

K3 DOUBLE STRUCTURES ON ENRIQUES SURFACES AND THEIR SMOOTHINGS

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ABSTRACT. Let Y be a smooth Enriques surface. A $K3$ carpet on Y is a double structure on Y with the same invariants as a smooth $K3$ surface (i.e., regular and with trivial canonical sheaf). The surface Y possesses an étale $K3$ double cover $X \xrightarrow{\pi} Y$. We prove that π can be deformed to a family $\mathcal{X} \rightarrow \mathbf{P}_{T^*}^N$ of projective embeddings of $K3$ surfaces and that any projective $K3$ carpet on Y arises from such a family as the flat limit of smooth, embedded $K3$ surfaces.

INTRODUCTION

In this article we continue to study the relation between double covers and double structures. This relation was first studied in [Fon93], for hyperelliptic canonical morphisms and the so-called canonical ribbons and in [GP97] for hyperelliptic $K3$ surfaces and $K3$ carpets on rational normal scrolls. Recently, M. González in [Gon06] and the authors in [GGP07] studied this relation in a much more general setting, namely, finite covers of curves of arbitrary degree on the one hand and one dimensional, locally Cohen–Macaulay multiple structures of arbitrary multiplicity on the other hand. In the present work we look at the relation between a natural and particularly nice double cover, the étale $K3$ double cover of an Enriques surface, and an interesting class of double structures, the *$K3$ carpets on Enriques surfaces*.

Multiple structures have appeared, among other settings, in the study of vector bundles. For example, double structures on surfaces can be found in [HV85], where Hulek and Van de Ven use the doubling construction done by Fossum and Ferrand (see [Fos73] and [Fer77]) in connection with the study of the zero locus of sections of the Horrocks–Mumford vector bundle (see also [BM85], [Har78] and [Man81]). On the other hand, $K3$ carpets on rational normal scrolls have been considered in the study of degenerations of smooth $K3$ surfaces (see for instance [BE95] and [GP97]). In this article we study another kind of $K3$ carpets, namely, those supported on Enriques surfaces.

A $K3$ carpet on a smooth Enriques surface Y will be a locally Cohen–Macaulay double structure on Y with the same invariants as a smooth $K3$ surface (i.e., regular and with trivial canonical sheaf). The surface Y possesses an étale $K3$ double cover $X \xrightarrow{\pi} Y$

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associated to the canonical bundle of Y , which is 2-torsion. We prove that any projective $K3$ carpet on Y arises from a family $\mathcal{X} \rightarrow \mathbf{P}_{T^*}^N$ of projective embeddings of $K3$ surfaces that degenerates to π . As a consequence of this, we show that any projective $K3$ carpet on Y can be *smoothed*, i.e., obtained as the flat limit of a family of smooth, irreducible (projective $K3$) surfaces.

The reader might probably have noted in the previous paragraph the phrase “projective $K3$ carpet”. $K3$ carpets on an Enriques surfaces (like indeed double structures on any other surface) need not be projective, unlike ribbons on curves. Thus our first task is to characterize (see Theorem 2.5) those $K3$ carpets which are projective. This is accomplished in Section 2. There we also see “how many” projective $K3$ carpets there are. We do this in two settings. On the one hand, we compare the sizes of the families of projective $K3$ surfaces on a given (abstract) Enriques surface Y and the size of the family of non-projective $K3$ carpets (see Theorem 2.5). This situation has some strong resemblance to the case of projective and non-projective smooth $K3$ surfaces, where the former lie on infinite, countably many codimension 1 families in the moduli space of $K3$ surfaces. On the other hand, we also compute the dimension of the space that parametrizes the family of projective $K3$ surfaces supported on a given Enriques surface which is embedded in a projective space (see Theorem 2.4).

In Section 3 we prove the results regarding deformation of morphisms and smoothings of carpets. First we show (see Theorem 3.2) that the cover π can be deformed to a family of embeddings of $K3$ surfaces to projective space. Then, in order to obtain a smoothing of a projective $K3$ carpet \tilde{Y} , one considers a suitable embedding of \tilde{Y} in projective space, then one chooses the family of embeddings of Theorem 3.2 suitably, in order to obtain a family of projective schemes consisting of the images of smooth $K3$ surfaces degenerating to \tilde{Y} . From these theorems we obtain a smoothing result for most of the embedded $K3$ carpets (see Theorem 3.5) and subsequently we show that any (abstract) projective $K3$ carpet can be smoothed (see Theorem 3.6).

Finally we devote Section 4 to study the Hilbert points of projective $K3$ carpets. We prove that their Hilbert points are always smooth (see Theorem 4.2), unlike the case of $K3$ carpets on rational normal scrolls. In that case some Hilbert points are smooth and some are not (see [GP97, Section 4]). There were previous results about smoothness of the Hilbert scheme at points corresponding to certain ribbons on curves: Bănică and Manolache, in [BM85], prove that the Hilbert points of ribbons in \mathbf{P}^3 supported on conics are smooth; Bayer and Eisenbud, in [BE95], prove that the Hilbert points of canonically embedded ribbons on \mathbf{P}^1 are smooth; and González proved, in [Gon04], the smoothness of the Hilbert point for most ribbons on curves of arbitrary genus.

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Convention. We work over \mathbf{C} . Recall that a surface Y , proper over \mathbf{C} , is said to be regular if $H^1(\mathcal{O}_Y) = 0$. An Enriques surface is a regular surface whose canonical bundle is 2-torsion. A $K3$ surface is a regular surface whose canonical bundle is trivial. Throughout this article, whenever we use the phrases Enriques surface, $K3$ surface or regular surface we will always mean in addition smooth, irreducible and proper over \mathbf{C} .

1. $K3$ CARPETS. CHARACTERIZATION

Among carpets on an Enriques surface Y , we single out a family which deserve special attention as far as they share the invariants of smooth $K3$ surfaces. We call them $K3$ carpets. In fact, we will give a more general definition: a $K3$ carpet on any regular surface will be a carpet with the same invariants of a smooth $K3$ surface (i.e., trivial dualizing sheaf and irregularity $q = 0$; see Definition 1.2 and Proposition 1.6). Gallego and Purnaprajna, in [GP97], studied $K3$ carpets supported on rational normal scrolls. In this paper we consider carpets on a different type of surfaces possessing a double covering from a smooth $K3$ surface, namely Enriques surfaces. In this new case, as in [GP97], the adjective $K3$ is not only justified by the fact that these carpets have the same invariants as smooth $K3$ surfaces, but also from the fact that projective $K3$ carpets are degenerations of smooth $K3$ surfaces, as we shall prove in this paper.

We start by recalling the definition of a carpet on a smooth surface.

Definition 1.1. *Let Y be a reduced connected scheme and let \mathcal{E} be a line bundle on Y . A ribbon on Y with conormal bundle \mathcal{E} is a scheme \tilde{Y} with $\tilde{Y}_{\text{red}} = Y$, such that*

- (1) $\mathcal{I}_{Y, \tilde{Y}}^2 = 0$ and
- (2) $\mathcal{I}_{Y, \tilde{Y}} \simeq \mathcal{E}$ as \mathcal{O}_Y -modules.

When Y is a surface, \tilde{Y} is called a carpet on Y .

We now give the definition of a $K3$ carpet supported on a regular surface. Although our definition does not require the carpet to be a regular scheme, we will see in Proposition 1.6 that a $K3$ carpet defined according to Definition 1.2 is always regular.

Definition 1.2. *Let Y be a regular surface. A $K3$ carpet \tilde{Y} on Y is a carpet on Y such that its dualizing sheaf $\omega_{\tilde{Y}} \simeq \mathcal{O}_{\tilde{Y}}$.*

Remark 1.3. The existence of a dualizing sheaf with nice functorial properties on a proper scheme is well known (see e.g., [Kle80, (7), p. 46]). Any ribbon \tilde{Y} on a smooth irreducible proper variety Y is a proper scheme over \mathbf{C} . This justifies the existence of the dualizing sheaf in Definition 1.2.

Any ribbon \tilde{Y} on Y is a locally Gorenstein (in fact, locally a complete intersection) scheme. Therefore the dualizing sheaf $\omega_{\tilde{Y}}$ is an invertible sheaf (see e.g. [Har66, V 9.3, 9.7, VII 3.4] or [Con00, p. 157]). \square

Next we point out some nice properties of the dualizing sheaf on \tilde{Y} . The assertions in Lemma 1.4 are also well known (see e.g. [BF86, Proposition 2.3], [BM85, Lemma 7] or [Man94, Remark 4.12]) and follow quickly from the definition of the dualizing sheaf (see [Har66, p. 241] or [Kle80, (1),(6)]). They are valid for ribbons in general.

Lemma 1.4. *Let Y be a smooth irreducible proper variety. Let \tilde{Y} be a ribbon on Y with conormal bundle \mathcal{E} . Then the dualizing sheaf $\omega_{\tilde{Y}}$ fits into an extension*

$$(1.4.1) \quad 0 \longrightarrow \omega_Y \longrightarrow \omega_{\tilde{Y}} \longrightarrow \mathcal{E}^{-1} \otimes \omega_Y \longrightarrow 0,$$

and, therefore, there is an isomorphism

$$(1.4.2) \quad \omega_{\tilde{Y}}|_Y = \mathcal{E}^{-1} \otimes \omega_Y.$$

Now we characterize $K3$ carpets from its conormal bundle.

Proposition 1.5. *Let Y be a regular surface and let \tilde{Y} be a carpet whose reduced part is Y . Let \mathcal{E} be the ideal sheaf of Y in \tilde{Y} . Then \tilde{Y} is a $K3$ carpet iff $\mathcal{E} \simeq \omega_Y$.*

Proof. Let $\mathcal{E} \simeq \omega_Y$. Look at (1.4.1). Since $H^1(\omega_Y) = 0$, the section $1 \in H^0(\mathcal{O}_Y)$ can be lifted to $H^0(\omega_{\tilde{Y}})$, and hence, $\omega_{\tilde{Y}}$ being invertible, we have $\omega_{\tilde{Y}} \simeq \mathcal{O}_{\tilde{Y}}$. Now assume $\omega_{\tilde{Y}} \simeq \mathcal{O}_{\tilde{Y}}$. Then, from (1.4.2), we get $\mathcal{E} \simeq \omega_Y$. \square

As a consequence of Proposition 1.5 we see that a $K3$ carpet, as defined in Definition 1.2, is a regular scheme, as is the case of smooth $K3$ surfaces.

Proposition 1.6. *Let \tilde{Y} be a $K3$ carpet on a regular surface Y . Then $H^1(\mathcal{O}_{\tilde{Y}}) = 0$.*

Proof. From Proposition 1.5, the conormal bundle is $\mathcal{E} = \omega_Y$. Since Y is a regular surface $H^1(\omega_Y) = H^1(\mathcal{O}_Y) = 0$ and hence, from the exact sequence $0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{\tilde{Y}} \rightarrow \mathcal{O}_Y \rightarrow 0$, we obtain $H^1(\mathcal{O}_{\tilde{Y}}) = 0$. \square

Remark 1.7. We have seen that the $K3$ carpets on a given regular surface are the carpets with conormal bundle ω_Y . Thus (see [BE95, 1.4]) the space of non-split $K3$ carpets on a given regular surface Y is the projective space of lines in $\text{Ext}_Y^1(\Omega_Y, \omega_Y)$.

Notice that, when Y is an Enriques surface the dimension of $\text{Ext}_Y^1(\Omega_Y, \omega_Y)$ is the Hodge number $h^{1,1} = 10$. \square

2. PROJECTIVE AND NON-PROJECTIVE $K3$ CARPETS

In contrast to ribbons on curves, not all carpets are projective, (see [Har77, III Ex. 5.9]) even if all of them are proper or if, as is the case with Enriques surfaces, they are supported on a projective surface. Thus the very first question about the $K3$ carpets on Enriques surfaces is whether there exist families of projective $K3$ carpets. This question has a positive answer as is illustrated in Theorem 2.2 and Theorem 2.3. Next step is to compute the dimension of the space parametrizing $K3$ carpets on a given Enriques surface. This is settled in Theorem 2.4 for the dimension of the family of embedded (projective) carpets on a given embedded Enriques surface, and in Theorem 2.5, where we compute the size of the space of projective $K3$ carpets supported on a given (abstract) Enriques surface Y , comparing it also with the space of all $K3$ carpets on Y . As we will see, the situation somehow resembles that of smooth $K3$ surfaces.

To start searching for embedded $K3$ carpets we need to look first for embeddings of Enriques surfaces in projective space. We recall some well known facts about this:

Remark 2.1. Let Y be an Enriques surface.

- (1) If Y is embedded in \mathbf{P}^N , then $N \geq 5$.
- (2) A very ample line bundle on Y has sectional genus $g \geq 6$ and degree $d \geq 10$.
- (3) If $N \geq 5$, then the surface Y can be embedded in \mathbf{P}^N .

Proof. By adjunction, there do not exist Enriques surfaces in \mathbf{P}^3 . On the other hand, applying the formula for the numerical invariants of a smooth surface Y in \mathbf{P}^4 (see [Har77, A.4.1.3]),

$$d^2 - 10d - 5HK_Y - 2K_Y^2 + 12 + 12p_a = 0,$$

we see at once that there do not exist Enriques surfaces in \mathbf{P}^4 either. This completes the proof of (1). Now, a line bundle on Y with sectional genus g has g linearly independent global sections. Then, if the line bundle is very ample, (1) implies that $g \geq 6$, so its degree is $2g - 2 \geq 10$. This proves (2). Finally, since Y is projective, Y can be embedded in \mathbf{P}^M , with $M \gg 0$ and we project it isomorphically into \mathbf{P}^N as far as $N \geq 5$. \square

Now we want to know how many $K3$ carpets are supported on a given embedded Enriques surfaces. This will do in Theorem 2.4. To do this we will need to know the dimension of the space of first-order infinitesimal deformations of a morphism from a $K3$ surface to projective space. Given a morphism φ from a variety X to \mathbf{P}^N , the normal sheaf \mathcal{N}_φ is defined as the cokernel of the natural map $\mathcal{T}_X \rightarrow \varphi^* \mathcal{T}_{\mathbf{P}^N}$. Then the first-order infinitesimal deformations of φ , up to isomorphism, are parametrized by $H^0(\mathcal{N}_\varphi)$ (see [Hor74, 4.2]). In our setting since X is a smooth $K3$ surface, it is a smooth variety. Then, if the image of φ has the same dimension as X , we have the following exact sequence:

$$(2.1.1) \quad 0 \rightarrow \mathcal{T}_X \rightarrow \varphi^* \mathcal{T}_{\mathbf{P}^N} \rightarrow \mathcal{N}_\varphi \rightarrow 0.$$

Theorem 2.2. *Let X be a smooth projective $K3$ surface and let $X \xrightarrow{\varphi} \mathbf{P}^N$ be a morphism whose image is a surface. Let \mathcal{N}_φ be the normal sheaf of φ . Then,*

- (1) *the dimension of the image of the connecting map*

$$H^0(\mathcal{N}_\varphi) \rightarrow H^1(\mathcal{T}_X)$$

of the long exact sequence of cohomology of (2.1.1) is 19;

- (2) $H^1(\mathcal{N}_\varphi) = 0$; and
 (3) $H^2(\mathcal{N}_\varphi) = 0$.

Proof. Let us denote $L = \varphi^* \mathcal{O}_{\mathbf{P}^N}(1)$ and let us consider the Atiyah extension of L

$$(2.2.1) \quad 0 \rightarrow \mathcal{O}_X \rightarrow \Sigma_L \rightarrow \mathcal{T}_X \rightarrow 0.$$

The space $H^1(\Sigma_L)$ parametrizes first-order infinitesimal deformations of the pair (X, L) up to isomorphism (see [Zar95, pp. 126–128] or [Ser06, II.2.2]) and the map $H^0(\mathcal{N}_\varphi) \rightarrow H^1(\mathcal{T}_X)$ factors through $H^1(\Sigma_L)$. Taking cohomology on (2.2.1) yields the exact sequence

$$H^1(\mathcal{O}_X) \rightarrow H^1(\Sigma_L) \rightarrow H^1(\mathcal{T}_X) \rightarrow H^2(\mathcal{O}_X).$$

Since X is a $K3$ surface, $h^1(\mathcal{O}_X) = 0$, $h^2(\mathcal{O}_X) = 1$ and $h^1(\mathcal{T}_X)$ is the same as the Hodge number $h^{1,1}$ of X , hence

$$(2.2.2) \quad \dim H^1(\mathcal{T}_X) = 20.$$

On the other hand, $H^1(\mathcal{T}_X) \rightarrow H^2(\mathcal{O}_X)$ is induced by cup product with the cohomology class $c(L) \in H^1(\Omega_X)$ (see [Ser06, Proposition II.2.2]), so it is surjective, for L is non-trivial (see [Ser06, p. 57]). Then

$$(2.2.3) \quad \dim H^1(\Sigma_L) = 19.$$

Then, going back to (2.1.1) we have the long exact sequence

$$H^0(\mathcal{N}_\varphi) \xrightarrow{\nu} H^1(\mathcal{T}_X) \rightarrow H^1(\varphi^* \mathcal{T}_{\mathbf{P}^N}) \rightarrow H^1(\mathcal{N}_\varphi) \rightarrow 0,$$

where the exactness on the far right comes from $h^2(\mathcal{T}_X) = h^{0,1} = 0$. Then (2.2.3) implies that the image of ν has dimension less than or equal to 19. On the other hand, taking cohomology on the dual of the Euler sequence restricted to X yields $h^1(\varphi^* \mathcal{T}_{\mathbf{P}^N}) = 1$, for

$H^1(L) = H^2(L) = 0$ since L is ample. All this together with (2.2.2) implies that the image of ν has dimension 19 and $H^1(\mathcal{N}_\varphi) = 0$.

To prove (3) note that taking cohomology on the dual of the Euler sequence restricted to X yields $H^2(\varphi^* \mathcal{T}_{\mathbf{P}^N}) = 0$, for $H^2(L) = 0$. Then it follows that $H^2(\mathcal{N}_\varphi) = 0$. \square

We will use Theorem 2.2 in this situation (see e.g. (2.4.6) in the proof of Theorem 2.4): we set φ to be the composition of the étale $K3$ double cover $X \xrightarrow{\pi} Y$ of an Enriques surface Y followed by an embedding $Y \xrightarrow{i} \mathbf{P}^N$. On the other hand, Theorem 2.2 can be also used if φ is an embedding into projective space, so we recover the following result:

Corollary 2.3. *If X is a smooth projective $K3$ surface embedded in projective space, (not necessarily as a linearly normal variety nor as a non-degenerate variety), then the point of X in the Hilbert scheme is smooth.*

Next theorem gives a quantitative measure on the $K3$ carpets supported on an embedded Enriques surfaces. Precisely, given an embedded Enriques surface $Y \xrightarrow{i} \mathbf{P}^N$, we find the dimension of the variety that parametrizes the $K3$ carpets in \mathbf{P}^N , supported on $i(Y)$.

Theorem 2.4. *Let Y be an Enriques surface and let $Y \xrightarrow{i} \mathbf{P}^N$ be an embedding of Y . Let g be the sectional genus of $i(Y)$. The $K3$ carpets embedded in \mathbf{P}^N and supported on $i(Y)$ are parametrized by a non-empty open set in the projective space of lines in $H^0(\mathcal{N}_{Y, \mathbf{P}^N} \otimes \omega_Y)$, whose dimension is $g(N+1)+8$. In particular, if i is induced by the complete linear series of $\mathcal{O}_Y(1)$, then the dimension of this open set is g^2+8 .*

Proof. Denote $\mathcal{I} = \mathcal{I}_{i(Y), \mathbf{P}^N}$. The $K3$ carpets in \mathbf{P}^N which are supported on $i(Y)$ are in one-to-one correspondence with the surjective elements in $\text{Hom}(\mathcal{I}/\mathcal{I}^2, \omega_Y)$, up to nonzero scalar multiple (see [Gon06, Proposition 2.1.(2)]; see also [GP97, Lemma 1.4] or [HV85]). We start computing the dimension of $\text{Hom}(\mathcal{I}/\mathcal{I}^2, \omega_Y)$. Recall that $\Omega_Y^* \otimes \omega_Y \simeq \Omega_Y$. Then, since Y is regular, and by Serre duality and Hodge Theory, we have $h^0(\Omega_Y^* \otimes \omega_Y) = h^2(\Omega_Y^* \otimes \omega_Y) = 0$. Then, taking cohomology on the conormal sequence of $i(Y)$, we get

$$(2.4.1) \quad 0 \rightarrow \text{Hom}(\Omega_{\mathbf{P}^N} \otimes \mathcal{O}_Y, \omega_Y) \rightarrow \text{Hom}(\mathcal{I}/\mathcal{I}^2, \omega_Y) \xrightarrow{\delta} \text{Ext}^1(\Omega_Y, \omega_Y) \rightarrow \text{Ext}^1(\Omega_{\mathbf{P}^N} \otimes \mathcal{O}_Y, \omega_Y) \rightarrow \text{Ext}^1(\mathcal{I}/\mathcal{I}^2, \omega_Y) \rightarrow 0.$$

To find the dimension of $\text{Hom}(\mathcal{I}/\mathcal{I}^2, \omega_Y)$ we need to compute the dimensions of the other terms of the sequence (2.4.1). Dualizing the restriction to Y of the Euler sequence and tensoring by ω_Y , we have the exact sequence

$$(2.4.2) \quad 0 \longrightarrow \omega_Y \longrightarrow \mathcal{O}_Y^{\oplus N+1}(1) \otimes \omega_Y \longrightarrow \Omega_{\mathbf{P}^N}^* \otimes \omega_Y \longrightarrow 0.$$

Since $h^1(\mathcal{O}_Y(1) \otimes \omega_Y) = h^2(\mathcal{O}_Y(1) \otimes \omega_Y) = 0$, it follows that $h^1(\Omega_{\mathbf{P}^N}^* \otimes \omega_Y) = h^2(\omega_Y) = 1$. So

$$(2.4.3) \quad \dim \text{Ext}^1(\Omega_{\mathbf{P}^N} \otimes \mathcal{O}_Y, \omega_Y) = h^1(\Omega_{\mathbf{P}^N}^* \otimes \omega_Y) = 1.$$

Also, $h^0(\omega_Y) = h^1(\omega_Y) = 0$, so we have

$$(2.4.4) \quad \dim \text{Hom}(\Omega_{\mathbf{P}^N} \otimes \mathcal{O}_Y, \omega_Y) = (N+1) \cdot h^0(\mathcal{O}_Y(1) \otimes \omega_Y) = g(N+1).$$

On the other hand (see Remark 1.7)

$$(2.4.5) \quad \dim \text{Ext}^1(\Omega_Y, \omega_Y) = 10.$$

Finally we will see that $\text{Ext}^1(\mathcal{I}/\mathcal{I}^2, \omega_Y) = 0$. To do this, let $X \xrightarrow{\pi} Y$ be the étale $K3$ double cover of Y . Denote $i \circ \pi = \varphi$. From Theorem 2.2, (2), for the normal sheaf of φ we have

$$(2.4.6) \quad H^1(\mathcal{N}_\varphi) = 0.$$

We will see that $\text{Ext}^1(\mathcal{I}/\mathcal{I}^2, \omega_Y)$ is a direct summand of $H^1(\mathcal{N}_\varphi)$.

Let \mathcal{F} be the kernel of $\varphi^*\Omega_{\mathbf{P}^N} \rightarrow \Omega_X$. Since π is étale, it follows that $\Omega_{X/Y}$ and Ω_{X/\mathbf{P}^N} are both 0, so we have the following commutative diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \pi^*(\mathcal{I}/\mathcal{I}^2) = \pi^*(\mathcal{I}/\mathcal{I}^2) & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \varphi^*\Omega_{\mathbf{P}^N} & \longrightarrow & \Omega_X \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ & & 0 & \longrightarrow & \pi^*\Omega_Y & \longrightarrow & \Omega_X \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & 0. & & \end{array}$$

Therefore there is an isomorphism

$$\mathcal{N}_\varphi \simeq \mathcal{H}om(\pi^*\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X).$$

Since $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \omega_Y$, taking cohomology and using the adjunction isomorphism we get

$$\begin{aligned} H^1(\mathcal{N}_\varphi) &= H^1(\mathcal{H}om(\pi^*\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)) = \text{Ext}^1(\pi^*\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) = \\ &= \text{Ext}^1(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \oplus \text{Ext}^1(\mathcal{I}/\mathcal{I}^2, \omega_Y). \end{aligned}$$

Then Theorem 2.2, (2) implies

$$(2.4.7) \quad \text{Ext}^1(\mathcal{I}/\mathcal{I}^2, \omega_Y) = 0.$$

Then, from (2.4.1), (2.4.3), (2.4.4), (2.4.5), and (2.4.7), we see at once that

$$\dim \text{Hom}(\mathcal{I}/\mathcal{I}^2, \omega_Y) = g(N+1) + 9.$$

Recall that the $K3$ carpets on Y embedded in \mathbf{P}^N are in one-to-one correspondence with the surjective homomorphisms in $\text{Hom}(\mathcal{I}/\mathcal{I}^2, \omega_Y)$, up to nonzero scalar multiple, or equivalently, with the nowhere vanishing global sections of the $(N-2)$ -rank vector bundle $\mathcal{N}_{Y, \mathbf{P}^N} \otimes \omega_Y$, up to nonzero scalar multiple. Recall also that the elements of $\text{Hom}(\mathcal{I}/\mathcal{I}^2, \omega_Y)$ corresponding to surjective homomorphisms form an open set (see [Gon06, Lemma4.1]). Therefore, to finish the proof we need to show that there is a nowhere vanishing section in the space $H^0(\mathcal{N}_{Y, \mathbf{P}^N} \otimes \omega_Y)$. Observe first that $\mathcal{N}_{Y, \mathbf{P}^N} \otimes \omega_Y$ is globally generated. To see this note that we have a surjection $\Omega_{\mathbf{P}^N}^* \otimes \omega_Y \rightarrow \mathcal{N}_{Y, \mathbf{P}^N} \otimes \omega_Y$ so, from (2.4.2), we see that $\mathcal{N}_{Y, \mathbf{P}^N} \otimes \omega_Y$ is globally generated as long as $\mathcal{O}_Y(1) \otimes \omega_Y$ is globally generated. This follows from Reider's theorem ([Rei88]), since $\mathcal{O}_Y(1)$ is very ample and its degree $d = 2g - 2 \geq 10$ (see Remark 2.1, (2)). Finally since the rank of $\mathcal{N}_{Y, \mathbf{P}^N} \otimes \omega_Y$ is $N - 2 > \dim Y$ (see Remark 2.1, (1)) and it is a globally generated vector bundle, it has a nowhere vanishing section. Thus the $K3$ carpets inside \mathbf{P}^N , supported on $i(Y)$ in \mathbf{P}^N , are parametrized by a non-empty open set in the projective space of lines in $H^0(\mathcal{N}_{Y, \mathbf{P}^N} \otimes \omega_Y)$, whose dimension is $g(N+1) + 8$. \square

The following theorem is a refinement of [Har77, III Ex. 5.9] to characterize non-projective $K3$ carpets. As result of this theorem, we can say more about the size of the families of projective $K3$ carpets on a given (abstract) Enriques surface, compared to the set of non-projective $K3$ carpets.

Theorem 2.5. *Let Y be an Enriques surface and let \tilde{Y} be a $K3$ carpet on Y corresponding to an element $\tau \in \text{Ext}^1(\Omega_Y, \omega_Y)$.*

- (1) *The carpet \tilde{Y} is projective if and only if there exists an ample divisor D on Y such that $\int_D \tau = 0$, when τ is thought as an element of $H^{1,1}(Y) = H^2(Y, \mathbf{C})$.*
- (2) *Non-split projective $K3$ carpets on Y are parametrized by a union of (countably infinitely many distinct) hyperplanes of the 9-dimensional projective space of lines in $\text{Ext}^1(\Omega_Y, \omega_Y)$. These hyperplanes are in one-to-one correspondence with the set of classes in $NS(Y)$ of primitive ample divisors on Y .*

Proof. Recall (see Remark 1.7) that a $K3$ carpet on Y corresponds to an element

$$\tau \in \text{Ext}^1(\Omega_Y, \omega_Y) \simeq H^1(\Omega_Y^* \otimes \omega_Y) \simeq H^1(\Omega_Y) = H^{1,1}(Y) = H^2(Y, \mathbf{C}).$$

Since the ideal of Y inside \tilde{Y} is a square zero ideal, we have an exact sequence

$$0 \rightarrow \omega_Y \rightarrow \mathcal{O}_{\tilde{Y}}^* \rightarrow \mathcal{O}_Y^* \rightarrow 1.$$

This yields

$$0 \rightarrow \text{Pic } \tilde{Y} \xrightarrow{\gamma} \text{Pic } Y \xrightarrow{\lambda} H^2(\omega_Y) \rightarrow H^2(\mathcal{O}_{\tilde{Y}}^*) \rightarrow H^2(\mathcal{O}_Y^*).$$

The map λ works as follows: if D is a divisor on Y , then $\lambda(\mathcal{O}_Y(D)) = \int_D \tau$. The map γ sends each line bundle on \tilde{Y} to its restriction to Y . The carpet \tilde{Y} is projective if and only if it possesses an ample line bundle. On the other hand, a line bundle on \tilde{Y} is ample if and only if its restriction to Y is ample. Therefore \tilde{Y} is projective if and only if there exists an ample line bundle on Y that can be lifted by γ to \tilde{Y} . This is the same as saying that there exists an ample line bundle on Y lying in the kernel of λ . Thus \tilde{Y} is projective if and only if there exists an ample divisor D on Y such that $\int_D \tau = 0$. Then, given an ample divisor D on Y , the elements $\tau \in \text{Ext}^1(\Omega_Y, \omega_Y) \simeq H^{1,1}(Y)$ with $\int_D \tau = 0$ form a hyperplane H_D of $\text{Ext}^1(\Omega_Y, \omega_Y)$, whose elements correspond to projective $K3$ carpets. Then projective $K3$ carpets are parametrized by the projective lines in

$$\bigcup_D H_D,$$

where D ranges over the set of primitive ample divisors on Y . □

Remark 2.6. Let Y be an Enriques surface. Theorem 2.5 shows in particular the existence of non-projective $K3$ carpets on a given Enriques surface Y . Indeed, the non-split non-projective $K3$ carpets on Y are parametrized by the complement of a union of countably many hyperplanes of the 9-dimensional projective space of lines in $\text{Ext}^1(\Omega_Y, \omega_Y)$. There are “more” non-projective $K3$ carpets than projective $K3$ carpets.

The arguments of the proof of Theorem 2.4 give another way of looking at Theorem 2.5:

Proposition 2.7. *Let Y be an Enriques surface. Associated to every embedding i of Y into some projective space \mathbf{P}^N , there is a sequence (2.4.1), arising from the conormal sequence of $i(Y)$ in \mathbf{P}^N . For the sequence (2.4.1) associated to i , we will denote by δ_i the map δ . Let*

$\mathbf{P}(\mathrm{Im} \delta_i)$ be the projective space of lines in $\mathrm{Im} \delta_i$. Then the non-split projective K3 carpets on Y are parametrized by

$$\bigcup_i \mathbf{P}(\mathrm{Im} \delta_i),$$

where i ranges among all the embeddings of Y into some projective space. For each i , $\mathbf{P}(\mathrm{Im} \delta_i)$ is a hyperplane in the 9-dimensional projective space of lines in $\mathrm{Ext}^1(\Omega_Y, \omega_Y)$.

Proof. If a K3 carpet \tilde{Y} on Y is projective, it can be embedded in some projective space \mathbf{P}^N by the complete linear series of a very ample line bundle. This embedding induces an embedding i of Y as (a degenerate) subvariety of \mathbf{P}^N . Let \mathcal{I} be the ideal sheaf of Y in \mathbf{P}^N . Then, the carpet \tilde{Y} embedded in \mathbf{P}^N corresponds to an element of $\mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \omega_Y)$. Thus, the K3 carpet \tilde{Y} , considered as an abstract scheme, corresponds to a point lying in the image of the map δ_i . From (2.4.3) and (2.4.7) we gather that the cokernel of δ_i has dimension 1, hence the image of δ_i in $\mathrm{Ext}^1(\Omega_Y, \omega_Y)$ is a hyperplane. Thus the class in $\mathrm{Ext}^1(\Omega_Y, \omega_Y)$ of every projective K3 carpet lies in the image of the map δ_i associated to some embedding i of Y into some projective space. Since obviously the classes lying in the image of any of the maps δ_i correspond to projective K3 carpets, we see that non-split projective K3 are parametrized by

$$\bigcup_i \mathbf{P}(\mathrm{Im} \delta_i),$$

where i ranges among all the embeddings of Y into some projective space. \square

In Theorem 2.4 and Theorem 2.5 we saw how many projective K3 carpets there are supported on an Enriques surface. In the next observation, we describe how embeddings by a complete linear series of a K3 carpet look like.

Remark 2.8. Let Y be an Enriques surface and let \tilde{Y} be a projective K3 carpet on Y . Assume that \tilde{Y} is embedded, as a non-degenerate subscheme into some projective space, by the complete linear series of a very ample line bundle. Let g be the sectional genus of $\mathcal{O}_Y(1) = \mathcal{O}_{\tilde{Y}}(1) \otimes \mathcal{O}_Y$. Then, from $H^1(\mathcal{O}_Y(1) \otimes \omega_Y) = 0$ and the exact sequence

$$0 \longrightarrow \omega_Y(1) \longrightarrow \mathcal{O}_{\tilde{Y}}(1) \longrightarrow \mathcal{O}_Y(1) \longrightarrow 0,$$

we have

$$H^0(\mathcal{O}_{\tilde{Y}}(1)) = H^0(\mathcal{O}_Y(1)) \oplus H^0(\omega_Y(1)).$$

Therefore the embedding induced on Y is also given by the complete linear series of $\mathcal{O}_Y(1)$ and there is a diagram

$$\begin{array}{ccc} \tilde{Y} & \hookrightarrow & \mathbf{P}^{2g-1} = \mathbf{P}(H^0(\mathcal{O}_Y(1)) \oplus H^0(\omega_Y(1))) \\ \uparrow & & \uparrow \\ Y & \hookrightarrow & \mathbf{P}^{g-1} = \mathbf{P}(H^0(\mathcal{O}_Y(1))). \end{array}$$

3. DEFORMATION OF MORPHISMS AND SMOOTHING OF PROJECTIVE $K3$ CARPETS

In this section we prove two results. First we show in Theorem 3.2 that the étale $K3$ double cover π of an Enriques surface can be deformed, in many different ways, to a family of projective embeddings. Second, as a consequence of Theorem 3.2 we show (see Theorem 3.5 and Corollary 3.6) that every projective $K3$ carpet \tilde{Y} on an Enriques surface can be smoothed. By this we mean that we can find a flat, proper, integral family \mathcal{Y} over a smooth affine curve T , such that over for $0 \in T$, $\mathcal{Y}_0 = \tilde{Y}$ and for $t \in T, t \neq 0$, \mathcal{Y}_t is a smooth, irreducible, and, in our case, projective $K3$ surface.

The key point that connects Theorems 3.2 and 3.5 is the fact that \tilde{Y} , after being embedded in some projective space \mathbf{P}^N , arises as the central fiber of the image of a first-order infinitesimal deformation of the composition of π with the inclusion of Y in \mathbf{P}^N :

Theorem 3.1. *Let $\tilde{Y} \subset \mathbf{P}^N$ be a projective $K3$ carpet on a smooth Enriques surface Y . Let $X \xrightarrow{\pi} Y$ be the étale $K3$ double cover of Y and let $X \xrightarrow{\varphi} \mathbf{P}^N$ be the morphism obtained by composing π with the inclusion of Y in \mathbf{P}^N . Then \tilde{Y} is the central fiber of the image of some first-order infinitesimal deformation of φ .*

Proof. Since π is étale, we have $\mathcal{N}_\pi = 0$. Then the result follows from [Gon06, Theorem 3.9]. \square

Next we show that φ can be deformed to a family of embeddings to \mathbf{P}^N . We do so by proving something stronger, namely, that any infinitesimal deformation of φ can be extended to a family of embeddings of smooth $K3$ surfaces in \mathbf{P}^N . Theorem 3.2 is, in the present setting, the counterpart of [GGP07, Theorem 2.1], where the authors showed that a finite cover of a curve can be deformed to a family of embeddings.

Theorem 3.2. *Let $X \xrightarrow{\pi} Y$ be the étale $K3$ double cover of an Enriques surface Y , embedded in \mathbf{P}^N with sectional genus g and satisfying $N \leq 2g - 1$. Let φ denote the composition of π with the inclusion of Y in \mathbf{P}^N . Let $\Delta = \text{Spec } \mathbf{k}[\epsilon]/\epsilon^2$. Then for every first-order infinitesimal deformation*

$$\tilde{X} \xrightarrow{\tilde{\varphi}} \mathbf{P}_\Delta^N$$

of $X \xrightarrow{\varphi} \mathbf{P}^N$, there exists a smooth irreducible family \mathcal{X} , proper and flat over a smooth pointed affine curve $(T, 0)$, and a T -morphism $\mathcal{X} \xrightarrow{\Phi} \mathbf{P}_T^N$ with the following features:

- (1) *the general fiber $\mathcal{X}_t \xrightarrow{\Phi_t} \mathbf{P}^N$, $t \in T - 0$, is a closed immersion of a smooth $K3$ surface; and*
- (2) *the fiber of $\mathcal{X} \xrightarrow{\Phi} \mathbf{P}_T^N$ over the tangent vector at $0 \in T$ is $\tilde{X} \xrightarrow{\tilde{\varphi}} \mathbf{P}_\Delta^N$; in particular, the central fiber $\mathcal{X}_0 \xrightarrow{\Phi_0} \mathbf{P}^N$ is $X \xrightarrow{\varphi} \mathbf{P}^N$*

Remark 3.3. We require $N \leq 2g - 1$ in the statement of Theorem 3.2. This hypothesis is, in fact, quite natural. Indeed, if $\tilde{Y} \subset \mathbf{P}^N$ is non-degenerate (i.e., not contained in a hyperplane), then $N \leq 2g - 1$ (see Remark 2.8). The hypothesis is used in Step 2 of the proof of Theorem 3.2 (see (3.4.1)).

Before proving Theorem 3.2 we need the following lemma:

Lemma 3.4. *Let Y be an Enriques surface, embedded in projective space with sectional genus g , and let $X \xrightarrow{\pi} Y$ be its étale $K3$ double cover. Then, if $L = \pi^* \mathcal{O}_Y(1)$, L is very ample.*

Proof. From Remark 2.1 it follows that $L^2 = 4g - 4 \geq 20$. Then, to prove that L is very ample, it suffices to check the following (see [SD74, 4.2, 5.2, 6.1]):

- (1) there is no irreducible curve E such that $p_a(E) = 1$ and $L \cdot E = 2$, and
- (2) there is no smooth rational curve E such that $L \cdot E = 0$.

The first condition holds because L is base-point-free and the second condition holds because L is ample. \square

Proof of Theorem 3.2. Step 1. To obtain Φ we first construct, in a suitable way, a pair $(\mathcal{X}, \mathcal{L})$, where \mathcal{X} is a family of smooth $K3$ surfaces and \mathcal{L} is a family of very ample line bundles.

Let us denote $\tilde{L} = \tilde{\varphi}^* \mathcal{O}_{\mathbf{P}_{\Delta}^N}(1)$. Then \tilde{L} restricts to L on X and the Δ -module $\Gamma(\tilde{L})$ is free of rank $h^0(L)$ and $\Gamma(\tilde{L}) \otimes k[\epsilon]/\epsilon k[\epsilon] = H^0(L)$.

Now we want to obtain a family $(\mathcal{X}, \mathcal{L})$, proper and flat over a smooth pointed affine curve $(T, 0)$, whose central fiber is (X, L) , whose restriction to the tangent vector to T at 0 is (\tilde{X}, \tilde{L}) and whose general member $(\mathcal{X}_t, \mathcal{L}_t)$ consists of a smooth irreducible $K3$ surface and a very ample line bundle \mathcal{L}_t .

Note that L has degree $4g - 4$ and $h^0(L) = 2g$. Then, from Lemma 3.4 we know that L is very ample and, by Corollary 2.3, its complete linear series $|L|$ defines an embedding which determines a smooth point $[X]$ in a single component of the Hilbert scheme of surfaces of degree $4g - 4$ in \mathbf{P}^{2g-1} . The general point $[X']$ in this component represents a smooth irreducible $K3$ surface. Then we may consider an open neighborhood H of $[X]$ in its Hilbert component, with H parametrizing only smooth $K3$ surfaces. Moreover, since L is very ample and $H^1(L) = 0$, also \tilde{L} is very ample relative to Δ and the embedding $X \xrightarrow{|L|} \mathbf{P}^{2g-1}$ extends to an embedding $\tilde{X} \hookrightarrow \mathbf{P}_{\Delta}^{2g-1}$. So the image of $\tilde{X} \hookrightarrow \mathbf{P}_{\Delta}^{2g-1}$ is a flat family over Δ which corresponds to a tangent vector to H at $[X]$. We can take the embedding $\tilde{X} \hookrightarrow \mathbf{P}_{\Delta}^{2g-1}$ so that this tangent vector is nonzero. Now, since $[X]$ is a smooth point in H , we can take a smooth irreducible affine curve T in H passing through $[X]$ with tangent direction the given tangent vector.

Let $0 \in T$ denote the point corresponding to $[X]$. Then the pullback to T of the universal family provides a family $(\mathcal{X}, \mathcal{L})$, proper and flat over T , whose central fiber is (X, L) , whose restriction to the tangent vector to T at 0 is (\tilde{X}, \tilde{L}) and whose general member $(\mathcal{X}_t, \mathcal{L}_t)$ consists of a smooth irreducible $K3$ surface and a very ample line bundle \mathcal{L}_t , with $H^1(\mathcal{L}_t) = H^2(\mathcal{L}_t) = 0$, and hence, with $h^0(\mathcal{L}_t) = h^0(L) = 2g$.

Step 2. Once we have the pair $(\mathcal{X}, \mathcal{L})$, we are going to use it to construct a relative morphism

$$\mathcal{X} \xrightarrow{\Phi} \mathbf{P}_T^N$$

with the properties described in the statement.

Recall that \mathcal{L} is very ample relative to T and that $h^0(\mathcal{L}_t) = h^0(L) = 2g$ and $h^1(\mathcal{L}_t) = 0$ for all $t \in T$. Then formation of p_* commutes with base extension and, after shrinking T ,

we may assume that $\Gamma(\mathcal{L})$ is a free \mathcal{O}_T -module. Then \mathcal{L} induces a morphism

$$\mathcal{X} \xrightarrow{\Psi} \mathbf{P}_T^{2g-1}$$

which is a closed immersion at each fiber. The morphism $\tilde{\varphi}$ is the composition $\tilde{\rho} \circ \Psi_\Delta$, for some linear projection $\mathbf{P}_\Delta^{2g-1} \xrightarrow{\tilde{\rho}} \mathbf{P}_\Delta^N$. Now we look at some t near (but different from) 0. Since

$$(3.4.1) \quad N \leq 2g - 1 = \dim |\mathcal{L}_t|,$$

we can find a linear projection ρ_t mapping $\Psi_t(\mathcal{X}_t)$ to \mathbf{P}^N . On the other hand, Remark 2.1 implies $N \geq 5$. Then choosing ρ_t sufficiently general, we may assume the composition $\rho_t \circ \Psi_t$ to be a closed immersion. We lift $\tilde{\rho}$ and ρ_t to a linear projection ρ to \mathbf{P}_T^N . Finally we define Φ as the composition $\rho \circ \Psi$. Since the restriction Φ_t is a closed immersion, by [Gro61, 4.6.7] so are the restrictions of Φ to the nearby fibers. Then, maybe shrinking T we can conclude that the restriction of Φ to Δ is $\tilde{\varphi}$ and that the restrictions Φ_t are closed immersions for all $t \in T$, $t \neq 0$. \square

Now we use Theorems 3.1 and 3.2 to show that \tilde{Y} is the limit of the images of a family of embeddings Φ_t of smooth $K3$ surfaces, degenerating to φ . Precisely, we want to extend the infinitesimal deformation of φ in such a way that, if we call the image of the family of morphisms $\mathcal{Y} \subset \mathbf{P}^N \times T$, then $\mathcal{Y}_0 = \tilde{Y}$. All this is done in the next theorem:

Theorem 3.5. *Let \tilde{Y} be a projective $K3$ carpet embedded in \mathbf{P}^N , and supported on an Enriques surface Y embedded in \mathbf{P}^N with sectional genus g and $N \leq 2g - 1$. Then there exists a family of morphisms Φ over an affine curve T as described in Theorem 3.2 such that the image \mathcal{Y} of Φ is a closed integral subscheme $\mathcal{Y} \subset \mathbf{P}_T^N$, flat over T , with the following features:*

- (1) *the general fiber \mathcal{Y}_t , $t \in T - 0$, is a smooth irreducible projective non-degenerate $K3$ surface in \mathbf{P}^N ,*
- (2) *the central fiber $\mathcal{Y}_0 \subset \mathbf{P}^N$ is $\tilde{Y} \subset \mathbf{P}^N$.*

Proof. We use the notations of the proof of the Theorem 3.2.

From Theorem 3.1 we know that there exists a first order infinitesimal deformation

$$\tilde{X} \xrightarrow{\tilde{\varphi}} \mathbf{P}_\Delta^N$$

of φ such that the central fiber of the image of $\tilde{\varphi}$ is equal to \tilde{Y} . Therefore there is a family $\mathcal{X} \rightarrow T$ and a T -morphism $\mathcal{X} \xrightarrow{\Phi} \mathbf{P}_T^N$ as in Theorem 3.2. Let \mathcal{Y} be the image of the T -morphism $\mathcal{X} \xrightarrow{\Phi} \mathbf{P}_T^N$. The total family \mathcal{X} is smooth and irreducible so \mathcal{Y} is integral. Furthermore, Φ is a closed immersion over $T - 0$ since, by Theorem 3.2, Φ_t is a closed immersion for every $t \in T - 0$ (see e.g. [Gro61, 4.6.7]). Therefore for $t \in T - 0$ we have the equality $\mathcal{Y}_t = \text{im}(\Phi_t)$. Since \mathcal{X}_t is smooth, this proves (1). Finally, the fact that T is an integral smooth curve and \mathcal{Y} is integral and dominates T implies that \mathcal{Y} is flat over T . So the fiber \mathcal{Y}_0 of \mathcal{Y} at $0 \in T$ is the flat limit of the images of $\mathcal{X}_t \xrightarrow{\Phi_t} \mathbf{P}^N$ for $t \neq 0$. Moreover, this fiber \mathcal{Y}_0 contains the central fiber $(\text{im } \tilde{\varphi})_0$ of the image of $\tilde{\varphi}$. Since \tilde{Y} has conormal bundle \mathcal{E} and π has trace zero module \mathcal{E} , both \mathcal{Y}_0 and $(\text{im } \tilde{\varphi})_0$ have the same Hilbert polynomial, so they are equal. \square

We highlight this consequence of Theorem 3.5:

Theorem 3.6. *Any projective K3 carpet \tilde{Y} on a Enriques surface Y is smoothable.*

Proof. Let us embed \tilde{Y} in projective space by the complete linear series of a very ample line bundle. Then Remark 2.8 implies that the condition $N \leq 2g - 1$ is satisfied, so the result follows from Theorem 3.5. \square

4. THE HILBERT POINT OF A PROJECTIVE K3 CARPET

In this section we prove, in Theorem 4.2, that the Hilbert point of a projective K3 carpet on an Enriques surface is smooth. This is in sharp contrast with the result of Gallego and Purnaprajna on Hilbert points corresponding to K3 carpets on a rational normal scroll. In that case, some of the Hilbert points are smooth and others are not, as shown in [GP97].

First we state a preliminary result valid in general for ribbons.

Lemma 4.1. *Let $Y \subset \tilde{Y} \subset \mathbf{P}^N$ be an embedded ribbon, with conormal bundle \mathcal{E} , on a smooth irreducible projective variety Y . Then there are exact sequences*

$$(4.1.1) \quad 0 \longrightarrow \mathcal{N}_{\tilde{Y}, \mathbf{P}^N}|_Y \otimes \mathcal{E} \longrightarrow \mathcal{N}_{\tilde{Y}, \mathbf{P}^N} \longrightarrow \mathcal{N}_{\tilde{Y}, \mathbf{P}^N}|_Y \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{E}^{-1} \longrightarrow \mathcal{N}_{Y, \mathbf{P}^N} \longrightarrow \mathcal{H}om_Y(\mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2, \mathcal{O}_Y) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{H}om_Y(\mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2, \mathcal{O}_Y) \longrightarrow \mathcal{N}_{\tilde{Y}, \mathbf{P}^N}|_Y \longrightarrow \mathcal{E}^{-2} \longrightarrow 0.$$

Proof. We know that \tilde{Y} is a local complete intersection so $\mathcal{N}_{\tilde{Y}, \mathbf{P}^N}$ is locally free. Therefore from

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\tilde{Y}} \longrightarrow \mathcal{O}_Y \longrightarrow 0,$$

we obtain the sequence (4.1.1). Also $\mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2$ is locally free so we have

$$\mathcal{H}om_{\tilde{Y}}(\mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2, \mathcal{O}_{\tilde{Y}})|_Y = \mathcal{H}om_Y(\mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2|_Y, \mathcal{O}_Y).$$

Furthermore $\mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2|_Y = \mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N} \mathcal{I}_{\tilde{Y}, \mathbf{P}^N}$ so we have an exact sequence

$$0 \longrightarrow (\mathcal{E}^l)^{-1} \longrightarrow \mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2|_Y \longrightarrow \mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N} \longrightarrow 0,$$

where \mathcal{E}^l is an invertible sheaf on Y . So there is an exact sequence

$$(4.1.2) \quad 0 \longrightarrow \mathcal{H}om_Y(\mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2, \mathcal{O}_Y) \longrightarrow \mathcal{N}_{\tilde{Y}, \mathbf{P}^N}|_Y \longrightarrow \mathcal{E}^l \longrightarrow 0.$$

Furthermore from

$$0 \longrightarrow \mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2 \longrightarrow \mathcal{I}_{Y, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2 \longrightarrow \mathcal{E} \longrightarrow 0,$$

we obtain the exact sequence

$$(4.1.3) \quad 0 \longrightarrow \mathcal{E}^{-1} \longrightarrow \mathcal{N}_{Y, \mathbf{P}^N} \longrightarrow \mathcal{H}om_Y(\mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2, \mathcal{O}_Y) \longrightarrow 0.$$

Moreover, since \tilde{Y} is a local complete intersection, we have

$$\bigwedge^c \mathcal{N}_{\tilde{Y}, \mathbf{P}^N} = \omega_{\tilde{Y}} \otimes \omega_{\mathbf{P}^N}^{-1} = \omega_{\tilde{Y}} \otimes \mathcal{O}_{\tilde{Y}}(N+1),$$

where c is the codimension of Y .

So

$$\bigwedge^c \mathcal{N}_{\tilde{Y}, \mathbf{P}^N}|_Y = \omega_{\tilde{Y}}|_Y \otimes \mathcal{O}_Y(N+1),$$

and from the isomorphism (1.4.2)

$$\bigwedge^c \mathcal{N}_{\tilde{Y}, \mathbf{P}^N}|_Y = \omega_Y \otimes \mathcal{E}^{-1} \otimes \mathcal{O}_Y(N+1).$$

Moreover

$$\bigwedge^c \mathcal{N}_{Y, \mathbf{P}^N} = \omega_Y \otimes \mathcal{O}_Y(N+1),$$

so

$$(4.1.4) \quad \bigwedge^c \mathcal{N}_{\tilde{Y}, \mathbf{P}^N}|_Y = \bigwedge^c \mathcal{N}_{Y, \mathbf{P}^N} \otimes \mathcal{E}^{-1}.$$

I claim that

$$(4.1.5) \quad \mathcal{E}' = \mathcal{E}^{-2}.$$

Indeed, from (4.1.2) we obtain

$$\bigwedge^c \mathcal{N}_{\tilde{Y}, \mathbf{P}^N}|_Y = \bigwedge^{c-1} \mathcal{H}om_Y(\mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2, \mathcal{O}_Y) \otimes \mathcal{E}',$$

and from (4.1.3)

$$\bigwedge^c \mathcal{N}_{Y, \mathbf{P}^N} = \bigwedge^{c-1} \mathcal{H}om_Y(\mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2, \mathcal{O}_Y) \otimes \mathcal{E}^{-1}.$$

So from (4.1.4) we obtain (4.1.5). \square

Theorem 4.2. *Let \tilde{Y} be a projective $K3$ carpet on an Enriques surface Y embedded in \mathbf{P}^N as in Theorem 3.5. Then the Hilbert point of \tilde{Y} is nonsingular.*

Proof. We have proved in Theorem 3.5 that \tilde{Y} admits an embedded smoothing. Moreover, from Theorem 2.2, we know that for any $K3$ surface $X \subset \mathbf{P}^N$ we have $H^1(\mathcal{N}_{X, \mathbf{P}^N}) = H^2(\mathcal{N}_{X, \mathbf{P}^N}) = 0$. So, from an straightforward computation, we see that the dimension of a component parametrizing $K3$ surfaces in \mathbf{P}^N is $18 + 2g(N+1)$. Therefore the $K3$ carpet \tilde{Y} represents a smooth point in the Hilbert scheme iff $h^0(\mathcal{N}_{\tilde{Y}, \mathbf{P}^N}) = 18 + 2g(N+1)$.

As a consequence of Theorem 3.5 or by direct computation using the sequences in Lemma 4.1, we see that the Euler characteristic $\chi(\mathcal{N}_{\tilde{Y}, \mathbf{P}^N}) = 18 + 2g(N+1)$. Therefore we have to show that

$$h^1(\mathcal{N}_{\tilde{Y}, \mathbf{P}^N}) - h^2(\mathcal{N}_{\tilde{Y}, \mathbf{P}^N}) = 0.$$

Indeed, first we see at once that

$$H^1(\mathcal{N}_{Y, \mathbf{P}^N}) = H^2(\mathcal{N}_{Y, \mathbf{P}^N}) = H^2(\mathcal{N}_{Y, \mathbf{P}^N} \otimes \omega_Y) = 0.$$

In addition, (2.4.7) says that

$$H^1(\mathcal{N}_{Y, \mathbf{P}^N} \otimes \omega_Y) = 0.$$

Therefore, from the sequences in Lemma 4.1, we obtain

$$H^2(\mathcal{H}om_Y(\mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2, \mathcal{O}_Y)) = 0,$$

$$H^2(\mathcal{H}om_Y(\mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2, \mathcal{O}_Y) \otimes \omega_Y) = 0,$$

$$H^1(\mathcal{H}om_Y(\mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2, \mathcal{O}_Y)) = H^2(\omega_Y^{-1}) = \mathbf{C},$$

$$H^1(\mathcal{H}om_Y(\mathcal{I}_{\tilde{Y}, \mathbf{P}^N} / \mathcal{I}_{Y, \mathbf{P}^N}^2, \mathcal{O}_Y) \otimes \omega_Y) = H^2(\mathcal{O}_Y) = 0.$$

Then we obtain

$$H^1(\mathcal{N}_{\tilde{Y}, \mathbf{P}^N}|_Y \otimes \omega_Y) = 0,$$

$$H^2(\mathcal{N}_{\tilde{Y}, \mathbf{P}^N}|_Y \otimes \omega_Y) = H^2(\omega_Y^{-1}) = \mathbf{C},$$

$$H^2(\mathcal{N}_{\tilde{Y}, \mathbf{P}^N}|_Y) = 0,$$

and

$$H^1(\mathcal{N}_{\tilde{Y}, \mathbf{P}^N}|_Y) = 0 \text{ or } \mathbf{C}.$$

Finally, from sequence (4.1.1), we see that $h^1(\mathcal{N}_{\tilde{Y}, \mathbf{P}^N}) - h^2(\mathcal{N}_{\tilde{Y}, \mathbf{P}^N}) = 0$ or -1 , but now observe that, since our component has dimension $18 + 2g(N+1)$, we know that $h^0(\mathcal{N}_{\tilde{Y}, \mathbf{P}^N}) \geq \chi(\mathcal{N}_{\tilde{Y}, \mathbf{P}^N})$, so $h^1(\mathcal{N}_{\tilde{Y}, \mathbf{P}^N}) - h^2(\mathcal{N}_{\tilde{Y}, \mathbf{P}^N}) \geq 0$. \square

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