On the triviality of flows in Alexandroff spaces

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Abstract

We prove that the unique possible flow in an Alexandroff T_0-space is the trivial one. On the way of motivation, we relate Alexandroff spaces with topological hyperspaces.

1 Introduction and motivation

In [1], P.S. Alexandroff introduced what he called "Diskrete Räume" ("discrete spaces"), which are nowadays known as Alexandroff spaces. Today, the term discrete space refers to a topological space (X, T), where the topology T is the power set of X. An Alexandroff space (X, T) is a topological space, where the topology T satisfies the following stronger axiom: the arbitrary intersection of open sets is again an open set. P.S. Alexandroff also related the spaces introduced in [1] to preordered sets. In particular, we have the following results.

Lemma 1.1. *An Alexandroff space satisfies the Kolgomorov separation axiom (also known as T_0 axiom) if and only if the related preorder is a partial order.*

Lemma 1.2. *An Alexandroff space satisfies the separation axiom T_1 if and only if it is a discrete space.*

Besides this facts, the class of Alexandroff spaces contains the class of all finite topological spaces, i.e., topological spaces having finite cardinal.

Some important properties of Alexandroff spaces were established by M.C. McCord in [9].

Theorem 1.3. *For any Alexandroff space X, there exist a CW-complex K and a weak homotopy equivalence $f_K : |K| \to X$. Moreover, if X is path-connected, K can be chosen path-connected.*

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Flows (or continuous dynamical systems) are another central concept of this note. Recall that a flow on a topological space X is a continuous map $\varphi : \mathbb{R} \times X \to X$ satisfying that $\varphi(0, x) = x$ for every $x \in X$ and $\varphi(s + t, x) = \varphi(s, \varphi(t, x))$ for every $t, s \in \mathbb{R}$ and $x \in X$, where $\mathbb{R}$ represents the real numbers with its usual topology. Note that if $\varphi_t : X \to X$ is the

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map $\varphi_t(x) = \varphi(t, x)$, then φ_0 is the identity map and $\varphi_s \circ \varphi_t = \varphi_{s+t}$. Consequently, φ_t is a homeomorphism for every $t \in \mathbb{R}$, where the inverse of φ_t is given by φ_{-t}.

Note that if a topological space (X, T) is discrete or finite and $\varphi : \mathbb{R} \times X \to X$ is a flow, then φ is trivial, that is to say, $\varphi(t, x) = x$ for all $x \in X$, $t \in \mathbb{R}$. If (X, T) is finite, it is a consequence of the finitude of the group of homeomorphisms of X. If (X, T) is discrete , the above result about the triviality of flows is a direct consequence of the fact that the unique continuous paths in (X, T) are the constant ones.

Recently, it has been proved that topological spaces can be approximated by Alexandroff spaces using the concepts of inverse systems (inverse sequences) and inverse limits, see [12] for compact metric spaces and [3] for locally compact, paracompact and Hausdorff spaces. One could ask if flows on a general space (X, T) can be in some sense approximated by flows in sufficiently close elements in the corresponding approach of inverse systems mentioned above. Or, in other words, can Alexandroff spaces have sufficiently enough topological richness to admit different flows on them in order to approximate different flows in the general topological space (X, T)? We were led to this question motivated by some ideas developed in the memoir [5] that will be the Ph. D. thesis of the first author co-advised by the second and third authors of this note. We recommend J.P. May's notes [8] for general knowledge on Alexandorff spaces.

2 Some previous related results and the answer to the question

In view of Theorem 1.3 and Theorem 1.4, the above question has some interest because there are a lot of non-finite Alexandroff spaces and not discrete with very rich structure of continuous paths into them. In particular, the fundamental group of such spaces can be chosen to be isomorphic to any prefixed group. On the other hand every group can be realized as the group of autohomeomorphisms of an Alexandroff space, see [4].

Sometime ago, part of the authors of this note studied the so-called upper semicontinuous or lower semicontinuous topologies in hyperspaces of closed sets in a topological space, see for instance [6, 7, 2]. The corresponding definitions appear in [10, Definition 9.1]. Specifically, they proved in [7, Proposition 1] that the unique possible flows in the lower hyperspace of closed set of a topological space (X, T), denoted by 2^X_L, are those constructed in a natural way by means of genuine flows on (X, T). As an immediate consequence, we get the following result.

Corollary 2.1. *Let X_d be a topological space with the discrete topology. Then, the unique possible flow on $2^{X_d}_L$ is the trivial one.*

The reason why Corollary 2.1 holds is because any flow $\varphi : \mathbb{R} \times 2^{X_d}_L \to 2^{X_d}_L$ comes naturally from a flow $\pi : \mathbb{R} \times X_d \to X_d$. Therefore, for any $t \in \mathbb{R}$ and $x \in X$, we have a continuous path $\pi^x_t : [0, t] \to X$ given by $\pi^x_z(s) = \pi(s, x)$, $\pi^x_t(0) = x$ and $\pi^x_t = \pi(t, x)$. Since X_d is a discrete topological space, the unique possible paths are just $\pi^x_t(s) = x$ for all $t \in \mathbb{R}$ and $s \in [0, t]$. Then, the flow π is trivial and consequently φ is trivial.

To relate the above result with our current framework, we have the following proposition.

Proposition 2.2. *Let X_d be a discrete topological space. Then, $2^{X_d}_L$ is an Alexandroff space if and only if X is a finite set.*

Proof. Suppose that X is a finite set and $C \subset X$ is a non-empty subset. Then, $C = \{c_1, ..., c_k\}$, where $k \in \mathbb{N}$. Consequently, $U_C = L_{\{c_1\}} \cap ... \cap L_{\{c_k\}} = \{D \subset X | C \subset D\}$ is open in $2^{X_d}_L$ and obviously is the minimal open neighborhood of C in $2_L^{X_d}$. In the above notation, L_U for an open set U in X means $L_U = \{F \subset X | F \neq \emptyset \text{ and } F \cap U \neq \emptyset\}$. Now, suppose $2_L^{X_d}$ is an Alexandroff space and consider $X \in 2_L^{X_d}$. Note firstly that for any non-empty open set $\mathcal{U} \subset 2_L^{X_d}$, we have $X \in \mathcal{U}$. This is because if $C \in \mathcal{U}$, then there are non-empty open subsets $U_1, ..., U_m$ of X such that $C \in L_{U_1} \cap ... \cap L_{U_m} \subset \mathcal{U}$. Obviously, by definition, $X \in L_{U_1} \cap ... \cap L_{U_m}$. Then X is an element in the intersection of all non-empty open sets in $2_L^{X_d}$. In fact, we can prove that $\{X\} = \bigcap_{\mathcal{U} \neq \emptyset \text{ open in } 2_L^{X_d}} \mathcal{U}$. This is because if $F \subset X$, where $F \neq X$, then there is an element $f \in X$ such that $f \notin F$, which means that $F \notin L_{\{f\}}$. Note that $L_{\{f\}}$ is open in $2_L^{X_d}$ because $\{f\}$ is open in X_d. We have proved that $\{X\} = \bigcap_{\mathcal{U} \neq \emptyset \text{ open in } 2_L^{X_d}} \mathcal{U}$. Hence, the unitary subset $\{X\}$ is open in $2_L^{X_d}$ if $2_L^{X_d}$ were an Alexandroff space. Moreover, in this case, $\{X\}$ is the minimal open neighborhood of the point X in $2_L^{X_d}$. Following [10, Definition 9.1], we have that there exist non-empty subsets $U_1, ..., U_l$ of X with $\{X\} = L_{U_1} \cap ... \cap L_{U_l}$, where we can assume without loss of generality that $U_i \neq U_j$ if $j \neq i$. This condition implies that there is a finite subset $F \subset \bigcup_{i=1}^{l} U_i$ with $F \cap U_j \neq \emptyset$ for all $j = 1, ..., l$. Consequently, $F \in L_{U_1} \cap ... \cap L_{U_l}$ and then $F = X$. Thus, X is finite. □

Proposition 2.2 motivates us to give the following definition.

Definition 2.3. *Suppose X_d is a discrete topological space, $U \subset X$ and $L_U = \{F \subset X | F \neq \emptyset$ and $F \cap U \neq \emptyset\}$. The family $SL = \{\bigcap_{i \in I} L_{U_i} | I$ is any set with $Card(I) \leq Card(X)$ $\}$ is a base for a topology in 2^X. We represent by 2_{SL}^X the corresponding topological space and we call it the hyperspace of X with the strong lower semifinite topology.*

Proposition 2.4. *Suppose X is any set and consider it as a topological space with the discrete topology, denoted by X_d. Then,*

1. *$2_{SL}^{X_d}$ is always an Alexandroff T_0-space.*

2. *For any $C \in 2^X$, the minimal neighborhood of C in $2_{SL}^{X_d}$ is given by $\bigcap_{c \in C} L_{\{c\}} = \{D \in 2^X | C \subset D\}$.*

3. *The partial ordered set associated to the Alexandroff T_0-space $2_{SL}^{X_d}$ is given by $C \geq D$ if and only if $C \subseteq D$.*

4. *The identity map $I : 2_{SL}^{X_d} \to 2_L^{X_d}$ is continuous and the inverse is continuous at $C \in 2^X$ if and only if C is a finite set.*

5. *If we consider in 2^X the opposite order to that considered in 3., the corresponding Alexandroff T_0-space we get is $2_U^{X_d}$, that is to say, the hyperspace of the discrete space X_d with the upper semifinite topology.*

With all above, we get the following four families of Alexandroff T_0-spaces:

- $\mathcal{F}_1 = \{2_{SL}^{X_d}$ for a discrete topological space $X_d\}$.
- $\mathcal{F}_2 = \{2_U^{X_d}$ for a discrete topological space $X_d\}$.
- $\mathcal{F}_3 = \{2_{F_L}^{X_d}$ for a discrete topological space $X_d\}$.

- $\mathcal{F}_4 = \{2^{X_d}_{F_U}$ for a discrete topological space $X_d\}$.

Where $2^X_F = \{C \subset X | C \neq \emptyset$ and $card(C) < \infty\}$ and $2^{X_d}_{F_L}$ is the topological subspace restricting the lower semifinite topology to $2^{X_d}_F$. Analogously, we have $2^{X_d}_{F_U}$ for the upper semifinite topology.

Remark 2.5.
- *The families $\mathcal{F}_1$ and $\mathcal{F}_2$ can be used to embed, topologically, any Alexandroff T_0-space.*
- *The families $\mathcal{F}_3$ and $\mathcal{F}_4$ can be used to embed, topologically, any locally finite Alexandroff T_0-space.*

Results related to above statements can be found in [11, Chapter 4].

One could follow proving [7, Proposition 1] for spaces in any of the families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ obtaining that any space in $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$ is an example of an Alexandroff space admitting only trivial flows.Instead of this, we finish this note giving the following general result.

Theorem 2.6. *If X is any Alexandroff T_0 space and $\varphi : \mathbb{R} \times X \to X$ is a flow. Then, φ is trivial, i.e., $\varphi(t, x) = x$ for any $t \in \mathbb{R}$ and $x \in X$.*

Proof. For any $x \in X$, consider U_x as the minimal open neighborhood of x in X. In terms of the corresponding partial order $\leq$ in X, we have $U_x = \{y \in X | y \leq x\}$. Using the product topology in $\mathbb{R} \times X$, the continuity of φ and the fact that $\varphi(0, x) = x$ for all $x \in X$, we deduce that for any $x \in X$, there is a positive value t_x such that $\varphi((-t_x, t_x) \times U_x) \subset U_x$. Consider now t such that $|t| < t_x$. Then, the map $\varphi_{t|U_x}$ given by $\varphi_{t|U_x}(y) = \varphi(t, y)$ for $y \in U_x$ is surjective, in fact, a homeomorphism from U_x onto itself. To see this, suppose $y \in U_x$, $y = \varphi_{0|U_x}(y) = \varphi_{t|U_x} \circ \varphi_{-t|U_x}(y)$ but $\varphi_{-t|U_x}(y) \in U_x$ because $|t| < t_x$ and then $\varphi_{t|U_x}(\varphi_{-t|U_x}(y)) = y$.

Now, consider the point $(t, x) \in (-t_x, t_x) \times U_x$. We are going to prove that $\varphi(t, x) = x$. For this, we have that $x = \varphi_t(\varphi_{-t}(x))$ and $\varphi_{-t}(x) \in U_x$. If $\varphi_{-t}(x) \neq x$ we get that $\varphi_{-t}(x) < x$ and we also have that $\varphi_t(x) \neq x$ due to the injectivity of φ_t. By continuity φ_t preserves the order, so $x = \varphi_t(\varphi_{-t}(x)) < \varphi_t(x)$ and then $x \in U_{\varphi_t(x)}$. But $\varphi_t(x) \in U_x$ because $|t| < t_x$. This implies that $U_{\varphi_t(x)} = U_x$ and it is a contradiction because X is a T_0-space. So we have $\varphi_{-t}(x) = x$. Since $\varphi_t^{-1} = \varphi_{-t}$, we have $\varphi_t(x) = \varphi(t, x) = x$ for any t satisfying $|t| < t_x$. Finally, fix now $(s, x) \in \mathbb{R} \times X$. Then, there is a natural number n such that $|\frac{s}{n}| < t_x$. So $\varphi(\frac{s}{n}, x) = x$ but $s = \sum_{k=1}^{n} \frac{s}{n}$. Since φ is a flow, and using the associativity of the sum of real numbers, we have $\varphi(s, x) = \varphi(\sum_{k=1}^{n} \frac{s}{n}, x) = \varphi(\sum_{k=1}^{n-1} \frac{s}{n}, \varphi(\frac{s}{n}, x)) = \cdots = \varphi(\frac{s}{n}, x) = x$ and then φ is trivial . □

In fact, we have the following result.

Corollary 2.7. *Let X be an Alexandroff space. Then X satisfies the T_0 separation axiom if and only if X does not admit non-trivial flows.*

Proof. Suppose X is an Alexandroff space which is not T_0. Then there are two different points x and y in X with equal minimal neighborhoods. Consider the function $I_{x,y} : X \longrightarrow X$ given by, $I_{x,y}(z) = z$ if $z \in X - \{x, y\}$, $I_{x,y}(x) = y$ and $I_{x,y}(y) = x$. Then the map $\varphi : \mathbb{R} \times X \to X$ given by $\varphi(t, z) = I_{x,y}(z)$ if $t \neq 0$ and $\varphi(0, z) = z$ for any $z \in X$ is a non-trivial flow. From the theorem above, we have the equivalence. □

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