples measuring the Reid Vapor Pressure (RVP) value are selected. However, the measures obtained at the gasoline pump might lack accuracy and to achieve a good accuracy in the measures it is necessary to send the samples to the laboratory, which is expensive. Hence, in (Nussbaum and Sinha, 1997) a RSS is proposed to reduce the number of samples to be sent to laboratory and obtain good

8.98	7.90	**7.85**	8.63	**8.62**	8.17	8.25	**9.25**	7.92
8.28	10.72	8.32	9.21	7.80	**8.89**	7.95	**8.89**	8.01
7.95	8.01	**7.85**	7.86	8.25	**7.88**	8.01	7.89	**9.15**
7.73	7.80	7.96	9.14	7.60	**8.88**	7.73	7.86	**7.98**

Table 4: RVP for a sample of $n = 36$ data. Twelve samples of size 3 are considered randomly and four groups (cycles) of three samples of size three are built, each cycle is written in a row. Data in boldtype are the selected data to be sent to laboratory, i.e. they are the data considered in the estimation.

τ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
α	8.418	8.418	8.417	8.416	8.415	8.414	8.412	8.410	8.407
β	24.674	23.512	22.535	21.709	21.010	20.413	19.901	19.459	19.076

Table 5: Estimation results for the parameters of the log-logistic for different estimators from data of Table 3.

estimations.

Following (Wolfe, 2012), we consider a sample size of 36. These data are randomly ordered and next, they are clustered in 12 sets of size 3, as if we have 12 samples of size 3. Finally, four groups of three samples of size three are considered. The set of data is given in Table 3.

Next, for each group, we consider a RSS. Hence, for the first group of three samples of size three, we consider the smallest for the first sample, the middle datum for the second and the largest for the third sample. These data are written in boldtype in the table.

Assume now that RVP can be measured via a log-logistic distribution and that we aim to estimate the parameters α, β. In next Table, we consider the corresponding estimations according the different estimators.

As it can be seen in Table 6, the estimations for α are very similar, although differences for the estimating values of β arise. Let us now modify one of the values; we have turned the first value 7.85 into 78.5 and repeated the estimation. The results appear in Table 6.

It can be seen comparing both tables that the estimations for MDPDE are much more stable than the corresponding estimations for MLE. This can be seen especially for the case of parameter β. And this holds even for a small sample size. This seems to confirm that MDPDE are an appealing choice for estimating the parameters of the log-logistic distribution.

τ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
α	8.789	8.496	8.501	8.507	8.512	8.518	8.523	8.529	8.535
β	5.134	23.500	22.816	21.906	21.133	20.479	19.922	19.447	19.040

Table 6: Estimation results for the parameters of the log-logistic for different estimators from data of Table 3 with an outlier.

7 Conclusions and open problems

In this paper we have introduced a new family of robust estimators based on the DPD for the parameters of a log-logistic distribution considering RSS. The choice of this family of divergences is justified by its successful application in other statistical models, where it has led to robust estimators with an small loss of efficiency. Besides, the DPD includes the MLE and so the proposed family can be considered as an generalization of the likelihood approach depending on a tuning parameter determining the compromise between robustness and efficiency. Moreover, we have developed the estimators of the parameters and we have obtained their corresponding asymptotic distribution, for each model parameter separately and also for the joint estimators. An extensive simulation study has been carried out to evaluate the performance of the method in practice. The results show that this new family of estimators are competitive to the MLE and other estimators appearing in the literature in terms of efficiency, whilst it has a better performance when data contamination arises for moderate values of the tuning parameter, ranging from 0.2 to 0.5. As a conclusion, we feel that the MDPDE family is an appealing alternative to estimate the parameters of the log-logistic distribution under ranked sampling. Besides, the real data example considered in Section 6 shows a potential use of MDPDE in this situation, as it provides similar estimations when no contamination arises, even for small sample size, and it remains very stable even for heavy contamination.

The DPD has been proved to lead to robust estimators in many other statistical models. However, it is not the only family of divergences leading robust estimators. Another appealing choice as a measure of dissimilarity is the family of Rényi's pseudo-distances, also providing robust estimators. Developing these estimators and comparing them with the MDPDEs presented in this paper is a topic we intend to address in future research. In this linea, the problem of finding estimators based on minimizing Rényi's pseudodistance in the context of independent but not identically distributed data has been discussed e.g. in (Castilla et al., 2022; Felipe et al., 2024).

After introducing the MDPDE, it could be interesting to consider the restricted MDPDE following the work of (Basu et al., 2022b). The restricted MDPDE was considered in Statistical Information Theory in (Pardo et al., 2002) in relation to ϕ-divergences or in (Jaenada et al., 2022) for Rényi's pseudodistance. The restricted MDPDE is necessary to define Rao-type test statistics based on the DPD, first considered in (Basu et al., 2022b), for testing composite hypothesis for the parameters of the log-logistic distribution. Similarly, it is also possible to consider Wald-type test statistics based on MDPDE for testing simple and composite null hypothesis for the parameters of the log-logistic distribution (see e.g. (Basu et al., 2016, 2017, 2018, 2019, 2021; Ghosh et al., 2016, 2021) for Wald-type tests statistics based on MDPDE).

Acknowledgements

This work was supported by the Spanish Grant PID2021-124933NB-I00. M. Jaenada, P. Miranda and L. Pardo are members of the Interdisciplinary Mathematics Institute (IMI).

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