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Breve Introducción a las Funciones Elíptica

Elliptic Functions: a brief introduction

Verónica Errasti Díez

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Elliptic Functions: a brief introduction and advent in the zero-field eight-vertex model

Resumen:

Esta tesis está dedicada a la elucidación de la aparición de funciones elípticas de Jacobi en la parametrización del límite de campo cero del modelo de ocho vértices en mecánica estadística. Dicha parametrización constituyó un paso clave en la resolución del modelo por parte de Baxter, en 1972. Sin embargo, esta solo se ha presentado implícita o parcialmente en la literatura, que nos conste.

Nuestra exposición se articula en dos partes. En primer lugar, repasamos las funciones elípticas de Jacobi y derivamos en detalle sus identidades relevantes. Acto seguido, revisamos el modelo de ocho vértices en su límite de campo cero y explicamos exhaustivamente la idoneidad de las funciones elípticas en la parametrización propuesta por Baxter.

Abstract:

This thesis is devoted to the elucidation of the advent of Jacobi elliptic functions in the parametrization of the zero-field limit of the eight-vertex statistical mechanical model. Such a parametrization was a crucial step in the resolution of the model by Baxter, back in 1972. However, it has only been presented implicitly or partially in the literature, to the best of our knowledge.

Our exposition is articulated into two parts. First, we review Jacobi elliptic functions and derive in detail their relevant identities. Then, we recount the zero-field eight-vertex model and comprehensively account for the suitability of elliptic functions in the parametrization proposed by Baxter.

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Part I

On elliptic functions

The origins of elliptic functions can be traced to the calculation of arc lengths of ellipses circa 1750. At the beginning of the XIX century, these functions were analytically continued into the complex plane. Therein, they were identified as meromorphic doubly periodic functions, thus acquiring their modern understanding and scope. Elliptic functions have since proven to be intrinsically interesting objects. Moreover, they have become a common tool in several branches of mathematics and they have been recognized to admit multifaceted applications.

In this thesis, we shall be concerned with the presentation of elliptic functions (part I) and their application in uniformization theory of algebraic curves. This theory of algebraic geometry seeks to parametrize curves defined by polynomials. In particular, it is instrumental to the resolution of intricate statistical mechanics models, including the specific model we will survey in part II.

The first part I of the thesis is arranged as follows. In section 1, we provide an intuition about Jacobi elliptic functions, in relation to trigonometric functions. In section 2, we formalize the definition of these elliptic functions in terms of theta functions. Sections 3 and 4 are devoted to deriving certain (self-)relations between theta functions, directly from their definition. Such relations are instrumental in section 5, where we prove specific identities of Jacobi elliptic functions. The proven identities shall be employed for the convenient parametrization of the model of interest in the second part II of this thesis.

1 Trigonometric introduction to Jacobi elliptic functions

As an intuitive introduction to elliptic functions, we shall begin by focusing on Jacobi elliptic functions and presenting them as generalizations of the sine and cosine trigonometric functions.

Let $x, y \in \mathbb{R}$. Consider the unit circle

$$r^2 = x^2 + y^2 = 1. \tag{1.1}$$

Let ϕ denote the angle from the positive x axis in the counterclockwise direction. Then, the unit circle admits the following parametrization in terms of the basic trigonometric functions:

$$\cos\phi = x, \qquad \sin\phi = y. \tag{1.2}$$

These are periodic functions in ϕ, with period 2π.

Next, consider the ellipse

$$\frac{x^2}{a^2} + y^2 = 1, \qquad a > 1. \tag{1.3}$$

It admits the following parametrization in terms of two of the Jacobi elliptic functions:

$$\mathrm{cn}u = \frac{x}{a}, \qquad \mathrm{sn}u = y, \tag{1.4}$$

where u relates to the angle ϕ through the incomplete elliptic integral of the first kind

$$u = \int_0^{\phi} r(\phi', e)d\phi', \tag{1.5}$$

here implicitly portrayed, and where e denotes the eccentricity of the ellipse

$$e = \sqrt{1 - \frac{1}{a^2}}. \tag{1.6}$$

A third Jacobi elliptic function exists,

$$\mathrm{dn}u = \frac{r}{a}, \tag{1.7}$$

which accounts for the non-constant radius of the ellipse and, as such, has no analogue in the circular counterpart. Indeed, $a = 1$ for the circle and so $\mathrm{dn}u = 1$ in this case.

In short, a first understanding of Jacobi elliptic functions is provided by their parametrization of an ellipse, in analogy to the role of the cosine and sine trigonometric functions as regards the circle. For greater clarity, the resemblance is portrayed in figure 1.

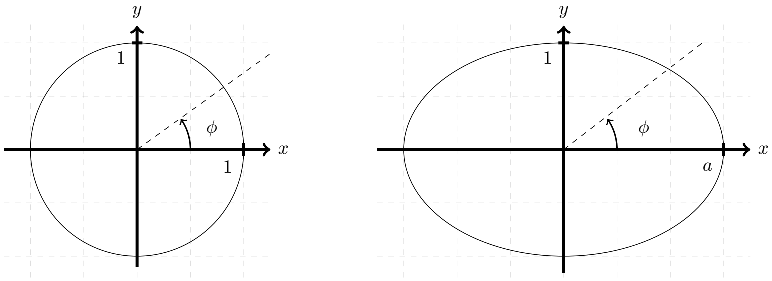

Figure 1: The unit circle and an oblong ellipse.

The analytic continuation of the variables u and e into the complex plane, pursued independently by Abel [1] in 1827 and by Jacobi [2] in 1829, catalyzed the theory of elliptic functions. In this moment, it was unveiled that Jacobi elliptic functions are doubly periodic functions in u.

2 Definition and selected relations between Jacobi elliptic functions

Elliptic functions are doubly periodic meromorphic functions. As a reminder, meromorphic functions are holomorphic functions on the whole of their domain, except for at the poles of the function. In turn, a function is holomorphic if it is complex differentiable everywhere on its domain. Equivalently, it is differentiable in the reals and satisfies the Cauchy-Riemann equations:

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (u, v) \end{aligned} \qquad \text{such that} \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.1}$$

Hence, the distinctive feature of elliptic functions is their double periodicity.

Jacobi elliptic functions are three basic elliptic functions which, together with the constant function, form a basis for the vector space of all elliptic functions with fixed periods and a certain pole structure. Owing to their plentiful and multifaceted properties, they can be defined in several equivalent manners. Jacobi himself introduced them in [2] in relation to the inverse of the *incomplete elliptic integral of the first kind*. Following Lawden [3] (p.24-26), we shall consider their definition in terms of ratios of the basic theta functions:

$$\mathrm{sn}u = \frac{\theta_1(z)}{\theta_4(z)}\frac{\theta_3(0)}{\theta_2(0)}, \qquad \mathrm{cn}u = \frac{\theta_2(z)}{\theta_4(z)}\frac{\theta_4(0)}{\theta_2(0)}, \qquad \mathrm{dn}u = \frac{\theta_3(z)}{\theta_4(z)}\frac{\theta_4(0)}{\theta_3(0)}, \tag{2.2}$$

where $z = u\theta_3^{-2}(0)$ is a complex variable. We shall soon turn to the definition of the theta functions above. It is customary and convenient to further introduce the *modulus*

$$k = \frac{\theta_2^2(0)}{\theta_3^2(0)}, \tag{2.3}$$

which is the analytically continued eccentricity parameter e in (1.6) earlier on.

Two of the most well-known identities between Jacobi elliptic functions are

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1, \tag{2.4}$$

which is the elliptic analogue to the trigonometric identity $\sin^2\phi + \cos^2\phi = 1$, and

$$\mathrm{dn}^2 u + k^2 \mathrm{sn}^2 u = 1. \tag{2.5}$$

In the following two sections 3 and 4, we will deploy the artillery that will allow us to readily prove (2.4) and (2.5). We shall provide purely algebraic proofs for both of these identities, after Lawden [3], who in turn emulates Jacobi [2]. Whittaker and Watson [4] (p.486) point out that a “more powerful” approach consists in making use of Cauchy’s theorem, a method pioneered by Liouville in his lectures [5] and abundantly used thereafter, e.g. by Whittaker and Watson themselves [4] (ch.21) and by Baxter [6] (ch. 15). Our choice allows for self-contained derivations that delve into the fundamental double periodicity property, without appeal to extrinsic theorems. Beyond their intrinsic interest, these identities will be pivotal in the second part II of this thesis.

3 Definition and selected relations between theta functions

Theta functions are typically attributed to Jacobi [2], who pioneered their systematic investigation. Interestingly, Whittaker and Watson [4] (p. 486) note that it was Euler [7] (p. 259-260) who first considered them. We remark that Euler’s treatment is brief and only implicitly related to Jacobi’s theta functions. Since the definition of and notation for theta functions are far from unique, for clarity we note that we shall abide by the conventions in Lawden [3] (p. 4-11).

Consider the four basic theta functions, in the form

$$\begin{aligned} \theta_1(z,q) &= -i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2} e^{i(2n+1)z}, &\qquad \theta_2(z,q) &= \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e^{i(2n+1)z}, \\ \theta_3(z,q) &= \sum_{n=-\infty}^{\infty} q^{n^2} e^{2inz}, & \theta_4(z,q) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2inz}, \end{aligned} \tag{3.1}$$

where z denotes an arbitrary complex variable and the complex valued *nome* q satisfies $|q| < 1$. Unlike in the previous section 2, we now make explicit the dependence on the nome because we shall consider variations thereof. An alternative notation which shall come in handy is in terms of the complex valued *fundamental parameter* τ, whose imaginary part is taken to be positive definite:

$$q = e^{i\pi\tau}. \tag{3.2}$$

It can be shown that the above theta functions are regular, i.e. analytic and single-valued in their domain. As shall be made abundantly clear below, the four basic theta functions are not independent.

It is obvious from their very definition that $\theta_1(z,q)$ and $\theta_2(z,q)$ are periodic in z with period 2π, while $\theta_3(z,q)$ and $\theta_4(z,q)$ are periodic in z with period π. For later convenience, we single out the periodicity relation

$$\theta_3(z-\pi,q)=\theta_3(z,q). \tag{3.3}$$

Also straightforward yet relevant to us and hence worthy of mention are the following relations between the theta functions, obtained by considering an increase of half the period:

$$\theta_1\left(z+\frac{\pi}{2},q\right)=\theta_2(z,q), \qquad \theta_3\left(z+\frac{\pi}{2},q\right)=\theta_4(z,q). \tag{3.4}$$

Additionally, it is not hard to show that theta functions possess the quasi-period $\pi\tau$ in z, with the periodicity factor $\pm(qe^{2iz})$. Here, the sign depends on the specific theta function under consideration.

Apposite to our ulterior purposes are some of the relations between theta functions that can be obtained via the increment of half periods plus half quasi-periods. In particular,

$$\mu\theta_1\left(z+\frac{\pi}{2}+\frac{\pi}{2}\tau,q\right)=\theta_3(z,q), \qquad i\mu\theta_2\left(z+\frac{\pi}{2}+\frac{\pi}{2}\tau,q\right)=\theta_4(z,q), \tag{3.5}$$

where we have introduced

$$\mu=q^{\frac{1}{4}}e^{iz}. \tag{3.6}$$

We proceed to proving the first such relation:

$$\begin{aligned}\mu\theta_1\left(z+\frac{\pi}{2}+\frac{\pi}{2}\tau,q\right)&=-i\mu\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}e^{i(2n+1)(z+\frac{\pi}{2}+\frac{\pi}{2}\tau)}\\&=-i\mu\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}q^{n+\frac{1}{2}}e^{i(2n+1)(z+\frac{\pi}{2})}\\&=\mu\sum_{n=-\infty}^{\infty}q^{(n+\frac{1}{2})^2}q^{n+\frac{1}{2}}e^{i(2n+1)z}\\&=\sum_{n=-\infty}^{\infty}q^{(n+1)^2}e^{2i(n+1)z}=\sum_{n=-\infty}^{\infty}q^{n^2}e^{2inz}=\theta_3(z,q). \quad \square\end{aligned} \tag{3.7}$$

Here, we have used (3.1) and (3.2), operated $e^{i(2n+1)\frac{\pi}{2}}$, employed (3.6), shifted $n\to n-1$ and again used (3.1), respectively. The second relation in (3.5) can be obtained by following the exact same steps, the only difference being the prefactors involved.

4 Selected relations between products of theta functions

Having established linear type of relations between the theta functions, we now proceed to establishing quadratic type of relations among them.

To begin with, we shall be interested in the subsequent three relations:

$$\begin{aligned}\theta_1(x,q)\theta_1(y,q)&=-\theta_2(x+y,q^2)\theta_3(x-y,q^2)+\theta_3(x+y,q^2)\theta_2(x-y,q^2),\\\theta_2(x,q)\theta_2(y,q)&=\theta_2(x+y,q^2)\theta_3(x-y,q^2)+\theta_3(x+y,q^2)\theta_2(x-y,q^2),\\\theta_3(x,q)\theta_3(y,q)&=\theta_2(x+y,q^2)\theta_2(x-y,q^2)+\theta_3(x+y,q^2)\theta_3(x-y,q^2).\end{aligned} \tag{4.1}$$

To prove the first one, we begin by making repeated use of (3.1):

$$\theta_1(x,q)\theta_1(y,q) = -\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(-1)^{m+n}q^{(m+\frac{1}{2})^2+(n+\frac{1}{2})^2}e^{i(2m+1)x+i(2n+1)y}. \tag{4.2}$$

Next, we introduce the bijection

$$\begin{cases} r = m+n \\ s = m-n \end{cases} \implies \begin{cases} m = \dfrac{r+s}{2} \\ n = \dfrac{r-s}{2} \end{cases} \tag{4.3}$$

and notice that, for same/different parity of m and n, there corresponds even/odd r and s. This observation, together with minor algebraic effort yields

$$\theta_1(x,q)\theta_1(y,q) = -\sum_{\substack{(r,s)\\ \text{even}}}\sum_{\substack{(r,s)\\ \text{odd}}}(-1)^r q^{\frac{1}{2}[(r+1)^2+s^2]}e^{i(r+1)(x+y)+is(x-y)}. \tag{4.4}$$

In order to bring the sums from parity definite pairs to all integers, we consider the following rewritings

$$\begin{cases} r \to 2r \text{ and } s \to 2s & \text{for even } (r,s) \text{ pairs} \\ r \to 2r-1 \text{ and } s \to 2s+1 & \text{for odd } (r,s) \text{ pairs} \end{cases} \tag{4.5}$$

and thus obtain

$$\begin{aligned} \theta_1(x,q)\theta_1(y,q) = & -\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} q^{\frac{1}{2}[(2r+1)^2+4s^2]}e^{i(2r+1)(x+y)+2is(x-y)} \\ & +\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} q^{\frac{1}{2}[(2r)^2+(2s+1)^2]}e^{2ir(x+y)+i(2s+1)(x-y)}. \end{aligned} \tag{4.6}$$

Operating through and collecting terms, the relation we wished to prove follows readily:

$$\begin{aligned} \theta_1(x,q)\theta_1(y,q) &= -\sum_{r=-\infty}^{\infty} q^{2(r+\frac{1}{2})^2}e^{i(2r+1)(x+y)}\sum_{s=-\infty}^{\infty} q^{2s^2}e^{2is(x-y)} \\ &\quad +\sum_{r=-\infty}^{\infty} q^{2r^2}e^{2ir(x+y)}\sum_{s=-\infty}^{\infty} q^{2(s+\frac{1}{2})^2}e^{i(2s+1)(x-y)} \\ &= \theta_2(x+y,q^2)\theta_3(x-y,q^2)+\theta_3(x+y,q^2)\theta_2(x-y,q^2). \quad \square \end{aligned} \tag{4.7}$$

The second and third relations in (4.1) can be obtained by following the exact same steps. In the case of the second relation, the only difference is in the prefactors. In the case of the third relation, beyond the difference in prefactors, the adequate shift for odd (r,s) pairs is not that in (4.5) but rather $r \to 2r+1$ and $s \to 2s+1$.

Particular instances of the just derived relations (4.1) soon to be useful are

$$\theta_2(x,q)\theta_2(0,q) = 2\theta_2(x,q^2)\theta_3(x,q^2), \qquad \theta_3(x,q)\theta_3(0,q) = \theta_2^2(x,q^2)+\theta_3^2(x,q^2). \tag{4.8}$$

These are simply obtained by setting $y=0$ in the second and third relations in (4.1).

All the above can be combined to yield further relations of interest. First, consider the difference of the squares of the second and first relations in (4.1):

$$\begin{aligned}\theta_2^2(x,q)\theta_2^2(y,q)-\theta_1^2(x,q)\theta_1^2(y,q) &= 4\theta_2(x+y,q^2)\theta_3(x+y,q^2)\theta_2(x-y,q^2)\theta_3(x-y,q^2)\\ &= \theta_2(x+y,q)\theta_2(x-y,q)\theta_2^2(0,q),\end{aligned} \tag{4.9}$$

where in the last step we have made repeated use of the first relation in (4.8). Next, consider the addition of the squares of the first and third relations in (4.1). Uncomplicated and quick algebra yields

$$\begin{aligned}\theta_1^2(x,q)\theta_1^2(y,q)+\theta_3^2(x,q)\theta_3^2(y,q) &= \left[\theta_2^2(x+y,q^2)+\theta_3^2(x+y,q^2)\right]\left[\theta_2^2(x-y,q^2)+\theta_3^2(x-y,q^2)\right]\\ &= \theta_3(x+y,q)\theta_3(x-y,q)\theta_3^2(0,q),\end{aligned} \tag{4.10}$$

where in the last step we have made repeated use of the second relation in (4.8).

To conclude this section, we will entertain specific increments of half a period (plus half quasi-period) on the relations (4.9) and (4.10), together with particular instances thereof. We start by applying the shift

$$x \to x+\frac{\pi}{2}+\frac{\pi}{2}\tau \tag{4.11}$$

to (4.9). Using once the first relation and thrice the second relation in (3.5), it follows that

$$\theta_4(x+y,q)\theta_4(x-y,q)\theta_2^2(0,q)=\theta_4^2(x,q)\theta_2^2(y,q)+\theta_3^2(x,q)\theta_1^2(y,q), \tag{4.12}$$

after simplification of the overall factor $-q^{-\frac{1}{2}}e^{-2ix}$. Exchanging $x \leftrightarrow y$ in the above and taking into consideration the inherent parity in the complex variable of the fourth theta function, i.e. $\theta_4(z,q)=\theta_4(-z,q)$ by virtue of its definition in (3.1), we arrive at the desired expression:

$$\theta_4(x+y,q)\theta_4(x-y,q)\theta_2^2(0,q)=\theta_1^2(x,q)\theta_3^2(y,q)+\theta_2^2(x,q)\theta_4^2(y,q). \tag{4.13}$$

On the other hand, we will apply the shift

$$y \to y+\frac{\pi}{2} \tag{4.14}$$

to (4.10). Availing ourselves once of the first relation and thrice of the second relation in (3.4), we obtain

$$\theta_4(x+y,q)\theta_4(x-y,q)\theta_3^2(0,q)=\theta_1^2(x,q)\theta_2^2(y,q)+\theta_3^2(x,q)\theta_4^2(y,q). \tag{4.15}$$

In the derivation of the above, it is perhaps worthy of enhanced detail to notice that

$$\theta_3(x-y,q)\to\theta_3\left(x-y-\frac{\pi}{2}\right)=\theta_3\left(x-y-\pi+\frac{\pi}{2}\right)=\theta_3\left(x-y+\frac{\pi}{2}\right)=\theta_4(x-y), \tag{4.16}$$

where in the second and last equalities we have employed (3.3) and the second relation in (3.4), respectively. At last, setting $y=0$ in (4.13) and (4.15) yields the particular cases that shall render the proofs of (2.4) and (2.5) immediate:

$$\begin{aligned}\theta_4^2(x,q)\theta_2^2(0,q) &= \theta_1^2(x,q)\theta_3^2(0,q)+\theta_2^2(x,q)\theta_4^2(0,q),\\ \theta_4^2(x,q)\theta_3^2(0,q) &= \theta_1^2(x,q)\theta_2^2(0,q)+\theta_3^2(x,q)\theta_4^2(0,q).\end{aligned} \tag{4.17}$$

5 Fleeting rebound into Jacobi elliptic functions

In light of our above results, it is now easy to show the two key identities of Jacobi elliptic functions (2.4) and (2.5). Indeed,

$$\mathrm{sn}^2u+\mathrm{cn}^2u=\frac{1}{\theta_4^2(z)}\frac{1}{\theta_2^2(0)}\left[\theta_1^2(z)\theta_3^2(0)+\theta_2^2(z)\theta_4^2(0)\right]=\frac{\theta_4^2(z)}{\theta_4^2(z)}\frac{\theta_2^2(0)}{\theta_2^2(0)}=1,\quad\square \tag{5.1}$$

where we have made use of the definitions (2.2) and the just proven first relation in (4.17). Likewise,

$$\mathrm{dn}^2u+k^2\mathrm{sn}^2u=\frac{1}{\theta_4^2(z)}\frac{1}{\theta_3^2(0)}\left[\theta_1^2(z)\theta_2^2(0)+\theta_3^2(z)\theta_4^2(0)\right]=\frac{\theta_4^2(z)}{\theta_4^2(z)}\frac{\theta_3^2(0)}{\theta_3^2(0)}=1.\quad\square \tag{5.2}$$

While we shall not make constructive use of them, it will eventually prove enlightening to explicitly mention the two periods of the Jacobi elliptic functions. Define

$$K=\frac{\pi}{2}\theta_3^2(0),\qquad iK'=\tau K, \tag{5.3}$$

in terms of the third basic theta function $\theta_3(z)$ in (3.1) and the fundamental parameter τ in (3.2). The convention is such that, for purely imaginary τ, K and K' are both positive and real. By means of the (quasi-)periodic relations of the basic theta functions, it can be verified that

$$\begin{aligned}\mathrm{sn}u&=\mathrm{sn}(u+4K)=\mathrm{sn}(u+2iK'),\\ \mathrm{cn}u&=\mathrm{cn}(u+4K)=\mathrm{cn}(u+2K+2iK'),\\ \mathrm{dn}u&=\mathrm{dn}(u+2K)=\mathrm{dn}(u+4iK').\end{aligned} \tag{5.4}$$

For instance, see Lawden [3] (p.27).

Part II

An application to statistical mechanics

Statistical mechanics is a branch of physics whose origin is typically attributed to the cumulative work of Maxwell, Boltzmann and Gibbs in the later half of the XIX century. In broad strokes, it can be understood as the application of probability theory to large assemblies of microscopic constituents in order to account for macroscopic properties of matter. As such, statistical mechanics is closely related to classical thermodynamics, from where it engendered. Moreover, statistical mechanics can be applied to quantum mechanical systems, which gives rise to the very active field of quantum statistical mechanics.

In this thesis, we shall focus on a particular statistical mechanical model, known as the eight-vertex model. More specifically, we shall consider a certain limit of the model, the zero-field limit. In this context, we will elucidate the emergence of the Jacobi elliptic functions introduced in part I as befitting parametrization objects. While the elliptic parametrization we will present is well-known, to the best of our knowledge it has not been worked out in detail anywhere in the literature.

The second part II of the thesis is arranged as follows. In section 6, we provide a quick introduction to the eight-vertex model. We specialize to its zero-field limit in section 7. Section 8 is devoted to changing the parametrization of the zero-field eight-vertex model from physically intuitive variables to the variables in terms of which the model was first resolved. In section 9, we notice that the desired change of variables is subject to non-trivial restrictions, and we characterize them. Finally, in section 10, we specify the reparametrization, with the previously unveiled restrictions implemented. This is achieved by means of Jacobi elliptic functions, particularly the identities proven in the first part I of the thesis.

6 Lightning introduction to the eight-vertex model

Consider a two-dimensional square lattice with $(N+1)\times(N+1)$ sites. Impose periodic boundary conditions in both the horizontal and vertical directions. By definition, the topology of the system is thus set to be that of a torus, see figure 2.

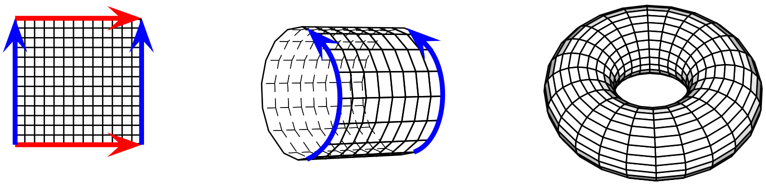

Figure 2: The torus as a square with periodic boundary conditions.

On every edge of such a lattice, consider an arrow. Clearly, on the horizontal edges, any fixed arrow can point either left or right. Likewise, on the vertical edges, any fixed arrow can point either up or down. Such arrow configurations constitute toy-models for dipole distributions in thin-film molecular arrangements. Accordingly, they are of relevance in both physics and chemistry.

Let the system be in thermal equilibrium, at a fixed temperature T. Then, the possible arrow configurations conform to a canonical ensemble that is classical and discrete.

In every site, demand that an even number of arrows point inwards (equivalently, an even number of arrows point outwards). This system is known as the *eight-vertex model*, in agreement with the total number of permitted arrow configurations per site, shown in figure 3.

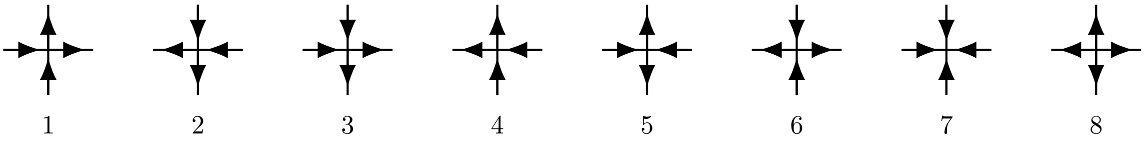

Figure 3: Permitted site configurations in the eight-vertex model.

The eight-vertex model is a prominent system in statistical mechanics, of distinctive difficulty. Indeed, the model remains unsolved, except for particular cases. The model was independently proposed by Sutherland [8] and Fan and Wu [9] in 1970. It was conceived as a generalization of the *six-vertex model*, wherein the latter two site configurations of figure 3, the so-called *sink* and *source*, do not occur. The six-vertex model had been introduced by Pauling [10] in 1935. It was gradually solved throughout 1967 by Lieb [11], Sutherland [12] and Yang [13]. Two years later, Nagle [14] partially solved the three-dimensional generalization. The eight-vertex model does not seem to have been meaningfully extended. Instead, a different model by the name *XYZ chain*, first proposed by Heisenberg [15] in 1928, seems to have overshadowed it in the long run. Interestingly, the eight-vertex model and the XYZ chain share a common subcase of certain notoriety: the *XXZ model.*

7 Physical parametrization and the zero-field limit

Let C denote a fixed arrow configuration in a site of the previously described eight-vertex model. Let E_C denote its energy. Recall that T refers to the fixed temperature the system is kept at. Then, the canonical partition function of the model can be written in terms of these quantities as

$$Z = \sum_C \exp(-\beta E_C), \qquad \beta = (k_B T)^{-1}, \tag{7.1}$$

where k_B is the Boltzmann constant. It is worth bearing in mind the central importance of Z: all thermodynamic observables can be obtained from Z and its derivatives.

E_C can be understood as the sum of the energies in each site of the lattice. Since only eight arrow configurations are allowed per site, those in figure 3, we have that

$$E_C = \sum_{i=1}^{8} n_i \epsilon_i, \tag{7.2}$$

where n_i denotes the number of sites with configuration i and ϵ_i is the corresponding site energy. The chosen periodic conditions on the lattice require an equal number of sinks and sources: $n_7 = n_8$. Moreover, since reversal of the vertical arrows in the configurations 5 and 6 of figure 3 yield a sink and a source respectively, $n_5 = n_6$ also holds true in the toroidal topology. It follows that E_C and hence Z only depend on $(\epsilon_5, \dots, \epsilon_8)$ via the combinations $\epsilon_5 + \epsilon_6$ and $\epsilon_7 + \epsilon_8$. Without loss of generality, we set

$$\epsilon_5 = \epsilon_6, \qquad \epsilon_7 = \epsilon_8. \tag{7.3}$$

Henceforth, we restrict attention to the *zero-field limit* of the eight-vertex model, wherein

$$\epsilon_1 = \epsilon_2, \qquad \epsilon_3 = \epsilon_4, \tag{7.4}$$

Notice that (7.3) and (7.4) render the system invariant under reversal of all arrows in the lattice. Regarding the arrows as electric dipoles, this means no external electric field is applied. Hence, the name of this particular case. Introduce the positive-definite real parameters

$$a = \exp(-\beta\epsilon_1), \qquad b = \exp(-\beta\epsilon_3), \qquad c = \exp(-\beta\epsilon_5), \qquad d = \exp(-\beta\epsilon_7), \tag{7.5}$$

and rename

$$n_a = n_1 + n_2, \qquad n_b = n_3 + n_4, \qquad n_c = n_5 + n_6, \qquad n_d = n_7 + n_8, \tag{7.6}$$

in order to bring the partition function to the readily physically meaningful form

$$Z = \sum_C a^{n_a} b^{n_b} c^{n_c} d^{n_d}. \tag{7.7}$$

The zero-field eight-vertex model enjoys a considerable number of symmetries. For instance, the reversal of all horizontal arrows leaves the system invariant. Applied in figure 3, this reversal exchanges the following site configurations: (1,4), (2,3), (5,8) and (6,7). Plugged in (7.6), this exchanges (n_a, n_b) and (n_c, n_d). Used in (7.7), this entails

$$Z(a, b, c, d) = Z(b, a, d, c). \tag{7.8}$$

An exhaustive discussion of the symmetries is beyond the scope of this thesis. The interested reader is referred to [6] (ch.10.2).

The interest of the zero-field eight-vertex model resides in that it was solved by Baxter [16] in 1972. It is worth noting that Kumar [17] found an alternative, markedly simpler resolution two years later, including a generalization to a certain non-zero-field scenario. In both cases, solving the model is to be understood as finding the largest magnitude eigenvalue of the partition function (7.7). The existence, uniqueness and reality of such largest eigenvalue is ensured by virtue of the Perron-Frobenius theorems, dating back to 1907-1912. As a reminder, the Perron theorem [18] states that, given a real square matrix with positive definite entries, there exists a unique eigenvalue of largest modulus, and it is real. The Frobenius theorems [19, 20] generalize this result to irreducible, non-negative real square matrices. Both theorems are stated and proven in a modern language in Gantmacher's book [21] (p.53-62). For a friendly, eloquent review of the relevance of these theorems, see [22] – in Spanish.

Another appealing feature of the zero-field limit of the eight-vertex model is that it admits a one-on-one correspondence with two coupled copies of the Ising model [23, 24]. For a lucid explanation of the correspondence, see [6] (ch.10.3).

8 Mathematical reparametrization

Next, we wish to change the physically meaningful parametrization of the zero-field eight-vertex model in (7.5) for a mathematically more convenient one. Specifically, we wish to change from coordinates (a, b, c, d) to coordinates

$$\Delta = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}, \qquad \Gamma = \frac{ab - cd}{ab + cd}, \tag{8.1}$$

u and some irrelevant normalization factor, such that (a, b, c, d) are entire functions of u, while (Δ, Γ) are independent of u. Bear in mind that an entire function is a complex-valued function that is holomorphic on the whole complex plane. The quantities Δ and Γ are important auxiliary objects in the resolution of the zero-field eight-vertex model. They appear in both the original approach by Baxter [16] and in the ulterior independent and generalizable procedure by Kumar [17]. For a more sapid presentation of the former, see Baxter's book [6] (ch.9.6 and 10.4). In both cases, these quantities play a role in the construction of as simple as possible so-called *transfer matrices*. Transfer matrices in turn allow to bring the partition function into a form that renders feasible the computation of its largest magnitude eigenvalue, which is regarded as the solution of the system.

Let

$$\mathcal{D} = \{a, b, c, d \mid a, b, c, d \in \mathbb{R}_{>0}\} \tag{8.2}$$

denote the domain of physically meaningful parameter values, see (7.5). On $\mathcal{D}$, $\Delta \in \mathbb{R}$ and $\Gamma \in (-1, 1)$. To see the former, first consider fixing (a, b, c) and varying d; then, consider fixing (b, c, d) and varying a. The whole real line is thus spanned. To see the latter, rewrite it as

$$\Gamma = \frac{1-\gamma}{1+\gamma}, \tag{8.3}$$

where we have introduced

$$\gamma = \frac{cd}{ab}. \tag{8.4}$$

By definition, $\gamma \in \mathbb{R}_{>0}$. On its domain, (8.3) is obviously continuous and strictly decreasing (thus, univalued) function of γ and such that

$$\lim_{\gamma\to 0} \Gamma = 1, \qquad \lim_{\gamma\to\infty} \Gamma = -1. \tag{8.5}$$

Moreover, the relation (8.3) manifestly maps one-to-one the possible values of γ and Γ on their domain $\mathcal{D}$. Therefore, we can use either γ or Γ for reparametrization purposes. Without loss of generality, we will opt for γ henceforth. Furthermore, let Δ and γ take fixed values in their respective ranges hereafter.

Removal of d

For starters, we wish to trade the dependence of Δ on d by γ. From (8.4), it readily follows that

$$d = \gamma\frac{ab}{c}, \qquad ab(1 \pm \gamma) = ab \pm cd. \tag{8.6}$$

A minimal manipulation of (8.1) yields

$$2(ab+cd)\Delta = a^2 + b^2 - c^2 - d^2, \qquad (ab+cd)\Gamma = ab - cd. \tag{8.7}$$

Substituting (8.6) into (8.7), we obtain

$$2ab(1+\gamma)\Delta = a^2 + b^2 - c^2 - \frac{a^2b^2}{c^2}\gamma^2. \tag{8.8}$$

This expression does not dependent on d, other than implicitly via the fixed parameters Δ and γ. We have thus accomplished our first goal.

Removal of a

Next, we wish to express a in terms of b, Δ and γ. The dependence on c is irrelevant. Indeed, there exists a redundant parameter among a, b, c, d in both the partition function (7.7) and in (8.8), which admits the interpretation of a normalization factor that we are free to fix to any positive real value of our convenience. For (8.8), the redundancy can be made apparent by dividing by c^2 and noting that the result depends only on the ratios

$$x = \frac{a}{c}, \qquad y = \frac{b}{c}. \tag{8.9}$$

By definition, on $\mathcal{D}$, these ratios take positive definite values: $x, y \in \mathbb{R}_{>0}$. Explicitly, we have that

$$2(1+\gamma)\Delta xy = x^2 + y^2 - 1 - \gamma^2x^2y^2. \tag{8.10}$$

An elementary rearrangement of terms presents the above relation as a quadratic equation for x:

$$Ax^2 + Bx + C = 0, \tag{8.11}$$

where

$$A = 1 - \gamma^2 y^2, \qquad B = -2(1+\gamma)\Delta y, \qquad C = y^2 - 1. \tag{8.12}$$

It can be readily verified that, on the domain $\mathcal{D}$, $A \in (-\infty, 1)$, $B \in \mathbb{R}$ and $C \in (-1, \infty)$. The solutions to a quadratic function like (8.11) are well-known:

$$x = \frac{-B \pm 2\sqrt{\delta}}{2A}, \tag{8.13}$$

where δ is the discriminant, conveniently divided by four:

$$4\delta = B^2 - 4AC. \tag{8.14}$$

Using (8.12), it can be written as

$$\delta = (1+\gamma)^2 \Delta^2 y^2 + (1 - y^2)(1 - \gamma^2 y^2). \tag{8.15}$$

Substitution of (8.12) and (8.15) into (8.13), yields the desired result: an expression wherein the physical parameter a is expressed in terms of the physical parameter b, with a dependence on the physical parameter d that is only implicit, via the fixed quantities Δ and γ. The explicit expression, in its full generality, is unenlightening and so we do not show it here.

Overall, we have now formally accomplished the sought reparametrization. However, further work is due to render the reparametrization practical.

9 Restrictions for a non-negative discriminant δ

For the self-consistency of our proposed reparametrization, given fixed Δ and γ, we must ensure that $\delta \geqslant 0$ on the domain $\mathcal{D}$. Otherwise, x in (8.12), and therefore the physical parameter a, will be allowed to take on complex values, resulting in a contradiction, see (7.5) and (8.9). As we will prove, this turns out to be a non-trivial requirement in general.

Let us begin by rewriting (8.15) in the more suggestive form of a product of quadratic functions without linear terms:

$$\delta = (1 - \alpha y^2)(1 - \beta y^2). \tag{9.1}$$

Expanding the right-hand sides of (8.15) and (9.1) and matching terms, we see that

$$\alpha + \beta = (1 + \gamma^2) - (1+\gamma)^2 \Delta^2 \equiv R, \qquad \alpha\beta = \gamma^2. \tag{9.2}$$

On the domain of interest $\mathcal{D}$, the latter equality prevents α and β from vanishing: $\alpha, \beta \neq 0$. Hence, we can solve (9.2) for α and β as follows. From the latter equality, we have that

$$\beta = \frac{\gamma^2}{\alpha}. \tag{9.3}$$

Substituting the above into the former equality, multiplying by α and rearranging terms, we get

$$\alpha^2 - R\alpha + \gamma^2 = 0 \qquad \Longrightarrow \qquad \alpha = \frac{R \pm \sqrt{S}}{2}, \qquad S = R^2 - 4\gamma^2. \tag{9.4}$$

Explicitly,

$$S = (1+\gamma)^4\Delta^4 - 2(1+\gamma^2)(1+\gamma)^2\Delta^2 + (1-\gamma^2)^2. \tag{9.5}$$

Hence, in order to determine the parametric regions wherein the discriminant (9.1) is positive definite, we must investigate the nature of α and β in light of R and S. We proceed to do so.

View S as a function of Δ^2. Then, S is a convex (equivalently, concave up) quadratic function. It is a matter of minor algebraic effort to verify that the minimum of S lies at

$$\left.\frac{\partial S}{\partial \Delta^2}\right|_{\Delta_0^2} = 0 \qquad \Longrightarrow \qquad \Delta_0^2 = 1 - \frac{2\gamma}{(1+\gamma)^2}, \qquad \text{such that} \quad \frac{\partial^2 S}{(\partial \Delta^2)^2} = 2(1+\gamma)^4 > 0 \quad \forall \Delta^2, \tag{9.6}$$

and that it is negative valued for all $\gamma \in \mathbb{R}_{>0}$:

$$\left. S \right|_{\Delta_0^2} = -4\gamma^2 < 0. \tag{9.7}$$

Moreover, S achieves zero value when

$$\Delta_\pm^2 = \frac{(1 \pm \gamma)^2}{(1+\gamma)^2}. \tag{9.8}$$

Recalling (8.3), we can write these two limiting values as

$$\Delta_-^2 = \Gamma^2 \in [0,1), \qquad \Delta_+^2 = 1. \tag{9.9}$$

Overall, S takes positive (negative) values outside (inside) the interval (Δ_-^2, Δ_+^2), for any allowed value of $\gamma \in \mathbb{R}_{>0}$. Notice that this interval cannot have zero length. In other words, there does not exist any value of Δ and γ such that S is non-negative on its whole domain.

Given a fixed value of Δ^2, a definite value of S in (9.5) follows. Moreover, the value of R in (9.2), α in (9.4) and β in (9.3) can be calculated. The latter two, when plugged in (9.1), lead to the conclusion that the discriminant δ is positive definite in most but not all of the cases. Specifically, positive definiteness of δ is guaranteed whenever $\Delta^2 \geqslant \Gamma^2$, while the case $\Delta < \Gamma^2$ requires further investigation. The situation is graphically summarized in figure 4. Further quantitative details are provided in table 1.

For completeness, we note that

$$\delta = \begin{cases} (1-\gamma y^2)^2 & \text{if } \Delta^2 = \Gamma^2, \\ (1-\mathcal{A}_+ y^2)^2 + \mathcal{B}^2 y^4 & \text{if } \Delta^2 \in (\Gamma^2, \Delta_0^2), \\ 1+\gamma^2 y^4 & \text{if } \Delta^2 = \Delta_0^2, \\ (1-\mathcal{A}_- y^2)^2 + \mathcal{B}^2 y^4 & \text{if } \Delta^2 \in (\Delta_0^2, 1), \\ (1+\gamma y^2)^2 & \text{if } \Delta^2 = 1, \\ (1+|\alpha| y^2)(1+|\beta| y^2) & \text{if } \Delta^2 > 1, \end{cases} \tag{9.10}$$

which is manifestly non-negative for any allowed γ and y, as previously announced. Here, we have introduced

$$\mathcal{A} = \mathrm{Re}(\alpha), \qquad \mathcal{B} = \mathrm{Im}(\alpha), \tag{9.11}$$

in the relevant parametric regions wherein α and β take complex conjugate values $\overline{\alpha} = \beta$. A subscript plus (minus) indicates when $\mathcal{A}$ takes positive (negative) definite values.

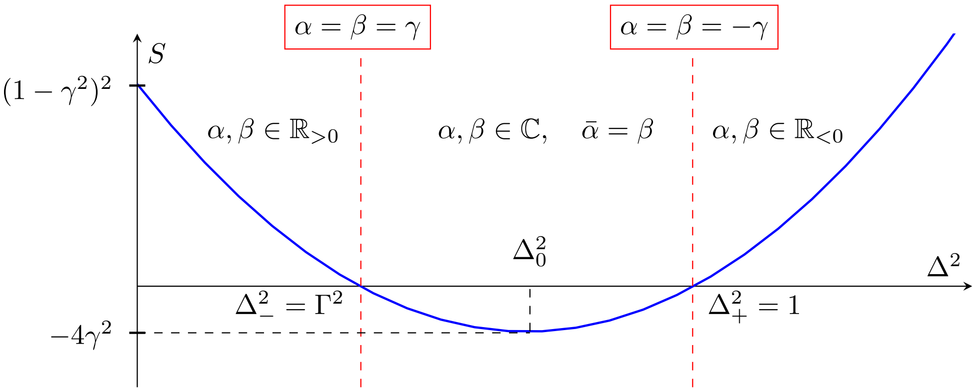

Figure 4: In blue, S in (9.5) as a function of Δ^2. S takes negative values if $\Delta^2 \in (\Gamma^2, 1)$, wherein α and β in (9.2) take complex conjugate values. Else, S is non-negative, while α and β are real valued: positive if $\Delta^2 \leqslant \Gamma^2$ and negative if $\Delta^2 \geqslant 1$. Notice that $\Gamma^2 \in [0, 1)$, the exact value depending on the specific system as per (8.1). The positive definiteness of the discriminant δ in (9.1) is only guaranteed for $\Delta^2 \geqslant \Gamma^2$.

$\boldsymbol{\Delta^2}$	**S**	**R**	$\boldsymbol{\alpha}$ **and** $\boldsymbol{\beta}$	$\boldsymbol{\delta}$
$\Delta^2 < \Gamma^2$	$S > 0$	$R > 2\gamma$	$\alpha, \beta \in \mathbb{R}_{>0}, \quad \alpha \neq \beta$	$\delta \in \mathbb{R}$
$\Delta^2 = \Gamma^2$	$S = 0$	$R = 2\gamma$	$\alpha = \beta = \gamma$	$\delta \in \mathbb{R}_{>0}$
$\Delta^2 \in (\Gamma^2, \Delta_0^2)$	$S < 0$	$R \in (2\gamma, 0)$	$\alpha, \beta \in \mathbb{C}, \quad \overline{\alpha} = \beta, \quad \mathrm{Re}(\alpha) > 0$	
$\Delta^2 = \Delta_0^2$		$R = 0$	$\alpha = -\beta = \pm i\gamma$	
$\Delta^2 \in (\Delta_0^2, 1)$		$R \in (0, -2\gamma)$	$\alpha, \beta \in \mathbb{C}, \quad \overline{\alpha} = \beta, \quad \mathrm{Re}(\alpha) < 0$	
$\Delta^2 = 1$	$S = 0$	$R = -2\gamma$	$\alpha = \beta = -\gamma$	
$\Delta^2 > 1$	$S > 0$	$R < -2\gamma$	$\alpha, \beta \in \mathbb{R}_{<0}, \quad \alpha \neq \beta$	

Table 1: For a fixed value of Δ^2, tabulation of the values or relevant ranges of S in (9.5), R in (9.2), α in (9.4), β in (9.3) and δ in (9.1). The discriminant δ takes physically meaningful positive definite values if $\Delta^2 \geqslant \Gamma^2$. For $\Delta^2 < \Gamma^2$, δ can also take unphysical negative values.

The case $\Delta^2 < \Gamma^2$

We proceed to determining when the discriminant δ in (9.1) does take physically meaningful positive values in the non-trivial parametric region wherein $\Delta^2 < \Gamma^2$. To this aim, we shall perform another uncomplicated extremization exercise.

View δ as a function of y. Recall that, in the regime of current interest, $\alpha, \beta \in \mathbb{R}_{>0}$ are distinct. Then, we easily find two critical values δ_c exist, located at:

$$\left.\frac{d\delta}{dy}\right|_{y_c} = 0 \quad \Longrightarrow \quad y_c = 0, \qquad y_c = +\sqrt{\frac{\alpha+\beta}{2\alpha\beta}} \equiv y_0. \tag{9.12}$$

The negative square root $y_c = -y_0$ also locates an extremal value of δ, but we ignore it because it lies outside the physical domain $\mathcal{D}$. The former is a maximum, while the latter is a minimum:

$$\left.\frac{d^2\delta}{dy^2}\right|_{y_c=0} = -2(\alpha+\beta) < 0, \qquad \left.\frac{d^2\delta}{dy^2}\right|_{y_c=y_0} = 4(\alpha+\beta) > 0. \tag{9.13}$$

The respective critical values of the discriminant are

$$\delta_c = \delta|_{y_c=0} = 1 > 0, \qquad \delta_{c0} = \delta|_{y_c=y_0} = -\frac{(\alpha-\beta)^2}{4\alpha\beta} < 0. \tag{9.14}$$

The above implies that the a priori allowed region $y \in \mathbb{R}_{>0}$ is to be further restricted so that $\delta \geqslant 0$ and physically meaningful solutions are obtained for x in terms of y – recall (8.13).

To do so, let us reconsider the discriminant δ. Provided that we are in the region wherein $\alpha \neq \beta$ are positive definite, we can absorb one of these two parameters into y^2 in (9.1). We are thus led to consider the simpler form of the discriminant

$$\delta = (1-t^2)(1-k^2t^2), \tag{9.15}$$

where

$$t^2 = \alpha y^2, \qquad k^2 = \frac{\beta}{\alpha}. \tag{9.16}$$

Notice that, by definition, $t^2, k^2 \in \mathbb{R}_{>0}$. The discriminant (9.15) vanishes when $t^2 = 1$ and $t^2 = k^{-2}$. It takes unphysical negative values in the open interval between these two limiting values and physically acceptable positive values in the complementary interval. Mathematically, we are interested in

$$\begin{cases} t^2 \in [0,1] \cup [k^{-2},\infty) & \text{if } \alpha > \beta, \\ t^2 \in (0,k^{-2}] \cup [1,\infty) & \text{if } \alpha < \beta. \end{cases} \tag{9.17}$$

Without loss of generality, we shall restrict attention to the case $\alpha > \beta$ henceforth, wherein $k^2 \in (0,1)$. The situation is portrayed in figure 5.

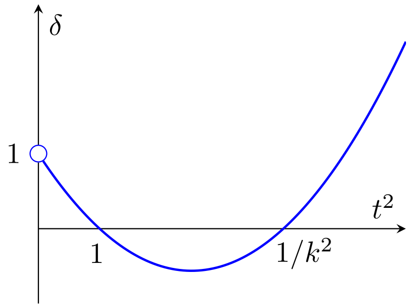

Figure 5: In blue, the discriminant δ in (9.15) as a function of t^2. δ takes unphysical negative values if $t^2 \in (1, k^{-2})$. Jacobi elliptic functions provide for a cogent parametrization of the non-negative regions of interest.

10 Elliptic parametrization of the relevant, non-trivial regions

Subsequently, we shall be concerned with the parametrization of exclusively the non-negative parts of the discriminant δ in (9.1), in the regime wherein $\Delta^2 < \Gamma^2$. This is equivalent to parametrizing δ in (9.15) for $k^2 \in (0,1)$ in the two disjoint regions $t^2 \in (0,1]$ and $t^2 \in [k^{-2}, \infty)$. In both instances, a natural parametrization will be in terms of elliptic functions. We stress that such a rewriting constitutes a key step in the original solution of the zero-field eight-vertex model by Baxter [16]. In fact, it constitutes the point wherein elliptic functions first make their appearance.

The case $t^2 \in [0,1]$

Consider the befitting rewriting (9.15) of the discriminant (9.1). A natural parametrization in the regime of present interest is in terms of the Jacobi elliptic sine function, simply as

$$t = \mathrm{sn} u, \tag{10.1}$$

where u is the argument, taken to be real so that $t \in [-1,1]$. Then, $k \in (0,1)$ is to be understood as the modulus (2.3), with $k = 0\,(1)$ corresponding to the trigonometric (hyperbolic) limit. Figure 6 depicts sn for values of the modulus near these two limits. The rescaling of the sine portrayed in dashed red therein stems from the following considerations. It can be shown that fixing the modulus k fixes the period $4K$ of the elliptic sine, see (5.4). We consulted the mathematical handbook [25] to find the correspondence between $k^2 = 0.5$ and $K = 1.85$ (p.108). The product of such $4K$ and 0.847 approximately yields the period 2π of the trigonometric sine.

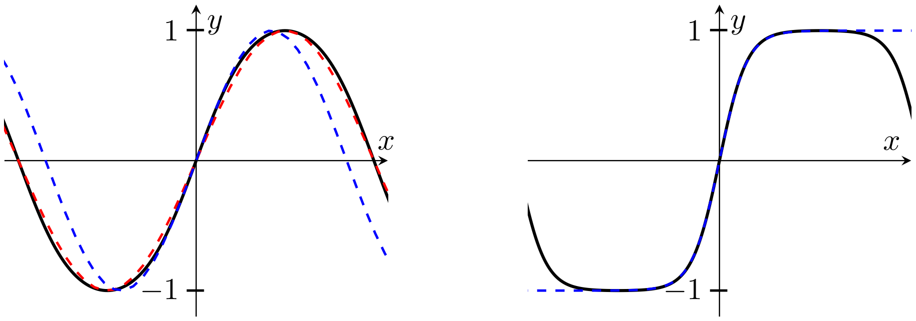

Figure 6: The Jacobi elliptic sine function is plotted in black for a fixed value k_0 of the modulus and real values of the variable: $\mathrm{sn}(x, k_0)$ with $x \in \mathbb{R}$. On the left, the modulus is set to $k_0^2 = 0.5$, (visually) near the trigonometric limit $k \to 0$. On the right, the modulus is set to $k_0^2 = 0.99$, near the hyperbolic limit $k \to 1$. In dashed blue, we plot $\sin x$ (left) and $\tanh x$ (right), for comparison purposes. In dashed red, we plot $\sin 0.847x$ (left), thereby adjusting the periods of the Jacobi elliptic and trigonometric sines. The period adjustment is essentially imperceptible on the right and hence omitted.

Under (10.1), the discriminant (9.15) becomes

$$\delta = (1 - \mathrm{sn}^2 u)(1 - k^2 \mathrm{sn}^2 u). \tag{10.2}$$

The fact that, as desired, the discriminant is positive definite in this parametrization becomes evident through the sum of squares formulae (2.4) and (2.5), proven in the first part I of this work.

Indeed, their use immediately yields

$$\delta = \mathrm{cn}^2 u \cdot \mathrm{dn}^2 u \geqslant 0. \tag{10.3}$$

The case $t^2 \in [k^{-2}, \infty)$

Still considering (9.15), but turning attention to the second regime of interest, we restore to the natural parametrization

$$t = \frac{1}{k \cdot \mathrm{sn} u}, \tag{10.4}$$

again in terms of a real argument u and with $k \in (0, 1)$ acting as the modulus. Upon employing the sum of squares formulae (2.4) and (2.5), the above painlessly yields a manifestly positive definite discriminant (9.15):

$$\delta = \left(1 - \frac{1}{k^2 \cdot \mathrm{sn}^2 u}\right)\left(1 - \frac{1}{\mathrm{sn}^2 u}\right) = \frac{\left(k^2 \mathrm{sn}^2 u - 1\right)\left(\mathrm{sn}^2 u - 1\right)}{k^2 \cdot \mathrm{sn}^4 u} = \frac{\mathrm{cn}^2 u \cdot \mathrm{dn}^2 u}{k^2 \cdot \mathrm{sn}^4 u} \geqslant 0. \tag{10.5}$$

Conclusions

In this thesis, we have provided a comprehensive explanation of the appearance of Jacobi elliptic functions in Baxter's original approach to the zero-field eight-vertex model [6, 16], dating back to 1972. In part I, we have put forward an accessible presentation of Jacobi elliptic functions and expanded on a self-contained derivation of consequential identities thereof. To this aim, we have availed ourselves of theta functions. In part II, we have introduced the zero-field limit of the eight-vertex statistical mechanical model and elaborated on the technicalities underlying the change of parametrization proposed by Baxter, from physically intuitive variables to mathematically motivated ones. The latter were instrumental in the first-ever resolution of the said model.

At the present time, elliptic functions continue to flourish. A well-established and fecund generalization is given by Abelian functions, which depend on several complex variables and exhibit twice as many periods. For a lucid and engaging albeit not contemporary review, see [26]. Closely related and also worthy of mention are hyperelliptic functions, adapted to the parametrization of algebraic curves with genus $g > 1$ of a fixed form. The interested reader is referred to [27] for an overview of their use in the computation of related arithmetic invariants. Jacobi elliptic functions themselves have been extended by allowing arbitrary integer powers in the radicand of the incomplete elliptic integral of the first kind, e.g. [28] and references therein.

Contrastingly, the eight-vertex model seems to have received limited attention since its apogee in the decade of the 1970s. Perhaps most remarkably, it has been extended to supersymmetric scenarios by Hagendorf and collaborators [29–32], wherein elliptic functions still play a pivotal role in establishing a cogent parametrization.

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