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The real genus of the groups of order 48

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Abstract Every finite group G acts as an automorphism group of several bordered compact Klein surfaces. The minimal genus of these surfaces is called the real genus of G, and it is denoted $\rho(G)$. The systematical study of this parameter was begun by May and continued by him in several papers about the topic. As a consequence of these works, he and other authors obtained the groups such that $0 \leq \rho(G) \leq 16$. On the other hand, the real genus of the groups of order less than 64, excepting 48, is already known. The orders 48 and 64 are just the first obstructions in the search of the groups of real genus greater than 16. In this work we complete the calculation of the real genus of all the fifty two groups of order 48.

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1 Introduction and preliminaries

A Klein surface X is a compact surface endowed with a dianalytic structure [1]. Klein surfaces may be seen as a generalization of compact Riemann surfaces including bordered and non-orientable surfaces. An orientable unbordered Klein surface is a Riemann surface. Given a Klein surface X of topological genus g with $k > 0$ boundary components the number

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$p = \eta g + k - 1$ is called the algebraic genus of X, where $\eta = 2$ if X is an orientable surface and $\eta = 1$ otherwise.

In the study of Klein surfaces and their automorphism groups the non-Euclidean crystallographic groups (NEC groups in short) play an essential role. An NEC group Γ is a discrete subgroup of $\mathcal{G}$ (the full group of isometries of the hyperbolic plane $\mathcal{H}$) with compact quotient $\mathcal{H}/\Gamma$. For a Klein surface X with $p \geq 2$ there exists an NEC group Γ, such that $X = \mathcal{H}/\Gamma$, [22].

A finite group G of order N is an automorphism group of a Klein surface $X = \mathcal{H}/\Gamma$ if and only if there exists an NEC group Λ such that Γ is a normal subgroup of Λ with index N and $G = \Lambda/\Gamma$. If G is a finite group there exists a bordered Klein surface X such that G is an automorphism group of X, [12].

A finite group G may act as an automorphism group of different bordered Klein surfaces. The minimum algebraic genus of these surfaces is called the real genus of G and it is denoted by $\rho(G)$. The systematical study of the real genus was begun by May in [14]. Two kinds of problems appear. Firstly to obtain the groups with real genus n for each natural number n. On the other hand for a given family of finite groups, to calculate the real genus of each member in that family.

The groups of real genus 0 are C_n and D_n (cyclic groups and dihedral groups, respectively). The groups with real genus 1 are $C_2 \times C_n$ for $n \geq 4$ even, and $C_2 \times D_n$ for n even. If $\rho \geq 2$, the number of groups G with real genus $\rho(G) = \rho$ is finite. There is no group with real genus $2,\ 12$ or 24. The groups such that $\rho(G) \leq 16$ are already known, see [18].

The real genus of finite Abelian groups [21], and of several families of groups is also known. In particular, the real genus of the groups $C_m \times D_n$ was obtained by the authors in [7,8] and the real genus of the groups $D_m \times D_n$ in the last quoted paper.

Other results about several families of groups are known and we shall quote them along this work. In particular the real genus of each group of order less than 48 has been determined, see [18] and the references therein. The groups of order n, $49 \leq n \leq 63$ are considered in [4]. Observe that the orders 48 and 64 are the first obstructions in order to continue the search of those groups G such that $\rho(G) \geq 17$.

In this work we fill the gap by giving the real genus of each group of order 48, calculating those that were still unknown.

First, we give some preliminaries about NEC groups and Klein surfaces.

An NEC group Γ is a discrete subgroup of isometries of the hyperbolic plane $\mathcal{H}$, including orientation-reversing elements, with compact quotient $X = \mathcal{H}/\Gamma$. Each NEC group Γ has associated a signature [11]:

$$\sigma(\Gamma) = (g, \pm, [m_1, \ldots, m_r], \{(n_{i,1}, \ldots, n_{i,s_i}), i = 1, \ldots k\}), \tag{1}$$

where $g,\ k,\ r,\ m_i,\ n_{i,j}$ are integers satisfying $g,\ k,\ r \geq 0,\ m_i,\ n_{i,j} \geq 2$. The number g is the topological genus of X. The sign determines the orientability of X. The numbers m_i are the *proper periods*. The brackets $(n_{i,1}, \ldots, n_{i,s_i})$ are the *period-cycles*. The number k of period-cycles is equal to the number of boundary components of X. Numbers $n_{i,j}$ are the periods of the period-cycle $(n_{i,1}, \ldots, n_{i,s_i})$, also called *link-periods*. We will denote by $[-],\ (-)$ and $\{-\}$ the cases when $r = 0,\ s_i = 0$ and $k = 0$, respectively.

The signature determines the following presentation of Γ, [24]:

Generators:

$$x_i \quad i = 1, \ldots, r;$$
$$e_i \quad i = 1, \ldots, k;$$

$$\begin{aligned} c_{i,j} &\quad i=1,\ldots,k;\ j=0,\ldots,s_i;\\ a_i,b_i &\quad i=1,\ldots,g;\ (\text{if}\quad \sigma \text{ has sign``}+\text{''});\\ d_i &\quad i=1,\ldots,g;\ (\text{if}\quad \sigma \text{ has sign``}-\text{''}). \end{aligned}$$

Relations:

$$\begin{aligned} &x_i^{m_i}=1;\quad i=1,\ldots,r;\\ &c_{i,j-1}^2=c_{i,j}^2=(c_{i,j-1}c_{i,j})^{n_{i,j}}=1;\quad i=1,\ldots,k;\ j=1,\ldots,s_i;\\ &e_i^{-1}c_{i,0}e_ic_{i,s_i}=1;\quad i=1,\ldots,k;\\ &\prod_{i=1}^{r}x_i\prod_{i=1}^{k}e_i\prod_{i=1}^{g}(a_ib_ia_i^{-1}b_i^{-1})=1;\ (\text{if } \sigma \text{ has sign ``}+\text{''});\\ &\prod_{i=1}^{r}x_i\prod_{i=1}^{k}e_i\prod_{i=1}^{g}d_i^2=1;\ (\text{if } \sigma \text{ has sign ``}-\text{''}). \end{aligned}$$

The isometries x_i are elliptic, e_i, a_i, b_i are hyperbolic, $c_{i,j}$ are reflections and d_i are glide-reflections.

Every NEC group Γ with signature (1) has associated a fundamental region whose area $\mu(\Gamma)$, called *area of* Γ, is

$$\mu(\Gamma)=2\pi\left(\eta g+k-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1-\frac{1}{n_{i,j}}\right)\right), \tag{2}$$

with $\eta=2$ or 1 according to the sign be "+" or "−". An NEC group with signature (1) actually exists if and only if the right hand side of (2) is greater than 0.

We denote by $|\Gamma|$ the expression $\mu(\Gamma)/2\pi$ and call it the *reduced area* of Γ.

If Γ is a subgroup of an NEC group Γ' of finite index N, then also Γ is an NEC group and the following Riemann-Hurwitz formula holds:

$$|\Gamma|=N|\Gamma'|. \tag{3}$$

Let X be a Klein surface of topological genus g with $k>0$ boundary components. Then by [22] there exists an NEC group Γ with signature

$$\sigma(\Gamma)=(g,\pm,[-],\{(-),\overset{k}{\ldots},(-)\}), \tag{4}$$

such that $X=\mathcal{H}/\Gamma$. An NEC group with this signature is called a *bordered surface group*. Notice that $|\Gamma|=p-1$ in this case.

For each automorphism group G of a surface $X=\mathcal{H}/\Gamma$ of algebraic genus $p\geq 2$ there exists an NEC group Γ' such that $G=\Gamma'/\Gamma$ where $\Gamma\subset\Gamma'\subset N_{\mathcal{G}}$, [12], and $N_{\mathcal{G}}$ denotes the normalizer of Γ in the full group $\mathcal{G}$ of isometries of $\mathcal{H}$.

Every finite group G acts as an automorphism group of different bordered Klein surfaces. The minimum genus of these surfaces is called the *real genus* of G, and it is denoted by $\rho(G)$. Let $X=\mathcal{H}/\Gamma$ be a bordered Klein surface on which G acts as an automorphism group. Then there exists another NEC group Λ such that $G=\Lambda/\Gamma$. From the relation between areas we have

Table 1

	Group $G = C_3 \times H$	$\rho(G)$	GAP
1	$C_3 \times C_{16} \approx C_{48}$	0	[48,2]
2	$C_3 \times (C_2 \times C_8) \approx C_2 \times C_{24}$	1	[48,23]
3	$C_3 \times (C_2 \times C_2 \times C_4) \approx C_2 \times C_2 \times C_{12}$	25	[48,44]
4	$C_3 \times (C_4 \times C_4) \approx C_4 \times C_{12}$	33	[48,20]
5	$C_3 \times (C_2 \times C_2 \times C_2 \times C_2) \approx C_2 \times C_2 \times C_2 \times C_6$	37	[48,52]

$$p - 1 = o(G)|\Lambda|,$$

where $o(G)$ denotes the order of G, and p is the algebraic genus of X. Then

$$\rho(G) \leq p = 1 + o(G)|\Lambda|,$$

and so $\rho(G) = 1 + o(G)|\Lambda_0|$ if the reduced area $|\Lambda_0|$ is minimal among the reduced areas of those groups admitting an epimorphism onto G such that the signature of its kernel has the form (4). We recall that the signature of Λ must contain an empty period-cycle or a period cycle with two consecutive link-periods equal to 2, see [3].

We now start the study of groups of order 48. For some of them the real genus is already known. If it is the case we explicitly give the group Λ and the epimorphism from Λ onto G with kernel Γ, when it has not been published.

Let us fix some notation concerning finite groups. We call C_n the cyclic group of order n, D_n the dihedral group of order $2n$, DC_n the dicyclic group of order $4n$, and A_n and S_n the alternating and symmetric group on n letters. For other groups we follow the names given by Coxeter in [5,6]. Also, given permutations σ and τ, we write $\sigma\tau$ the permutation which sends m to $\sigma(\tau(m))$.

There are fifty two groups to be considered. Twenty nine of them are the direct product of two groups of lesser order. So we divide the study of these groups into four sections according to they have as a factor the group C_3, C_4, C_2 or D_3, respectively. The final section will deal with the groups which are not a direct product.

2 The real genus of the direct product of C_3 and a group H of order 16

There exist fourteen groups of order 16, five of which are Abelian, namely $H = C_{16}, C_2 \times C_8, C_2 \times C_2 \times C_4, C_4 \times C_4$ and $C_2 \times C_2 \times C_2 \times C_2$. Since the real genus of Abelian groups was obtained in [21], we summarize the results in Table 1. In this Table and in the other ones, the last column gives the GAP number of each group, see [10].

This Table gives the real genus of all Abelian groups of order 48.

Besides there are two groups H such that $G = C_3 \times H$ is a direct product of a cyclic and a dihedral group, namely $H = D_8$ and $H = C_2 \times D_4$. The real genus of these groups G was obtained in [7,8] as follows in Table 2.

The seven other groups of order 16 will be dealt separately. Since 3 and 16 are coprime we can apply in some cases the following nice result from [17, Theorem 3]:

Proposition 2.1 *Let the group H be generated by a pair of elements one of which is an involution. Let t be the smallest integer such that X and Y generate H, with $o(X) = 2$ and*

Table 2

	Group $G = C_3 \times H$	$\rho(G)$	GAP
6	$C_3 \times D_8$	17	[48,25]
7	$C_3 \times (C_2 \times D_4) \approx C_6 \times D_4$	25	[48,45]

$o(Y) = t$; and k the smallest integer such that V and W generate H, with $o(V) = 2$ and $o([V, W]) = k$. If n and $o(H)$ are coprime then $\rho(C_n \times H)$ is equal to

(i) $1 + o(H)(nt - 2)/2t$ *if* $n < 2k/t$,
(ii) $1 + o(H)n(k - 1)/2k$ *otherwise.*

The seven remaining groups H of order 16 are the semidihedral group L_4 which is called $\langle -2, 4 | 2\rangle$ in [6]; $\langle 2, 2, 2\rangle_2$; $(4, 4 | 2, 2)$; the quasiabelian group QA_4; the dicyclic group DC_4; $\langle 2, 2 | 4; 2\rangle$; and finally $C_2 \times Q$, where Q is the quaternion group of order 8. We study them one by one. We call Z a generator of C_3.

(i) Let H be the semidihedral group L_4. This group admits a presentation with generators X, Y, and relations $X^8 = Y^2 = 1,\ YXY = X^3$. Hence it is generated by two elements one of which is an involution and we can apply Proposition 2.1. Obviously $t \geq 4$ and we are going to see that $k = 4$. The group is generated by elements Y and XY. Notice that $o(XY) = 4$ and the commutator

$$[Y, XY] = YXYY(XY)^3 = YX(XY)^3$$
$$= X^3Y(XY)^3 = X^3YXYXYXY = X^7YXY = X^{10} = X^2$$

has order 4. Note that there are no elements $V, W \in H$ such that $H = \langle V, W\rangle$ and $o(V) = o([V, W]) = 2$. Hence with the notations in Proposition 2.1, $t = k = 4$ and $n = 3$. Thus, $\rho(C_3 \times H) = 1 + o(H)n(k - 1)/2k = 19$.
We now exhibit the epimorphism that yields the real genus. Let Λ be an NEC group of signature $(0, +, [-], \{(-), (4)\})$ and define the homomorphism θ from Λ into $C_3 \times H$ by

$$\theta(e_1) = XYZ,\ \theta(c_{1,0}) = 1,\ \theta(e_2) = (XYZ)^{-1},\ \theta(c_{2,0}) = Y,\ \theta(c_{2,1}) = XYX^{-1}.$$

By the results in Chapter 2 of [2] the above assignment induces a homomorphism $\theta : \Lambda \to C_3 \times H$ whose kernel is a surface NEC group. Let us check that it is surjective. Since $o(XY) = 4$ and $o(Z) = 3$, we have $\theta(e_1^4) = Z$ and $\theta(e_1^3) = (XY)^{-1}$. Hence Z, XY and Y belong to the image of θ, and so it is an epimorphism.
Besides

$$\theta(e_2^{-1}c_{2,0}e_2c_{2,1}) = XYZY(XYZ)^{-1}XYX^{-1} = 1,$$

and finally

$$\theta(c_{2,0}c_{2,1}) = YXYX^{-1} = X^2$$

has order 4.
Observe that the reduced area of the group Λ is $3/8$ and this gives, as we obtained before using Proposition 2.1, $\rho(C_3 \times H) = 1 + o(C_3 \times H)|\Lambda| = 1 + 48 \cdot \frac{3}{8} = 19$.
In what follows we shall omit some details concerning the checking of the surjectivity of the homomorphism θ, and that its kernel is a surface group, if they are similar to this case.

(ii) Let now H be the group $\langle 2, 2, 2\rangle_2$ generated by three elements R, S, T subjected to the complete set of relations $R^2 = S^2 = T^2 = 1$ and $RST = STR = TRS$. These relations imply $o(RS) = o(ST) = 4$, see for instance in [23] the table of the group, which is there called 16/8. The elements R, S and T play the same role and they can be determined freely. This group does not have generating systems of two elements and hence the same happens to $C_3 \times H$. Besides it is not generated by involutions. Summing up these conditions, the reduced area of Λ cannot be lesser than $1/2$, see for instance [8, p. 151].

Let Λ be an NEC group with signature $(0, +, [2], \{(-), (-)\})$, whose reduced area is $|\Lambda| = 1/2$. Then take the epimorphism θ from Λ onto $C_3 \times H$ defined by

$$\theta(x_1) = R,\ \theta(e_1) = SZ,\ \theta(c_{1,0}) = 1,\ \theta(e_2) = SRZ^{-1},\ \theta(c_{2,0}) = T.$$

Observe that $\theta(e_2^{-1}c_{2,0}e_2c_{2,0}) = RSTSRT = TRSSRT = 1$. Thus the real genus of the group $C_3 \times H$ is 25.

(iii) Let now H be the group $(4, 4|\ 2, 2)$. It has a presentation with two generators R and S, and relations $R^4 = S^4 = (RS)^2 = (R^{-1}S)^2 = 1$. Hence the elements R^2 and S^2 belong to the center of H. Evidently the group H is generated by R and RS which have orders 4 and 2 respectively. Hence we may apply Proposition 2.1. Consider the element RSR. Obviously the pair RS and RSR also generates H. The commutator $[RS, RSR] = R^2S^2$ has order 2. Then $t = 4$ and $k = 2$, and so the real genus of $C_3 \times H$ is 13, as it is mentioned in [18]. We exhibit the corresponding epimorphism. Take a group Λ of signature $(0, +, [-], \{(-), (2)\})$ and define θ from Λ onto $C_3 \times H$ as follows:

$$\theta(e_1) = RSRZ,\ \theta(c_{1,0}) = 1,\ \theta(e_2) = (RSRZ)^{-1},\ \theta(c_{2,0}) = RS,\ \theta(c_{2,1}) = SR.$$

(iv) Now let H be the quasiabelian group QA_4. It can be generated by two elements X and Y satisfying $X^8 = Y^2 = 1$ and $YXY = X^5$. We can apply again Proposition 2.1. Since the order of the commutator $[X, Y]$ is 2, we have $k = 2$ and hence the real genus is 13. (See also [18]). Take a group Λ of signature $(0, +, [-], \{(-), (2)\})$ and define θ from Λ onto $C_3 \times QA_4$ as follows:

$$\theta(e_1) = XZ,\ \theta(c_{1,0}) = 1,\ \theta(e_2) = (XZ)^{-1},\ \theta(c_{2,0}) = Y,\ \theta(c_{2,1}) = YX^4.$$

(v) Now let H be the dicyclic group DC_4. This group can be generated by two elements X and Y of order 4. Let Λ be an NEC group with signature $(0, +, [4, 12], \{(-)\})$ and the epimorphism defined by

$$\theta(x_1) = X,\ \theta(x_2) = YZ,\ \theta(e_1) = (XYZ)^{-1},\ \theta(c_{1,0}) = 1.$$

The reduced area of Λ is $2/3$. This area cannot be lowered because the elements in any generating pair of DC_4 have order 4. Hence in order to generate $C_3 \times DC_4$ with two elements, both of them have order a multiple of 4, and one of them has order a multiple of 3. In these conditions Λ is the group with minimal reduced area. So we have $\rho(C_3 \times DC_4) = 33$.

(vi) Let H be the group $\langle 2, 2|\ 4; 2\rangle$. This group has a presentation by generators S and T satisfying $S^4 = T^4 = 1$, $T^{-1}ST = S^{-1}$. Let Λ be an NEC group with signature $(0, +, [4, 12], \{(-)\})$ and the epimorphism defined by

$$\theta(x_1) = S,\ \theta(x_2) = TZ,\ \theta(e_1) = (STZ)^{-1},\ \theta(c_{1,0}) = 1.$$

Again the reduced area of Λ is $2/3$ and this area cannot be lowered by a similar argument as in the previous group. We have so $\rho(C_3 \times H) = 33$.

Table 3

	Group $G = C_3 \times H$	$\rho(G)$	GAP
8	$C_3 \times L_4$	19	[48,26]
9	$C_3 \times \langle 2, 2, 2\rangle_2$	25	[48,47]
10	$C_3 \times (4, 4 \mid 2, 2)$	13	[48,21]
11	$C_3 \times QA_4$	13	[48,24]
12	$C_3 \times DC_4$	33	[48,27]
13	$C_3 \times \langle 2, 2 \mid 4; 2\rangle$	33	[48,22]
14	$C_3 \times (C_2 \times Q) \approx C_6 \times Q$	37	[48,46]

(vii) Finally let $H = C_2 \times Q$. The real genus of this group was obtained by May in [16] for a lowest reduced area $3/4$. Since H is an image of $C_3 \times H$ this number is a lower bound for our study of the group $C_3 \times H$. We show that this bound is actually attained. The generators of Q are i and j of order 4 and let X be the generator of $C_2 \times C_3$. Let Λ be an NEC group of signature $(0, +, [4], \{(-), (-)\})$ and let $\theta : \Lambda \to C_3 \times H$ be defined by

$$\theta(x_1) = i,\ \theta(e_1) = jX^2,\ \theta(c_{1,0}) = X^3,\ \theta(e_2) = (ij)^{-1}X^4,\ \theta(c_{2,0}) = 1.$$

Since the reduced area of Λ is $3/4$ we have realized that $\rho(C_3 \times H) = 37$.
In Table 3 we display the real genus of these seven groups.

3 The real genus of the direct product of C_4 and a group H of order 12

In this Section we consider the groups which are the product of C_4 and a group H of order 12. There are five groups of order 12. Two of them are Abelian and the respective $C_4 \times H$ were already seen in the previous Section (see Table 1, groups **3** and **4**). The three remaining groups H are the dihedral group D_6, the alternating group A_4 and the dicyclic group DC_3.

The real genus of $C_4 \times D_6$ is 25 (see [7]). We now deal with the two other groups. We shall call Z a generator of C_4.

(i) Let H be the alternating group A_4. Let $\sigma = (1, 2)(3, 4)$ and $\tau = (1, 2, 3)$. These permutations generate A_4. We are going to see that the real genus of $C_4 \times A_4$ is 13 as shown in [18].
Let us consider an NEC group Λ of signature $(0, +, [-], \{(2), (-)\})$ and the epimorphism θ from Λ onto $C_4 \times A_4$ defined by

$$\theta(e_1) = \tau Z, \theta(c_{1,0}) = \sigma, \theta(c_{1,1}) = (1, 4)(2, 3), \theta(e_2) = (\tau Z)^{-1}, \theta(c_{2,0}) = 1.$$

The reduced area of Λ is $1/4$ and so the real genus is 13.

(ii) Let H be the dicyclic group DC_3. It has a presentation with two generators X, Y, and relations $X^6 = 1$, $X^3 = Y^2$ and $Y^{-1}XY = X^{-1}$. Then the elements Y and XY have order 4 and generate H. Moreover, to check the surjectivity of θ it is useful to notice that XY^2 has order 3. Let Λ be an NEC group of signature $(0, +, [4, 4], \{(-)\})$ and the epimorphism θ from Λ onto $C_4 \times DC_3$ defined by

$$\theta(x_1) = XYZ,\ \theta(x_2) = Y,\ \theta(e_1) = (XY^2Z)^{-1},\ \theta(c_{1,0}) = 1.$$

Table 4

	Group $G = C_4 \times H$	$\rho(G)$	GAP
15	$C_4 \times D_6$	25	[48,35]
16	$C_4 \times A_4$	13	[48,31]
17	$C_4 \times DC_3$	25	[48,11]

Table 5

	Group $G = C_2 \times H$	$\rho(G)$	GAP
18	$C_2 \times D_{12}$	1	[48,36]
19	$C_2 \times (C_2 \times D_6) \approx D_2 \times D_6$	13	[48,51]

The reduced area of Λ is $1/2$. Notice also that $C_2 \times H$ is a homomorphic image of $C_4 \times H$, and it was proved in [15] that $1/2$ is the lowest reduced area for that group. Hence it is also the lowest bound for our group $C_4 \times H$ and so its real genus is 25.

We resume these results in Table 4.

4 The real genus of the direct product of C_2 and a group H of order 24

We devote this Section to the groups which are the product of C_2 and a group H of order 24. There exist fifteen groups of order 24. Among them there are three Abelian groups. The corresponding groups $C_2 \times H$ are the groups **2**, **3** and **5** in Table 1.

Among the twelve remaining groups there are four such that $C_2 \times H$ is either a product of two dihedral groups or a product of a cyclic and a dihedral group. These groups are the following. The group $C_2 \times (C_3 \times D_4) \approx C_6 \times D_4$, which is the group **7** in Table 2. The group $C_2 \times (C_4 \times D_3) \approx C_4 \times D_6$ which is the group **15** in Table 4. We are left with two groups, one of them has real genus 1, and the other was studied in [8]. They are shown in Table 5.

The eight remaining groups H of order 24 are the following: the symmetric group S_4; $C_2 \times A_4$; $(4, 6|\, 2, 2)$; $\langle 2, 3, 3\rangle$; the dicyclic group DC_6; $C_2 \times DC_3$; $\langle -2, 2, 3\rangle$; and $C_3 \times Q$. Since $C_2 \times (C_3 \times Q) \approx C_6 \times Q$ is the group **14** in Table 3, we need to study the seven other products what shall be done one by one. We shall call Z a generator of C_2.

(i) Let H be the symmetric group S_4. The group $C_2 \times S_4$ is well known to be an M^*-group and hence its real genus is 5. See for example [6,13].

(ii) Let now H be $C_2 \times A_4$ generated by a generator W of C_2, $\sigma = (1, 2)(3, 4)$ and $\tau = (1, 2, 3)$. The group $C_2 \times H$ has real genus 13, see [18]. In order to have the corresponding epimorphism, consider an NEC group Λ with signature $(0, +, [-], \{(-), (2)\})$ and define θ from Λ to G by

$$\theta(e_1) = \tau W,\ \theta(c_{1,0}) = 1,\ \theta(e_2) = \tau^2 W,\ \theta(c_{2,0}) = \sigma Z,\ \theta(c_{2,1}) = (1, 3)(2, 4)Z.$$

Since the product of σ and τ has order 3 the homomorphism $\theta : \Lambda \to G$ is surjective. The reduced area of Λ is $1/4$, and so $\rho(G) = 13$.

(iii) Let H be the group $(4, 6|\, 2, 2)$. This group has two generators X, Y satisfying the relations $X^4 = Y^6 = (XY)^2 = (X^{-1}Y)^2 = 1$. The group $C_2 \times H$ has real genus 13,

Table 6

	Group $G = C_2 \times H$	$\rho(G)$	GAP
20	$C_2 \times S_4$	5	[48,48]
21	$C_2 \times (C_2 \times A_4)$	13	[48,49]
22	$C_2 \times (4, 6 \vert\, 2, 2)$	13	[48,43]
23	$C_2 \times \langle 2, 3, 3\rangle$	21	[48,32]
24	$C_2 \times DC_6$	37	[48,34]
25	$C_2 \times (C_2 \times DC_3)$	37	[48,42]
26	$C_2 \times \langle -2, 2, 3\rangle$	33	[48,9]

see [18]. We now exhibit a suitable epimorphism. Let Λ be an NEC group of signature $(0, +, [-], \{(2, 2, 4, 4)\})$ and define θ from Λ onto $C_2 \times H$ by

$$\theta(c_{1,0}) = X^{-1}YZ,\ \theta(c_{1,1}) = 1,\ \theta(c_{1,2}) = YX,\ \theta(c_{1,3}) = Y^3,\ \theta(c_{1,4}) = X^{-1}YZ.$$

The elements $X^{-1}Y$, YX and Y^3 generate H. Besides $\theta((c_{1,2}c_{1,0})^3) = Z$. One may check that $\theta(c_{1,2}c_{1,3})$ and $\theta(c_{1,3}c_{1,4})$ have order 4. Since the reduced area of Λ is $1/4$ we have the required real genus.

(iv) Let H be the binary tetrahedral group $\langle 2, 3, 3\rangle$. This group has two generators A, B satisfying the relations $A^3 = 1$, $ABA = BAB$. Then the element ABA has order 4 and B has order 3, see the table of the group 24/13 in [23]. Let Λ be an NEC group of signature $(0, +, [3, 4], \{(-)\})$ and define θ from Λ onto $C_2 \times H$ by

$$\theta(x_1) = A,\ \theta(x_2) = ABAZ,\ \theta(e_1) = A^{-1}B^{-1}AZ,\ \theta(c_{1,0}) = 1.$$

Since B has order 3 and $\theta(x_1^2x_2x_1^2) = BZ$ the map θ is an epimorphism. The reduced area of Λ is $5/12$ and the real genus is 21.

(v) Let H be the dicyclic group DC_6. A general result for groups $C_2 \times DC_n$ was obtained by May in [16, Theorem 2]. In particular the real genus of $C_2 \times DC_6$ is 37.

(vi) Let now H be $C_2 \times DC_3$. The group $C_2 \times H$ does not admit a system of two generators. Besides in order to generate DC_3 we need a generator of order 4. Let Λ be an NEC group of signature $(0, +, [4], \{(-), (-)\})$. We take the presentation of DC_3 as in Sect. 3. (ii) and in this case we call W the generator of the factor C_2 of H. Define θ from Λ onto $C_2 \times H$ by

$$\theta(x_1) = XYZ, \theta(e_1) = Y, \theta(e_2) = (XY^2Z)^{-1},\ \theta(c_{1,0}) = 1, \theta(c_{2,0}) = W.$$

The group Λ has reduced area $3/4$ which is the lowest possible. Hence $\rho(C_2 \times C_2 \times DC_3) = 37$.

(vii) Let H be the group $\langle -2, 2, 3\rangle$ with two generators S and T of order 8, such that S^3T has order 3 (see [15]). Let Λ be an NEC group of signature $(0, +, [3], \{(-), (-)\})$. Define θ from Λ onto $C_2 \times H$ by

$$\theta(x_1) = S^3T,\ \theta(e_1) = T,\ \theta(e_2) = (S^3T^2)^{-1},\ \theta(c_{1,0}) = 1,\ \theta(c_{2,0}) = Z.$$

The group Λ has reduced area $2/3$ which is the lowest possible. Hence $\rho(C_2 \times H) = 33$. We resume these results in Table 6.

Table 7

	Group $G = D_3 \times H$	$\rho(G)$	GAP
27	$D_3 \times C_8$	17	[48,4]
28	$D_3 \times D_4$	7	[48,38]
29	$D_3 \times Q$	37	[48,40]

Table 8

	Group G	$\rho(G)$	GAP
30	D_{24}	0	[48,7]
31	$GL(2,3)$	9	[48,29]
32	P_{48}	9	[48,33]
33	$C_{24} \times_\phi C_2$	13	[48,6]
34	$C_3 \times_\theta (4,4\| 2,2)$	13	[48,14]
35	$C_3 \times_\psi \langle 2,2,2\rangle_2$	13	[48,37]
36	$C_3 \times_\phi D_8 \approx (6,8\|2,2)$	14	[48,15]
37	DC_{12}	25	[48,8]

5 The real genus of the direct product of D_3 and a group H of order 8

There are five groups of order 8. Since $D_3 \times (C_2 \times C_4) \approx C_4 \times D_6$ (group **15** in Table 4) and $D_3 \times (C_2 \times C_2 \times C_2) \approx C_2 \times C_2 \times D_6$ (group **19** in Table 5), we only need to consider three groups H, namely C_8, D_4 and Q.

It is already known that $\rho(D_3 \times C_8) = 17$ (see [7]) and $\rho(D_3 \times D_4) = 7$ (see [8]). We deal with the group $D_3 \times Q$. Let us call i and j the generators of order 4 of Q and A and B the generators of D_3 satisfying $A^2 = B^2 = (AB)^3 = 1$.

Let Λ be an NEC group of signature $(0, +, [4], \{(-), (-)\})$ and define θ by

$$\theta(x_1) = iA,\ \theta(e_1) = j,\ \theta(e_2) = (ij)^{-1}A,\ \theta(c_{1,0}) = B,\ \theta(c_{2,0}) = 1.$$

The group Λ has reduced area 3/4, and this is the lowest possible. Indeed 3/4 is the smallest value of the reduced areas of those NEC groups Λ' which project epimorphically, with surface group kernel, onto the product $C_2 \times Q$, and this last group is a homomorphic image of $G = D_3 \times Q$. Hence $\rho(D_3 \times Q) = 37$.

We resume these results in Table 7.

6 The real genus of the remaining groups of order 48

We are left with twenty three groups of order 48 which are not a direct product. The real genus of some of them is already known. We give in Table 8 the list of groups G of order 48 with $\rho(G) \leq 16$ (see [18]). Besides the real genus of all dicyclic groups was obtained by May in [14] and so we know the real genus of DC_{12}.

In this Table we denote by $\times_\phi$, $\times_\theta$, $\times_\psi$, semidirect products in order to preserve the notation in [18].

Presentations for the groups **31** and **32** appear in [19]. Both groups admit a pair of generators of order 2 and 3 and so they are images of an NEC group of signature $(0, +, [2, 3], \{(-)\})$, with reduced area 1/6. Thus, $\rho(G) = 1 + 48 \cdot \frac{1}{6} = 9$ in both cases.

The group **33** has a presentation with generators X and T and relations $X^{24} = T^2 = 1$, $TXT = X^{11}$. Hence XT has order 4 and this group can be generated by T and XT. So it is a homomorphic image of an NEC group of signature $(0, +, [2, 4], \{(-)\})$, whose reduced area is $1/4$.

The group **34** has a presentation with generators X, R and S and relations $X^3 = R^4 = S^4 = (RS)^2 = (R^{-1}S)^2 = 1$, $XR = RX$, $S^{-1}XS = X^{-1}$. In Sect. 2. (iii) we observed that RS and RSR generate the group $(4, 4|\, 2, 2)$. Notice that $Y = XRS$ and $Z = RSR$ have orders 2 and 4 respectively. Moreover, since $(RS)^2 = 1$ and $XR = RX$ we have

$$(YZ)^2 = (XRS)(RSR)(XRS)(RSR) = X(RS)^2RX(RS)^2R = (XR)^2 = X^2R^2,$$

and so $(YZ)^4 = X^4R^4 = X^4 = X$. Thus the group G is generated by Y and Z, and this implies that it is a homomorphic image of an NEC group with signature $(0, +, [2, 4], \{(-)\})$, with reduced area $1/4$.

The group **35** has a presentation with generators X, R and S and relations $X^3 = R^2 = S^2 = T^2 = 1$, $RST = STR = TRS$, $RXR = X^{-1}$, $XS = SX$, $TXT = X^{-1}$. We proved in Sect. 2. (ii) that these relations imply that $o(RS) = o(ST) = 4$. Consider an NEC group Λ of signature $(0, +, [-], \{(2, 2, 4, 4)\})$, with reduced area $1/4$ and define θ from Λ onto G as follows

$$\theta(c_{1,0}) = RX, \theta(c_{1,1}) = 1, \theta(c_{1,2}) = STS, \theta(c_{1,3}) = S, \theta(c_{1,4}) = RX.$$

Observe that $(STS)S = ST$ has order 4, and $S(RX)$ has also order 4 because $(SRX)^2 = (SR)^2$ which has order 2. Finally we see that θ is an epimorphism. First

$$\begin{aligned}\theta(c_{1,3}c_{1,2}c_{1,3}c_{1,0}c_{1,2}c_{1,0}) &= S(STS)S(RX)(STS)(RX)\\ &= TRXSTSRX = TRSTSRXX\\ &= TTRSSRXX = XX = X^{-1},\end{aligned}$$

and so X belongs to the image of θ. Thus, also $R = \theta(c_{1,4})X^{-1}$ belongs to the image of θ. This together with the equalities $S = \theta(c_{1,3})$ and $T = S^{-1}\theta(c_{1,2})S^{-1}$ implies the surjectivity of θ.

Now we deal with the group **36**, whose real genus is 14, which has a presentation with generators X, A, B and relations $X^3 = A^2 = B^8 = (AB)^2 = 1$, $AXA = X^{-1}$, $B^{-1}XB = X^{-1}$, see [18]. Consider an NEC group Λ of signature $(0, +, [-], \{(2, 2, 3, 8)\})$, with reduced area $13/48$, and define θ from Λ onto G as follows

$$\theta(c_{1,0}) = AB,\ \theta(c_{1,1}) = 1,\ \theta(c_{1,2}) = XA,\ \theta(c_{1,3}) = A,\ \theta(c_{1,4}) = AB.$$

It is immediate that XA has order 2 and that θ is a well defined epimorphism. Observe that, (see [10]) there is another presentation of the group [48, 15] with generators A and B and relations $A^6 = B^8 = (A^{-1}B^{-1})^2 = (AB^{-1})^2 = 1$. This group is called $(6, 8|\, 2, 2)$ in [5].

Finally we shall study the fifteen remaining groups case by case according to their GAP number. We use presentations for the groups obtained from the Small GroupsLibrary. Recall that all of them have real genus greater than 16.

(i) Let G be the group [48, 1] whose structure is $C_3 \rtimes C_{16}$. This group admits a presentation with generators A and B and relations $A^{16} = B^3 = 1$, $BA = AB^2$. Since $BA^2 =$

A^2B, A^2 is central in G. Now we calculate the orders of the elements BA^k and B^2A^k. First $o(BA^{2t}) = o(B)o(A^{2t}) = 3 \cdot 16/(\gcd(16, 2t)) = 24/\gcd(8, t)$ because B and A^{2t} commute and have coprime orders. Since $C = A^2$ is a central element and $BAB = A$,

$$(BA^{2t+1})^2 = BC^tABC^tA = C^{2t}(BAB)A = C^{2t}A^2 = C^{2t+1} = A^{4t+2},$$

and so

$$o((BA^{2t+1})^2) = o(A^{4t+2}) = o(A)/\gcd(o(A), 4t+2) = 16/\gcd(16, 4t+2) = 16/2 = 8.$$

Hence $o(BA^{2t+1}) = 16$.
Now we calculate the orders of the elements B^2A^{2t} and B^2A^{2t+1}. Since B^2 and A^{2t} commute and their orders are coprime, we have

$$\begin{aligned} o(B^2A^{2t}) &= 1\,\mathrm{cm}(o(B^2), o(A^{2t})) = 1\,\mathrm{cm}(3, o(A)/\gcd(o(A), 2t)) \\ &= 1\,\mathrm{cm}(3, 16/\gcd(16, 2t)) = 1\,\mathrm{cm}(3, 8/\gcd(8, t)) = 24/\gcd(8, t). \end{aligned}$$

On the other hand, $B^2AB^2 = B(BAB)B = BAB = A$, and so, with the notations as above,

$$(B^2A^{2t+1})^2 = B^2C^tAB^2C^tA = C^{2t}(B^2AB^2)A = C^{2t}A^2 = C^{2t+1} = A^{4t+2}.$$

Thus, $o((BA^{2t+1})^2) = o(A^{4t+2}) = 8$.
Since the subgroup generated by B and BA^{2t} does not contain A, the signature of the NEC group Λ with lesser reduced area admitting an epimorphism $\theta : \Lambda \to G$ whose kernel is a surface group is $(0, +, [3, 16], \{(-)\})$, whose reduced area is $29/48$. Therefore $\rho(G) = 1 + 48 \cdot \frac{29}{48} = 30$.

(ii) Let now G be the group [48, 3] which has structure $(C_4 \times C_4) \rtimes C_3$. This group was called $(3, 3|\ 3, 4)$ in [5]. It has a presentation with generators R and S and relations $R^3 = S^3 = (RS)^3 = (R^{-1}S)^4 = 1$. Hence G is an image of an NEC group of signature $(0, +, [3, 3], \{(-)\})$ with reduced area $1/3$. So the real genus of G is 17.

(iii) Take now G the group [48, 5]. It has structure $C_{24} \rtimes C_2$ and presentation with two generators of orders 2 and 8. Hence G is an image of an NEC group with signature $(0, +, [2, 8], \{(-)\})$ and reduced area $3/8$. So the real genus of G is 19.

(iv) Let now G be the group [48, 10]. It has structure $(C_3 \rtimes C_8) \rtimes C_2$. Recall that $C_3 \rtimes C_8$ is the group $\langle -2, 2, 3\rangle$. The group G has a presentation with generators A and B and relations $A^8 = B^{-1}A^2B^{-1} = (B^{-1}ABA^{-1}B^{-1}A^{-1})^2 = 1$. The element AB has then order 6. Since AB and A generate G, this is an image of an NEC group Λ with signature $(0, +, [6, 8], \{(-)\})$. The reduced area of Λ is $17/24$ and so $\rho(G) = 35$.

(v) Consider now the group G with number [48, 12]. It has structure $(C_3 \rtimes C_4) \rtimes C_4$, that is to say, it is a semidirect product of DC_3 and C_4. This group is generated by two elements A and B of order 4. So G is an image of an NEC group Λ with signature $(0, +, [4, 4], \{(-)\})$. The reduced area of Λ is $1/2$ and so $\rho(G) = 25$.

(vi) The next group G to be considered is [48, 13]. It has structure $C_{12} \rtimes C_4$. A presentation for G is given by generators A and B and relations $A^4 = B^{12} = A^{-1}B^{-1}AB^{-1} = 1$. Then $A = B^{-1}AB^{-1}$ and so

$$(AB)^2 = A(BAB) = (B^{-1}AB^{-1})(BAB) = B^{-1}A^2B.$$

Thus $(AB)^2$ and A^2 are conjugate and so $o((AB)^2) = o(A^2) = 2$, that is, $o(AB) = 4$. Evidently A and AB generate the group G, and so this is an image of an NEC group with signature $(0, +, [4, 4], \{(-)\})$ and reduced area $1/2$. So again $\rho(G) = 25$.

(vii) We now deal with the group G with number [48, 16]. It is a semidirect product $(C_3 \rtimes C_8) \rtimes C_2$, different to the group considered in **iv)**. Recall that the group $(C_3 \rtimes C_8)$ is called $\langle -2, 2, 3\rangle$. The group G has a presentation [10] with generators A and B and relations

$$\begin{aligned} A^4 &= A^{-1}B^{-1}A^2BA^{-1} = B^{-2}A^2B^{-2} \\ &= A^{-1}B^{-2}AB^{-2} = (A^{-1}B^{-1})^3(AB)^2A^{-1}B^{-1} = 1. \end{aligned}$$

First of all let us observe that the relation $A^{-1}B^{-1}A^2BA^{-1} = 1$ is redundant. In fact, from $B^{-2}A^2B^{-2} = 1$ we have $A^2 = B^4$, and so $A^2B = B^5 = BA^2$. Hence A^2 is central and so the considered relation follows at once. In the same way, also the relation $A^{-1}B^{-2}AB^{-2} = 1$ is redundant. So G is determined by just three relations, namely

$$A^4 = B^{-2}A^2B^{-2} = (A^{-1}B^{-1})^3(AB)^2A^{-1}B^{-1} = 1.$$

We are going to prove that AB has order 6. First, since the last relation in the presentation implies $(AB)^2 = (BA)^4$,

$$\begin{aligned} (AB)^6 &= ABA(BA)^4B = ABA(AB)^2B = ABA^2(BAB)B = ABB^4BAB^2 \\ &= AB^4(B^2AB^2) = AA^2A = A^4 = 1. \end{aligned}$$

Suppose now that $(AB)^3 = 1$. Then also $(BA)^3 = 1$ and this implies

$$B^6 = BB^4B = BA^2B = (BA)(AB) = (BA)^4(AB) = (AB)^2(AB) = (AB)^3 = 1,$$

but this is false, because $o(B) = 8$.

Finally suppose that $o(AB) = 2$, which implies $(BA)^2 = 1$ and it follows that $(A^{-1}B^{-1})^4 = 1$. Then the relation $(A^{-1}B^{-1})^3(AB)^2A^{-1}B^{-1} = 1$ is a consequence of $(AB)^2 = 1$ and so it would not appear in the presentation of the group G. We have proved that $o(AB) = 6$.

Since A and AB generate G, this group is an image of an NEC group with signature $(0, +, [4, 6], \{(-)\})$ and reduced area $7/12$ and so have $\rho(G) = 29$.

(viii) Let G be the group with number [48, 17]. It is a semidirect product of $C_3 \times Q$ and C_2. The group G is generated by two elements A and B of orders 2 and 8. Hence it is an image of an NEC group Λ with signature $(0, +, [2, 8], \{(-)\})$. The reduced area of Λ is $3/8$ and so $\rho(G) = 19$.

(ix) Take now the group G with number [48, 18] with structure $C_3 \rtimes DC_4$. This group has rank 2 and a partial presentation with generators A and B satisfying $B^4 = A^6B^2 = 1$. The last relation implies $A^6 = B^2$ and so A has order 12. Since the strong symmetric genus of G is 14, see [9], one deduces that AB has order 8. Since AB and B generate the group G, this is an image of an NEC group Λ with signature $(0, +, [4, 8], \{(-)\})$. The reduced area of Λ is $5/8$ and so $\rho(G) = 31$.

(x) Let G be the group with number [48, 19] with structure $(C_2 \times DC_3) \rtimes C_2$. This group has a presentation with generators A and B and relations

$$A^4 = B^6 = A^{-1}B^{-1}A^2BA^{-1} = A^{-1}B^{-2}AB^2 = (A^{-1}B^{-1})^2(AB)^2 = 1.$$

The strong symmetric genus of G is 9, see [9]. Hence this group is an image of a Fuchsian group with signature $(0, +, [4, 4, 6], \{-\})$ and so it is generated by two elements of order 4. Thus G is an image of an NEC group with signature $(0, +, [4, 4], \{(-)\})$ and reduced area $1/2$, so we have $\rho(G) = 25$.

(xi) Consider now the group G with number [48, 28]. The structure of this group is an extension $C_2.S_4$, and it can be generated by two elements of orders 3 and 4, see [20, p. 4094]. Hence it is an image of an NEC group with signature $(0, +, [3, 4], \{(-)\})$ and reduced area $5/12$. So $\rho(G) = 21$.

(xii) Let G be the group with number [48, 30]. We may give two interesting kinds of structures for this group. Namely, $A_4 \rtimes C_4$ and $(C_2 \times C_2) \rtimes DC_3$. Also this group has two generators of orders 3 and 4, see [20, p. 4091], and we have as above $\rho(G) = 21$.

We have finished with groups of rank 2 and now we deal with those of rank 3.

(xiii) Now G is the group with number [48, 39] with structure $(C_2 \times (C_3 \rtimes C_4)) \rtimes C_2$, that is to say $(C_2 \times DC_3) \rtimes C_2$. This group has a presentation with generators A, B and C, and relations

$$\begin{aligned} A^2 &= B^4 = C^{-1}B^2C^{-1} \\ &= B^{-1}ABA = B^{-1}C^{-1}ABC^{-1}A \\ &= (B^{-1}C)^2AC^{-1}B^{-1}C^{-1}BCAC^{-1} = 1. \end{aligned}$$

We are going to prove that the element $ACACBC$ has order 2. First of all the first four relations in the above presentation can be rewritten $A^2 = 1$, $B^4 = 1$, $B^2 = C^2$ and $AB = BA$. This implies that $C^4 = 1$ and that $B^2 = C^2$ belongs to the center of G. From the fifth relation we have $ABC^3A = ABC^{-1}A = CB$, hence $C^2ABCA = CB$, and $CABCA = B$. From this equality we have $BCA = A^{-1}C^{-1}B = AC^3B = AC^2CB = C^2ACB$. On the other hand the last relation in the presentation can be rewritten as

$$C = B^{-1}CB^{-1}CA(C^3B^3C^3)BCA.$$

Using that B^2 and C^2 are central in G we have

$$C^3B^3C^3 = CC^2BB^2CC^2 = CBCC^4B^2 = (CBC)B^2 = (CBC)C^2.$$

So, multiplying the previous equality by B on the left and using $B^2 = C^2$ we have

$$\begin{aligned} BC &= CB^3CA(CBC)C^2BCA = CBC^2CACBCC^2BCA \\ &= (CB)(CA)(CB)(CB)(CA). \end{aligned}$$

Multiplying both sides by CB on the left we get

$$(CB)^2(CA)(CB)^2CA = CBBC = CB^2C = CC^2C = C^4 = 1.$$

Using in this equality that $BCA = C^2ACB$, we have

$$\begin{aligned} 1 &= CBC(BCA)CBC(BCA) = CBC(C^2ACB)CBC(C^2ACB) \\ &= CBC(ACB)CBC(ACB) = (CB)(CACBCBCA)(CB). \end{aligned}$$

Hence using again that $B^2 = C^2$ is central, $CACBCBCA = (CB)^{-1}(CB)^{-1} = B^3C^3B^3C^3 = BCBC$. Now multiplying both members by CA on the right we have $CACBCBCACA = BCBCCA = C^2BCBA$, and so

$$\begin{aligned} CBCACA &= (CACB)^{-1}C^2BCBA = B^{-1}C^{-1}A^{-1}C^{-1}C^2BCBA \\ &= B^3C^3ACBCBA = BCACBCBA. \end{aligned}$$

We now multiply both members by CA on the left, and keeping in mind that $CABCA = B$, we have $CACBCACA = CABCACBCBA = BCBCBA$. Multiplying again by CA on the left we get $CACACBCACA = CABCBCBA = CABCAABCBA = BABCBA$. Now multiplying by CB on the left we obtain $CBCACACBCACA = CBBABCBA = CB^2ABCBA = CABCBAB^2 = CABCAABAB^2 = BABAB^2$. Using now that $AB = BA$ we deduce that

$$(CBCACA)^2 = BABAB^2 = BBAAB^2 = B^2A^2B^2 = 1.$$

Thus the element $CBCACA$ has order 2 and hence $CACACB$ has also order 2. Since $ACACBC = C^{-1}(CACACB)C$, the order of $ACACBC$ is 2 as we wanted to prove. Now we prove that $C^{-1}(ACACBC)C(ACACBC) = 1$. Since $C^{-1} = C^3$ and C^2 is a central element of G, we have

$$\begin{aligned}C^{-1}ACACBCCACACBC &= C^3ACACBCCACACBC\\ &= CC^2ACACBCCACACBC\\ &= CACACB(C^2CC)ACACBC = CACACBACACBC.\end{aligned}$$

Using now that $AB = BA$ and that $CABCA = B$, we get

$$\begin{aligned}C^{-1}ACACBCCACACBC &= CACACBACACBC = (CACAC)AB(CACBC)\\ &= CACA(CABCA)CBC = CACABCBC\\ &= CACABCAABC = CA(CABCA)ABC\\ &= CABABC = CA(AB)BC = CA^2B^2C\\ &= CB^2C = CC^2C = C^4 = 1.\end{aligned}$$

It is clear that $A,\ C$ and $ACACBC$ generate the group G. Take an NEC group Λ with signature $(0, +, [2], \{(-), (-)\})$ and define θ from Λ onto G by

$$\theta(x_1) = A,\ \theta(e_1) = C,\ \theta(c_{1,0}) = ACACBC,\ \theta(e_2) = C^{-1}A,\ \theta(c_{2,0}) = 1.$$

The kernel of the homomorphism θ is a bordered surface group because

$$\theta(e_1^{-1}c_{1,0}e_1c_{1,0}) = C^{-1}(ACACBC)C(ACACBC) = 1.$$

The reduced area of Λ is $1/2$ and so $\rho(G) = 25$.

(xiv) Take the group G with number [48, 41] and structure $(C_4 \times D_3) \rtimes C_2$. This group has a presentation with generators $A,\ B$ and C and relations,

$$\begin{aligned}&A^4 = B^2 = C^2 = A^{-1}BA^2BA^{-1} = A^{-1}CA^2CA^{-1}\\ &= (A^{-1}CB)^2 = (A^{-1}B)^2(AC)^2 = C(BA^{-1})^2BCB = 1.\end{aligned}$$

Since $C,\ B$ and $A^{-1}CB$ generate the group G, this is generated by three involutions. Let Λ be an NEC group with signature $(0, +, [-], \{(2, 2, 4, 12)\})$. Define θ from Λ onto G as follows:

$$\theta(c_{1,0}) = B,\ \theta(c_{1,1}) = 1,\ \theta(c_{1,2}) = C,\ \theta(c_{1,3}) = A^{-1}CB,\ \theta(c_{1,4}) = B.$$

We are going to prove that the element $C(A^{-1}CB)$ has order 4, and that $(A^{-1}CB)B = A^{-1}C$ has order 12, and so θ is a well-defined epimorphism.
First we deal with the element $C(A^{-1}CB)$. We can rewrite the first five relations of the presentation as $A^4 = B^2 = C^2 = 1,\ A^2B = BA^2,\ A^2C = CA^2$. Hence A^2 is a

central element of G. Since $A^{-1} = A^3$, and A^2 is central, we have

$$1 = (A^{-1}B)^2(AC)^2 = A^3BA^3BACAC = ABABACAC = (AB)^2(AC)^2.$$

On the other hand we rewrite the last relation of the presentation as

$$1 = CBA^{-1}BA^{-1}BCB = CBA^3BA^3BCB = CBABABCB = (CBABAB)CB.$$

Hence $(CB)(CBABAB) = 1$, and so $(CB)^2(AB)^2 = 1$.
Also, $1 = (A^{-1}CB)^2 = A^3CBA^3CB = (ACB)(ACB)$. Hence,

$$ACB = (ACB)^{-1} = B^{-1}C^{-1}A^{-1} = BCA^3.$$

We are going to prove that $X = CA^{-1}CB = CA^3CB = A^2CACB$ has order 4. In fact we shall prove that $o(X^2) = 2$. Since $X^2 = (A^2CACB)(A^2CACB) = (CACBCA)(CB)$, the order of X^2 is $o((CACBCA)(CB)) = o(CBCACBCA)$. Using that $(AB)^2(AC)^2 = 1$ and that $ACB = BCA^3$, we have

$$\begin{aligned} CBCACBCA &= CBC(ACB)CA = CBCBCA^3CA = CBCBAACACA \\ &= (CB)^2A(AC)^2A = (CB)^2A(AB)^{-2}A \\ &= (CB)^2AB^{-1}A^{-1}B^{-1}A^{-1}A = (CB)^2ABA^3B \\ &= (CB)^2(AB)^2A^2 = A^2. \end{aligned}$$

Thus $o(X^2) = o(A^2) = 2$, and so $o(X) = o(CA^{-1}CB) = 4$.
Let us see now that $o(A^{-1}C) = 12$. Since

$$o(A^{-1}C) = o((A^{-1}C)^{-1}) = o(C^{-1}A) = o(CA) = o(AC),$$

we will prove that $o(AC) = 12$. First,

$$(AC)^2 = (AB)^{-2} = B^{-1}A^{-1}B^{-1}A^{-1} = BA^3BA^3 = BABA = (BA)^2.$$

Now,

$$\begin{aligned} (AC)^3 &= (AC)(AC)^2 = (AC)(BA)^2 = ACBABA \\ &= (ACB)(ABA) = (BCA^3)(ABA) = BCBA. \end{aligned}$$

Let us call $Y = BCBA = (AC)^3$. We are going to prove that $o(Y^2) = 2$ and so $o(Y) = 4$.

$$\begin{aligned} o(Y^2) &= o((BCBABCB)A) = o(ABCBABCB) \\ &= o((AB)(CB)^2(CB)^{-1}ABCB) = o((AB)(AB)^{-2}(CB)^{-1}ABCB)) \\ &= o((AB)^{-1}BCABCB) = o(BA^{-1}BCABCB) \\ &= o(B(A^{-1}BCABC)B^{-1}) = o(A^{-1}BCABC) = o(BCABCA^{-1}) \\ &= o(BCABCA^3) = o(BCA^3BCA) = o(ACBBCA) = o(A^2) = 2. \end{aligned}$$

So that $4 = o((AC)^3) = o(AC)/\gcd(3, o(AC))$. Thus $o(AC)$ is 4 or 12, and we are going to see that $o(AC) = 4$ is not possible. Since $(AC)^3 = BCBA$, if $(AC)^4 = 1$ it follows

$$1 = (AC)^4 = (AC)^3(AC) = BCBAAC = BCBA^2C = A^2(BC)^2,$$

and so $(CB)^2 = (BC)^{-2} = A^2$. So it follows that $(AB)^2 = (AC)^2 = A^2$. From that we have $BAB = A$ and $CAC = A$, and so $BA = AB$ and $CA = AC$. Thus A is a central element of G and hence $K = \langle A \rangle$ is a normal subgroup of G. If we call

Table 9

	Group G	$\rho(G)$	GAP
38	$C_3 \rtimes C_{16}$	30	[48,1]
39	$(C_4 \times C_4) \rtimes C_3 \approx (3, 3\| 3, 4)$	17	[48,3]
40	$C_{24} \rtimes C_2$	19	[48,5]
41	$(C_3 \rtimes C_8) \rtimes C_2$	35	[48,10]
42	$(C_3 \rtimes C_4) \times C_4$	25	[48,12]
43	$C_{12} \rtimes C_4$	25	[48,13]
44	$(C_3 \rtimes C_8) \rtimes C_2$	29	[48,16]
45	$(C_3 \times Q) \rtimes C_2$	19	[48,17]
46	$C_3 \rtimes DC_4$	31	[48,18]
47	$(C_2 \times DC_3) \rtimes C_2$	25	[48,19]
48	$C_2.S_4$	21	[48,28]
49	$A_4 \rtimes C_4$	21	[48,30]
50	$(C_2 \times (C_3 \rtimes C_4)) \rtimes C_2$	25	[48,39]
51	$(C_4 \times D_3) \rtimes C_2$	17	[48,41]
52	$(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3$	21	[48,50]

$[B] = KB$, $[C] = KC$, we have $G/K = \langle [B], [C] \rangle$. Then $ACB = BCA^3$ implies $[C][B] = [B][C]$, and so G/K is an abelian group generated by two elements of order 2. Therefore $o(G/K) \leq 4$, and since $o(A) = 4$, we would have $o(G) \leq 16$, and this is false. Thus $o(AC) = o(A^{-1}C) = 12$.

We have proved that θ is an epimorphism whose kernel is a bordered surface group. The reduced area of Λ is $1/3$ and so $\rho(G) = 17$.

(xv) Finally we deal with the group G with number [48, 50]. It has structure $(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3$, and a presentation with generators A, B and C, and relations

$$A^2 = B^3 = C^3 = (CB)^2 = (AB^{-1})^3$$
$$= C^{-1}B^{-1}ABCA = CBAB^{-1}C^{-1}A = 1.$$

Then A, C and CB generate G. Consider an NEC group Λ with signature $(0, +, [3], \{(2, 2, 2)\})$ and reduced area $5/12$. Define θ from Λ onto G by

$$\theta(x_1) = C,\ \theta(e_1) = C^2,\ \theta(c_{1,0}) = A,\ \theta(c_{1,1}) = 1,\ \theta(c_{1,2}) = CB,\ \theta(c_{1,3}) = CAC^2.$$

We are going to prove that the element $CBCAC^2$ has order 2 and so θ is an epimorphism. In fact, $o(CBCAC^2) = o(CBCAC^{-1}) = o(BCA)$. On the other hand, from $(CB)^2 = 1$, we have $BC = C^{-1}B^{-1}$, and so the sixth relation in the presentation can be rewritten as $1 = (BC)ABCA = (BCA)^2$. Hence $o(CBCAC^2) = o(BCA) = 2$. Therefore $\rho(G) = 21$.

We resume the results of the last fifteen groups in the Table 9.

Theorem 6.1 *The real genus of all groups of order* 48 *is given in Tables 1–9 above.*

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