Research Article

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The Singular Perturbation Problem for a Class of Generalized Logistic Equations Under Non-classical Mixed Boundary Conditions

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Abstract: This paper studies a singular perturbation result for a class of generalized diffusive logistic equations, $d\mathcal{L}u = uh(u, x)$, under non-classical mixed boundary conditions, $\mathcal{B}u = 0$ on $\partial\Omega$. Most of the precursors of this result dealt with Dirichlet boundary conditions and self-adjoint second order elliptic operators. To overcome the new technical difficulties originated by the generality of the new setting, we have characterized the regularity of $\partial\Omega$ through the regularity of the associated *conormal projections* and *conormal distances*. This seems to be a new result of a huge relevance on its own. It actually complements some classical findings of Serrin, Gilbarg and Trudinger, Krantz and Parks, Foote, and Li and Nirenberg concerning the regularity of the inner distance function to the boundary.

Keywords: Singular Perturbation, Positive Solution, Generalized Logistic Equation, Non-classical Mixed Boundary Conditions, Conormal Vector Field, Conormal Projection, Conormal Distance, Regularity

MSC 2010: Primary 35B25, 35B09; secondary 35J25, 35Q92

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1 Introduction

The main goal of this paper is to analyze the limiting behavior as $d \downarrow 0$ of the positive solutions of

$$\begin{cases} d\mathcal{L}u = uh(u, x) & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, $d > 0$ is a positive constant, and the differential operator $\mathcal{L}$ is uniformly elliptic in Ω and has the form

$$\mathcal{L} = -\operatorname{div}(\mathrm{A}\nabla\,\cdot\,) + \mathrm{b}\nabla + \mathrm{c}, \tag{1.2}$$

with $A \in \mathcal{M}_N^{\mathrm{sym}}(\mathcal{C}^1(\bar{\Omega}))$, $b \in \mathcal{M}_{1\times N}(\mathcal{C}(\bar{\Omega}))$ and $c \in \mathcal{C}(\bar{\Omega})$. Given any Banach space X and two integers $n, m \geq 1$, $\mathcal{M}_{n\times m}(X)$ stands for the vector space of the matrices with n rows and m columns with entries in X. Naturally, we set $\mathcal{M}_n(X) := \mathcal{M}_{n\times n}(X)$, and $\mathcal{M}_n^{\mathrm{sym}}(X)$ denotes the subset of symmetric matrices.

Except in Section 2, $\partial\Omega$ is assumed to be an $(N-1)$-dimensional manifold of class $\mathcal{C}^2$ consisting of finitely many (connected) components

$$\Gamma_{\mathrm{D}}^j, \quad 1 \leq j \leq n_{\mathrm{D}}, \qquad \Gamma_{\mathrm{R}}^k, \quad 1 \leq k \leq n_{\mathrm{R}},$$

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for some integers $n_{\mathrm{D}}, n_{\mathrm{R}} \geq 1$, and we denote by

$$\Gamma_{\mathrm{D}} := \bigcup_{j=1}^{n_{\mathrm{D}}} \Gamma_{\mathrm{D}}^{j}, \quad \Gamma_{\mathrm{R}} := \bigcup_{j=1}^{n_{\mathrm{R}}} \Gamma_{\mathrm{R}}^{j},$$

the Dirichlet and Robin portions of $\partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{R}}$. It should be noted that either Γ_{D} or Γ_{R} might be empty. Associated with this decomposition of $\partial\Omega$, arises in a rather natural way the boundary operator $\mathcal{B}$ defined by

$$\mathcal{B}u = \begin{cases} \mathrm{D}u := u & \text{on } \Gamma_{\mathrm{D}}, \\ \mathrm{R}u := \dfrac{\partial u}{\partial \boldsymbol{\nu}} + \beta u & \text{on } \Gamma_{\mathrm{R}}, \end{cases} \qquad \text{for every } u \in W^{2,p}(\Omega),\ p > N,$$

where $\beta \in \mathcal{C}(\partial\Omega)$, $\mathbf{n}$ stands for the outward normal vector field along $\partial\Omega$, and $\boldsymbol{\nu} := A\mathbf{n}$ is the conormal vector field associated to $\mathcal{L}$.

As far as concerns the nonlinearity of (1.1), the function $h(u,x)$ is assumed to satisfy the following:

(H1) $h\colon \mathbb{R} \times \bar{\Omega} \to \mathbb{R}$ is of class $\mathcal{C}^1$ in $u \geq 0$ and continuous in $x \in \bar{\Omega}$.

(H2) $\partial_u h(u,x) < 0$ for all $u > 0$ and $x \in \bar{\Omega}$.

In addition, throughout this paper, we will impose that, for some $d > 0$,

(H3) there exists a positive constant $M > 0$ such that $h(M,x) < dc(x)$ for all $x \in \bar{\Omega}$.

In particular, we eventually can assume (H3) with $d = 0$, i.e., that

(H4) there exists a positive constant $M > 0$ such that $h(M,x) < 0$ for all $x \in \bar{\Omega}$.

Note that (H4) implies (H3) for sufficiently small $d > 0$, regardless the sign of $c(x)$. A prototypic example of admissible h, for which (1.1) becomes a generalized diffusive logistic equation, is given by

$$h(u,x) = \ell(x) - a(x)f(u), \quad u \in \mathbb{R},\ x \in \bar{\Omega},$$

where $\ell \in \mathcal{C}(\bar{\Omega})$ can change of sign, $a \in \mathcal{C}(\bar{\Omega})$ and $f \in \mathcal{C}^1(\mathbb{R})$ satisfy $\min_{\bar{\Omega}} a > 0$, $f(0) = 0$, $f'(u) > 0$ for all $u > 0$ and $\lim_{u\uparrow\infty} f(u) = +\infty$. For this choice, it is easily seen that (H1) and (H2) hold. As far as concerns (H3), note that, for every $d > 0$,

$$h(M,x) = \ell(x) - a(x)f(M) \leq \max_{\bar{\Omega}} \ell - f(M) \min_{\bar{\Omega}} a < d \min_{\bar{\Omega}} c, \quad x \in \bar{\Omega},$$

provided $M = M(d) > 0$ is sufficiently large, because $f(M) \uparrow \infty$ as $M \uparrow \infty$. Thus, (H3) holds for all $d > 0$. Moreover, by taking a sufficiently large $M > 0$ so that $f(M) > \max_{\bar{\Omega}} \ell / \min_{\bar{\Omega}} a$, it is clear that (H4) also holds.

Under the general conditions (H1), (H2) and (H4), it is easily seen that the maximal non-negative solution of the non-spatial equation $uh(u,x) = 0$,

$$\Theta_h(x) := \begin{cases} 0 & \text{if } h(\xi,x) < 0 \text{ for all } \xi > 0, \\ \xi & \text{if } \xi > 0 \text{ exists such that } h(\xi,x) = 0, \end{cases}$$

is continuous in $\bar{\Omega}$. Actually, for every $x \in \bar{\Omega}$, $\Theta_h(x)$ is the unique non-negative linearly stable, or linearly neutrally stable, steady state of the ordinary differential equation

$$u'(t) = u(t)h(u(t),x), \quad t \geq 0,$$

which is throughout referred as the *kinetic* model associated to (1.1).

According to the next theorem, which is the main existence result of this paper, for sufficiently small $d > 0$, (1.1) possesses, at most, one positive solution. Throughout this paper, for any given $V \in \mathcal{C}(\bar{\Omega})$, we will denote by $\sigma_1[d\mathcal{L} + V; \mathcal{B}, \Omega]$ the *principal eigenvalue* of the linear eigenvalue problem

$$\begin{cases} d\mathcal{L}\varphi + V(x)\varphi = \sigma\varphi & \text{in } \Omega, \\ \mathcal{B}\varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

i.e., its *lowest real eigenvalue*. According to [31, Theorem 7.7], it is algebraically simple and it provides us with the unique eigenvalue that is associated with a positive (principal) eigenfunction.

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Theorem 1.1. *Assume that $h(u,x)$ satisfies* (H1), (H2) *and* (H3) *for some $d > 0$. Then problem* (1.1) *has a positive solution $u \in \bigcap_{p>N} W^{2,p}(\Omega)$ if and only if*

$$\sigma_1[d\mathcal{L} - h(0,\cdot); \mathcal{B}, \Omega] < 0.$$

Moreover, it is unique if it exists.

Note that it complements [19, Lemma 3.4]. The main goal of this paper is to establish the next singular perturbation result, where $\theta_{\{d,h\}}$ stands for the maximal non-negative solution of (1.1).

Theorem 1.2. *Assume that h satisfies* (H1), (H2) *and* (H4), *and let Γ_{R}^+ denote the union of the components of Γ_{R}, where Θ_h is everywhere positive. Then, for any compact subset K of $\Omega \cup \Gamma_{\mathrm{R}}^+ \cup \Theta_h^{-1}(0)$,*

$$\lim_{d \downarrow 0} \theta_{\{d,h\}} = \Theta_h \quad \textit{uniformly in } K.$$

In other words, the maximal non-negative solution of (1.1) approximates Θ_h as $d \downarrow 0$ uniformly on compact subsets of $\Omega \cup \Gamma_{\mathrm{R}}^+ \cup \Theta_h^{-1}(0)$.

To the best of our knowledge, the most pioneering version of this result goes back to [5], where the singular perturbation problem

$$\begin{cases} -d\Delta u = u(1 - a(x)u^2) & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

with Ω and $a(x)$ of class $\mathcal{C}^\infty$ and $\min_{\bar{\Omega}} a > 0$, was analyzed in dimension $N \leq 3$. Precisely, in [5], Berger and Fraenkel established that for sufficiently small $d > 0$, problem (1.3) possesses a unique smooth positive solution $u_d(x)$, which converges to $1/\sqrt{a(x)}$ as $d \downarrow 0$, outside a boundary layer of width $O(\sqrt{d})$. Moreover, a global continuation of u_d in d was performed up to the critical value of the diffusion, where u_d bifurcates from $u = 0$. The main technical tool of [5] relies on the method of matched asymptotic expansions, applied to approximate the positive solution. The global existence of the positive solution was derived from some classical results in critical point theory. An abstract version of this singular perturbation result for autonomous equations was given by the same authors in [6]. Two years later, De Villiers [10] sharpened these findings up to cover a general class of $\mathcal{C}^\infty$ functions, $g(u,x)$, instead of $u - a(x)u^3$. Almost simultaneously, Fife [16], and Fife and Greenlee [17] extended these results to a general class of nonhomogeneous Dirichlet boundary value problems, including

$$\begin{cases} -d\,\mathrm{div}(A(x,d)\nabla u) = g(u,x,d) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

with Ω, $A(x,d)$ and $g(u,x,d)$ of class $\mathcal{C}^\infty$ and such that, for every $x \in \bar{\Omega}$, the equation $g(u,x,0) = 0$ has a solution $u_0(x)$, for which $\partial_u g(u_0(x),x,0) < 0$. This negativity entails the linearized stability of the equilibrium solution $u_0(x)$ of the associated kinetic model

$$u'(t) = g(u(t),x,0), \quad t \geq 0, \tag{1.5}$$

for all $x \in \bar{\Omega}$. Much like in [5], the singular perturbations results of [16, 17] are based on a bound for the inverse of the linearization about the formal solution constructed with the matched asymptotic expansion. Fife and Greenlee [17] also analyzed the more general case when $g(u,x,0) = 0$ possesses two $\mathcal{C}^\infty$-curves of solutions, $u_{0,1}(x)$ and $u_{0,2}(x)$, $x \in \bar{\Omega}$, which are linearly stable as steady-state solutions of (1.5) and separated away from each other.

Essentially, all these monographs adapted the former asymptotic expansion methods developed in the context of ODEs by the Russian School (e.g., see [7, 40]) to a PDE's framework. Naturally, working with ODEs many of the underlying technicalities can be easily overcome.

The first papers where some intrinsic techniques of the theory of PDEs, like the method of sub and supersolutions, were used to obtain singular perturbation results were those of Howes [22–24]. As a result, the previous restrictive regularity assumptions were relaxed. Precisely, Howes [23] considered a general class of problems, including (1.4) with $A = I$ and $g(u,x,d) = g(u,x)$ of class $\mathcal{C}^m$ for sufficiently large $m \geq 1$. Essentially, assuming that Ω is sufficiently smooth and that, for every $x \in \bar{\Omega}$, $g(u_0(x),x) = 0$ for some smooth $u_0(x)$

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which is linearly stable as an equilibrium of (1.5), Howes found some sufficient conditions for the existence of a classical solution u_d of (1.4) such that

$$\lim_{d \downarrow 0} u_d = u_0 \quad \text{uniformly on compact subsets of } \Omega.$$

Almost simultaneously, Howes [22] extended these results to cover the following very special class of Robin problems:

$$\begin{cases} -d\Delta u = g(u, x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{\nu}}(x) + \beta(x)u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\beta \geq 0$ on $\partial\Omega$ and $\beta \in \mathcal{C}^{2,\mu}(\partial\Omega)$ for some $\mu \in (0,1)$. As a consequence, e.g., of [22, Theorem 2.1], Howes could infer in [22, Example 2.2] that in the special case when $g(u) = u - u^3$, $u_{0,\pm} \equiv \pm 1$ are I_0-stable zeroes of $g(u,x) = 0$, because $g'(\pm 1) = -2 < 0$, and therefore (1.6) has two solutions $u_{d,\pm}(x)$, such that

$$\lim_{d \downarrow 0} u_{d,\pm}(x) = \pm 1 \quad \text{uniformly in } \bar{\Omega}.$$

In these papers, the regularity of the support domain Ω is imposed through the existence of a function $F \in \mathcal{C}^{2,\mu}(\mathbb{R}^N; \mathbb{R})$ such that $|\nabla F(x)| = 1$ for all $x \in \partial\Omega$ and

$$\Omega = \{x \in \mathbb{R}^N : F(x) < 0\}, \quad \partial\Omega = F^{-1}(0). \tag{1.7}$$

Incidentally, in the papers of Howes the problem of ascertaining whether, or not, a function F satisfying (1.7) exists, with the required regularity, remained open. Except for some pioneering results of Oleĭnik [35–37] for linear problems with transport terms, [22] seems to be the first paper dealing with the singular perturbation problem for a semilinear equation under Neumann or (classical) Robin boundary conditions with $\beta \geq 0$. The singular perturbation results of Howes for essential nonlinearities involving transport terms, like those of [22, Sections 3 and 4] and [24], remain outside the general scope of this paper.

Some time later, these pioneering findings were slightly, and occasionally substantially, improved by Angenent [4], De Santi [11], Clément and Sweers [9], and Kelley and Ko [26], among many others, who dealt with the singular perturbation problem under Dirichlet boundary conditions through some comparison techniques based on the synthesis of Amann [1, 2], Sattinger [38] and Matano [33].

As shown by the simplest examples of truly spatially heterogeneous semilinear elliptic equations in the context of population dynamics, the most serious shortcoming of the classical singular perturbation theory is caused by the fact that the curves, $u_{0,j}(x)$, $1 \leq j \leq q$, $q = 1, 2$, solving the equation $g(u,x) = 0$ must preserve their stability character for all $x \in \bar{\Omega}$, regarded as steady-state solutions of (1.5). For example, even in the simplest case situation when $g(u,x)$ inherits a logistic structure,

$$g(u,x) = \ell(x)u - a(x)u^2$$

for some functions $\ell, a \in \mathcal{C}(\bar{\Omega})$ such that $\ell(x)$ changes sign in Ω and $\min_{\bar{\Omega}} a > 0$, most of the assumptions imposed in the previous references fail to be true. Indeed, although $u_{0,1}(x) \equiv 0$ and $u_{0,2}(x) := \ell(x)/a(x)$, $x \in \bar{\Omega}$, might provide us with two smooth curves of $g(u,x) = 0$ for sufficiently smooth $\ell(x)$ and $a(x)$, it becomes apparent from $\partial_u g(u,x) = \ell(x) - 2a(x)u$ that

- $u_{0,1}(x) = 0$ is linearly stable, as a steady-state solution of (1.5) if and only if $\ell(x) < 0$,
- $u_{0,2}(x) = \ell(x)/a(x)$ is linearly stable if and only if $\ell(x) > 0$.

Therefore, the curves $u_{0,i}(x)$, $i = 1, 2$, cannot satisfy the requirements of the previous references, because they have a different stability character if $\ell(x) \neq 0$. Even considering the 'mixed interlaced branches' constructed from $u_{0,1}(x)$ and $u_{0,2}(x)$ through

$$\tilde{u}_{0,1}(x) := \max\{u_{0,1}(x), u_{0,2}(x)\}, \quad \tilde{u}_{0,2}(x) := \min\{u_{0,1}(x), u_{0,2}(x)\}, \quad x \in \bar{\Omega},$$

it is apparent that $\tilde{u}_{0,1}(x)$ is linearly stable if and only if $\ell(x) \neq 0$, and hence the classical theory cannot be applied neither, because the linearized stability fails at $\ell^{-1}(0)$ and, in general, these curves are far from

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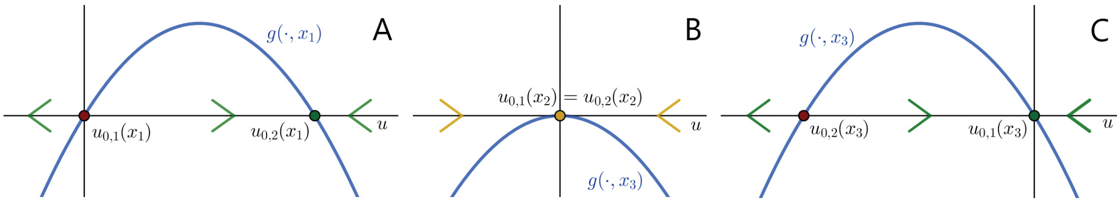

Figure 1: Plots of $u \mapsto g(u, x_i) := \ell(x_i)u - a(x_i)u^2$, $i \in \{1, 2, 3\}$, for a function $\ell \in \mathcal{C}(\bar{\Omega})$ that changes sign in Ω with $\ell(x_1) > 0$, $\ell(x_2) = 0$ and $\ell(x_3) < 0$. In the central case, (B), $u = 0$ must be a double zero of $g(\cdot, x_2)$. In each of these plots we have superimposed the 1-dimensional dynamics of (1.5) on the horizontal axis.

smooth. In these degenerate situations, not previously considered in the specialized literature, Furter and López-Gómez [20] established that the unique positive solution u_d of

$$\begin{cases} -d\Delta u = u(\ell(x) - a(x)u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies

$$\lim_{d \downarrow 0} u_d = \ell_+/a = \max\{0, \ell/a\} = \tilde{u}_{0,1} \quad \text{uniformly on compact subsets of } \Omega$$

(see [20, Theorem 3.5]), which suggests the validity of the next general principle in the context of (1.1):

> **Principle of Singular Perturbation (PSP).** If for every $x \in \bar{\Omega}$, the associated kinetic problem possesses a unique linearly stable, or linearly neutrally stable, non-negative steady-state solution $\Theta(x)$, which is somewhere positive in Ω, then, for sufficiently small $d > 0$, the associated parabolic problem possesses a unique positive steady-state solution θ_d. Moreover, $\lim_{d\downarrow 0} \theta_d = \Theta$ uniformly on any compact subset of Ω, where $\Theta(x)$ is continuous.

This is widely confirmed by Theorems 1.1 and 1.2. The same principle was already shown to hold under homogeneous Neumann boundary conditions by Hutson et al. [25, Lemma 2.4], as well as in the context of competitive systems (see [25, Theorem 4.1] and [13, Theorem 1], [15, Theorem 1.2], [14, Theorem 4.1] for some special cases when $\mathcal{L} = -\Delta$ or $b = 0$).

A problem of a different nature was studied by Nakashima, Ni and Su [34] for the special case when $\mathcal{L} = -\Delta$ and $g(u, x) = a(x)f(u)$, for the appropriate choices of the functions $a(x)$ and $f(u)$, under Neumann boundary conditions. In such case, the steady-state solutions of (1.5) are spatially homogeneous, though their linearized stabilities, regarded as equilibria of (1.5), vary with the location of x in $\bar{\Omega}$ according to the sign of $a(x)$. In spite of these differences, it turns out that this model also satisfies the *Principle of Singular Perturbation* formulated above (see [34, Theorem 1.3]).

Our Theorem 1.2 provides us with an extremely general version of all previous existing singular perturbation results for Kolmogorov nonlinearities of the form $g(u, x) = uh(u, x)$, where $h(u, x)$ satisfies (H1), (H2) and (H4). Actually, it is the first general result available for second order uniformly elliptic operators, $\mathcal{L}$, under general mixed boundary conditions of non-classical type. As the general linear existence theory developed in [31, Section 4.6] is only available for operators of the form (1.2), in this paper the principal part of $\mathcal{L}$ is required to be in divergence form. Nevertheless, even imposing this restriction, Theorem 1.2 is substantially sharper than most of the previous singular perturbation results for the generalized logistic equation.

The proof of Theorem 1.2 is based on the method of sub and supersolutions, which relies on the theorem of characterization of the Strong Maximum Principle of López-Gómez and Molina-Meyer [30, 32], and Amann and López-Gómez [3]. A comparison argument provides us with a global uniform supersolution of (1.1) on $\bar{\Omega}$, while the construction of the appropriate local subsolutions, combined with a compactness argument, provides us with the necessary lower estimates to get Theorem 1.2. The main technical difficulties that we must overcome in the proof of Theorem 1.2 come from the following facts:

(I) The principal eigenfunctions associated to $\mathcal{L}$ in interior balls do not enjoy the nice symmetry properties of the principal eigenfunctions of $-\Delta$, which take the maximum on the center of these balls. This dif-

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ficulty is overcome through a technical device introduced in [29], which facilitates the construction of local subsolutions in the general non-autonomous case.

(II) A more subtle difficulty relies on the construction of a global supersolution of (1.1) sufficiently close to Θ_h, which is far from obvious when dealing with general mixed boundary conditions. As no previous singular perturbation result is available under mixed boundary conditions, these difficulties have been overcomed for the first time here.

(III) In our general setting, the coefficient function $\beta(x)$ can change sign. Thus, we must perform a preliminary change of variables for transforming (1.1) into an equivalent problem of the same nature with $\beta \geq 0$.

The resolution of the technical difficulties sketched in (II) and (III) relies on the next theorem, which might be of independent interest in differential geometry.

Theorem 1.3. *Assume that Ω is an open subdomain of $\mathbb{R}^N$ such that $\partial\Omega$ is a topological $(N-1)$-manifold. Then, for every integer $r \geq 2$, the next assertions are equivalent:*

(a) *$\partial\Omega$ is of class $\mathcal{C}^r$.*

(b) *$\partial\Omega$ admits an outward vector field $\boldsymbol{v} \in \mathcal{C}^{r-1}(\partial\Omega; \mathbb{R}^N)$ and, for each of these vector fields, there exists an open subset $\mathcal{U}$ of $\mathbb{R}^N$, with $\partial\Omega \subset \mathcal{U}$, and a function $\Pi_{\boldsymbol{v}} \in \mathcal{C}^{r-1}(\mathcal{U}; \partial\Omega)$ such that*

 (i) *$\Pi_{\boldsymbol{v}}(x) = x$ for all $x \in \partial\Omega$,*

 (ii) *$\Pi_{\boldsymbol{v}}(x - \lambda\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))) = \Pi_{\boldsymbol{v}}(x)$ for every $x \in \mathcal{U}$ and $\lambda \in \mathbb{R}$ such that $x - \lambda\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x)) \in \mathcal{U}$. Thus,*

$$\frac{\partial \Pi_{\boldsymbol{v}}}{\partial \boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))}(x) = 0 \quad \text{for all } x \in \mathcal{U}.$$

Moreover, the function $\mathfrak{d}_{\boldsymbol{v}} : \mathcal{U} \to \mathbb{R}$ defined by

$$\mathfrak{d}_{\boldsymbol{v}}(x) := \begin{cases} |x - \Pi_{\boldsymbol{v}}(x)|/|\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))| & \text{if } x \in \mathcal{U} \cap \Omega, \\ -|x - \Pi_{\boldsymbol{v}}(x)|/|\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))| & \text{if } x \in \mathcal{U} \setminus \Omega, \end{cases} \tag{1.8}$$

is of class $\mathcal{C}^r$.

(c) *Property* (b) *holds for some outward vector field $\boldsymbol{v} \in \mathcal{C}^{r-1}(\partial\Omega; \mathbb{R}^N)$.*

(d) *$\partial\Omega$ admits an outward vector field $\boldsymbol{v} \in \mathcal{C}^{r-1}(\partial\Omega; \mathbb{R}^N)$ and, for each of these vector fields, there exists an open subset $\mathcal{U}$ of $\mathbb{R}^N$, with $\partial\Omega \subset \mathcal{U}$, and a function $\psi \in \mathcal{C}^r(\mathcal{U}; \mathbb{R})$ such that $\psi(x) < 0$ for all $x \in \Omega \cap \mathcal{U}$, $\psi(x) > 0$ for all $x \in \mathcal{U} \setminus \bar{\Omega}$ and*

$$\min_{x \in \partial\Omega} \frac{\partial \psi}{\partial \boldsymbol{v}}(x) > 0.$$

(e) *Property* (d) *holds for some outward vector field $\boldsymbol{v} \in \mathcal{C}^{r-1}(\partial\Omega; \mathbb{R}^N)$.*

(f) *There exist an open subset $\mathcal{U}$ of $\mathbb{R}^N$, with $\partial\Omega \subset \mathcal{U}$, and a function $\Psi \in \mathcal{C}^r(\mathcal{U}; \mathbb{R})$ such that $\Omega = \{x \in \mathcal{U} : \Psi(x) < 0\}$, $\partial\Omega = \Psi^{-1}(0)$, and $|\nabla\Psi(x)| = 1$ for all $x \in \partial\Omega$.*

A vector field $\boldsymbol{v}$ on $\partial\Omega$ is said to be an *outward vector field* if there exists $\varepsilon_0 > 0$ such that

$$x + \varepsilon\boldsymbol{v}(x) \in \mathbb{R}^N \setminus \bar{\Omega} \quad \text{and} \quad x - \varepsilon\boldsymbol{v}(x) \in \Omega$$

for all $x \in \partial\Omega$ and $0 < \varepsilon < \varepsilon_0$. The function $\Pi_{\boldsymbol{v}}$, whose existence is established by part (b), will be throughout called the *projection onto the boundary along the vector field* $\boldsymbol{v}$, or simply *coprojection* when $\boldsymbol{v}$ is the conormal vector field. Naturally, the *distance to the boundary along* $\boldsymbol{v}$, or *conormal distance*, is defined through

$$\operatorname{dist}_{\boldsymbol{v}}(x, \partial\Omega) := \frac{|x - \Pi_{\boldsymbol{v}}(x)|}{|\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))|}, \quad x \in \mathcal{U},$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^N$. According to Theorem 1.3, $\partial\Omega$ is of class $\mathcal{C}^r$ if and only if for some outward vector field $\boldsymbol{v} \in \mathcal{C}^{r-1}$, the function $\operatorname{dist}_{\boldsymbol{v}}$ is of class $\mathcal{C}^r$ in $\mathcal{U} \setminus \partial\Omega$. In part (d), the function ψ is given essentially by $-\operatorname{dist}_{\boldsymbol{v}}$ in $\mathcal{U} \cap \Omega$. Note that, by the continuity of ψ on $\mathcal{U}$, $\psi(x) = 0$ for all $x \in \partial\Omega$.

According to [31, Lemma 2.1], using a partition of the unity of class $\mathcal{C}^r$, or a cut-off function, the function $\psi(x)$ in part (d), as well as Ψ in part (f), can be assumed to be globally defined in a neighborhood of $\bar{\Omega}$, or even in $\mathbb{R}^N$, and in such case $\psi(x) < 0$ (resp. $\Psi(x) < 0$) for all $x \in \Omega$ and $\psi(x) > 0$ (resp. $\Psi(x) > 0$) for all $x \in \mathbb{R}^N \setminus \bar{\Omega}$.

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Note that (f) is the condition used in some of the classical papers discussed above, with $F := \Psi$. It is astonishing that, in spite of the equivalence between (a) and (f), yet the existence of Ψ of class $\mathcal{C}^r$ satisfying (f) is far from adopted in the specialized literature as the most natural, and simple, definition for a bounded domain of class $\mathcal{C}^r$. Indeed, the usual definition in the most paradigmatic textbooks, like [21] or [12], involves local charts at any point of the boundary, instead of the minimal requirements of (e). Theorem 1.3 might help to clarify all these – always very delicate – regularity issues, though, as pointed out to the authors by the reviewer: "It is legitimate to ask why a smooth domain is not defined through a smooth embedding. But it seems to me that to define a differential structure on a manifold you need the concept of local chart and atlas. So you cannot escape from the definition with local charts".

Nevertheless, to the best of our knowledge, the existence of the *conormal projection* and the *conormal distance* constructed in Theorem 1.3, as well as the proof of the fact that they inherit the regularity of $\partial\Omega$, seem completely new findings. Astonishingly, the Math. Sci. Net. of the Amer. Math. Soc. was unable to capture any entry with the words *conormal distance*, or *conormal projection*, though a huge list was given with *conormal*. Thus, Theorem 1.3 might be introducing these concepts into the debate of the characterization of the regularity of $\partial\Omega$ in terms of the regularity of the associated distance function. Note that $\mathcal{C}^2$ is the minimal regularity of $\partial\Omega$, required to guarantee that the distance function through the 'nearest point' is well defined (see [27, Example 4]).

Actually, although Gilbarg and Trudinger [21, Lemma 14.16] show that the distance function to the boundary, $\operatorname{dist}(x, \partial\Omega)$, is of class $\mathcal{C}^r$, $r \geq 2$, if $\partial\Omega$ is of class $\mathcal{C}^r$, and this result was later sharpened up to cover the case $r = 1$ by Krantz and Parks [27], even the problem of establishing the regularity of $\partial\Omega$ from the regularity of $\operatorname{dist}(x, \partial\Omega)$ remains open. These results actually sharpened a pioneering finding of Serrin [39], which established the $\mathcal{C}^{r-1}$-regularity of $\operatorname{dist}(x, \partial\Omega)$ from the $\mathcal{C}^r$-regularity of $\partial\Omega$. Some time later, Foote [18] generalized some of the results of [27] by establishing that, for every compact submanifold M of class $\mathcal{C}^k$, $k \geq 2$, there exists a neighborhood U such that the distance function $d(x, M)$ is $\mathcal{C}^k$ in $U \setminus M$. Under these assumptions, the fact that M has a neighborhood U with the unique nearest point property, as well as the fact that the projection map $\Pi\colon U \to M$ is $\mathcal{C}^{k-1}$, relies on the *tubular neighborhood theorem* with the added observation that Π factors through the map that creates the neighborhood. More recently, almost twenty years later, Li and Nirenberg [28] established that if Ω is a domain in a smooth complete Finsler manifold, and G stands for the largest open subset of Ω with the nearest point property in the Finsler metric, then the distance function from $\partial\Omega$ is in $\mathcal{C}^{k,a}_{\mathrm{loc}}(G \cup \partial\Omega)$, $k \geq 2$ and $0 < a \leq 1$, if $\partial\Omega$ is of class $\mathcal{C}^{k,a}$. But no converse result, within the vain of the characterization provided by Theorem 1.3, seems to be available in the literature.

This paper is distributed as follows. Section 2 proves Theorem 1.3, Section 3 uses Theorem 1.3 to reduce the general case when β changes sign to the classical case when $\beta \geq 0$. This simplifies substantially the underlying analysis and, in particular, the proof of Theorem 1.2. Section 4 establishes some important monotonicity properties of the associated principal eigenvalues with respect to the domain and the potential, Section 5 proves Theorem 1.1 and derives from it some important monotonicity properties, and Section 6 delivers the proof of Theorem 1.2.

2 Proof of Theorem 1.3

It suffices to prove the following implications: (a) implies (b), (b) implies (c), (d) and (f), (c), or (d), or (f), implies (e), and (e) implies (a). First, we will prove that (a) implies (b). Note that the normal vector field is of class $\mathcal{C}^{r-1}$ as soon as $\partial\Omega$ is of class $\mathcal{C}^r$. Now, consider a field $\boldsymbol{\nu}$ satisfying the requirements of part (b). For each $\varepsilon > 0$, let us denote by $Q_{\boldsymbol{\nu}} \in \mathcal{C}^{r-1}((-\varepsilon, \varepsilon) \times \partial\Omega; \mathbb{R}^N)$ the function defined by

$$Q_{\boldsymbol{\nu}}\colon (-\varepsilon, \varepsilon) \times \partial\Omega \to \mathcal{U}_\varepsilon := \operatorname{Im} Q_{\boldsymbol{\nu}} \subset \mathbb{R}^N, \quad (s, x) \mapsto x - s\boldsymbol{\nu}(x),$$

which establishes a bijection over its image for sufficiently small $\varepsilon > 0$. Moreover, shortening $\varepsilon > 0$, if necessary, $Q_{\boldsymbol{\nu}}^{-1}$ also is of class $\mathcal{C}^{r-1}$-regularity. Indeed, the proof of the injectivity proceeds by contradiction.

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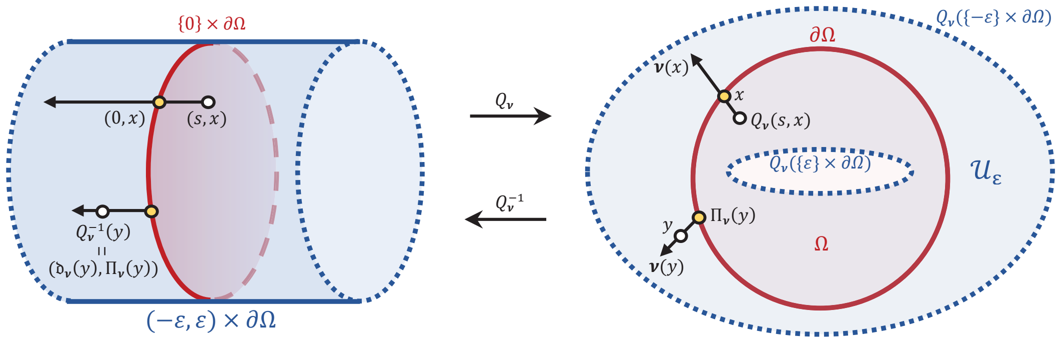

Figure 2: Scheme for the realization of Q_ν and Q_ν^{-1} and their relationships with the projection Π_ν and the distance function $\mathfrak{d}_\nu$.

Suppose that Q_ν is not injective for sufficiently small $\varepsilon > 0$. Then there exist $\{s_n^1\}_{n\geq 1}, \{s_n^2\}_{n\geq 1} \subset \mathbb{R}$, with $s_n^1 \to 0$ and $s_n^2 \to 0$ as $n \uparrow \infty$, and $\{x_n^1\}_{n\geq 1}, \{x_n^2\}_{n\geq 1} \subset \partial\Omega$, such that

$$(s_n^1, x_n^1) \neq (s_n^2, x_n^2) \quad \text{and} \quad Q_\nu(s_n^1, x_n^1) = Q_\nu(s_n^2, x_n^2) \quad \text{for all } n \geq 1.$$

In other words,

$$x_n^1 - s_n^1 \nu(x_n^1) = x_n^2 - s_n^2 \nu(x_n^2) \quad \text{for all } n \geq 1. \tag{2.1}$$

Moreover, without lost of generality, we can assume that $s_n^1 s_n^2 > 0$ for all $n \geq 1$. Otherwise, since ν is an outward vector field, for sufficiently large $n \geq 1$, we have that

$$x_n^1 - s_n^1 \nu(x_n^1) = x_n^2 - s_n^2 \nu(x_n^2)$$

should lie, simultaneously, in Ω and in $\mathbb{R}^N \setminus \bar{\Omega}$, which is impossible.

Suppose $x_n^1 = x_n^2$ for some $n \geq 1$. Then, since $\nu(x_n^1) \neq 0$, (2.1) implies that $s_n^1 = s_n^2$, which cannot hold. Hence, $x_n^1 \neq x_n^2$ for all $n \geq 1$. Since $\partial\Omega$ is compact, along some subsequences of $\{x_n^1\}$ and $\{x_n^2\}$, relabeled by n, we have that

$$\lim_{n\to\infty} x_n^j = x_\infty^j, \quad j = 1, 2,$$

for some $x_\infty^1, x_\infty^2 \in \partial\Omega$. Subsequently, we are renaming by $\{s_n^1\}_{n\geq 1}$, $\{s_n^2\}_{n\geq 1}$, $\{x_n^1\}_{n\geq 1}$ and $\{x_n^2\}_{n\geq 1}$ the new subsequences. Letting $n \uparrow \infty$ in (2.1) yields $x_\infty^1 = x_\infty^2 =: x_\infty$. Now, for each $j = 1, 2$, we consider the sequence $\{\varsigma_n^j\}_{n\geq 1}$ defined through

$$\varsigma_n^j := s_n^i |\nu(x_n^j)|, \quad n \geq 1.$$

Then, by the continuity of ν, the new sequences still satisfy

$$\lim_{n\to\infty} \varsigma_n^1 = \lim_{n\to\infty} \varsigma_n^2 = 0, \tag{2.2}$$

and, setting $\xi := \nu/|\nu|$ for the unitary outward vector field, (2.1) can be equivalently expressed as

$$x_n^1 - x_n^2 = \varsigma_n^1 \xi(x_n^1) - \varsigma_n^2 \xi(x_n^2) = (\varsigma_n^1 - \varsigma_n^2)\xi(x_n^1) + \varsigma_n^2(\xi(x_n^1) - \xi(x_n^2)) \tag{2.3}$$

for all $n \geq 1$. On the other hand, since $x_n^1 \neq x_n^2$, we have that

$$\frac{x_n^1 - x_n^2}{|x_n^1 - x_n^2|} \in \mathbb{S}^{N-1} \subset \mathbb{R}^N \quad \text{for all } n \geq 1,$$

where $\mathbb{S}^{N-1}$ stands for the $(N-1)$-dimensional sphere. As the sphere is compact, we can extract subsequences of $\{\varsigma_n^1\}_{n\geq 1}$, $\{\varsigma_n^2\}_{n\geq 1}$, $\{x_n^1\}_{n\geq 1}$ and $\{x_n^2\}_{n\geq 1}$, again labeled by n, such that

$$\tau_\infty := \lim_{n\to\infty} \frac{x_n^1 - x_n^2}{|x_n^1 - x_n^2|} \in T_{x_\infty}\partial\Omega,$$

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where $T_{x_\infty}\partial\Omega$ stands for the tangent hyperplane of $\partial\Omega$ at x_∞. Note that $|\tau_\infty| = 1$. Moreover, by construction, we have that

$$|\varsigma_n^j| = |x_n^j - Q_{\mathbf{v}}(s_n^j, x_n^j)|, \quad Q_{\mathbf{v}}(s_n^1, x_n^1) = Q_{\mathbf{v}}(s_n^2, x_n^2), \quad j = 1, 2, \; n \geq 1.$$

Thus, since $s_n^1 s_n^2 > 0$ for all $n \geq 1$, the triangular inequality yields

$$|\varsigma_n^1 - \varsigma_n^2| = \big||\varsigma_n^1| - |\varsigma_n^2|\big| = \big||x_n^1 - Q_{\mathbf{v}}(s_n^1, x_n^1)| - |x_n^2 - Q_{\mathbf{v}}(s_n^2, x_n^2)|\big| \leq |x_n^1 - x_n^2|$$

for all $n \geq 1$. Consequently, by the Bolzano–Weierstrass theorem, there exist $\eta \in [-1, 1]$ and subsequences of $\{\varsigma_n^1\}_{n\geq1}$, $\{\varsigma_n^2\}_{n\geq1}$, $\{x_n^1\}_{n\geq1}$ and $\{x_n^2\}_{n\geq1}$, relabeled by n, such that

$$\lim_{n\to\infty} \frac{\varsigma_n^1 - \varsigma_n^2}{|x_n^1 - x_n^2|} = \eta. \tag{2.4}$$

Now we will show that, as a consequence of the regularity of ξ, the limit

$$\lim_{n\to\infty} \frac{\xi(x_n^1) - \xi(x_n^2)}{|x_n^1 - x_n^2|}$$

is well defined in $\mathbb{R}^N$. Indeed, since $\partial\Omega$ is a $\mathcal{C}^r$-manifold, there exist $\delta > 0$ and a local chart of $\partial\Omega$ on a neighborhood of x_∞, $\Phi \in \mathcal{C}^r(B_\delta(0); \mathbb{R}^N)$ with $\Phi(0) = x_\infty$. Subsequently, we set $y_n^j := \Phi^{-1}(x_n^j)$ for $j = 1, 2$ and sufficiently large $n \geq 1$. By the continuity of Φ^{-1},

$$\lim_{n\to\infty} y_n^j = 0, \quad j = 1, 2.$$

Since $x_n^1 \neq x_n^2$ and Φ is a local diffeomorphism, $y_n^1 \neq y_n^2$ and hence

$$\frac{y_n^1 - y_n^2}{|y_n^1 - y_n^2|} \in \mathbb{S}^{N-2}, \quad n \geq n_0.$$

Thus, by compactness, we can extract subsequences, relabeled by n, such that

$$\tilde{\tau}_\infty := \lim_{n\to\infty} \frac{y_n^1 - y_n^2}{|y_n^1 - y_n^2|} \in \mathbb{S}^{N-2}. \tag{2.5}$$

Then, for every $\varphi \in \mathcal{C}^1(B_\delta(0); \mathbb{R}^N)$, we have that

$$\lim_{n\to\infty} \frac{\varphi(y_n^1) - \varphi(y_n^2)}{|y_n^1 - y_n^2|} = \mathcal{D}\varphi(0)\tilde{\tau}_\infty = \frac{\partial\varphi}{\partial\tilde{\tau}_\infty}(0).$$

Indeed,

$$\begin{aligned}
\left|\frac{\varphi(y_n^1) - \varphi(y_n^2)}{|y_n^1 - y_n^2|} - \mathcal{D}\varphi(0)\tilde{\tau}_\infty\right| &= \left|\frac{\varphi(y_n^2 + (y_n^1 - y_n^2)) - \varphi(y_n^2)}{|y_n^1 - y_n^2|} - \mathcal{D}\varphi(0)\tilde{\tau}_\infty\right| \\
&= \left|\frac{1}{|y_n^1 - y_n^2|}\int_0^1 \mathcal{D}\varphi(y_n^2 + t(y_n^1 - y_n^2))(y_n^1 - y_n^2)\,dt - \int_0^1 \mathcal{D}\varphi(0)\tilde{\tau}_\infty\,dt\right| \\
&= \left|\int_0^1 \left(\mathcal{D}\varphi(y_n^2 + t(y_n^1 - y_n^2))\frac{y_n^1 - y_n^2}{|y_n^1 - y_n^2|} - \mathcal{D}\varphi(0)\tilde{\tau}_\infty\right)dt\right| \\
&\leq \int_0^1 \left|\mathcal{D}\varphi(y_n^2 + t(y_n^1 - y_n^2))\left(\frac{y_n^1 - y_n^2}{|y_n^1 - y_n^2|} - \tilde{\tau}_\infty\right)\right|dt \\
&\quad + \int_0^1 \left|(\mathcal{D}\varphi(y_n^2 + t(y_n^1 - y_n^2)) - \mathcal{D}\varphi(0))\tilde{\tau}_\infty\right|dt,
\end{aligned}$$

which, thanks to (2.5) and the uniform continuity of $\mathcal{D}\varphi$ in $B_{\delta/2}(0)$, converges to 0 as $n \uparrow \infty$. Hence, by the

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regularity of $\boldsymbol{\nu}$, and so of ξ, we have that

$$\begin{aligned}\lim_{n\to\infty}\frac{\xi(x_n^1)-\xi(x_n^2)}{|x_n^1-x_n^2|}&=\lim_{n\to\infty}\frac{(\xi\circ\Phi)(y_n^1)-(\xi\circ\Phi)(y_n^2)}{|\Phi(y_n^1)-\Phi(y_n^2)|}\\&=\lim_{n\to\infty}\frac{|y_n^1-y_n^2|}{|\Phi(y_n^1)-\Phi(y_n^2)|}\frac{(\xi\circ\Phi)(y_n^1)-(\xi\circ\Phi)(y_n^2)}{|y_n^1-y_n^2|}\\&=\frac{1}{|\mathcal{D}\Phi(0)\tilde{\tau}_\infty|}\mathcal{D}(\xi\circ\Phi)(0)\tilde{\tau}_\infty\in\mathbb{R}^N.\end{aligned}\tag{2.6}$$

Therefore, thanks to (2.2), (2.4) and (2.6), dividing by $|x_n^1-x_n^2|$ in (2.3) and letting $n\uparrow+\infty$ yields

$$\tau_\infty=\eta\xi(x_\infty)=\eta\frac{\boldsymbol{\nu}(x_\infty)}{|\boldsymbol{\nu}(x_\infty)|}.$$

Since $\tau_\infty\in\mathbb{S}^{N-1}$, taking norms in both sides provides us with $|\eta|=1$. However, since $\tau_\infty\in T_{x_\infty}\partial\Omega$ and ξ is an outward unit vector field along $\partial\Omega$, we have that

$$\langle\tau_\infty,\mathbf{n}(x_\infty)\rangle=0\quad\text{and}\quad\langle\xi(x_\infty),\mathbf{n}(x_\infty)\rangle>0,$$

respectively, which implies $\eta=0$, driving to a contradiction. Thus, there exists $\varepsilon>0$ such that $Q_{\boldsymbol{\nu}}:(-\varepsilon,\varepsilon)\to\mathcal{U}_\varepsilon$ is bijective. Note that $Q_{\boldsymbol{\nu}}$ inherits the regularity of $\boldsymbol{\nu}$. So, it is of class $\mathcal{C}^{r-1}((-\varepsilon,\varepsilon)\times\partial\Omega;\mathcal{U}_\varepsilon)$, and $Q_{\boldsymbol{\nu}}(0,x)=x$ for all $x\in\partial\Omega$.

It remains to show the regularity of $Q_{\boldsymbol{\nu}}^{-1}:\mathcal{U}_\varepsilon\to(-\varepsilon,\varepsilon)\times\partial\Omega$ for sufficiently small $\varepsilon>0$. This is a consequence of the inverse function theorem. By continuity and compactness, it suffices to establish that $\mathcal{D}Q_{\boldsymbol{\nu}}$ is non-degenerate on $\{0\}\times\partial\Omega$. Indeed, since $\partial\Omega$ is a class $\mathcal{C}^r$ manifold, for each $x\in\partial\Omega$, there exist $\delta_x>0$ and a homeomorphism onto its image $\Phi_x\in\mathcal{C}^r(B_{\delta_x}(0)\subset\mathbb{R}^{N-1};\mathbb{R}^N)$, with $\Phi_x(0)=x$ and $\Phi_x(B_{\delta_x}(0))\subset\partial\Omega$. Actually, Φ_x parameterizes $\partial\Omega$ in a neighborhood of x. Consider the function $\tilde{Q}_{\boldsymbol{\nu}}:(-\varepsilon,\varepsilon)\times B_{\delta_x}(0)\to\mathcal{U}_\varepsilon$ defined by

$$\tilde{Q}_{\boldsymbol{\nu}}(s,y):=Q_{\boldsymbol{\nu}}(s,\Phi_x(y))=\Phi_x(y)-s\boldsymbol{\nu}(\Phi_x(y)).$$

Then, for every $s\in(-\varepsilon,\varepsilon)$ and $y\in B_{\delta_x}(0)$, $\mathcal{D}Q_{\boldsymbol{\nu}}(s,\Phi_x(y))$ is represented by

$$\mathcal{D}\tilde{Q}_{\boldsymbol{\nu}}(s,y)=[-\boldsymbol{\nu}(\Phi_x(y)),\ \mathcal{D}\Phi_x(y)-s\mathcal{D}(\boldsymbol{\nu}\circ\Phi_x)(y)].$$

In particular,

$$\mathcal{D}\tilde{Q}_{\boldsymbol{\nu}}(0,y)=[-\boldsymbol{\nu}(\Phi_x(y)),\ \mathcal{D}\Phi_x(y)].$$

Since Φ_x is a local chart of a $\mathcal{C}^r$ $(N-1)$-dimensional manifold, rank $\mathcal{D}\Phi_x(y)=N-1$ for all $y\in B_{\delta_x}(0)$, and hence it generates the tangent space at $\Phi_x(y)$. Thus, since $\boldsymbol{\nu}(\Phi_x(y))$ is a non-tangential vector field, it becomes apparent that

$$\operatorname{rank}\mathcal{D}\tilde{Q}_{\boldsymbol{\nu}}(0,y)=N.$$

Consequently, $\mathcal{D}\tilde{Q}_{\boldsymbol{\nu}}(0,y)$ is an isomorphism. Therefore, $Q_{\boldsymbol{\nu}}$ establishes a $\mathcal{C}^{r-1}$-diffeomorphism onto its image for sufficiently small $\varepsilon>0$. In order to complete the proof of (a) implies (b), it remains to construct the projection $\Pi_{\boldsymbol{\nu}}$ and show that the function $\mathfrak{d}_{\boldsymbol{\nu}}$ defined in (1.8) is of class $\mathcal{C}^r$. Let $P_1:\mathbb{R}\times\partial\Omega\to\mathbb{R}$ and $P_2:\mathbb{R}\times\partial\Omega\to\partial\Omega$ denote the projections on the first and the second component, respectively, i.e.,

$$\begin{aligned}P_1&:\mathbb{R}\times\partial\Omega\to\mathbb{R},&&(s,x)\mapsto s,\\P_2&:\mathbb{R}\times\partial\Omega\to\partial\Omega,&&(s,x)\mapsto x.\end{aligned}$$

Obviously, P_1 and P_2 are of class $\mathcal{C}^\infty$ and, by construction, it is easily seen that the map

$$\Pi_{\boldsymbol{\nu}}:=P_2\circ Q_{\boldsymbol{\nu}}^{-1}:\mathcal{U}_\varepsilon\to\partial\Omega\subset\mathbb{R}^N$$

satisfies all the requirements of part (b). Indeed, $\Pi_{\boldsymbol{\nu}}$ also is of class $\mathcal{C}^{r-1}$, as $Q_{\boldsymbol{\nu}}^{-1}$ and P_2. Moreover, for every $x\in\partial\Omega$, we have that

$$\Pi_{\boldsymbol{\nu}}(x)=P_2\circ Q_{\boldsymbol{\nu}}^{-1}(x)=P_2(0,x)=x.$$

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Since $Q_{\boldsymbol{v}}$ is a diffeomorphism, for every $x \in \mathcal{U}_\varepsilon$, there exists $s \in (-\varepsilon, \varepsilon)$ such that

$$x = Q_{\boldsymbol{v}}(s, \Pi_{\boldsymbol{v}}(x)) = \Pi_{\boldsymbol{v}}(x) - s\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x)).$$

Hence, if $\lambda \in \mathbb{R}$ satisfies $x - \lambda\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x)) \in \mathcal{U}_\varepsilon$, we find that

$$\Pi_{\boldsymbol{v}}(x - \lambda\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))) = \Pi_{\boldsymbol{v}}\big(\Pi_{\boldsymbol{v}}(x) - s\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x)) - \lambda\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))\big) = P_2 \circ Q_{\boldsymbol{v}}^{-1}(Q_{\boldsymbol{v}}(s + \lambda, \Pi_{\boldsymbol{v}}(x))) = \Pi_{\boldsymbol{v}}(x).$$

In particular, this entails that $\frac{\partial \Pi_{\boldsymbol{v}}}{\partial \boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))}(x) = 0$ for all $x \in \mathcal{U}_\varepsilon$. By the definition of $Q_{\boldsymbol{v}}$, $\eth_{\boldsymbol{v}} = P_1 \circ Q_{\boldsymbol{v}}^{-1}$, and so it is of class $\mathcal{C}^{r-1}(\mathcal{U}_\varepsilon)$. Moreover, for every $x \in \mathcal{U}_\varepsilon$,

$$x = \Pi_{\boldsymbol{v}}(x) - \eth_{\boldsymbol{v}}(x)\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x)).$$

Thus,

$$\eth_{\boldsymbol{v}}(x) = \frac{1}{|\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))|^2}\big\langle \Pi_{\boldsymbol{v}}(x) - x, \boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))\big\rangle,$$

and hence, combining the Leibniz rule with the properties of the projection $\Pi_{\boldsymbol{v}}$, we find that, for every $x \in \mathcal{U}_\varepsilon$,

$$\begin{aligned}
\mathcal{D}\eth_{\boldsymbol{v}}(x) &= -\frac{2\langle \boldsymbol{v}(\Pi_{\boldsymbol{v}}(x)), \mathcal{D}(\boldsymbol{v}\circ\Pi_{\boldsymbol{v}})(x)\rangle}{|\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))|^4}\langle \Pi_{\boldsymbol{v}}(x) - x, \boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))\rangle \\
&\quad + \frac{1}{|\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))|^2}\big(\langle \mathcal{D}\Pi_{\boldsymbol{v}}(x) - \mathrm{Id}, \boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))\rangle + \langle \Pi_{\boldsymbol{v}}(x) - x, \mathcal{D}(\boldsymbol{v}\circ\Pi_{\boldsymbol{v}})(x)\rangle\big) \\
&= \frac{1}{|\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))|^2}\big(\langle \mathcal{D}\Pi_{\boldsymbol{v}}(x), \boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))\rangle - \eth_{\boldsymbol{v}}(x)\langle \boldsymbol{v}(\Pi_{\boldsymbol{v}}(x)), \mathcal{D}(\boldsymbol{v}\circ\Pi_{\boldsymbol{v}})(x)\rangle - \langle \mathrm{Id}, \boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))\rangle\big) \\
&= \frac{1}{|\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))|^2}\bigg(\frac{\partial \Pi_{\boldsymbol{v}}}{\partial \boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))}(x) - \eth_{\boldsymbol{v}}(x)\frac{\partial(\boldsymbol{v}\circ\Pi_{\boldsymbol{v}})}{\partial \boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))}(x) - \boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))\bigg) \\
&= -\frac{\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))}{|\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))|^2},
\end{aligned}$$

because $\Pi_{\boldsymbol{v}}$ and $\boldsymbol{v}\circ\Pi_{\boldsymbol{v}}$ are constant along each direction $\boldsymbol{v}(\Pi_{\boldsymbol{v}}(x))$. Therefore, $\mathcal{D}\eth_{\boldsymbol{v}} \in \mathcal{C}^{r-1}$, which entails $\eth_{\boldsymbol{v}} \in \mathcal{C}^r$ and ends the proof of (a) implies (b).

The fact that part (b) implies part (c) is immediate. Next, we will prove that (b) implies (d) and (f). Suppose (b) and consider any outward vector field $\boldsymbol{v} \in \mathcal{C}^{r-1}$. Then $\tilde{\boldsymbol{v}} := \boldsymbol{v}/|\boldsymbol{v}| \in \mathcal{C}^{r-1}$. Let $\mathcal{U}$, $\Pi_{\tilde{\boldsymbol{v}}}$ and $\eth_{\tilde{\boldsymbol{v}}}$ denote, respectively, the open set, the projection and the 'regularized distance' (1.8) provided by part (b). Then the function $\psi_{\boldsymbol{v}} : \mathcal{U} \to \mathbb{R}$ defined by $\psi_{\boldsymbol{v}} := -\eth_{\tilde{\boldsymbol{v}}}$ satisfies

$$\nabla\psi_{\boldsymbol{v}}(x) = \mathcal{D}\psi_{\boldsymbol{v}}(x) = -\mathcal{D}\eth_{\tilde{\boldsymbol{v}}} = \tilde{\boldsymbol{v}}(\Pi_{\tilde{\boldsymbol{v}}}(x))$$

for all $x \in \mathcal{U}$. In particular, $\nabla\psi_{\boldsymbol{v}}(x) = \tilde{\boldsymbol{v}}(x)$ for every $x \in \partial\Omega$, and hence

$$\frac{\partial\psi_{\boldsymbol{v}}}{\partial\boldsymbol{v}}(x) = \langle \nabla\psi_{\boldsymbol{v}}(x), \boldsymbol{v}(x)\rangle = \langle \tilde{\boldsymbol{v}}(x), \boldsymbol{v}(x)\rangle = |\boldsymbol{v}(x)| > 0,$$

which ends the proof of (b) implies (d). Actually, since $|\nabla\psi_{\boldsymbol{v}}(x)| = |\tilde{\boldsymbol{v}}(x)| = 1$ for all $x \in \partial\Omega$, $\Psi := \psi_{\boldsymbol{v}}$ satisfies the requirements of part (f).

The fact that (d) implies (e) is trivial, and the proof of (c) implies (e) follows the same patterns as the proof of (b) implies (d). The fact that (f) implies (e) follows from the fact that $\boldsymbol{v}(x) := \nabla\Psi(x)$ is an outward vector field of class $\mathcal{C}^{r-1}$ satisfying

$$\frac{\partial\Psi}{\partial\boldsymbol{v}}(x) = |\nabla\Psi(x)|^2 = 1 > 0$$

for all $x \in \partial\Omega$. Thus, part (e) holds by choosing $\psi := \Psi$.

It remains to prove that (e) implies (a). By the properties of the function ψ guaranteed by part (e), it is apparent that $\partial\Omega := \psi^{-1}(0)$. Let us consider $x_0 \in \partial\Omega$ and $\boldsymbol{v}(x_0)$, and let $\{\mathbf{e}_j\}_{j=1}^{N-1}$ be an orthonormal basis of span $[\boldsymbol{v}(x_0)]^\perp$ in $\mathbb{R}^N$. Subsequently, for every $\delta > 0$, we denote by $F_\delta : (-\delta, \delta) \times (-\delta, \delta)^{N-1} \to \mathbb{R}^N$ the $\mathcal{C}^\infty$ map defined through

$$F_\delta(z, \mathbf{y}) := x_0 + z\boldsymbol{v}(x_0) + \sum_{j=1}^{N-1} y_j\mathbf{e}_j, \quad \mathbf{y} = (y_1, \dots, y_{N-1}).$$

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This map establishes a diffeomorphism onto its image, which is an open neighborhood of x_0 denoted by $\mathcal{W}_\delta$. Note that $F_\delta(0,0)=x_0$. Choose $\delta>0$ such that $\mathcal{W}_\delta\subset\mathcal{U}$, where $\mathcal{U}$ is the open neighborhood of $\partial\Omega$ guaranteed by part (c). Lastly, consider the function

$$G_\delta := \psi\circ F_\delta \in \mathcal{C}^r((-\delta,\delta)^N;\mathbb{R}).$$

Obviously, $G_\delta(0,0)=0$. Moreover,

$$\frac{\partial G_\delta}{\partial z}(0,0)=[\mathcal{D}\psi(x_0)]\Big(\frac{\partial F_\delta}{\partial z}(0,0)\Big)=\mathcal{D}\psi(x_0)(\mathbf{v}(x_0))=\frac{\partial\psi}{\partial\mathbf{v}}(x_0)>0.$$

Thus, according to the implicit function theorem, there exists $\delta_0>0$ and $\zeta\in\mathcal{C}^r((-\delta_0,\delta_0)^{N-1};\mathbb{R})$ such that

$$G_{\delta_0}^{-1}(0)=\{(\zeta(\mathbf{y}),\mathbf{y})\in\mathbb{R}^N : \mathbf{y}\in(-\delta_0,\delta_0)^{N-1}\}.$$

In particular, the function $(-\delta_0,\delta_0)^{N-1}\ni\mathbf{y}\mapsto F_{\delta_0}(\zeta(\mathbf{y}),\mathbf{y})$ provides us with a class $\mathcal{C}^r$ parametrization of $\partial\Omega\cap\mathcal{W}_{\delta_0}$. Since x_0 was arbitrary, $\partial\Omega$ is an $(N-1)$-manifold of class $\mathcal{C}^r$. This ends the proof of Theorem 1.3.

A further (deeper) analysis of the role played by the regularity of the outward vector field reveals the validity of the next result.

Corollary 2.1. *If $\partial\Omega$ is an $(N-1)$-dimensional manifold of class $\mathcal{C}^r$, $r\geq 1$, and $\mathbf{v}\in\mathcal{C}^k(\partial\Omega;\mathbb{R}^N)$, $k\geq 1$, is an outward vector field, then there exist an open subset $\mathcal{U}$ of $\mathbb{R}^N$, with $\partial\Omega\subset\mathcal{U}$, and a function $\Pi_{\mathbf{v}}\in\mathcal{C}^{\min\{r,k\}}(\mathcal{U};\partial\Omega)$ satisfying the requirements of $\Pi_{\mathbf{v}}$ in the statement of Theorem 1.3* (b). *In particular, the function $\mathfrak{d}_{\mathbf{v}}:\mathcal{U}\to\mathbb{R}$ defined in* (1.8) *is of class $\mathcal{C}^{\min\{r,k+1\}}$.*

3 A Canonical Transformation

As a byproduct of Theorem 1.3, the next result holds. It allows transforming the original problem into a problem with $\beta\geq 0$. So, without lost of generality, we can assume that $\beta\geq 0$ for the remaining of this paper.

Theorem 3.1. *Assume that $\partial\Omega$ is of class $\mathcal{C}^2$. Then there exists $E\in\mathcal{C}^2(\bar\Omega)$, with $E(x)>0$ for all $x\in\bar\Omega$, such that* (1.1) *can be equivalently expressed as*

$$\begin{cases} d\mathcal{L}_E w = h_E(w,x) & \text{in } \Omega,\\ \mathcal{B}_E w = 0 & \text{in } \partial\Omega,\end{cases}$$

where
(i) $h_E(w,x)=\frac{1}{E(x)}h(E(x)w,x)$ *for all* $w\geq 0$ *and* $x\in\bar\Omega$,
(ii) $\mathcal{L}_E=-\mathrm{div}(A\nabla\cdot)+b_E\nabla+c_E$, *with*

$$b_E := b-2A\frac{\nabla E}{E}\in\mathcal{M}_{1\times N}(\mathcal{C}(\bar\Omega)),\qquad c_E:=\frac{\mathcal{L}E}{E}\in\mathcal{C}(\bar\Omega),$$

(iii) $\mathcal{B}_E=\mathrm{D}$ *on* Γ_{D} *and* $\mathcal{B}_E=\frac{\partial}{\partial\mathbf{v}}+\beta_E$ *on* Γ_{R}, *with* $\beta_E:=\frac{\mathcal{B}E}{E}\geq 0$.
Moreover, h_E satisfies (H1), (H2) *and* (H4) *if h does too.*

Proof. First, let us consider an arbitrary $E\in\mathcal{C}^2(\bar\Omega)$ such that $E(x)>0$ for all $x\in\bar\Omega$. Suppose that u is a non-negative solution of (1.1). Then $w:=u/E$ satisfies

$$\begin{aligned}\mathcal{L}u=\mathcal{L}(Ew)&=-\mathrm{div}(A\nabla(Ew))+b\nabla(Ew)+cEw\\ &=-\mathrm{div}(EA\nabla w)-\mathrm{div}(wA\nabla E)+Eb\nabla w+wb\nabla E+wcE\\ &=-\nabla EA\nabla w-E\,\mathrm{div}(A\nabla w)-\nabla wA\nabla E-w\,\mathrm{div}(A\nabla E)+Eb\nabla w+wb\nabla E+wcE\\ &=-E\,\mathrm{div}(A\nabla w)+Eb\nabla w-\nabla EA\nabla w-\nabla wA\nabla E+w(-\mathrm{div}(A\nabla E)+b\nabla E+cE).\end{aligned}$$

By the symmetry of A, we have that $\nabla wA\nabla E=\nabla EA\nabla w$, and thus

$$\mathcal{L}u=E\Big(-\mathrm{div}(A\nabla w)+\Big(b-2A\frac{\nabla E}{E}\Big)\nabla w+\frac{\mathcal{L}E}{E}w\Big)=E\mathcal{L}_E w\quad\text{in }\Omega.$$

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Hence,

$$d\mathcal{L}_E w = \frac{1}{E} d\mathcal{L} u = \frac{1}{E} h(u,\cdot) = \frac{1}{E} h(Ew,\cdot) = h_E(w,\cdot) \quad \text{in } \Omega.$$

As for the boundary, we find that $\mathcal{B}_E w(x) = w(x) = u(x)/E(x) = 0$ for all $x \in \Gamma_D$, whereas

$$\begin{aligned} 0 = \mathcal{B}u(x) = \mathcal{B}(Ew)(x) &= \frac{\partial(Ew)}{\partial \boldsymbol{\nu}}(x) + \beta(x)E(x)w(x) \\ &= E(x)\frac{\partial w}{\partial \boldsymbol{\nu}}(x) + \Big(\frac{\partial E}{\partial \boldsymbol{\nu}}(x) + \beta(x)E(x)\Big) w(x) = E(x)\mathcal{B}_E w(x) \end{aligned}$$

for all $x \in \Gamma_R$. In order to choose E such that $\beta_E \geq 0$, note that, according to Theorem 1.3 and the remarks after it, there exist an open set $\mathcal{U}$, $\bar{\Omega} \subset \mathcal{U} \subset \mathbb{R}^N$, and a function $\psi \in \mathcal{C}^2(\mathcal{U})$ such that $\psi(x) < 0$ for all $x \in \Omega$, $\psi(x) = 0$ for all $x \in \partial\Omega$ and $\min_{\Gamma_R} \frac{\partial \psi}{\partial \boldsymbol{\nu}} > 0$. Consider

$$E := \exp(\mu\psi),$$

with $\mu > 0$ to be determined. Then, for each $x \in \Gamma_R$, $E(x) = 1$, and hence

$$\beta_E(x) = \frac{\mathcal{B}E(x)}{E(x)} = \beta(x) + \frac{1}{E(x)}\frac{\partial E}{\partial \boldsymbol{\nu}}(x) = \beta(x) + \mu \frac{\partial E}{\partial \boldsymbol{\nu}}(x).$$

Thus, since $\min_{\Gamma_R} \frac{\partial \psi}{\partial \boldsymbol{\nu}} > 0$, it becomes apparent that $\beta_E \geq 0$ on Γ_R for sufficiently large $\mu > 0$.

Now, let us analyze the properties of h_E. The regularity required for (H1) is a byproduct of the regularity of both h and E. On the other hand, for every $u > 0$ and $x \in \bar{\Omega}$, we have that

$$\frac{\partial h_E}{\partial w} = \frac{\partial}{\partial w}\Big(\frac{1}{E(x)} h(E(x)w, x)\Big) = \frac{1}{E(x)} E(x) \frac{\partial h}{\partial u}(E(x)w, x) = \frac{\partial h}{\partial u}(E(x)w, x) < 0.$$

Hence, h_E satisfies (H2). To conclude, since h satisfies (H4), there exists $M > 0$ such that $\max_{\bar{\Omega}} h(M,\cdot) < 0$. Therefore, setting

$$M_E := \frac{M}{\min_{\bar{\Omega}} E} > 0$$

and taking into account that h is decreasing in u by (H2), we conclude that, for every $x \in \bar{\Omega}$,

$$h_E(M_E, x) = \frac{1}{E(x)} h(M_E E(x), x) = \frac{1}{E(x)} h\Big(E(x)\frac{M}{\min_{\bar{\Omega}} E}, x\Big) \leq \frac{1}{E(x)} h(M, x) < 0,$$

which ends the proof. □

Remark 3.2. It should be noted that one can achieve $\beta_E(x) > 0$ for all $x \in \Gamma_R$ by choosing a sufficiently large $\mu > 0$ in the previous proof.

4 Monotonicity Properties of the Principal Eigenvalue

Throughout this section, for every $d > 0$ and $V \in \mathcal{C}(\bar{\Omega})$, we will denote by

$$\Sigma(d, V) := \sigma_1[d\mathcal{L} + V; \mathcal{B}, \Omega]$$

the principal eigenvalue of $[d\mathcal{L} + V; \mathcal{B}, \Omega]$ in $W^{2,\infty}_{\mathcal{B}}(\Omega) := \bigcap_{p>N} W^{2,p}_{\mathcal{B}}(\Omega)$, The next result collects the main properties of $\Sigma(d, V)$. It extends [14, Theorem 2.1] to deal with general differential operators, $\mathcal{L}$, not necessarily self-adjoint. Part (a) provides us with the monotony of the principal eigenvalue with respect to the potential.

Theorem 4.1. *$\Sigma(d, V)$ has the following properties:*

(a) *For every $d > 0$, the map $\Sigma(d,\cdot)\colon C(\bar{\Omega}) \to \mathbb{R}$ is strictly increasing, i.e., $\Sigma(d, V_1) < \Sigma(d, V_2)$ if $V_1, V_2 \in \mathcal{C}(\bar{\Omega})$ with $V_1 \lneq V_2$.*

(b) *For every $V \in \mathcal{C}(\bar{\Omega})$,*

$$\Sigma(0, V) := \lim_{d\to 0} \Sigma(d, V) = \min_{\bar{\Omega}} V.$$

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Proof. Let $\varphi_1 \gg 0$ denote the (unique) principal eigenfunction associated to $\sigma_1[d\mathcal{L}+V_1;\mathcal{B},\Omega]$ such that $\|\varphi_1\|_\infty = 1$. Then

$$\begin{cases} (d\mathcal{L}+V_2-\sigma_1[d\mathcal{L}+V_1;\mathcal{B},\Omega])\varphi_1 \gneq (d\mathcal{L}+V_1-\sigma_1[d\mathcal{L}+V_1;\mathcal{B},\Omega])\varphi_1 = 0 & \text{in } \Omega, \\ \mathcal{B}\varphi_1 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Therefore, the function φ_1 provides us with a positive strict supersolution of the differential operator $d\mathcal{L}+V_2-\sigma_1[d\mathcal{L}+V_1;\mathcal{B},\Omega]$ subject to the boundary operator $\mathcal{B}$ on $\partial\Omega$, and hence, thanks to the theorem of characterization provided by [31, Theorem 7.10], its principal eigenvalue must be positive. Thus,

$$\sigma_1[d\mathcal{L}+V_2;\mathcal{B},\Omega]-\sigma_1[d\mathcal{L}+V_1;\mathcal{B},\Omega] = \sigma_1[d\mathcal{L}+V_2-\sigma_1[d\mathcal{L}+V_1;\mathcal{B},\Omega];\mathcal{B},\Omega] > 0,$$

which ends the proof of part (a).

For the convergence in part (b), we first note that, thanks to part (a),

$$\sigma_1[d\mathcal{L}+V;\mathcal{B},\Omega] \geq d\sigma_1[\mathcal{L};\mathcal{B},\Omega]+\min_{\bar\Omega} V.$$

Thus,

$$\liminf_{d\to 0}\sigma_1[d\mathcal{L}+V;\mathcal{B},\Omega] \geq \min_{\bar\Omega} V.$$

Now, arguing by contradiction, suppose that

$$\limsup_{d\to 0}\sigma_1[d\mathcal{L}+V;\mathcal{B},\Omega] > \min_{\bar\Omega} V.$$

Then there exist $\varepsilon > 0$ and a sequence $\{d_n\}_{n\geq 1}\subset(0,+\infty)$, with $\lim_{n\to\infty} d_n = 0$, such that, for every $n \geq 1$,

$$\sigma_1[d_n\mathcal{L}+V;\mathcal{B},\Omega] > \min_{\bar\Omega} V+\varepsilon.$$

Equivalently,

$$\sigma_1\Big[d_n\mathcal{L}+V-\min_{\bar\Omega} V-\varepsilon;\mathcal{B},\Omega\Big] > 0,$$

and hence, by [31, Theorem 7.10], for every $n \geq 1$, the problem $[d_n\mathcal{L}+V-\min_{\bar\Omega} V-\varepsilon;\mathcal{B},\Omega]$ admits a strict supersolution $\varphi_n \gg 0$, i.e.,

$$\begin{cases} \Big(d_n\mathcal{L}+V-\min_{\bar\Omega} V-\varepsilon\Big)\varphi_n \geq 0 & \text{in } \Omega, \\ \mathcal{B}\varphi_n \geq 0 & \text{on } \partial\Omega, \end{cases}$$

with some of these inequalities strict. Let $x_0 \in \bar\Omega$ be such that $V(x_0) = \min_{\bar\Omega} V$. By continuity, there exists $\rho > 0$ such that

$$V(x) < \min_{\bar\Omega} V+\frac{\varepsilon}{2}$$

for all $x \in B_\rho(x_0)\cap\bar\Omega$. In particular, this estimate holds in an open ball $B \subset B_\rho(x_0)\cap\Omega$. Thus, for every $n \geq 1$, we have that

$$\begin{cases} \Big(d_n\mathcal{L}-\frac{\varepsilon}{2}\Big)\varphi_n \gneq 0 & \text{in } B, \\ \varphi_n > 0 & \text{on } \partial B. \end{cases}$$

Consequently, thanks again to [31, Theorem 7.10], we find that

$$\sigma_1\Big[d_n\mathcal{L}-\frac{\varepsilon}{2};\mathrm{D},B\Big] > 0,$$

which contradicts the fact that

$$\lim_{n\to\infty}\sigma_1\Big[d_n\mathcal{L}-\frac{\varepsilon}{2};\mathrm{D},B\Big] = \lim_{n\to\infty} d_n\sigma_1[\mathcal{L};\mathrm{D},\Omega]-\frac{\varepsilon}{2} = -\frac{\varepsilon}{2}.$$

This contradiction ends the proof. $\square$

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For establishing the monotonicity of the principal eigenvalue with respect to the underlying domain, we need to introduce some notations.

Definition 4.2. Let Ω_0 be a subdomain of class $\mathcal{C}^2$ of Ω and $\mathcal{B}_0$ a boundary operator on $\partial\Omega_0$. We will say that $(\mathcal{B}_0, \Omega_0)$ is comparable with $(\mathcal{B}, \Omega)$, and write $(\mathcal{B}_0, \Omega_0) \preceq (\mathcal{B}, \Omega)$, when the following conditions are satisfied:
(i) Each component Γ of $\partial\Omega_0$ is either a component of $\partial\Omega$, or $\Gamma \subset \Omega$.
(ii) The boundary operator $\mathcal{B}_0$ satisfies

$$\mathcal{B}_0 := \begin{cases} \mathrm{D} & \text{on } \partial\Omega_0 \cap \Omega, \\ \tilde{\mathcal{B}} & \text{on } \partial\Omega_0 \cap \partial\Omega, \end{cases}$$

where for every component Γ of $\partial\Omega_0 \cap \partial\Omega$, either $\tilde{\mathcal{B}} = \mathrm{D}$ on Γ, or $\Gamma \subset \Gamma_{\mathrm{R}}$ and there is $\beta_0 \in \mathcal{C}(\partial\Omega_0)$ with $\beta_0 \geq \beta$ such that

$$\tilde{\mathcal{B}} = \frac{\partial}{\partial \boldsymbol{\nu}} + \beta_0 \quad \text{on } \Gamma.$$

We will write $(\mathcal{B}_0, \Omega_0) \prec (\mathcal{B}, \Omega)$ if, in addition, $(\mathcal{B}_0, \Omega_0) \neq (\mathcal{B}, \Omega)$.

It should be noted that, according to [8, Theorem 9.1], the Dirichlet boundary operator on each component of $\partial\Omega$ can be approximated by letting $\min_\Gamma \beta \uparrow \infty$. Thus, the larger β_0, the closer are $\mathcal{B}_0$ and D. The next monotonicity result sharpens [14, Lemma 2.2].

Lemma 4.3. *Let Ω_0 be a subdomain of class $\mathcal{C}^2$ of Ω and $\mathcal{B}_0$ a boundary operator on $\partial\Omega_0$. If $(\mathcal{B}_0, \Omega_0) \prec (\mathcal{B}, \Omega)$, then*

$$\sigma_1[d\mathcal{L} + V; \mathcal{B}, \Omega] < \sigma_1[d\mathcal{L} + V; \mathcal{B}_0, \Omega_0] \quad \textit{for every } d > 0 \textit{ and } V \in \mathcal{C}(\bar{\Omega}).$$

Proof. Let $\varphi \gg 0$ be the principal eigenfunction associated to $\sigma_1[d\mathcal{L} + V; \mathcal{B}, \Omega]$, normalized so that $\|\varphi\|_\infty = 1$. Then, according to Definition 4.2, as long as $(\mathcal{B}_0, \Omega_0) \prec (\mathcal{B}, \Omega)$, there exist a component $\Gamma \neq \emptyset$ of $\partial\Omega$ for which some of the following alternatives hold:

- $\Gamma \subset \Omega$ and $\mathcal{B}_0\varphi = \varphi > 0$ on Γ. Actually, this occurs if Ω_0 is a proper subdomain of Ω.
- $\Gamma \subset \Gamma_{\mathrm{R}}$ and $\mathcal{B}_0\varphi = \varphi$ on Γ. Then, since $\varphi(x) > 0$ for all $x \in \Gamma_{\mathrm{R}}$, we have that $\mathcal{B}_0\varphi > 0$ on Γ.
- $\Gamma \subset \Gamma_{\mathrm{R}}$ and $\mathcal{B}_0 = \frac{\partial}{\partial \boldsymbol{\nu}} + \beta_0$, with $\beta_0 \gneq \beta$ on Γ. Then, since $\varphi(x) > 0$ for all $x \in \Gamma_{\mathrm{R}}$, we find that

$$\mathcal{B}_0\varphi = \frac{\partial\varphi}{\partial\boldsymbol{\nu}} + \beta_0\varphi \gneq \frac{\partial\varphi}{\partial\boldsymbol{\nu}} + \beta\varphi = \mathcal{B}\varphi = 0 \quad \text{on } \Gamma.$$

Hence, φ satisfies

$$\begin{cases} (d\mathcal{L} + V - \sigma_1[d\mathcal{L} + V; \mathcal{B}, \Omega])\varphi = 0 & \text{in } \Omega_0, \\ \mathcal{B}_0\varphi \gneq 0 & \text{on } \partial\Omega_0. \end{cases}$$

In particular, φ is a positive strict supersolution of $[d\mathcal{L} + V - \sigma_1[d\mathcal{L} + V; \mathcal{B}, \Omega]; \mathcal{B}_0, \Omega_0]$. Therefore, we can conclude from [31, Theorem 7.10] that

$$\sigma_1[d\mathcal{L} + V; \mathcal{B}_0, \Omega_0] - \sigma_1[d\mathcal{L} + V; \mathcal{B}, \Omega] = \sigma_1\big[d\mathcal{L} + V - \sigma_1[d\mathcal{L} + V; \mathcal{B}, \Omega]; \mathcal{B}_0, \Omega_0\big] > 0,$$

which ends the proof. □

5 The Generalized Diffusive Logistic Equation

We begin this section by proving Theorem 1.1, which characterizes the existence and establishes the uniqueness of the positive solution of (1.1) in terms of the linearized instability of $u = 0$ as a steady-state solution of its parabolic counterpart. As pointed out in Section 3, without lost of generality, we can assume that $\beta \geq 0$. Moreover, $h(u, x)$ is supposed to satisfy (H1), (H2) and (H3) for some $d > 0$.

Proof of Theorem 1.1. As a consequence of (H2) and (H3), and since β can be assumed to be non-negative, $\bar{u} := \kappa \geq M > 0$ is a supersolution of (1.1). Now, suppose that $\sigma_1[d\mathcal{L} - h(0, \cdot); \mathcal{B}, \Omega] < 0$ and let $\phi \gg 0$ be any

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associated eigenfunction. We claim that $\underline{u} := \varepsilon\phi$ is a subsolution of (1.1) for sufficiently small $\varepsilon > 0$. Since $\mathcal{B}(\varepsilon\phi) = \varepsilon\mathcal{B}\phi = 0$ on $\partial\Omega$, it suffices to show that

$$d\mathcal{L}(\varepsilon\phi) \leq \varepsilon\phi h(\varepsilon\phi, \cdot) \quad \text{in } \Omega.$$

By the choice of ϕ, we have that

$$d\mathcal{L}(\varepsilon\phi) = \varepsilon\big(\sigma_1[d\mathcal{L} - h(0,\cdot); \mathcal{B}, \Omega]\phi + h(0,\cdot)\phi\big) \quad \text{in } \Omega.$$

Hence, dividing by $\varepsilon\phi$, we should make sure that

$$\sigma_1[d\mathcal{L} - h(0,\cdot); \mathcal{B}, \Omega] \leq h(\varepsilon\phi, \cdot) - h(0,\cdot) \quad \text{in } \Omega. \tag{5.1}$$

Since h is uniformly continuous on $[0,1] \times \bar{\Omega}$ and $\varepsilon\phi$ converges to 0 uniformly in $\bar{\Omega}$ as $\varepsilon \downarrow 0$, we find that

$$\lim_{\varepsilon \to 0} \| h(\varepsilon\phi, \cdot) - h(0,\cdot) \|_\infty = 0.$$

Thus, condition (5.1) holds for sufficiently small ε, and hence $\underline{u} := \varepsilon\phi$ is a subsolution of (1.1). Since ε can be shortened up to get $\varepsilon\phi \leq \kappa$, (1.1) possesses a (strong) positive solution u such that $\varepsilon\phi \leq u \leq \kappa$.

Next, we will show that $\sigma_1[d\mathcal{L} - h(0,\cdot); \mathcal{B}, \Omega] < 0$ is necessary for the existence of a positive solutions. Indeed, if (1.1) admits a positive solution u, then $\sigma_1[d\mathcal{L} - h(u,\cdot); \mathcal{B}, \Omega] = 0$, by the uniqueness of the principal eigenvalue. Thus, by (H2), it follows from Theorem 4.1 (a) that

$$\sigma_1[d\mathcal{L} - h(0,\cdot); \mathcal{B}, \Omega] < \sigma_1[d\mathcal{L} - h(u,\cdot); \mathcal{B}, \Omega] = 0.$$

As for establishing the uniqueness, assume that $u_1, u_2 \in \bigcap_{p>N} W^{2,p}(\Omega)$ are two positive solutions of (1.1). In particular, $u_1, u_2 \gg 0$. Thanks to the first part of the proof, we already know that (1.1) admits a subsolution $\underline{u} = \varepsilon\phi$ and a supersolution $\bar{u} = \kappa > M$ such that

$$\underline{u} \leq u_1, u_2 \leq \bar{u}.$$

This can be easily obtained by shortening $\varepsilon > 0$ and enlarging κ as much as necessary. For these choices, thanks to [1, Theorem 3], problem (1.1) admits two strong solutions, $u_*, u^* \in \bigcap_{p>N} W^{2,p}(\Omega)$, which are the minimal and maximal solutions of (1.1), respectively, in the order interval $[\underline{u}, \bar{u}]$. In particular, we have that

$$\underline{u} \leq u_* \leq u_1, u_2 \leq u^* \leq \bar{u}$$

and, since $u_1 \neq u_2$, necessarily $u_* < u^*$. Since they are solutions of (1.1), we already know that

$$\sigma_1[d\mathcal{L} - h(u_*,\cdot); \mathcal{B}, \Omega] = \sigma_1[d\mathcal{L} - h(u^*,\cdot); \mathcal{B}, \Omega] = 0, \tag{5.2}$$

and, thanks to (H2),

$$h(u_*,\cdot) \gneq h(u^*,\cdot) \quad \text{in } \Omega.$$

Thus, by Theorem 4.1 (a),

$$\sigma_1[d\mathcal{L} - h(u_*,\cdot); \mathcal{B}, \Omega] < \sigma_1[d\mathcal{L} - h(u^*,\cdot); \mathcal{B}, \Omega],$$

which contradicts (5.2). Therefore, $u_1 = u_2$. This ends the proof. □

By linearizing (1.1) at $u = 0$, it is easily seen that $u = 0$ is linearly unstable if and only if

$$\sigma_1[d\mathcal{L} - h(0,\cdot); \mathcal{B}, \Omega] < 0,$$

while it is linearly stable, or linearly neutrally stable, in any other case.

Throughout the rest of this paper, we will denote by $\theta^{\mathcal{L},\mathcal{B},\Omega}_{\{d,h\}}$ the maximal non-negative solution of (1.1). By Theorem 1.1, $\theta^{\mathcal{L},\mathcal{B},\Omega}_{\{d,h\}} = 0$ if $\sigma_1[d\mathcal{L} - h(0,\cdot); \mathcal{B}, \Omega] \geq 0$, while $\theta^{\mathcal{L},\mathcal{B},\Omega}_{\{d,h\}} \gg 0$ if $\sigma_1[d\mathcal{L} - h(0,\cdot); \mathcal{B}, \Omega] < 0$. Should not exist any ambiguity, we will simply set

$$\theta_{\{d,h\}} := \theta^{\mathcal{L},\mathcal{B},\Omega}_{\{d,h\}},$$

or, alternatively, omit some of these indexes. As a byproduct of Theorems 4.1 (b) and 1.1, the positiveness of $\theta_{\{d,h\}}$ can be characterized for small $d > 0$ in terms of the sign of $\max_{\bar{\Omega}} h(0,\cdot)$, as established by the next result.

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Corollary 5.1. *Suppose that $h(u,x)$ satisfies* (H3) *for sufficiently small $d>0$. Then the following hold:*
(a) *If* $\max_{\bar\Omega} h(0,\cdot)<0$, *then a maximal $d_0\in(0,+\infty]$ exists such that $\theta_{\{d,h\}}=0$ for $d\in(0,d_0)$.*
(b) *If* $\max_{\bar\Omega} h(0,\cdot)>0$, *then a maximal $d_0\in(0,+\infty]$ exists such that $\theta_{\{d,h\}}\gg 0$ for $d\in(0,d_0)$.*

In the intermediate case when $\max_{\bar\Omega} h(0,\cdot)=0$, Theorem 4.1 (b) implies that

$$\lim_{d\downarrow 0}\sigma_1[d\mathcal{L}-h(0,\cdot);\mathcal{B},\Omega]=\min_{\bar\Omega}(-h(0,\cdot))=-\max_{\bar\Omega}h(0,\cdot)=0.$$

Thus, the sign of the principal eigenvalue $\sigma_1[d\mathcal{L}-h(0,\cdot);\mathcal{B},\Omega]$ for sufficiently small $d>0$ might depend on the nature of the coefficients of $\mathcal{L}$ as well as on the boundary operator $\mathcal{B}$, or even the geometry and the size of Ω. Indeed, if $\mathcal{L}=-\Delta$ is the Laplace operator and we assume that $\Gamma_{\mathrm{R}}=\emptyset$, i.e., $\mathcal{B}$ is the Dirichlet operator D and $h(0,\cdot)=0$, then

$$\sigma_1[-d\Delta;\mathrm{D},\Omega]=d\sigma_1[-\Delta;\mathrm{D},\Omega]>0$$

for all $d>0$ and hence, by Theorem 1.1, $\theta_{\{d,h\}}=0$ for all $d>0$. But if we assume that $\mathcal{L}=-\Delta-1$, $h(0,\cdot)=0$, $\Gamma_{\mathrm{D}}=\emptyset$ and $\beta\equiv 0$ on $\Gamma_{\mathrm{R}}=\partial\Omega$, i.e., $\mathcal{B}$ is the Neumann operator R_0, then

$$\sigma_1[d(-\Delta-1);\mathrm{R}_0,\Omega]=d\sigma_1[-\Delta;\mathrm{R}_0,\Omega]-d=-d<0$$

for all $d>0$. Therefore, due to Theorem 1.1, $\theta_{\{d,h\}}\gg 0$ for all $d>0$. Finally, note that, according to a celebrated variational inequality of Faber and Krahn (see, e.g., [31, Proposition 8.6]), the sign of

$$\sigma_1[d(-\Delta-1);\mathrm{D},\Omega]=d\,(\sigma_1[-\Delta;\mathrm{D},\Omega]-1)$$

depends on the Lebesgue measure of Ω. Indeed, for sufficiently small $|\Omega|$, $\sigma_1[-\Delta;\mathrm{D},\Omega]>1$, and hence $\theta_{\{d,h\}}=0$ for all $d>0$, while, for sufficiently large $|\Omega|$, $\sigma_1[-\Delta;\mathrm{D},\Omega]<1$, and therefore $\theta_{\{d,h\}}\gg 0$ for all $d>0$.

The following result provides us with a substantial sharpening of [14, Lemma 2.5].

Lemma 5.2. *Suppose that $h(u,x)$ satisfies* (H3) *for some $d>0$. Let Ω_0 be a subdomain of class $\mathcal{C}^2$ of Ω, $\mathcal{B}_0$ a boundary operator on $\partial\Omega_0$ such that, according to Definition 4.2, $(\mathcal{B}_0,\Omega_0)\preceq(\mathcal{B},\Omega)$, and suppose $h_0\in\mathcal{C}(\mathbb{R}\times\bar\Omega_0)$ satisfies* (H1), (H2), (H3) *and $h_0\le h$ in $[0,+\infty)\times\bar\Omega_0$. Then*

$$\theta^{\mathcal{L},\mathcal{B}_0,\Omega_0}_{\{d,h_0\}}\le\theta^{\mathcal{L},\mathcal{B},\Omega}_{\{d,h\}}\quad\textit{in }\Omega_0.$$

If, in addition, $(\mathcal{B}_0,\Omega_0)\prec(\mathcal{B},\Omega)$, or $h_0(u,\cdot)\neq h(u,\cdot)$ in Ω_0 for all $u\ge 0$, then

$$\theta^{\mathcal{L},\mathcal{B}_0,\Omega_0}_{\{d,h_0\}}\ll\theta^{\mathcal{L},\mathcal{B},\Omega}_{\{d,h\}}\quad\textit{in }\Omega_0$$

provided $\theta^{\mathcal{L},\mathcal{B},\Omega}_{\{d,h\}}>0$.

Proof. For the sake of simplicity, throughout this proof we will denote

$$\theta:=\theta^{\mathcal{L},\mathcal{B},\Omega}_{\{\nu,h\}},\qquad\theta_0:=\theta^{\mathcal{L},\mathcal{B}_0,\Omega_0}_{\{\nu,h_0\}}.$$

By Theorem 4.1 (a) and Lemma 4.3, we have that

$$\sigma_1[d\mathcal{L}-h(0,\cdot);\mathcal{B},\Omega]\le\sigma_1[d\mathcal{L}-h_0(0,\cdot);\mathcal{B}_0,\Omega_0].$$

Thus, due to Theorem 1.1,

$$\begin{aligned}&\theta=\theta_0=0\quad\text{if }\sigma_1[d\mathcal{L}-h(0,\cdot);\mathcal{B},\Omega]\ge 0,\\&\theta\gg\theta_0=0\quad\text{if }\sigma_1[d\mathcal{L}-h(0,\cdot);\mathcal{B},\Omega]<0\le\sigma_1[d\mathcal{L}-h_0(0,\cdot);\mathcal{B}_0,\Omega_0].\end{aligned}$$

Hence, it remains to study the case when

$$\sigma_1[d\mathcal{L}-h_0(0,\cdot);\mathcal{B}_0,\Omega_0]\le\sigma_1[d\mathcal{L}-h(0,\cdot);\mathcal{B},\Omega]<0.$$

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Then, by Theorem 1.1, $\theta, \theta_0 \gg 0$. Subsequently, we will consider the function $f \in \mathcal{C}(\bar{\Omega}_0)$ defined, for each $x \in \bar{\Omega}_0$, by

$$f(x) := \begin{cases} \dfrac{\theta(x)h_0(\theta(x),x) - \theta_0(x)h_0(\theta_0(x),x)}{\theta(x)-\theta_0(x)} & \text{if } \theta(x) \neq \theta_0(x), \\ h_0(\theta_0(x),x) + \theta_0(x)\dfrac{\partial}{\partial u}h_0(\theta_0(x),x) & \text{if } \theta(x) = \theta_0(x). \end{cases}$$

By definition, $\theta - \theta_0$ satisfies

$$d\mathcal{L}(\theta-\theta_0) = \theta h(\theta,\cdot) - \theta_0 h_0(\theta_0,\cdot) \geq \theta h_0(\theta,\cdot) - \theta_0 h_0(\theta_0,\cdot) = (\theta-\theta_0)f \quad \text{in } \Omega_0,$$

with strict inequality if $h(u,\cdot) \gneq h_0(u,\cdot)$ in Ω_0 for every $u > 0$. Moreover, since $(\mathcal{B}_0, \Omega_0) \preceq (\mathcal{B}, \Omega)$, we have that

$$\mathcal{B}_0(\theta-\theta_0) = \mathcal{B}_0\theta \geq 0 \quad \text{on } \partial\Omega_0,$$

with strict inequality if $(\mathcal{B}_0, \Omega_0) \prec (\mathcal{B}, \Omega)$. Thus, $\theta - \theta_0$ is a supersolution of

$$\begin{cases} (d\mathcal{L} - f)u = 0 & \text{in } \Omega_0, \\ \mathcal{B}_0 u = 0 & \text{on } \partial\Omega_0, \end{cases}$$

and, actually, it is a strict supersolution if $(\mathcal{B}_0, \Omega_0) \prec (\mathcal{B}, \Omega)$, or $h_0(u,\cdot) \neq h(u,\cdot)$ in Ω_0 for all $u \geq 0$. We claim that $\sigma_1[d\mathcal{L} - f; \mathcal{B}_0, \Omega_0] > 0$. Thanks to [31, Theorem 7.10], this entails that $\theta - \theta_0 \geq 0$ in Ω_0 and that, actually, $\theta \gg \theta_0$ if it is strict, and so concluding the proof.

To prove $\sigma_1[d\mathcal{L} - f; \mathcal{B}_0, \Omega_0] > 0$, we can argue as follows. Let $x \in \bar{\Omega}_0$ be such that $\theta(x) = \theta_0(x)$. Then, by definition, and thanks to (H2),

$$f(x) = h_0(\theta_0(x),x) + \theta_0(x)\frac{\partial h_0}{\partial u}(\theta_0(x),x) \leq h_0(\theta_0(x),x),$$

with strict inequality if $\theta_0(x) > 0$, while if $x \in \bar{\Omega}_0$, with $\theta(x) \neq \theta_0(x)$, then

$$\begin{aligned} f(x) &= \frac{\theta(x)h_0(\theta(x),x) - \theta_0(x)h_0(\theta_0(x),x)}{\theta(x)-\theta_0(x)} \\ &= h_0(\theta_0(x),x) + \theta(x)\frac{h_0(\theta(x),x) - h_0(\theta_0(x),x)}{\theta(x)-\theta_0(x)} \\ &\leq h_0(\theta_0(x),x), \end{aligned}$$

with strict inequality if $\theta(x) > 0$. Note that $\theta(x) > 0$ and $\theta_0(x) > 0$ for all $x \in \Omega_0$, and hence both inequalities are strict for all $x \in \Omega_0$. Therefore,

$$f \lneq h_0(\theta_0,\cdot) \quad \text{in } \bar{\Omega}_0,$$

and hence, owing to Theorem 4.1 (a),

$$\sigma_1[d\mathcal{L} - f; \mathcal{B}_0, \Omega_0] > \sigma_1[d\mathcal{L} - h_0(\theta_0,\cdot); \mathcal{B}_0, \Omega_0] = 0,$$

which ends the proof. □

6 Proof of Theorem 1.2

Throughout this section, we assume that h satisfies (H1), (H2) and (H4). Hence, (H3) holds for sufficiently small $d > 0$. The precise range of d where this occurs is unimportant for the proof, and so it is not specified. It should be remembered that the function

$$\Theta_h(x) := \begin{cases} 0 & \text{if } h(\xi,x) < 0 \text{ for all } \xi > 0, \\ \xi & \text{if there exists } \xi > 0 \text{ such that } h(\xi,x) = 0, \end{cases} \tag{6.1}$$

is well defined for all $x \in \bar{\Omega}$ and it is continuous in $\bar{\Omega}$. Let Γ_R^+ denote the union of the components of Γ_R where Θ_h is everywhere positive. This section gives the proof of Theorem 1.2.

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Remark 6.1. For every $x \in \bar{\Omega}$, $\Theta_h(x)$ provides us with the unique non-negative linearly stable, or linearly neutrally stable, steady-state solution of the associated kinetic model

$$\begin{cases} u'(t) = u(t)h(u(t), x), & t \in [0, +\infty), \\ u(0) = u_0 \geq 0. \end{cases}$$

Note that

$$\begin{aligned} &\Theta_h(x) = 0 && \text{if } h^{-1}(\cdot\,, x)(0) = \emptyset, \\ &\Theta_h(x) = \max\{0, h^{-1}(\cdot\,, x)(0)\} && \text{if } h^{-1}(\cdot\,, x)(0) \neq \emptyset. \end{aligned}$$

Remark 6.2. The condition (H4) is necessary for the continuity of Θ_h on $\bar{\Omega}$, as the following simple example shows:

$$\begin{cases} d(-\Delta u + u) = u(-x^2 + e^{-u}) & \text{in } \Omega = (-1, 1), \\ \mathcal{B}u = 0 & \text{on } \partial\Omega = \{-1, 1\}, \end{cases}$$

where $h(u, x) = -x^2 + e^{-u}$ for all $x \in (-1, 1)$ and $u \in \mathbb{R}$. According to (6.1), it becomes apparent that

$$\Theta_h(x) = -\log x^2, \quad x \in [-1, 1] \setminus \{0\},$$

which is discontinuous, and unbounded, at $x = 0$. It turns out that in this example the function $h(u, x)$ satisfies (H1), (H2) and (H3) for sufficiently small $d > 0$, however, it does not satisfies (H4). Therefore, condition (H4) is the minimal necessary condition required to guarantee the continuity of $\Theta_h(x)$.

The proof of Theorem 1.2 follows after a series of results of a technical nature, some of them of great interest on their own. The first one is a consequence of Theorem 1.3 in the special case $r = 2$.

Lemma 6.3. *Let $\xi_1, \xi_2 \in \mathcal{C}(\bar{\Omega})$ be such that $\xi_1(x) < \xi_2(x)$ for all $x \in \bar{\Omega}$. Then the following hold:*
(a) *There exists $\Phi \in \mathcal{C}^2(\bar{\Omega})$ such that $\xi_1 \leq \Phi \leq \xi_2$ in $\bar{\Omega}$ and $\mathrm{R}\Phi(x) > 0$ for all $x \in \Gamma_\mathrm{R}$.*
(b) *There exists $\Phi \in \mathcal{C}^2(\bar{\Omega})$ such that $\xi_1 \leq \Phi \leq \xi_2$ in $\bar{\Omega}$ and $\mathrm{R}\Phi(x) < 0$ for all $x \in \Gamma_\mathrm{R}$.*

Proof. By Theorem 1.3 applied to the conormal vector field, there exist an open neighborhood $\mathcal{U} \subset \mathbb{R}^N$ of $\partial\Omega$, a function $\psi \in \mathcal{C}^2(\mathcal{U}; \mathbb{R})$ and a constant $\tau > 0$ such that $\psi(x) < 0$ for all $x \in \mathcal{U} \cap \Omega$, $\psi(x) = 0$ for each $x \in \partial\Omega$ and

$$\frac{\partial \psi}{\partial \mathbf{v}}(x) \geq \tau \quad \text{for all } x \in \partial\Omega. \tag{6.2}$$

Let $\varepsilon > 0$ be such that

$$\varepsilon < \min_{\bar{\Omega}}(\xi_2 - \xi_1).$$

Then

$$\xi_1(x) + \frac{\varepsilon}{2} < \xi_2(x) - \frac{\varepsilon}{2} \quad \text{for all } x \in \bar{\Omega},$$

and hence there exists $\phi \in \mathcal{C}^\infty(\bar{\Omega})$ such that

$$\xi_1(x) + \frac{\varepsilon}{2} < \phi(x) < \xi_2(x) - \frac{\varepsilon}{2} \quad \text{for all } x \in \bar{\Omega}.$$

Consider, for each $M \in \mathbb{R}$, the map $\phi_M \in \mathcal{C}^2(\mathcal{U} \cap \bar{\Omega})$ defined by

$$\phi_M(x) := \phi(x) - 1 + e^{M\psi(x)}, \quad x \in \mathcal{U} \cap \bar{\Omega}.$$

By the continuity of ϕ_M, and the fact that $\phi_M(x) = \phi(x)$ for all $x \in \partial\Omega$, we can reduce $\mathcal{U}$ to some open set $\mathcal{U}_M$, with $\partial\Omega \subset \mathcal{U}_M \subset \mathcal{U}$, so that

$$\xi_1(x) + \frac{\varepsilon}{2} < \phi_M(x) < \xi_2(x) - \frac{\varepsilon}{2} \quad \text{for all } x \in \mathcal{U}_M \cap \bar{\Omega}.$$

On the other hand, since $\psi(x) = 0$ for all $x \in \partial\Omega$, it becomes apparent that, for every $x \in \Gamma_\mathrm{R}$,

$$\begin{aligned} \mathrm{R}\phi_M(x) &= \mathrm{R}\phi(x) + \mathrm{R}(e^{M\psi(x)} - 1) \\ &= \mathrm{R}\phi(x) + Me^{M\psi(x)}\frac{\partial \psi}{\partial \mathbf{v}}(x) + \beta(x)(e^{M\psi(x)} - 1) \\ &= \mathrm{R}\phi(x) + M\frac{\partial \psi}{\partial \mathbf{v}}(x). \end{aligned}$$

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According to (6.2), for sufficiently large $M > 0$, one can get $\mathrm{R}\phi_M(x) > 0$ for all $x \in \Gamma_{\mathrm{R}}$. So, in order to get part (a), it suffices to choose Φ equal to ϕ_M in a neighborhood of $\partial\Omega$. Similarly, by choosing $M < 0$ sufficiently large, part (b) can be easily accomplished.

In each of these cases, once we have fixed the appropriate M, it remains to take Φ as any smooth extension of ϕ_M from a neighborhood $\mathcal{V}$ of $\partial\Omega$, with $\mathcal{V} \subset \mathcal{U}_M$, to $\bar{\Omega}$ in such a way that $\xi_1(x) < \Phi(x) < \xi_2(x)$ for all $x \in \bar{\Omega}$. This can be accomplished through an appropriate cutoff function of class $\mathcal{C}^\infty$. □

Remark 6.4. Note that if $\xi_1 \geq 0$ in $\bar{\Omega}$, then the function Φ provided by Lemma 6.3 (a) satisfies $\mathcal{B}\Phi \geq 0$ on $\partial\Omega$, whereas if $\xi_2 \leq 0$ in $\bar{\Omega}$, then the function Φ provided by Lemma 6.3 (b) verifies $\mathcal{B}\Phi \leq 0$ on $\partial\Omega$.

The next result provides us with a global uniform estimate in $\bar{\Omega}$, when $d \sim 0$, for the non-negative solutions of (1.1).

Lemma 6.5. *For every $\varepsilon > 0$, there exists $d_0 = d_0(\varepsilon) > 0$ such that $\theta_{\{d,h\}} \leq \Theta_h + \varepsilon$ in $\bar{\Omega}$ for all $d \in (0, d_0)$.*

Proof. Subsequently, we suppose that d has been chosen sufficiently small so that (H3) holds. For a given $\varepsilon > 0$, set

$$\xi_1 := \Theta_h + \frac{\varepsilon}{2} > 0, \quad \xi_2 := \Theta_h + \varepsilon.$$

By Lemma 6.3 (a) and Remark 6.4, there exists $\Phi \in C^2(\bar{\Omega})$ such that

$$0 < \Theta_h + \frac{\varepsilon}{2} \leq \Phi \leq \Theta_h + \varepsilon \quad \text{in } \bar{\Omega} \qquad \text{and} \qquad \mathcal{B}\Phi \geq 0 \quad \text{on } \partial\Omega.$$

In particular, $\Phi(x) > \Theta_h(x)$ for all $x \in \bar{\Omega}$. Thus, since $h(\Theta_h(x), x) \leq 0$ for all $x \in \bar{\Omega}$ and, owing to (H2), it is strictly decreasing in the first variable, we find that

$$h(\Phi(x), x) < 0 \quad \text{for all } x \in \bar{\Omega}.$$

Hence, setting

$$d_0 := \frac{\max_{x\in\bar{\Omega}}(\Phi(x)h(\Phi(x), x))}{\min\{0, \min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x)\}} \in (0, +\infty],$$

it becomes apparent that, for every $d < d_0$,

$$\Phi(x)h(\Phi(x), x) \leq \max_{x\in\bar{\Omega}}(\Phi(x)h(\Phi(x), x)) \leq d \min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x) \leq d\mathcal{L}\Phi(x) \quad \text{in } \bar{\Omega}.$$

Note that this estimate holds true for all $d > 0$ if $\min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x) \geq 0$, because, by construction,

$$\Phi h(\Phi, \cdot) < 0 \quad \text{in } \bar{\Omega}.$$

This explains why we are setting $d_0 = +\infty$ when $\min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x) \geq 0$. On the other hand, when we have $\min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x) < 0$, the value of d_0 becomes

$$d_0 := \frac{\max_{x\in\bar{\Omega}}(\Phi(x)h(\Phi(x), x))}{\min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x)} = \frac{-\max_{x\in\bar{\Omega}}(\Phi(x)h(\Phi(x), x))}{-\min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x)} > 0.$$

Thus,

$$-d \min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x) < -\max_{x\in\bar{\Omega}}(\Phi(x)h(\Phi(x), x))$$

for all $d < d_0$, or, equivalently,

$$\max_{x\in\bar{\Omega}}(\Phi(x)h(\Phi(x), x)) < d \min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x),$$

which also shows the previous estimate in this case.

Consequently, Φ provides us with a positive supersolution of (1.1). Consider the function $f \in \mathcal{C}(\bar{\Omega})$ defined, for each $x \in \bar{\Omega}$, by

$$f(x) := \begin{cases} \dfrac{\Phi(x)h(\Phi(x), x) - \theta_{\{d,h\}}(x)h(\theta_{\{d,h\}}(x), x)}{\Phi(x) - \theta_{\{d,h\}}(x)} & \text{if } \Phi(x) \neq \theta_{\{d,h\}}(x), \\ h(\Phi(x), x) + \Phi(x)\dfrac{\partial h}{\partial u}(\Phi(x), x) & \text{if } \Phi(x) = \theta_{\{d,h\}}(x). \end{cases}$$

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Therefore, the function $\Phi - \theta_{\{d,h\}}$ is a supersolution of

$$\begin{cases} (d\mathcal{L} - f)u = 0 & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, either $\theta_{\{d,h\}} \equiv 0$, which ends the proof, or

$$\theta_{\{d,h\}} \gg 0, \quad \sigma_1[d\mathcal{L} - h(\theta_{\{d,h\}}, \cdot); \mathcal{B}, \Omega] = 0.$$

In the latter case, it is easily seen that (H2) implies $f \lneq h(\theta_{\{d,h\}}, \cdot)$ in $\bar{\Omega}$. Thus, for every $d \in (0, d_0)$, it follows from Theorem 4.1 (a) that

$$\sigma_1[d\mathcal{L} - f; \mathcal{B}, \Omega] > \sigma_1[d\mathcal{L} - h(\theta_{\{d,h\}}, \cdot); \mathcal{B}, \Omega] = 0.$$

By [31, Theorem 7.10], we may infer that, for every $0 < d < d_0$,

$$\theta_{\{d,h\}}(x) \le \Phi(x) \le \Theta_h(x) + \varepsilon \quad \text{for all } x \in \bar{\Omega}.$$

The proof is complete. □

The following result provides us with Theorem 1.2 in the special case when $\Gamma_R = \emptyset$.

Proposition 6.6. *For any compact subset K of $\Omega \cup \Theta_h^{-1}(0)$, we have that*

$$\lim_{d \downarrow 0} \theta^{\Omega}_{\{d,h\}} = \Theta_h \quad \textit{uniformly in } K.$$

Proof. Fix $\varepsilon > 0$. By Lemma 6.5, there exists $d_0 = d_0(\varepsilon) > 0$ such that

$$\theta_{\{d,h\}} \le \Theta_h + \varepsilon \quad \text{for all } x \in K \subset \bar{\Omega}, \ d \in (0, d_0).$$

In order to get a lower estimate, we will first assume $h(u, x)$ to be autonomous, i.e., $h(u, x) = h(u)$ for all $(u, x) \in \mathbb{R} \times \bar{\Omega}$. In such a case, Θ_h is a non-negative constant. Since $\theta_{\{d,h\}}$ is non negative, it is obvious that $\theta_{\{d,h\}} > \Theta_h - \varepsilon$ in $\bar{\Omega}$ for all $d > 0$ if $\Theta_h = 0$. Thus, the following estimate holds:

$$\Theta_h - \varepsilon \le \theta_{\{d,h\}} \le \Theta_h + \varepsilon \quad \text{for all } x \in K \subset \bar{\Omega}, \ d \in (0, d_0).$$

In order to get the lower estimate when Θ_h is a positive constant (necessarily, $h(0) > 0$, $h(\Theta_h) = 0$ and $K \subset \Omega$, because $\Theta_h^{-1}(0) = \emptyset$), we consider $\tilde{\varepsilon} \in (0, \min\{2\Theta_h, \varepsilon\})$, $x_0 \in K$, and $\rho > 0$ be such that $\rho < \operatorname{dist}(K, \partial\Omega)$. For these choices, $\bar{B}_\rho(x_0) \subset \Omega$. Let $\varphi \gg 0$ be the principal eigenfunction associated to $\sigma_1[\mathcal{L}; \mathrm{D}, B_\rho(x_0)]$ normalized so that $\|\varphi\|_\infty = 1/2$, and define the function $\phi \in \bigcap_{p>N} W^{2,p}(B_\rho(x_0))$ through

$$\phi := \begin{cases} \varphi & \text{in } \bar{B}_\rho(x_0) \setminus \bar{B}_{\rho/2}(x_0), \\ \tilde{\varphi} & \text{in } \bar{B}_{\rho/2}(x_0), \end{cases}$$

where $\tilde{\varphi}$ is any sufficiently smooth function chosen so that $\phi(x) > 0$ for all $x \in B_\rho(x_0)$, $\phi(x_0) = 1$ and $\|\phi\|_\infty = 1$. Set

$$\Phi := \Big(\Theta_h - \frac{\tilde{\varepsilon}}{2}\Big)\phi \quad \text{in } \bar{B}_\rho(x_0).$$

Then, by construction, $\mathrm{D}\Phi = 0$ on $\partial B_\rho(x_0)$ and

$$0 < \Phi(x) \le \Theta_h - \frac{\tilde{\varepsilon}}{2} \quad \text{for all } x \in B_\rho(x_0),$$

since Θ_h is a constant greater than $\tilde{\varepsilon}/2$ and $\|\phi\|_\infty = 1$. Thus, taking into account that, owing to (H2), $h(u)$ is strictly decreasing in $u > 0$, it is apparent that

$$h(\Phi(x)) \ge h\Big(\Theta_h - \frac{\tilde{\varepsilon}}{2}\Big) > h(\Theta_h) = 0 \quad \text{for all } x \in \bar{B}_\rho(x_0),$$

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and hence

$$\min_{x\in\bar{B}_\rho(x_0)} h(\Phi(x)) \geq h\Big(\Theta_h - \frac{\tilde{\varepsilon}}{2}\Big) > 0.$$

On the other hand, the function $\mathcal{L}\Phi/\Phi$ is continuous in $\bar{B}_\rho(x_0)$ because $\Phi(x) > 0$ for all $x \in B_\rho(x_0)$ and

$$\mathcal{L}\Phi(x)/\Phi(x) = \sigma_1[\mathcal{L}; \mathrm{D}, B_\rho(x_0)] \in \mathbb{R} \quad \text{for all } x \in \partial B_\rho(x_0).$$

Although unnecessary, $\rho(x_0)$ can be shortened so that $\sigma_1[\mathcal{L}; \mathrm{D}, B_\rho(x_0)] > 0$, because, due to the Faber–Krahn inequality, $\lim_{\rho\to 0}\sigma_1[\mathcal{L}; \mathrm{D}, B_\rho(x_0)] = +\infty$ (see, e.g., [31, Proposition 8.6]). Thus, setting

$$0 < d_{x_0} < \frac{\min_{\bar{B}_\rho(x_0)} h(\Phi)}{\max_{\bar{B}_\rho(x_0)}|\mathcal{L}\Phi/\Phi|},$$

we have that, for every $d \in (0, d_{x_0})$,

$$d \max_{\bar{B}_\rho(x_0)}|\mathcal{L}\Phi/\Phi| \lneq h(\Phi) \quad \text{in } \bar{B}_\rho(x_0),$$

and so

$$d\mathcal{L}\Phi = d\Phi\mathcal{L}\Phi/\Phi \leq d\Phi \max_{\bar{B}_\rho(x_0)} |\mathcal{L}\Phi/\Phi| \lneq \Phi h(\Phi) \quad \text{in } \bar{B}_\rho(x_0).$$

Therefore, Φ provides us with a strict subsolution of

$$\begin{cases} d\mathcal{L}u = uh(u) & \text{in } B_\rho(x_0), \\ u = 0 & \text{on } \partial B_\rho(x_0). \end{cases}$$

Equivalently, $\theta^{\mathrm{D},B_\rho(x_0)}_{\{d,h\}} - \Phi$ is a strict supersolution of

$$\begin{cases} (d\mathcal{L} - f)u = 0 & \text{in } B_\rho(x_0), \\ u = 0 & \text{on } \partial B_\rho(x_0), \end{cases}$$

where $f \in \mathcal{C}(\bar{B}_\rho(x_0))$ stands for the function defined, for every $x \in \bar{B}_\rho(x_0)$, by

$$f(x) := \begin{cases} \dfrac{\theta^{\mathrm{D},B_\rho(x_0)}_{\{d,h\}}(x)h(\theta^{\mathrm{D},B_\rho(x_0)}_{\{d,h\}}(x)) - \Phi(x)h(\Phi(x))}{\theta^{\mathrm{D},B_\rho(x_0)}_{\{d,h\}}(x) - \Phi(x)} & \text{if } \Phi(x) \neq \theta^{\mathrm{D},B_\rho(x_0)}_{\{d,h\}}(x), \\ h(\Phi(x)) + \Phi(x)h'(\Phi(x)) & \text{if } \Phi(x) = \theta^{\mathrm{D},B_\rho(x_0)}_{\{d,h\}}(x). \end{cases}$$

Moreover, thanks to (H2), $f \lneq h(\theta^{\mathrm{D},B_\rho(x_0)}_{\{d,h\}})$ in $\bar{B}_\rho(x_0)$ and thus, by the monotonicity of the principal eigenvalue with respect to the potential established by Theorem 4.1 (a), it becomes apparent that

$$\sigma_1[d\mathcal{L} - f; \mathrm{D}, B_\rho(x_0)] > \sigma_1[d\mathcal{L} - h(\theta^{\mathrm{D},B_\rho(x_0)}_{\{d,h\}}); \mathrm{D}, B_\rho(x_0)] = 0.$$

Note that $h(0) > 0$, and hence, owing to Corollary 5.1 (b), $\theta^{\mathrm{D},B_\rho(x_0)}_{\{d,h\}} \gg 0$ for sufficiently small $d > 0$. Consequently, by [31, Theorem 7.10], we find that, for every $d \in (0, d_{x_0})$,

$$\Phi(x) < \theta^{\mathrm{D},B_\rho(x_0)}_{\{d,h\}}(x) \quad \text{for all } x \in B_\rho(x_0). \tag{6.3}$$

Moreover, by Lemma 5.2,

$$\theta^{\mathrm{D},B_\rho(x_0)}_{\{d,h\}} \leq \theta^{\mathcal{B},\Omega}_{\{d,h\}} \quad \text{in } B_\rho(x_0). \tag{6.4}$$

On the other hand, since $\Phi \in \mathcal{C}(\bar{B}_\rho(x_0))$ and $\Phi(x_0) = \Theta_h - \tilde{\varepsilon}/2$, there exist $\rho_{x_0} \in (0, \rho)$ such that

$$\Phi(x) \geq \Theta_h - \tilde{\varepsilon} > \Theta_h - \varepsilon \quad \text{for all } x \in B_{\rho_{x_0}}(x_0). \tag{6.5}$$

According to (6.3)–(6.5), we find that

$$\theta^{\mathcal{B},\Omega}_{\{d,h\}} > \Theta_h - \varepsilon \quad \text{in } B_{\rho_{x_0}}(x_0) \text{ for all } d \in (0, d_{x_0}).$$

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As K is compact, we can extract $x_1, \dots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n B_{\rho_{x_i}}(x_i)$, and hence for every $d < \min_i d_{x_i}$, $d > 0$, the estimate $\theta_{\{d,h\}}^{\mathcal{B},\Omega} \geq \Theta_h - \varepsilon$ holds in K. This ends the proof when h is independent of x.

Subsequently, we will assume that $h(u,x)$ is a general function satisfying (H1), (H2) and (H4). Then, for sufficiently small $d > 0$, it is obvious that

$$\theta_{\{d,h\}}^{\mathcal{B},\Omega}(x) \geq 0 \geq \Theta_h(x) - \varepsilon \quad \text{for all } x \in \Theta_h^{-1}([0,\varepsilon]).$$

As this provides us with a satisfactory lower estimate in $K \cap \Theta_h^{-1}([0,\varepsilon])$, in order to extend it to K, it remains to show that there exists $d_1 > 0$ such that, for every $d \in (0, d_1)$,

$$\theta_{\{d,h\}}^{\mathcal{B},\Omega}(x) \geq \Theta_h(x) - \varepsilon \quad \text{for all } x \in K_0 := K \cap \Theta_h^{-1}([\varepsilon, \max_{\bar\Omega} \Theta_h]) \subset \Omega.$$

Should K_0 be empty, the proof is complete. So, suppose that K_0 is nonempty and pick $x_0 \in K_0$ and $\rho > 0$ such that

$$\bar B_\rho(x_0) \subset \Omega \cap \Theta_h^{-1}\Big(\frac{\varepsilon}{2}, +\infty\Big).$$

By construction, $\Theta_h(x) > \varepsilon/2 > 0$ for all $x \in \bar B_\rho(x_0)$ and hence, owing to (H2),

$$\min_{\bar B_\rho(x_0)} h(0,\cdot) > 0 \quad \text{and} \quad \min_{\bar B_\rho(x_0)} \Theta_h > \frac{\varepsilon}{2} > 0.$$

Actually, by continuity, $\rho > 0$ can be shortened, if necessary, so that

$$\min_{\bar B_\rho(x_0)} \Theta_h \geq \Theta_h(x) - \frac{\varepsilon}{2} \quad \text{for all } x \in \bar B_\rho(x_0). \tag{6.6}$$

The rest of the proof consists in reducing ourselves to the previous case, by establishing the existence of an autonomous function $H(u)$ satisfying (H1), (H2), (H4), and such that

$$H(u) \leq h(u,x) \quad \text{for all } u \geq 0,\ x \in \bar B_\rho(x_0), \tag{6.7}$$

and

$$\min_{x \in \bar B_\rho(x_0)} \Theta_h(x) - \frac{\varepsilon}{4} \leq \Theta_H \leq \min_{x \in \bar B_\rho(x_0)} \Theta_h(x). \tag{6.8}$$

The most natural candidate function for a (globally defined in $\mathbb{R}$) $H(u)$ is

$$h_{\min}(u) := \begin{cases} \min_{x \in \bar B_\rho(x_0)} h(u,x) & \text{if } u \geq 0, \\ \min_{x \in \bar B_\rho(x_0)} h(0,x) - u & \text{if } u < 0. \end{cases}$$

Obviously, $h_{\min} \in \mathcal{C}(\mathbb{R})$ and it is strictly decreasing, though, in general, it is not of class $\mathcal{C}^1(\mathbb{R})$. Thus, in order to construct $H(u)$ satisfying (6.7), (6.8), (H1), (H2) and (H4), we begin by considering the function

$$G(u) := \min\Big\{-\delta, \frac{4}{\varepsilon} h_{\min}\Big(u + \frac{\varepsilon}{4}\Big)\Big\} < 0, \quad u \in \mathbb{R},$$

with sufficiently small $\delta > 0$, to be chosen later, and then we take, for every $u \in \mathbb{R}$,

$$H(u) := \int_{\Theta_{\min} - \frac{\varepsilon}{4}}^{u} G(s)\,ds, \quad \text{where } \Theta_{\min} \equiv \min_{x \in \bar B_\rho(x_0)} \Theta_h(x).$$

Since G is a continuous function, H is a function of class $\mathcal{C}^1(\mathbb{R})$, and hence (H1) holds. Moreover, by definition, $H'(u) = G(u) < 0$ for all $u \in \mathbb{R}$. Thus, (H2) holds. Furthermore, since $H(\Theta_{\min} - \frac{\varepsilon}{4}) = 0$, (H4) also holds, because $H(u) < 0$ for all $u > \Theta_{\min} - \frac{\varepsilon}{4}$. Actually, (6.8) holds too, since $\Theta_H = \Theta_{\min} - \frac{\varepsilon}{4}$, by definition (see (6.1) if necessary). It remains to shorten δ, if necessary, to get (6.7). Suppose $u \leq \Theta_{\min} - \frac{\varepsilon}{4}$. Then $u + \frac{\varepsilon}{4} \leq \Theta_{\min}$, and hence $h_{\min}(u + \frac{\varepsilon}{4}) \geq 0$ and $G(u) = -\delta$, which implies

$$H(u) = -\delta\Big(u - \Theta_{\min} + \frac{\varepsilon}{4}\Big).$$

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Thus, for sufficiently small δ,

$$H(0) = \delta\Big(\Theta_{\min} - \frac{\varepsilon}{4}\Big) < h_{\min}\Big(\Theta_{\min} - \frac{\varepsilon}{4}\Big),$$

and therefore

$$H(u) \le H(0) < h_{\min}\Big(\Theta_{\min} - \frac{\varepsilon}{4}\Big) \le h_{\min}(u)$$

for all $u \in [0, \Theta_{\min} - \frac{\varepsilon}{4}]$. So, (6.7) holds in this interval. When $u \in (\Theta_{\min} - \frac{\varepsilon}{4}, \Theta_{\min})$, by construction,

$$H(u) = \int_{\Theta_{\min}-\frac{\varepsilon}{4}}^{u} G(s)\,ds < 0 < h_{\min}(u),$$

and hence (6.7) holds in $[0, \Theta_{\min})$. Finally, when $u \ge \Theta_{\min}$, we find that

$$G(u) = \min\Big\{-\delta, \frac{4}{\varepsilon} h_{\min}\Big(u + \frac{\varepsilon}{4}\Big)\Big\} \le \frac{4}{\varepsilon} h_{\min}\Big(u + \frac{\varepsilon}{4}\Big) < 0$$

and, consequently,

$$\begin{aligned} H(u) = \int_{\Theta_{\min}-\frac{\varepsilon}{4}}^{u} G(s)\,ds &\le \frac{4}{\varepsilon} \int_{\Theta_{\min}-\frac{\varepsilon}{4}}^{u} h_{\min}\Big(s + \frac{\varepsilon}{4}\Big)\,ds = \frac{4}{\varepsilon} \int_{\Theta_{\min}}^{u+\frac{\varepsilon}{4}} h_{\min}(t)\,dt \\ &\le \frac{4}{\varepsilon} \int_{u}^{u+\frac{\varepsilon}{4}} h_{\min}(t)\,dt < \frac{4}{\varepsilon} \int_{u}^{u+\frac{\varepsilon}{4}} h_{\min}(u)\,dt = h_{\min}(u), \end{aligned}$$

which shows (6.7).

By Lemma 5.2, for sufficiently small $d > 0$, the following estimate holds:

$$\theta_{\{d,H\}}^{\mathrm{D},B_\rho(x_0)} \le \theta_{\{d,h\}}^{\mathcal{B},\Omega} \quad \text{in } B_\rho(x_0). \tag{6.9}$$

By the first part of the proof, since $H(u)$ does not depend on $x \in \Omega$, there exists $d_{x_0,\varepsilon} > 0$ such that

$$\Theta_H - \frac{\varepsilon}{4} \le \theta_{\{d,H\}}^{\mathrm{D},B_\rho(x_0)} \quad \text{in } \bar{B}_{\rho/2}(x_0) \text{ for all } d \in (0, d_{x_0,\varepsilon}). \tag{6.10}$$

Combining (6.6), (6.8), (6.9) and (6.10) yields

$$\Theta_h - \varepsilon \le \min_{\bar{B}_\rho(x_0)} \Theta_h - \frac{\varepsilon}{2} \le \Theta_H - \frac{\varepsilon}{4} \le \theta_{\{d,H\}}^{\mathrm{D},B_\rho(x_0)} \le \theta_{\{d,h\}}^{\mathcal{B},\Omega} \quad \text{in } B_{\rho/2}(x_0)$$

for all $d \in (0, d_{x_0,\varepsilon})$. Lastly, since K_0 is compact, there exist $x_1, \dots, x_n \in K_0$ such that $K_0 \subset \bigcup_{i=1}^n B_{\rho_i/2}(x_i)$. Therefore,

$$\Theta_h - \varepsilon \le \theta_{\{d,h\}}^{\mathcal{B},\Omega} \quad \text{in } K_0 \text{ for all } d < d_0 := \min_{1\le i\le n} d_{x_i,\varepsilon},$$

which ends the proof. □

We already have all the necessary tools to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Since h satisfies (H2), $\Theta_h \equiv 0$ if $\max_{\bar{\Omega}} h(0,\cdot) \le 0$. Should it be the case, the result is a direct consequence from Proposition 6.6. So, subsequently, we assume that

$$\max_{\bar{\Omega}} h(0,\cdot) > 0.$$

Then, by Corollary 5.1 (b), $\theta_{\{d,h\}} \gg 0$ for sufficiently small $d > 0$.

Thanks to Proposition 6.6, Theorem 1.2 holds on any compact subset of $\Omega \cup \Theta_h^{-1}(0)$. Hence, it remains to prove the theorem on a neighborhood of Γ_{R}^+. Let γ be a component of Γ_{R}^+. By the definition of Γ_{R}^+, we have that $\Theta_h(x) > 0$ for all $x \in \gamma$. By the continuity of Θ_h, there exists $\rho > 0$ such that

$$\varepsilon_0 := \min_{\bar{\Omega}_{\gamma,\rho}} \Theta_h > 0, \quad \text{where } \Omega_{\gamma,\rho} \equiv \{x \in \Omega : \operatorname{dist}(x,\gamma) < \rho\}.$$

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Pick $\varepsilon \in (0, \varepsilon_0)$. By the proof of Theorem 1.3, we can shorten ρ, if necessary, so that

$$\{x \in \Omega : \operatorname{dist}(x, \gamma) = \rho\} = \partial\Omega_{\gamma,\rho} \cap \Omega$$

is diffeomorphic to γ, and so of class $\mathcal{C}^2$. Hence, $\Omega_{\gamma,\rho}$ is an open subdomain of Ω with boundary of class $\mathcal{C}^2$, consisting of two components $\partial\Omega_{\gamma,\rho} \cap \Omega$ and γ for sufficiently small $\rho > 0$.

Subsequently, we consider the compact subset of Ω

$$K_{\gamma,\rho} := \{x \in \Omega : \rho/2 \le \operatorname{dist}(x, \gamma) \le \rho\}.$$

By Proposition 6.6, there exists $d_\rho > 0$ such that

$$\Theta_h - \frac{\varepsilon}{2} \le \theta_{\{d,h\}} \quad \text{in } K_{\gamma,\rho} \text{ for all } d < d_\rho. \tag{6.11}$$

By applying Lemma 6.3 (a) and Remark 6.4 with the choices

$$\xi_1(x) := \Theta_h(x) - \varepsilon \ (\ge \varepsilon_0 - \varepsilon > 0)$$

and

$$\xi_2(x) := \Theta_h(x) - \frac{3\varepsilon}{4} < \Theta_h(x), \quad x \in \bar{\Omega}_{\gamma,\rho/2},$$

there exists $\Phi \in \mathcal{C}^2(\bar{\Omega}_{\gamma,\rho/2})$ such that

$$\Theta_h - \varepsilon \le \Phi \le \Theta_h - \frac{3\varepsilon}{4} \quad \text{in } \Omega_{\gamma,\rho/2} \text{ and } \mathrm{R}\Phi \le 0 \text{ on } \gamma. \tag{6.12}$$

In particular, since $\partial\Omega_{\gamma,\rho/2} \cap \Omega \subset K_{\gamma,\rho}$, we may infer from (6.11) and (6.12) that

$$\theta_{\{d,h\}} \ge \Theta_h - \frac{\varepsilon}{2} = \Theta_h - \frac{3\varepsilon}{4} + \frac{\varepsilon}{4} \ge \Phi + \frac{\varepsilon}{4} \quad \text{on } \partial\Omega_{\gamma,\rho/2} \cap \Omega \text{ for all } d < d_\rho. \tag{6.13}$$

Moreover, by (H2), since $\Phi(x) < \Theta_h(x)$ for all $x \in \bar{\Omega}_{\gamma,\rho/2}$, we have that

$$\min_{x \in \bar{\Omega}_{\gamma,\rho/2}} h(\Phi(x), x) > \min_{x \in \bar{\Omega}_{\gamma,\rho/2}} h(\Theta_h(x), x) = 0.$$

Thus, shortening d_ρ, if necessary, so that

$$d_\rho < \frac{\min_{x \in \bar{\Omega}_{\gamma,\rho/2}} \Phi(x) h(\Phi(x), x)}{\max\{0, \max_{\bar{\Omega}_{\gamma,\rho/2}} \mathcal{L}\Phi\}},$$

we are driven to

$$d\mathcal{L}\Phi \le \Phi h(\Phi, \cdot) \quad \text{in } \Omega_{\gamma,\rho/2} \text{ for all } d < d_\rho.$$

Let us denote by $f \in \mathcal{C}(\bar{\Omega}_{\gamma,\rho/2})$ the function defined, for every $x \in \bar{\Omega}_{\gamma,\rho/2}$, through

$$f(x) := \begin{cases} \dfrac{\theta_{\{d,h\}}(x) h(\theta_{\{d,h\}}(x), x) - \Phi(x) h(\Phi(x), x)}{\theta_{\{d,h\}}(x) - \Phi(x)} & \text{if } \theta_{\{d,h\}}(x) \ne \Phi(x), \\ h(\Phi(x), x) + \Phi(x) \dfrac{\partial h}{\partial u}(\Phi(x), x) & \text{if } \theta_{\{d,h\}}(x) \ne \Phi(x). \end{cases}$$

Then, for every $d < d_\rho$, taking into account (6.13), the function $w := \theta_{\{d,h\}} - \Phi$ satisfies

$$\begin{cases} (d\mathcal{L} - f)w \ge 0 & \text{in } \Omega_{\gamma,\rho/2}, \\ \mathcal{B}w = \mathrm{R}w > 0 & \text{on } \gamma, \\ w \ge \dfrac{\varepsilon}{4} > 0 & \text{on } \partial\Omega_{\gamma,\rho/2} \cap \Omega. \end{cases}$$

Therefore, w provides us with a strict supersolution of $[d\mathcal{L} - f; \mathcal{B}_0, \Omega_{\gamma,\rho/2}]$, where

$$\mathcal{B}_0 := \begin{cases} \mathcal{B} & \text{on } \gamma, \\ \mathrm{D} & \text{on } \partial\Omega_{\gamma,\rho/2} \setminus \gamma. \end{cases}$$

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Since, owing to (H2), h is strictly decreasing in the first variable, $f \lneq h(\theta_{\{d,h\}}, \cdot)$ in $\Omega_{\gamma,\rho/2}$. Moreover, we already know that $\theta_{\{d,h\}} \gg 0$ for sufficiently small $d > 0$. Thus, it follows from Theorem 4.1 (a) and Lemma 4.3 that

$$\begin{aligned}\sigma_1[d\mathcal{L} - f; \mathcal{B}_0, \Omega_{\gamma,\rho/2}] &> \sigma_1[d\mathcal{L} - h(\theta_{\{d,h\}}, \cdot); \mathcal{B}_0, \Omega_{\gamma,\rho/2}] \\ &> \sigma_1[d\mathcal{L} - h(\theta_{\{d,h\}}, \cdot); \mathcal{B}, \Omega] = 0\end{aligned}$$

for sufficiently small $d > 0$. Therefore, due to [31, Theorem 7.10], and taking into account (6.12), we conclude that

$$\theta_{\{d,h\}} \gg \Phi \geq \Theta_h - \varepsilon \quad \text{in } \Omega_{\gamma,\rho/2} \text{ for sufficiently small } d > 0.$$

The proof is complete. □

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