J. Math. Anal. Appl. 412 (2014) 1105–1134

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Smoothing and perturbation for some fourth order linear parabolic equations in $\mathbb{R}^N$ ☆

Carlos Quesada [a], Aníbal Rodríguez-Bernal [a,b,*]

[a] *Departamento de Matemática Aplicada, Universidad Complutense de Madrid, Madrid 28040, Spain*
[b] *Instituto de Ciencias Matemáticas, CSIC–UAM–UC3M–UCM, Spain*

ARTICLE INFO

Article history:
Received 18 December 2012
Available online 20 November 2013
Submitted by A. Lunardi

Keywords:
Bi-Laplacian
Analytic semigroups
Perturbation
Smoothing
Bessel spaces
Uniform spaces

ABSTRACT

We solve some fourth order parabolic equations, obtained from perturbations of the parabolic bi-Laplacian equation, with special focus on smoothing estimates. Several classes of initial data are considered including data in Lebesgue and Bessel–Lebesgue spaces. Robustness and convergence with respect to the perturbation are also obtained. Also, initial data in large uniform Bessel–Lebesgue spaces are considered as well as equations with higher order powers of the Laplacian.

1. Introduction

In this paper we address the solvability of some fourth order linear parabolic equations in $\mathbb{R}^N$. More precisely, we consider

$$\begin{cases} u_t + \Delta^2 u + Pu = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

with u_0 a suitable initial data defined in $\mathbb{R}^N$ and P a linear perturbation. We will consider space dependent perturbations of the form $Pu := \sum_{a,b} P_{a,b}u$ with

$$P_{a,b}u := D^b\big(d(x)D^a u\big), \quad x \in \mathbb{R}^N, \tag{1.2}$$

for some $a, b \in \{0, 1, 2, 3\}$ such that $a + b \leqslant 3$, where D^a, D^b denote any partial derivatives of order a, b, and $d(x)$ is a given function with $x \in \mathbb{R}^N$.

Our main goal is to consider in (1.1) some large classes of initial data u_0 in $\mathbb{R}^N$ as well as to consider wide classes of low regularity perturbations. For the latter we will consider classes of coefficients $d(x)$ with weak integrability properties. More precisely, we will assume below that the coefficient $d(x)$ belongs to some locally uniform space $L^p_U(\mathbb{R}^N)$, $1 \leqslant p < \infty$, composed of the functions $f \in L^p_{loc}(\mathbb{R}^N)$ such that there exists $C > 0$ such that for all $x_0 \in \mathbb{R}^N$

☆ Partially supported by Projects MTM2009-07540, MTM2012-31298, MEC and GR58/08 Grupo 920894, UCM, Spain.
* Corresponding author.

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.11.032

$$\int_{B(x_0,1)} |f|^p \leqslant C$$

endowed with the norm $\|f\|_{L^p_U(\mathbb{R}^N)} = \sup_{x_0\in\mathbb{R}^N} \|f\|_{L^p(B(x_0,1))}$. For $p=\infty$, $L^\infty_U(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$.

As for the initial data we will consider the standard Lebesgue space, $L^q(\mathbb{R}^N)$, $1<q<\infty$, or Bessel–Lebesgue spaces $H^{\alpha,q}(\mathbb{R}^N)$, with $1<q<\infty$, $\alpha\in\mathbb{R}$, and even uniform Bessel spaces $\dot{H}^{\alpha,q}_U(\mathbb{R}^N)$ to be introduced below.

Given such classes of initial data and perturbations we want to find suitable smoothing estimates on the solutions of (1.1) as will be explained below.

Note that for $P=0$ the solution of problem (1.1) can be described as the convolution of the initial data with the self-similar fundamental kernel for the bi-Laplacian operator, which satisfies suitable Gaussian bounds; see e.g. [12,13] and [11,7].

Recently, results in Bessel–Lebesgue spaces have been proved in [9] for $P\neq 0$. By means of resolvent estimates for Δ^2+P, the authors proved the well posedness of (1.1) with $Pu=d(x)u$, that is, a perturbation with $a,b=0$. They also found suitable smoothing estimates on the solutions as the ones we will find in (1.2).

Here, instead of relying on elliptic resolvent estimates for the operators Δ^2+P, with P as in (1.2), we rely on a more abstract "parabolic" argument developed in [16] and applied there to parabolic equations with second order elliptic operators. With this approach we consider a simpler problem, the one with $P=0$, that we can solve in several spaces simultaneously. That is, we consider a semigroup of solutions defined on a scale of spaces. For such simpler problem we start by proving suitable smoothing estimates on the spaces of the scale. Then we consider a suitable perturbation, P, that acts between two spaces of the scale. With these ingredients the abstract results in [16] allow to obtain a perturbed semigroup that corresponds to Eq. (1.1) with $P\neq 0$. Such a perturbed semigroup inherits some of the smoothing estimates of the original one in some of the spaces of the scale which are determined by the perturbation P itself.

Another important result that we are able to stablish using the tools developed in [16], is that of the robustness with respect to the perturbation. In this direction, we are able to prove two important results. First, we show that all constants involved in the smoothing estimates of the perturbed semigroups, including the exponential bounds on them, are bounded uniformly for bounded families of perturbations (i.e. for families of coefficients $d(x)$ as in (1.2) which are bounded in the uniform space $L^p_U(\mathbb{R}^N)$). Second, we prove that the perturbed semigroups obtained as above, continuously depend on the perturbation. That is, if the coefficients $d(x)$ depend on a parameter and converge in the space $L^p_U(\mathbb{R}^N)$, then the corresponding semigroups converge in norm.

As mentioned above this approach was applied in [16] to second order parabolic equations in bounded and unbounded domains, allowing perturbations in the equation and in the boundary conditions.

In this paper we carry out these ideas to fourth order parabolic equations in $\mathbb{R}^N$ as in (1.1). For that, we use an existence and regularity theory in suitable scales of spaces for the parabolic bi-Laplacian equation, i.e. (1.1) with $P=0$, in order to later introduce the perturbations. For this we use some available information about the heat equation $u_t-\Delta u=0$, in $\mathbb{R}^N$ and use that Δ^2 is the square operator of $-\Delta$. In particular, the same scales of spaces available for $-\Delta$ can be used for (1.1). In such scales suitable smoothing estimates for (1.1) with $P=0$ are obtained.

We now state one of the main results that we prove below, see Theorem 2.10. Note that this result applies in the Bessel–Lebesgue scale. A similar one, with technical differences, holds in the uniform Bessel scale, see Theorem 3.7.

Theorem 1.1. *Let $P_{a,b}$ be as in (1.2) with $k,a,b\in\{0,1,2,3\}$, $k=a+b$. Assume that $\|d\|_{L^p_U(\mathbb{R}^N)}\leqslant R_0$ with $p>\frac{N}{4-k}$.*

Then, for any $1<q<\infty$ and such $P_{a,b}$, there exists an interval $I(q,a,b)\subset(-1+\frac{a}{4},1-\frac{b}{4})$ containing $(-1+\frac{a}{4}+\frac{N}{4p},1-\frac{b}{4}-\frac{N}{4p})$, such that for any $\gamma\in I(q,a,b)$, we have a strongly continuous analytic semigroup, $S_{P_{a,b}}(t)$, in the space $H^{4\gamma,q}(\mathbb{R}^N)$, for the problem

$$\begin{cases} u_t+\Delta^2u+D^b\big(d(x)D^au\big)=0, & x\in\mathbb{R}^N,\ t>0,\\ u(0)=u_0 & in\ \mathbb{R}^N.\end{cases}$$

Moreover the semigroup has the smoothing estimates

$$\big\|S_{P_{a,b}}(t)u_0\big\|_{H^{4\gamma',q}(\mathbb{R}^N)}\leqslant\frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{H^{4\gamma,q}(\mathbb{R}^N)},\quad t>0,\ u_0\in H^{4\gamma,q}\big(\mathbb{R}^N\big),$$

for every $\gamma,\gamma'\in I(q,a,b)$ with $\gamma'\geqslant\gamma$, and

$$\big\|S_{P_{a,b}}(t)u_0\big\|_{L^r(\mathbb{R}^N)}\leqslant\frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q(\mathbb{R}^N)},\quad t>0,\ u_0\in L^q\big(\mathbb{R}^N\big),$$

for $1<q\leqslant r\leqslant\infty$, with some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu\in\mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q,a,b)$ is given by

$$I(q,a,b)=\left(-1+\frac{a}{4}+\frac{N}{4}\left(\frac{1}{p}-\frac{1}{q'}\right)_+,1-\frac{b}{4}-\frac{N}{4}\left(\frac{1}{p}-\frac{1}{q}\right)_+\right).$$

Finally, if

$$d_\varepsilon \to d \quad \text{in } L^p_U(\mathbb{R}^N),\ p > \frac{N}{4-k},$$

then for every $1 < q < \infty$ *and* $T > 0$ *there exists* $C(\varepsilon) \to 0$ *as* $\varepsilon \to 0$*, such that*

$$\left\| S_{P_\varepsilon}(t) - S_P(t) \right\|_{\mathcal{L}(H^{4\gamma,q}(\mathbb{R}^N), H^{4\gamma',q}(\mathbb{R}^N))} \leqslant \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \quad \forall 0 < t \leqslant T,$$

for every $\gamma, \gamma' \in I(q,a,b)$*,* $\gamma' \geqslant \gamma$ *and for every* $1 < q \leqslant r \leqslant \infty$

$$\left\| S_{P_\varepsilon}(t) - S_P(t) \right\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \leqslant \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \quad \forall 0 < t \leqslant T.$$

Note the ranges of spaces for which we can solve the equation are determined by the base space in terms of $1 < q < \infty$, the integrability p of the coefficient $d(x)$ and the order of derivatives a, b. Observe that in the theorem above just one perturbation $P_{a,b}$ is considered. Several perturbations can be thus combined together, although not all combinations are allowed. We discuss below a general procedure to determine whether or not some given perturbations can be combined together; see Proposition 2.12 and Remark 2.14. Also Eq. (1.1) is satisfied in the sense explained in Remark 2.11.

The paper is organized as follows. In Section 2 we prove that Δ^2 defines an analytic semigroup in the scales of Lebesgue and Bessel–Lebesgue spaces, which satisfy suitable smoothing estimates; see Lemma 2.2. Then using the results in [16] we are able to add perturbations to the equation along the lines described above, see Lemma 2.6 and Theorem 2.10. Some extension to fractional-like derivatives in (1.2) can be found in Theorem 2.15. In this case a, b are nonnegative real and $0 \leqslant a+b < 4$.

The same strategy is carried out in Section 3 for (1.1) in the uniform Bessel–Lebesgue scale. These spaces have been used for linear and nonlinear heat equations in [4,6,8]. Such spaces are very useful because, among other properties, they are very large spaces whose functions do not satisfy any smallness behavior at infinity and contain the standard Bessel–Lebesgue spaces as closed subspaces. Also, these spaces contain constant functions and are embedded as Lebesgue spaces are in a bounded domain.

After some result on these spaces in Proposition 3.1 that complements the ones in [4], we obtain resolvent estimates for the Laplacian operator that prove that it is sectorial. Then in Lemma 3.4 we show that the bi-Laplacian parabolic equation defines an analytic semigroup with suitable smoothing estimates in the uniform Bessel–Lebesgue scale. In Theorem 3.7 we introduce the perturbations and prove an analogous result to Theorem 1.1 in this scale. Note that since uniform spaces are not reflexive (even for $q = 2$) we can only consider perturbations as in (1.2) with $b = 0$, see Theorem 3.7.

Finally, in Section 4 we show how to obtain all the results in Sections 2 and 3 for other powers of the Laplacian $(-\Delta)^m$, $m \in \mathbb{N}$ as the main part in the elliptic operator.

At the end of the paper we have included four appendixes where we collect several results that we need for the main arguments in the previous sections. In particular, results from [16] are collected in Appendix A; some results from [2] are in Appendix B and some extension of them can be found in Appendix C. Appendix D contains some extensions to higher order operators.

The results presented here will be used somewhere else to study fourth order nonlinear problems.

2. Some fourth order equations in the Bessel–Lebesgue spaces in $\mathbb{R}^N$

We take, $A_0 = -\Delta$ in $L^q(\mathbb{R}^N)$, with $1 < q < \infty$ with domain $D(A_0) = H^{2,q}(\mathbb{R}^N)$, where $H^{k,q}(\mathbb{R}^N)$, $k \in \mathbb{N}$ denotes the standard Sobolev spaces (often denoted $W^{k,q}(\mathbb{R}^N)$). In this setting, $-\Delta$ is a sectorial operator, see [14,3], and $\text{type}(-\Delta) := \inf\{\text{Re}(\sigma(-\Delta))\} = 0$.

Using complex interpolation, these spaces can be extended to non-integer indexes, known as Bessel spaces. These spaces are very convenient because they satisfy the sharp Sobolev embeddings

$$H^{s,q}(\mathbb{R}^N) \subset \begin{cases} L^r(\mathbb{R}^N),\ s - \frac{N}{q} \geqslant -\frac{N}{r},\ q \leqslant r < \infty, & \text{if } s - \frac{N}{q} < 0, \\ L^r(\mathbb{R}^N),\ 1 \leqslant r < \infty, & \text{if } s - \frac{N}{q} = 0, \\ C^\eta(\mathbb{R}^N) & \text{if } s - \frac{N}{q} > \eta \geqslant 0. \end{cases}$$

Also, for the negative indexes, we have

$$H^{-s,q}(\mathbb{R}^N) = \left(H^{s,q'}(\mathbb{R}^N)\right)'. \tag{2.1}$$

For more details, see [14, p. 35], [1,3] [2, I.2] or [18]. In what follows we will denote $E_\alpha := H^{2\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, the Bessel–Lebesgue scale of spaces.

Also it is known, see [14] and [1], that for $1 < q < \infty$ the heat equation

$$\begin{cases} u_t - \Delta u = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(0) = u_0, & \text{in } \mathbb{R}^N, \end{cases} \tag{2.2}$$

defines a semigroup $S_{-\Delta}(t)$ in the scale of Bessel spaces $\{E_\alpha\}_{\alpha\in\mathbb{R}} := \{H^{2\alpha,q}(\mathbb{R}^N)\}_{\alpha\in\mathbb{R}}$ that satisfies the smoothing estimates

$$\left\| S_{-\Delta}(t)u_0 \right\|_{H^{2\alpha,q}(\mathbb{R}^N)} \leqslant \frac{M_{\alpha,\beta}e^{\mu_0 t}}{t^{\alpha-\beta}} \|u_0\|_{H^{2\beta,q}(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in H^{2\beta,q}\left(\mathbb{R}^N\right),$$

for $1 < q < \infty$, $\alpha, \beta \in \mathbb{R}$, $\alpha \geqslant \beta$, and

$$\left\| S_{-\Delta}(t)u_0 \right\|_{L^r(\mathbb{R}^N)} \leqslant \frac{M_{r,q}e^{\mu_0 t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{L^q(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in L^q\left(\mathbb{R}^N\right),$$

for $1 \leqslant q \leqslant r \leqslant \infty$ and some constant $M_{r,q}$. In both estimates above $\mu_0 > 0$ can be arbitrarily small, because $\text{type}(-\Delta) = 0$.

This as well as some other useful properties of $-\Delta$ and Δ^2 in $L^q(\mathbb{R}^N)$, $1 < q < \infty$, are collected in the next lemma.

Lemma 2.1. *Take $1 < q < \infty$ and denote $E_0 = L^q(\mathbb{R}^N)$.*

i) *The Laplace operator $-\Delta$ in E_0 with domain $E_1 = D(-\Delta) = H^{2,q}(\mathbb{R}^N)$ satisfies the estimate*

$$\left\| (-\Delta - \lambda)^{-1} \right\|_{\mathcal{L}(E_0)} \leqslant M|\lambda|^{-1} \quad \text{for all } \lambda \in S_{0,\phi}$$

for the sector

$$S_{a,\phi} = \left\{ z \in \mathbb{C}\colon\ \phi \leqslant \left|\arg(z-a)\right| \leqslant \pi,\ z \neq a \right\} \subset \rho(A_0) \tag{2.3}$$

with $\phi > 0$ arbitrarily small. Furthermore $\sigma(-\Delta) = [0, \infty)$ and therefore

$$\text{type}(-\Delta) = \inf\left\{ \text{Re}\left(\sigma(-\Delta)\right) \right\} = 0.$$

ii) *The bi-Laplacian operator Δ^2 in E_0 with domain $E_2 = D(\Delta^2) = H^{4,q}(\mathbb{R}^N)$ satisfies the estimate*

$$\left\| \left(\Delta^2 - \lambda\right)^{-1} \right\|_{\mathcal{L}(E_0)} \leqslant M|\lambda|^{-1} \quad \text{for all } \lambda \in S_{0,2\phi}$$

with $\phi > 0$ arbitrarily small. Furthermore $\sigma(\Delta^2) = [0, \infty)$ and therefore

$$\text{type}\left(\Delta^2\right) = \inf\left\{ \text{Re}\left(\sigma\left(\Delta^2\right)\right) \right\} = 0.$$

Proof. The first part, for the Laplacian, is well known. The resolvent estimate, in particular, can be found in pages 32 and 33 of [14].

For proving ii), since in i) $\phi > 0$ can be taken arbitrarily small, we can apply Proposition 10.5 in [15] (see also Proposition C.1) and we get that Δ^2 is sectorial with sector $S_{0,2\phi}$, where $2\phi > 0$ can be arbitrarily small. Then $\sigma(\Delta^2) \subset [0, \infty)$ is an immediate consequence of the fact that $\phi > 0$ is arbitrarily small. On the other hand, it can be proved that in the uniform space $\dot{L}^q_U(\mathbb{R}^N)$, we have $\sigma(\Delta^2) = [0, \infty)$, see Proposition 3.3 below for more details. Then, $\sigma(\Delta^2) = [0, \infty)$ in $L^q(\mathbb{R}^N)$ as well. From this, we get $\text{type}(\Delta^2) = 0$. □

Then we can prove the following.

Lemma 2.2. *Consider the problem*

$$\begin{cases} u_t + \Delta^2 u = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(0) = u_0, & \text{in } \mathbb{R}^N. \end{cases} \tag{2.4}$$

i) *Then for each $1 < q < \infty$, (2.4) defines an analytic semigroup, $S_{\Delta^2}(t)$, in the scale $X_\alpha = E_{2\alpha} = H^{4\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, such that for any $\mu_0 > 0$ there exists C such that*

$$\left\| S_{\Delta^2}(t) \right\|_{\mathcal{L}(H^{4\beta,q}(\mathbb{R}^N), H^{4\alpha,q}(\mathbb{R}^N))} \leqslant \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\mu_0 t}, \quad t > 0,\ \alpha, \beta \in \mathbb{R},\ \alpha \geqslant \beta.$$

ii) *The analytic semigroup $S_{\Delta^2}(t)$, in $L^q(\mathbb{R}^N)$, $1 < q < \infty$, satisfies*

$$\left\| S_{\Delta^2}(t) \right\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \leqslant \frac{M_{q,r}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} e^{\mu_0 t}, \quad t > 0,$$

for any $\mu_0 > 0$ and $1 < q \leqslant r \leqslant \infty$ and some $M_{q,r} > 0$ (which also depends on μ_0).

Proof. i) This is a consequence of Proposition C.2 for $A_0 = -\Delta$.

Note that from Lemma 2.1, type$(\Delta^2) = 0$ and then $\mu_0 > 0$ is arbitrary.

ii) For $1 < q < \infty$, we use i) with $\alpha = 0$ and we have that Δ^2 defines an analytic semigroup in $L^q(\mathbb{R}^N)$.

Now, if $r \geqslant q$ we use i) again, now with $\beta = 0$, and choosing α such that $-\frac{N}{r} = 4\alpha - \frac{N}{q}$ we get

$$\left\| S_{\Delta^2}(t)u_0 \right\|_{L^r(\mathbb{R}^N)} \leqslant \left\| S_{\Delta^2}(t)u_0 \right\|_{H^{4\alpha,q}(\mathbb{R}^N)} \leqslant \frac{M_\alpha e^{\mu_0 t}}{t^\alpha} \|u_0\|_{L^q(\mathbb{R}^N)},$$

which leads to

$$\left\| S_{\Delta^2}(t)u_0 \right\|_{L^r(\mathbb{R}^N)} \leqslant \frac{M_{r,q} e^{\mu_0 t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{L^q(\mathbb{R}^N)}.$$

Again, because of part ii) of Lemma 2.1, type$(\Delta^2) = 0$ and then $\mu_0 > 0$ is arbitrary. □

Remark 2.3. For $q = 1$, if we take any $r > 1$ and any $\beta > \frac{N}{4r'}$ then we have $H^{4\beta,r'}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ and therefore $L^1(\mathbb{R}^N) \hookrightarrow H^{-4\beta,r}(\mathbb{R}^N)$.

Now using i) with $\alpha = 0$ we get

$$\left\| S_{\Delta^2}(t)u_0 \right\|_{L^r(\mathbb{R}^N)} \leqslant \frac{M_{r,1} e^{\mu_0 t}}{t^\beta} \|u_0\|_{H^{-4\beta,r}(\mathbb{R}^N)} \leqslant \frac{M_{r,1} e^{\mu_0 t}}{t^\beta} \|u_0\|_{L^1(\mathbb{R}^N)}$$

for any $\beta > \frac{N}{4}(1 - \frac{1}{r})$. Hence we obtain the estimate in ii) for $q = 1$ and any $r > 1$, for an exponent as close as we want to $\frac{N}{4}(1 - \frac{1}{r})$.

Remark 2.4. Observe that the solution of problem (2.4) can be described as the convolution of the initial data with the self-similar kernel for the bi-Laplacian operator, which satisfies suitable Gaussian bounds; see e.g. [12,13] and [11,7].

Remark 2.5.

i) Observe that the Bessel spaces described above, naturally appear as a result of an abstract procedure, using complex interpolation, to construct spaces associated to sectorial operators; see e.g. [2,1] and Appendix B.

ii) Note that using [3, 9.7, p. 648] we get that $-\Delta$ has bounded imaginary powers in $L^q(\mathbb{R}^N)$ for $1 < q < \infty$. Hence, because of [2, V.1.5.13, p. 283], see also Remark B.4, the Bessel spaces described above coincide with the usual fractional power spaces of this operator, see [14].

iii) Also, some results on sectorial operators that apply to other higher order differential operators instead of Δ^2 in (2.4) can be found in Theorem 5.5 in [10]. These operators have always bounded imaginary powers (see (2.15), page 25 in [10]). Hence again their complex interpolation scale and their fractional power scale coincide, again by [2, V.1.5.13, p. 283] (see Remark B.4). Note however that Theorem 5.5 in [10] does not give the description of these spaces.

Now we can use the results in [16] to perturb Eq. (2.4). For this, let D^r denote any partial derivative of order $r \in \mathbb{N}$ and fix $m \in \mathbb{N}$. Then if $m \geqslant r$, we have $D^r : H^{m,q}(\mathbb{R}^N) \to H^{m-r,q}(\mathbb{R}^N)$. On the other hand, $D^r : H^{-m,q}(\mathbb{R}^N) \to H^{-m-r,q}(\mathbb{R}^N)$, is defined as

$$\langle D^r u, \varphi \rangle = (-1)^r \int_{\mathbb{R}^N} u D^r \varphi \quad \text{for all } \varphi \in H^{m+r,q'}(\mathbb{R}^N).$$

Finally, if $m < r$, $D^r : H^{m,q}(\mathbb{R}^N) \to H^{m-r,q}(\mathbb{R}^N)$ is defined as

$$\langle D^r u, \varphi \rangle = (-1)^{r-m} \int_{\mathbb{R}^N} D^m u D^{r-m} \varphi \quad \text{for all } \varphi \in H^{r-m,q'}(\mathbb{R}^N)$$

which corresponds to the composition $D^r = D^{r-m} D^m$, where $D^m : H^{m,q}(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ and $D^{r-m} : L^q(\mathbb{R}^N) \to H^{m-r,q}(\mathbb{R}^N)$.

Thus for any $1 < q < \infty$, $r \in \mathbb{N}$ and $m \in \mathbb{Z}$, we have

$$D^r \in \mathcal{L}\big(H^{m,q}(\mathbb{R}^N), H^{m-r,q}(\mathbb{R}^N)\big), \qquad \left\| D^r \right\|_{\mathcal{L}(H^{m,q}(\mathbb{R}^N), H^{m-r,q}(\mathbb{R}^N))} \leqslant C$$

for some C independent of r, m, q.

Now we extend this definition to non-integer m. For this take $m \in \mathbb{Z}$ and $s \in (m, m+1)$ and take $\theta \in (0,1)$ such that $s = \theta m + (1-\theta)(m+1)$. Then by interpolation

$$D^r : \big[H^{m+1,q}(\mathbb{R}^N), H^{m,q}(\mathbb{R}^N)\big]_\theta = H^{s,q}(\mathbb{R}^N) \to \big[H^{m+1-r,q}(\mathbb{R}^N), H^{m-r,q}(\mathbb{R}^N)\big]_\theta = H^{s-r,q}(\mathbb{R}^N),$$

and we get that for any $r \in \mathbb{N}$ and $s \in \mathbb{R}$

$$D^r \in \mathcal{L}\big(H^{s,q}\big(\mathbb{R}^N\big), H^{s-r,q}\big(\mathbb{R}^N\big)\big), \qquad \big\|D^r\big\|_{\mathcal{L}(H^{s,q}(\mathbb{R}^N),H^{s-r,q}(\mathbb{R}^N))} \leqslant C \tag{2.5}$$

for some C independent of r, s, q. Note that above we denoted by $[\cdot,\cdot]_\theta$ the complex interpolation functor, see [2] and [18].

Using this and the results in [16] we get the following result in which we allow perturbations with derivatives of order $k \leqslant 3$.

Proposition 2.6. *Take $J \in \mathbb{N}$ and $a_j \in \mathbb{R}$, $k_j \in \mathbb{N}$ for $j = 1,\dots,J$ with $\max_j |a_j| \leqslant R_0$ and $k = \max_j |k_j| \leqslant 3$. Then for each $1 < q < \infty$ the problem*

$$\begin{cases} u_t + \Delta^2 u + \displaystyle\sum_{j=1}^{J} a_j D^{k_j} u = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{2.6}$$

defines an analytic semigroup, $S(t)$, on the scale $\{X_\alpha\}_{\alpha\in\mathbb{R}}$ with $X_\alpha = E_{2\alpha} = H^{4\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, such that

$$\big\|S(t)\big\|_{\mathcal{L}(H^{4\gamma,q}(\mathbb{R}^N),H^{4\gamma',q}(\mathbb{R}^N))} \leqslant \frac{C(\gamma'-\gamma)}{t^{\gamma'-\gamma}} e^{\mu t}, \quad t > 0,\ \gamma,\gamma' \in \mathbb{R},\ \gamma' \geqslant \gamma,$$

and also

$$\big\|S(t)\big\|_{\mathcal{L}(L^q(\mathbb{R}^N),L^r(\mathbb{R}^N))} \leqslant \frac{C(q,r)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} e^{\mu t}, \quad t > 0,$$

for $1 < q \leqslant r \leqslant \infty$, with μ, $C(\gamma'-\gamma)$, $C(q,r)$ depending on $\{a_j\}$ only through R_0. The constant $C(\gamma'-\gamma)$ is bounded for γ,γ' in bounded sets of $\mathbb{R}$.

Furthermore, if for all $j = 1,\dots,J$, we have $a_j^\varepsilon \to a_j$ as $\varepsilon \to 0$ then for any $T > 0$, $\gamma' \geqslant \gamma$ or $r \geqslant q$, there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that the corresponding semigroups satisfy

$$\big\|S_\varepsilon(t) - S(t)\big\|_{\mathcal{L}(H^{4\gamma,q}(\mathbb{R}^N),H^{4\gamma',q}(\mathbb{R}^N))} \leqslant \frac{C(\varepsilon)}{t^{\alpha-\beta}}, \quad \forall 0 < t \leqslant T,$$

and

$$\big\|S_\varepsilon(t) - S(t)\big\|_{\mathcal{L}(L^q(\mathbb{R}^N),L^r(\mathbb{R}^N))} \leqslant \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \quad \forall 0 < t \leqslant T,$$

for $1 < q \leqslant r \leqslant \infty$.

Proof. Since $X_\alpha = E_{2\alpha} = H^{4\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, we get from Lemma 2.2 i) that

$$\big\|S_{\Delta^2}(t)\big\|_{\mathcal{L}(X_\beta,X_\alpha)} \leqslant \frac{C}{t^{\alpha-\beta}}, \quad 0 < t \leqslant 1,\ \alpha,\beta \in \mathbb{R},\ \alpha \geqslant \beta.$$

From (2.5) each of the perturbations $P_j = a_j D^{k_j}$ satisfies $\|P_j\|_{\mathcal{L}(X_\alpha,X_{\alpha-k_j/4})} \leqslant C$ for all $\alpha \in \mathbb{R}$ with $C = C(R_0)$ independent of j, and we have that

$$P = \sum_{j=1}^{J} P_j \in \mathcal{L}(X_\alpha, X_{\alpha-k/4}), \qquad \|P\|_{\mathcal{L}(X_\alpha,X_{\alpha-k/4})} \leqslant C(J,R_0), \quad \alpha \in \mathbb{R}.$$

Hence, we can apply [16, Proposition 10] (see also Theorem A.1) with $\alpha \in \mathbb{R}$, $\beta = \alpha - \frac{k}{4}$ and since the scale is nested, we get a semigroup $S(t) = S_P(t)$ in X_γ for any $\gamma \in E(\alpha) := (\alpha-1,\alpha]$ that satisfies the smoothing estimates

$$\big\|S(t)\big\|_{\mathcal{L}(X_\gamma,X_{\gamma'})} \leqslant \frac{M_{\gamma,\gamma'} e^{\mu t}}{t^{\gamma'-\gamma}} \tag{2.7}$$

with μ depending on R_0 and for every γ,γ' such that

$$\gamma \in E(\alpha) := (\alpha-1,\alpha], \qquad \gamma' \in R(\beta) := [\beta,\beta+1) = [\alpha-k/4, \alpha-k/4+1), \qquad \gamma' \geqslant \gamma.$$

We now want to see the largest range for γ and γ', in which (2.7) holds, that can be achieved in 2 "jumps" in the scale. For this we perform a bootstrap argument as follows. Given $\alpha \in \mathbb{R}$, take $\beta = \alpha - \frac{k}{4}$ as above. In this situation the semigroup transforms X_γ into $X_{\gamma'}$ for any $\gamma \in E(\alpha)$ to any $\gamma' \in R(\beta)$. We now choose an $\alpha' > \alpha$ such that $\beta < \alpha' - 1 < \beta + 1 =$

$\alpha - \frac{k}{4} + 1$, so $R(\beta) \cap E(\alpha') \neq \emptyset$. Then we can "jump" again, starting from any space $X_{\gamma'}$, $\gamma' \in R(\beta) \cap E(\alpha')$, into $X_{\gamma''}$ with $\gamma'' \in R(\beta')$, $\beta' = \alpha' - \frac{k}{4}$. Schematically, we write

$$\gamma \in E(\alpha) \to \gamma' \in R(\beta) \cap E(\alpha') \to \gamma'' \in R(\beta').$$

Then, using $S(t) = S(t/2) \circ S(t/2)$ we get

$$\|S(t)u_0\|_{\gamma''} \leqslant \frac{\tilde{M} e^{\mu(t/2)}}{(t/2)^{\gamma''-\gamma'}} \|S(t/2)u_0\|_{\gamma'} \leqslant \frac{\tilde{M} e^{\mu(t/2)}}{(t/2)^{\gamma''-\gamma'}} \frac{\tilde{M} e^{\mu(t/2)}}{(t/2)^{\gamma'-\gamma}} \|u_0\|_{\gamma} = \frac{M e^{\mu t}}{t^{\gamma''-\gamma}} \|u_0\|_{\gamma} \tag{2.8}$$

for $\gamma \in E(\alpha) = (\alpha - 1, \alpha]$ and $\gamma'' \in R(\beta') = [\beta', \beta' + 1)$ and M depending on γ and γ''. Note that the range for $R(\beta')$ moves continuously as we move α', thus

$$\gamma'' \in \bigcup_{\beta < \alpha' - 1 < \beta + 1} R(\beta') = \bigcup_{\beta < \alpha' - 1 < \beta + 1} [\beta', \beta' + 1) = \bigcup_{\beta < \alpha' - 1 < \beta + 1} \left[\alpha' - \frac{k}{4}, \alpha' - \frac{k}{4} + 1\right)$$
$$= \left(\beta + 1 - \frac{k}{4}, \beta + 3 - \frac{k}{4}\right) = \left(\alpha - \frac{2k}{4} + 1, \alpha - \frac{2k}{4} + 3\right).$$

Hence, after one or two "jumps" we get the estimate (2.7) for any $\gamma \in E(\alpha) := (\alpha - 1, \alpha]$ and $\gamma' \in [\alpha - \frac{k}{4}, \alpha - \frac{2k}{4} + 3)$ with $\gamma' \geqslant \gamma$.

Note that this argument can be repeated to obtain that we can have in (2.7) $\gamma' \in [\alpha - \frac{k}{4}, \alpha - \frac{nk}{4} + (2n-1))$ for any $n \in \mathbb{N}$. So, since $\alpha \in \mathbb{R}$ is arbitrary, after a finite number of iterations we get (2.7) for any $\gamma, \gamma' \in \mathbb{R}$, $\gamma' > \gamma$.

Now, if $1 < q < \infty$ and $r \geqslant q$ we take $\gamma = 0$ and γ' such that $H^{4\gamma',q}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$, that is $-\frac{N}{r} = 4\gamma' - \frac{N}{q}$. Then we get

$$\|S(t)u_0\|_{L^r(\mathbb{R}^N)} \leqslant C\|S(t)u_0\|_{H^{4\gamma',q}(\mathbb{R}^N)} \leqslant \frac{C(\gamma')e^{\mu t}}{t^{\gamma'}} \|u_0\|_{L^q(\mathbb{R}^N)} = \frac{C_{q,r} e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q} - \frac{1}{r})}} \|u_0\|_{L^q(\mathbb{R}^N)}.$$

The analyticity comes again from Lemma 2.2 and [16, Theorem 12] (see Theorem A.3).

The convergence of the semigroups is consequence of [16, Theorem 14] (see Theorem A.2) since if $a_j^\varepsilon \to a_j$ we would have $P_\varepsilon \to P$ in $\mathcal{L}(X_\alpha, X_{\alpha - k/4})$ as $\varepsilon \to 0$ for any $\alpha \in \mathbb{R}$. □

Remark 2.7. For a similar result with $q = 1$, we can proceed as in Remark 2.3.

Remark 2.8. Note that the estimates in Lemma 2.2 and Proposition 2.6 give that the solutions of problems (2.4) and (2.6) satisfy that $u(t) \in H^{4\gamma',r}(\mathbb{R}^N)$, for all $t > 0$, $\gamma' \in \mathbb{R}$ and $q \leqslant r < \infty$. Since the semigroups are analytic we have $u_t(t) \in H^{4\gamma',r}(\mathbb{R}^N)$ as well. Therefore, (2.4) and (2.6) are satisfied in a classical sense.

Finally, we study more general perturbations in which we allow a space dependence. For this, take $k \in \mathbb{N}$, which is the order of the perturbation, and take $a, b \in \mathbb{N}$ such that $a + b = k$. We define $P_{a,b}$ to be a perturbation of the form

$$P_{a,b}u = D^b\big(d(x)D^a u\big), \quad x \in \mathbb{R}^N,$$

for a given function $d(x)$ with $x \in \mathbb{R}^N$, in the sense that for any smooth enough φ

$$\langle P_{a,b}u, \varphi\rangle = (-1)^b \int_{\mathbb{R}^N} d(x) D^a u D^b \varphi. \tag{2.9}$$

We will assume below that the coefficient $d(x)$ belongs to the locally uniform space $L^p_U(\mathbb{R}^N)$ composed of the functions $f \in L^p_{loc}(\mathbb{R}^N)$ such that there exists $C > 0$ such that for all $x_0 \in \mathbb{R}^N$

$$\int_{B(x_0,1)} |f|^p \leqslant C \tag{2.10}$$

endowed with the norm

$$\|f\|_{L^p_U(\mathbb{R}^N)} = \sup_{x_0 \in \mathbb{R}^N} \|f\|_{L^p(B(x_0,1))}$$

(for $p = \infty$, $L^\infty_U(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$).

The following result states the spaces of the Bessel scale between which a perturbation $P_{a,b}$ is a well behaved linear operator. For this, below we will denote $(x)_- = \min\{0, x\}$ and $(x)_+ = \max\{0, x\}$, the negative and positive parts of $x \in \mathbb{R}$.

Proposition 2.9. *Let $P_{a,b}$ be as above, $d \in L^p_U(\mathbb{R}^N)$ and let $s \geqslant a$, $\sigma \geqslant b$. Then for $1 < q < \infty$ and*

$$\left(s - a - \frac{N}{q}\right)_- + \left(\sigma - b - \frac{N}{q'}\right)_- \geqslant -\frac{N}{p'} \tag{2.11}$$

we have

$$P_{a,b} \in \mathcal{L}\big(H^{s,q}(\mathbb{R}^N), H^{-\sigma,q}(\mathbb{R}^N)\big), \qquad \|P_{a,b}\|_{\mathcal{L}(H^{s,q}(\mathbb{R}^N),H^{-\sigma,q}(\mathbb{R}^N))} \leqslant C\|d\|_{L^p_U(\mathbb{R}^N)}.$$

Proof. Let $\{Q_i\}$, $i \in \mathbb{Z}^N$, be a partition of $\mathbb{R}^N$ in open disjoint cubes centered in $i \in \mathbb{Z}^N$ with sides of length 1, parallel to the axes. Note that $\mathbb{R}^N = \bigcup_{i\in\mathbb{Z}^N} \overline{Q_i}$ and $Q_i \cap Q_j = \emptyset$ for $i \neq j$. Then

$$\left|\int_{\mathbb{R}^N} dD^a u D^b\varphi\right| \leqslant \sum_i \left|\int_{Q_i} dD^a u D^b\varphi\right| \leqslant \sum_i \left(\int_{Q_i} |d|^p\right)^{\frac{1}{p}} \left(\int_{Q_i} \left|D^a u\right|^n\right)^{\frac{1}{n}} \left(\int_{Q_i} \left|D^b\varphi\right|^\tau\right)^{\frac{1}{\tau}}$$

where we have applied Hölder's inequality with $\frac{1}{p} + \frac{1}{n} + \frac{1}{\tau} = 1$. If (2.11) holds, we can choose n, τ as before such that $s - \frac{N}{q} \geqslant a - \frac{N}{n}$ and $\sigma - \frac{N}{q'} \geqslant b - \frac{N}{\tau}$. Now, we can use the embeddings of Bessel spaces and, for some C being independent of the cube Q_i, obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^N} dD^a u D^b\varphi\right| &\leqslant C\|d\|_{L^p_U(\mathbb{R}^N)} \sum_i \|u\|_{H^{s,q}(Q_i)} \|\varphi\|_{H^{\sigma,q'}(Q_i)} \\ &\leqslant C\|d\|_{L^p_U(\mathbb{R}^N)} \left(\sum_i \|u\|^q_{H^{s,q}(Q_i)}\right)^{1/q} \left(\sum_i \|\varphi\|^{q'}_{H^{\sigma,q'}(Q_i)}\right)^{1/q'}.\end{aligned} \tag{2.12}$$

Then, as in [5, Lemma 2.4], we get for any $0 \leqslant \alpha \leqslant 2$ and any $1 < q < \infty$

$$\sum_i \|\phi\|^q_{H^{2\alpha,q}(Q_i)} \leqslant C\|\phi\|^q_{H^{2\alpha,q}(\mathbb{R}^N)} \quad \text{for all } \phi \in H^{2\alpha,q}(\mathbb{R}^N),$$

and we obtain from (2.12)

$$\left|\int_{\mathbb{R}^N} dD^a u^b\varphi\right| \leqslant C\|d\|_{L^p_U(\mathbb{R}^N)} \|u\|_{H^{s,q}(\mathbb{R}^N)} \|\varphi\|_{H^{\sigma,q'}(\mathbb{R}^N)}$$

which gives the result. □

Now we can use again the results in [16] (see Appendix A) to obtain the following.

Theorem 2.10. *Let $P_{a,b}$ be as in (2.9) with $k, a, b \in \{0, 1, 2, 3\}$, $k = a + b$. Assume that $\|d\|_{L^p_U(\mathbb{R}^N)} \leqslant R_0$ with $p > \frac{N}{4-k}$, then for any $1 < q < \infty$ and such $P_{a,b}$, there exists an interval $I(q, a, b) \subset (-1 + \frac{a}{4}, 1 - \frac{b}{4})$ containing $(-1 + \frac{a}{4} + \frac{N}{4p}, 1 - \frac{b}{4} - \frac{N}{4p})$, such that for any $\gamma \in I(q, a, b)$, we have a strongly continuous analytic semigroup, $S_{P_{a,b}}(t)$, in the space $H^{4\gamma,q}(\mathbb{R}^N)$, for the problem*

$$\begin{cases} u_t + \Delta^2 u + D^b\big(d(x)D^a u\big) = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases} \tag{2.13}$$

Moreover the semigroup has the smoothing estimates

$$\left\|S_{P_{a,b}}(t)u_0\right\|_{H^{4\gamma',q}(\mathbb{R}^N)} \leqslant \frac{M_{\gamma',\gamma} e^{\mu t}}{t^{\gamma'-\gamma}} \|u_0\|_{H^{4\gamma,q}(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in H^{4\gamma,q}(\mathbb{R}^N),$$

for every $\gamma, \gamma' \in I(q, a, b)$ with $\gamma' \geqslant \gamma$, and

$$\left\|S_{P_{a,b}}(t)u_0\right\|_{L^r(\mathbb{R}^N)} \leqslant \frac{M_{q,r} e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{L^q(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in L^q(\mathbb{R}^N),$$

with $1 < q \leqslant r \leqslant \infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q, a, b)$ is given by

$$I(q, a, b) = \left(-1 + \frac{a}{4} + \frac{N}{4}\left(\frac{1}{p} - \frac{1}{q'}\right)_+, 1 - \frac{b}{4} - \frac{N}{4}\left(\frac{1}{p} - \frac{1}{q}\right)_+\right).$$

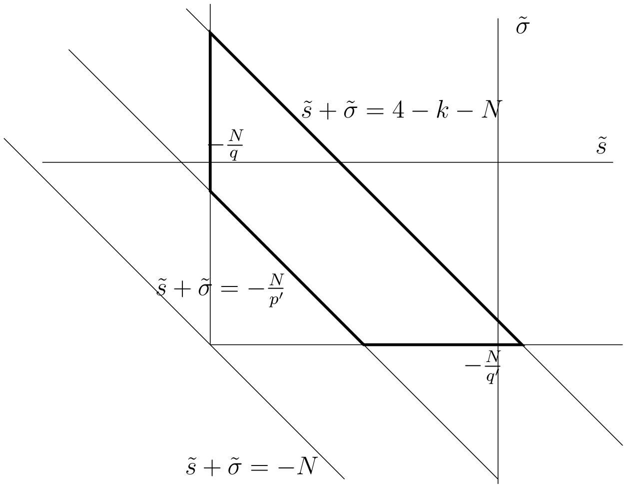

Fig. 1. Admissible $\tilde{s}$ and $\tilde{\sigma}$ with $p>q,q'$.

Finally, if

$$d_\varepsilon \to d \quad \text{in } L^p_U(\mathbb{R}^N),\ p>\frac{N}{4-k},$$

then for every $1<q<\infty$ *and* $T>0$ *there exists* $C(\varepsilon)\to 0$ *as* $\varepsilon\to 0$, *such that*

$$\left\| S_{P_\varepsilon}(t)-S_P(t)\right\|_{\mathcal{L}(H^{4\gamma,q}(\mathbb{R}^N),H^{4\gamma',q}(\mathbb{R}^N))} \leqslant \frac{C(\varepsilon)}{t^{\gamma'-\gamma}},\quad \forall 0<t\leqslant T,$$

for all $\gamma,\gamma'\in I(q,a,b)$, $\gamma'\geqslant\gamma$ *and for any* $1<q\leqslant r\leqslant\infty$

$$\left\| S_{P_\varepsilon}(t)-S_P(t)\right\|_{\mathcal{L}(L^q(\mathbb{R}^N),L^r(\mathbb{R}^N))} \leqslant \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}},\quad \forall 0<t\leqslant T.$$

Proof. By Proposition 2.9 and using $X_\alpha=E_{2\alpha}=H^{4\alpha,q}(\mathbb{R}^N)$, $\alpha\in\mathbb{R}$, if we assume for a moment that (2.11) is satisfied for some s and σ, then we would have

$$P\in\mathcal{L}(X_{s/4},X_{-\sigma/4}),\qquad \|P\|_{\mathcal{L}(X_{s/4},X_{-\sigma/4})}\leqslant C\|d\|_{L^p_U(\mathbb{R}^N)}.$$

Hence we can apply Theorem A.1 with $\alpha=s/4$ and $\beta=-\sigma/4$ provided $0\leqslant\alpha-\beta<1$, that is, $s+\sigma<4$.

Thus, we check now that (2.11) and $s+\sigma<4$ hold for suitable pairs (s,σ). For this we rewrite the ranges for s, σ in Proposition 2.9 in terms of $\tilde{s}=s-a-\frac{N}{q}$ and $\tilde{\sigma}=\sigma-b-\frac{N}{q'}$, so $\tilde{s}\geqslant-\frac{N}{q}$, $\tilde{\sigma}\geqslant-\frac{N}{q'}$ since $s\geqslant a$, $\sigma\geqslant b$. Then (2.11) and $s+\sigma<4$ read

$$\tilde{s}\geqslant-\frac{N}{q},\qquad \tilde{\sigma}\geqslant-\frac{N}{q'},\qquad -\frac{N}{p'}\leqslant\tilde{s}_-+\tilde{\sigma}_-,\qquad \tilde{s}+\tilde{\sigma}<4-k-N. \tag{2.14}$$

Note that since necessarily $-\frac{N}{p'}<4-k-N$, we get that $p>\frac{N}{4-k}$.

The set of admissible parameters $(\tilde{s},\tilde{\sigma})$ given by (2.14) depends on the relationship between q, q' and p. Note that (2.14) defines a planar trapezium-shaped polygon, $\tilde{\mathcal{P}}$, whose long base is on the line $\tilde{s}+\tilde{\sigma}=4-k-N$ and the short base is on the line $\tilde{s}+\tilde{\sigma}=-\frac{N}{p'}$ in the third quadrant. As for the lateral sides note that the restriction $-\frac{N}{p'}\leqslant\tilde{s}_-+\tilde{\sigma}_-$ adds the condition that $\tilde{s}\geqslant-\frac{N}{p'}$ in the second quadrant and $\tilde{\sigma}\geqslant-\frac{N}{p'}$ in the fourth. These have to be combined with $\tilde{s}\geqslant-\frac{N}{q}$ and $\tilde{\sigma}\geqslant-\frac{N}{q'}$. Therefore the lateral sides are given by the lines $\tilde{s}=\max\{-\frac{N}{p'},-\frac{N}{q}\}$ and $\tilde{\sigma}=\max\{-\frac{N}{p'},-\frac{N}{q'}\}$. One of the possible cases is depicted in Fig. 1.

Note that the polygon $\tilde{\mathcal{P}}$ transforms into a similar shaped polygon $\mathcal{P}$ which determines the region of admissible pairs (s,σ).

In any case, projecting $\tilde{\mathcal{P}}$ onto the axes gives the following ranges for $\tilde{s}$ and $\tilde{\sigma}$

$$\tilde{s}\in\left[\max\left\{-\frac{N}{p'},-\frac{N}{q}\right\},4-k-N-\max\left\{-\frac{N}{p'},-\frac{N}{q'}\right\}\right),$$
$$\tilde{\sigma}\in\left[\max\left\{-\frac{N}{p'},-\frac{N}{q'}\right\},4-k-N-\max\left\{-\frac{N}{p'},-\frac{N}{q}\right\}\right).$$

Thus the projection ranges for s and σ are given by

$$s \in J_1 = \left[a + \left(\frac{N}{q} - \frac{N}{p'}\right)_+, 4 - b - \left(\frac{N}{q'} - \frac{N}{p'}\right)_+\right), \tag{2.15}$$

$$\sigma \in J_2 = \left[b + \left(\frac{N}{q'} - \frac{N}{p'}\right)_+, 4 - a - \left(\frac{N}{q} - \frac{N}{p'}\right)_+\right). \tag{2.16}$$

For each pair of admissible pairs $(s,\sigma) \in \mathcal{P}$, by [16, Proposition 10] (see Theorem A.1) with $\alpha = \frac{s}{4}$ and $\beta = -\frac{\sigma}{4}$, we get a perturbed semigroup and smoothing estimates in the spaces corresponding to

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta + 1), \qquad \gamma' \geqslant \gamma.$$

Hence, as (s,σ) range in the region $\mathcal{P}$, a repeated bootstrap argument as in (2.8) gives that the smoothing estimates hold for $\gamma \in \bigcup_{(s,\sigma)\in\mathcal{P}} E(s/4)$ and $\gamma' \in \bigcup_{(s,\sigma)\in\mathcal{P}} R(\sigma/4)$, $\gamma' \geqslant \gamma$. This leads to

$$\gamma \in \left(\frac{\inf J_1}{4} - 1, \frac{\sup J_1}{4}\right], \qquad \gamma' \in \left[-\frac{\sup J_2}{4}, 1 - \frac{\inf J_2}{4}\right), \qquad \gamma' \geqslant \gamma,$$

which, after a simple calculation, reads

$$\gamma, \gamma' \in I(q,a,b) = \left(-1 + \frac{a}{4} + \frac{N}{4}\left(\frac{1}{q} - \frac{1}{p'}\right)_+, 1 - \frac{b}{4} - \frac{N}{4}\left(\frac{1}{q'} - \frac{1}{p'}\right)_+\right) = (\gamma_{min}, \gamma_{max}).$$

For the estimates in Lebesgue spaces we use the Sobolev inclusions. First note that for any $1 < q < \infty$, $I(q,a,b) \supset (-1 + \frac{a}{4} + \frac{N}{4p}, 1 - \frac{b}{4} - \frac{N}{4p})$ which does not depend on q and is not empty because $p > \frac{N}{4-k}$. Let $\tilde{\gamma} := 1 - \frac{b}{4} - \frac{N}{4p} > 0$ and take $0 \leqslant \gamma < \tilde{\gamma}$, then $H^{4\gamma,q}(\mathbb{R}^N) \hookrightarrow L^{\tilde{q}}(\mathbb{R}^N)$, for $\tilde{q} \geqslant q$ such that $-\frac{N}{\tilde{q}} = 4\gamma - \frac{N}{q}$, i.e. $\frac{1}{q} - \frac{1}{\tilde{q}} = \frac{4\gamma}{N}$ and we get

$$\left\|S_{P_{a,b}}(t)u_0\right\|_{L^{\tilde{q}}(\mathbb{R}^N)} \leqslant \left\|S_{P_{a,b}}(t)u_0\right\|_{H^{4\gamma,q}(\mathbb{R}^N)} \leqslant \frac{M_\gamma e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q} - \frac{1}{\tilde{q}})}} \|u_0\|_{L^q(\mathbb{R}^N)}.$$

In particular we can take $0 \leqslant \gamma \leqslant \frac{\tilde{\gamma}}{2}$ and we get the estimate above for all $\tilde{q} \geqslant q$ such that $\frac{1}{q} - \frac{1}{\tilde{q}} \in [0, \frac{2\tilde{\gamma}}{N}]$ and this interval does not depend on q.

We now use a bootstrap argument as in (2.8), jumping between different Lebesgue spaces at intermediate times. Starting with $r_0 := q$ and defining the numbers r_i, $i = 1, 2, 3, \ldots$, such that $\frac{1}{r_i} - \frac{1}{r_{i+1}} = \frac{2\tilde{\gamma}}{N}$, we obtain the estimate above for any $\tilde{q} \geqslant q$ such that $\tilde{q} \in [q, r_{i+1}]$. Hence in a finite number of steps we can reach any $\tilde{q}$ with $q < \tilde{q} \leqslant \infty$.

The convergence of the semigroups is a direct consequence of [16, Theorem 14] (see Theorem A.2), since Proposition 2.9 gives that if $d_\varepsilon \to d$ in $L^p_U(\mathbb{R}^N)$, then $P_\varepsilon \to P$ in $\mathcal{L}(X_{s/4}, X_{-\sigma/4})$ for any pair of admissible $(s,\sigma) \in \mathcal{P}$. The case of Lebesgue spaces follows from this as well.

Finally, the analyticity comes from Lemma 2.2 and [16, Theorem 12] (see Theorem A.3). □

Remark 2.11. Now we make precise in what sense Eq. (2.13) is satisfied.

i) First note that since $p > \frac{N}{4-k}$ we have $4\gamma_{max} > 4 - b - \frac{N}{p} > a$, and $4\gamma_{min} < -4 + a + \frac{N}{p} < -b$. Hence $[-\frac{b}{4}, \frac{a}{4}] \subset I(q,a,b)$.

ii) Because of the analyticity of the semigroup, and as in Remark 6 in [16], the equation $u_t + \Delta^2 u = Pu$ is satisfied in $H^{-b,q}(\mathbb{R}^N)$.
Therefore, we have that $u(t) \in H^{4-b,q}(\mathbb{R}^N)$, for all $t > 0$. In terms of the scale, $u(t) \in X_{\gamma^*}$, $\gamma^* = 1 - \frac{b}{4} \geqslant \gamma_{max}$. Note that in Theorem 2.10 we did not get an estimate of $u(t)$ in the space $H^{4-b,q}(\mathbb{R}^N)$ though.
Also, since the semigroup is analytic in X_γ, $u_t(t) \in X_\gamma$, for all $\gamma \in I(q,a,b)$ and $t > 0$.

iii) In particular, Eq. (2.13) is always satisfied as

$$\int_{\mathbb{R}^N} u_t\varphi + \int_{\mathbb{R}^N} u\Delta^2\varphi + \int_{\mathbb{R}^N} d(x)D^a u D^b\varphi = 0, \quad t > 0,$$

for any $\varphi \in H^{b,q'}(\mathbb{R}^N)$.
However, for $b = 3$, $a = 0$ we have $\gamma^* \geqslant \frac{1}{4}$, that is $u(t) \in H^{1,q}(\mathbb{R}^N)$, $t > 0$, and therefore

$$\int_{\mathbb{R}^N} u_t\varphi - \int_{\mathbb{R}^N} \nabla u\nabla(\Delta\varphi) - \int_{\mathbb{R}^N} d(x)uD^3\varphi = 0.$$

For $b=2$, $a\leqslant 1$ we have $\gamma^*\geqslant \frac{1}{2}$, that is $u(t)\in H^{2,q}(\mathbb{R}^N)$, $t>0$, and therefore

$$\int_{\mathbb{R}^N} u_t\varphi+\int_{\mathbb{R}^N}\Delta u\Delta\varphi+\int_{\mathbb{R}^N} d(x)D^a u D^2\varphi=0.$$

For $b=1$, $a\leqslant 2$ we have $\gamma^*\geqslant \frac{3}{4}$, that is $u(t)\in H^{3,q}(\mathbb{R}^N)$, $t>0$, and therefore

$$\int_{\mathbb{R}^N} u_t\varphi-\int_{\mathbb{R}^N}\nabla(\Delta u)\nabla\varphi-\int_{\mathbb{R}^N} d(x)D^a u D\varphi=0.$$

Finally, $b=0$, $a\leqslant 3$ we have $\gamma^*=4$, that is $u(t)\in H^{4,q}(\mathbb{R}^N)$, $t>0$, and therefore

$$\int_{\mathbb{R}^N} u_t\varphi+\int_{\mathbb{R}^N}\Delta^2 u\varphi+\int_{\mathbb{R}^N} d(x)D^a u\varphi=0.$$

Now we analyze which perturbations can be combined together. Notice that not all combinations are allowed.

Proposition 2.12. *Consider a finite family of perturbations $P_i:=P_{a_i,b_i}$ as in (2.9) with $\|d_i\|_{L^{p_i}_U(\mathbb{R}^N)}\leqslant R_0$, with $k_i,a_i,b_i\in\{0,1,2,3\}$, $k_i=a_i+b_i$, $p_i>\frac{N}{4-k_i}$, $i=1,\dots,J$. Denote $P:=\sum_i P_i$, then for any $1<q<\infty$, if*

$$\max_i\left\{a_i+\left(\frac{N}{p_i}-\frac{N}{q'}\right)_+\right\}+\max_i\left\{b_i+\left(\frac{N}{p_i}-\frac{N}{q}\right)_+\right\}<4 \tag{2.17}$$

then there exists an interval $I(q,P)\subset(-1+\frac{\max_i\{a_i\}}{4},1-\frac{\max_i\{b_i\}}{4})$ containing $(-1+\max_i\{\frac{a_i}{4}+\frac{N}{4p_i}\},1-\max_i\{\frac{b_i}{4}+\frac{N}{4p_i}\})$, such that for any $\gamma\in I(q,P)$, we have a strongly continuous, analytic semigroup, $S_P(t)$, in the space $H^{4\gamma,q}(\mathbb{R}^N)$, for the problem

$$\begin{cases} u_t+\Delta^2 u+Pu=0, & x\in\mathbb{R}^N,\ t>0,\\ u(0)=u_0 & \text{in } \mathbb{R}^N.\end{cases}$$

Moreover the semigroup has the smoothing estimates

$$\big\|S_P(t)u_0\big\|_{H^{4\gamma',q}(\mathbb{R}^N)}\leqslant\frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{H^{4\gamma,q}(\mathbb{R}^N)},\quad t>0,\ u_0\in H^{4\gamma,q}\big(\mathbb{R}^N\big),$$

for every $\gamma,\gamma'\in I(q,P)$ with $\gamma'\geqslant\gamma$, and

$$\big\|S_P(t)u_0\big\|_{L^r(\mathbb{R}^N)}\leqslant\frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q(\mathbb{R}^N)},\quad t>0,\ u_0\in L^q\big(\mathbb{R}^N\big),$$

with $1<q\leqslant r\leqslant\infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu\in\mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q,P)$ is given by

$$I(q,P)=\left(-1+\max_i\left\{\frac{a_i}{4}+\frac{N}{4}\left(\frac{1}{p_i}-\frac{1}{q'}\right)_+\right\},1-\max_i\left\{\frac{b_i}{4}+\frac{N}{4}\left(\frac{1}{p_i}-\frac{1}{q}\right)_+\right\}\right).$$

Finally, if as $\varepsilon\to 0$

$$d_i^\varepsilon\to d_i\quad\text{in } L^{p_i}_U\big(\mathbb{R}^N\big),\ p_i>\frac{N}{4-k_i},$$

then for every $1<q<\infty$ and $T>0$ there exists $C(\varepsilon)\to 0$ as $\varepsilon\to 0$, such that

$$\big\|S_{P_\varepsilon}(t)-S_P(t)\big\|_{\mathcal{L}(H^{4\gamma,q}(\mathbb{R}^N),H^{4\gamma',q}(\mathbb{R}^N))}\leqslant\frac{C(\varepsilon)}{t^{\gamma'-\gamma}},\quad\forall 0<t\leqslant T,$$

for all $\gamma,\gamma'\in I(q,P)$, $\gamma'\geqslant\gamma$ and for any $1<q\leqslant r\leqslant\infty$

$$\big\|S_{P_\varepsilon}(t)-S_P(t)\big\|_{\mathcal{L}(L^q(\mathbb{R}^N),L^r(\mathbb{R}^N))}\leqslant\frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}},\quad\forall 0<t\leqslant T.$$

Proof. From Theorem 2.10 we know that for each perturbation P_i there exists a nonempty trapezoidal polygon $\mathcal{P}_i$ of admissible pairs of spaces (s,σ) described in terms of $\tilde{s} = s - a_i - \frac{N}{q}$ and $\tilde{\sigma} = \sigma - b_i - \frac{N}{q'}$, see (2.14).

Therefore the polygon $\mathcal{P}_i$ of the perturbation P_i is given by a planar trapezium whose long base is on the line $s+\sigma = 4$ and the short base is on the line $s+\sigma = k_i + \frac{N}{p_i}$ in the third quadrant, with $k_i = a_i + b_i$. As for the lateral sides they are given by the lines $s = a_i + (\frac{N}{q} - \frac{N}{p_i'})_+$ and $\sigma = b_i + (\frac{N}{q'} - \frac{N}{p_i'})_+$. Thus the projection of $\mathcal{P}_i$ on the axes gives the intervals

$$s \in J_1^i = [s_{min}^i, 4 - \sigma_{min}^i) \quad \text{and} \quad \sigma \in J_2^i = [\sigma_{min}^i, 4 - s_{min}^i)$$

see (2.15) and (2.16).

According to Lemma 13, iii) in [16], we can consider $P := \sum_i P_i$, that is, all perturbations acting at the same time, if there exists a common region $\mathcal{P}$ of admissible pairs (s,σ), that is if $\mathcal{P} := \bigcap_i \mathcal{P}_i \neq \emptyset$.

Since the admissible sets $\mathcal{P}_i$ always have the long base on the line $s+\sigma = 4$ and the lateral sides are parallel to the axes, the set $\mathcal{P}$ is nonempty if and only if

$$\max_i\{\inf J_1^i\} < \min_i\{\sup J_1^i\} \quad \text{i.e.} \quad \max_i\{s_{min}^i\} < \min_i\{4 - \sigma_{min}^i\}$$

and

$$\max_i\{\inf J_2^i\} < \min_i\{\sup J_2^i\} \quad \text{i.e.} \quad \max_i\{\sigma_{min}^i\} < \min_i\{4 - s_{min}^i\}$$

which are equivalent to (2.17), that is

$$\max_i\left\{a_i + \left(\frac{N}{p_i} - \frac{N}{q'}\right)_+\right\} + \max_i\left\{b_i + \left(\frac{N}{p_i} - \frac{N}{q}\right)_+\right\} < 4.$$

In such a case the projection of $\mathcal{P} = \bigcap_i \mathcal{P}_i$ on the axes gives the intervals

$$s \in J_1 = \left[\max_i(\inf J_1^i), \min_i(\sup J_1^i)\right) = \left[\max_i\left\{a_i + \left(\frac{N}{p_i} - \frac{N}{q'}\right)_+\right\}, 4 - \max_i\left\{b_i + \left(\frac{N}{p_i} - \frac{N}{q}\right)_+\right\}\right),$$
$$\sigma \in J_2 = \left[\max_i(\inf J_2^i), \min_i(\sup J_2^i)\right) = \left[\max_i\left\{b_i + \left(\frac{N}{p_i} - \frac{N}{q}\right)_+\right\}, 4 - \max_i\left\{a_i + \left(\frac{N}{p_i} - \frac{N}{q}\right)_+\right\}\right).$$

For each pair of admissible pairs $(s,\sigma) \in \mathcal{P}$, by [16, Proposition 10] (see Theorem A.1) with $\alpha = \frac{s}{4}$ and $\beta = -\frac{\sigma}{4}$, we get a perturbed semigroup and smoothing estimates in the spaces corresponding to γ and γ' as in [16], i.e.

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta + 1), \qquad \gamma' \geqslant \gamma.$$

Hence as (s,σ) range in the region $\mathcal{P}$ a repeated bootstrap argument as in (2.8) gives that the smoothing estimates hold for $\gamma \in \bigcup_{(s,\sigma)\in\mathcal{P}} E(s/4)$ and $\gamma' \in \bigcup_{(s,\sigma)\in\mathcal{P}} R(-\sigma/4)$, $\gamma' \geqslant \gamma$, see also the proof of Theorem 2.10. This leads to

$$\gamma \in \left(\frac{\inf J_1}{4} - 1, \frac{\sup J_1}{4}\right], \qquad \gamma' \in \left[-\frac{\sup J_2}{4}, 1 - \frac{\inf J_2}{4}\right), \qquad \gamma' \geqslant \gamma,$$

which, after a simple calculation, reads

$$\gamma, \gamma' \in I(q, P) = \left(-1 + \max_i\left\{\frac{a_i}{4} + \frac{N}{4}\left(\frac{1}{p_i} - \frac{1}{q'}\right)_+\right\}, 1 - \max_i\left\{\frac{b_i}{4} + \frac{N}{4}\left(\frac{1}{p_i} - \frac{1}{q}\right)_+\right\}\right).$$

Note that this interval is contained in an interval $(-1 + \frac{\max_i\{a_i\}}{4}, 1 - \frac{\max_i\{b_i\}}{4})$ and contains $(-1 + \max_i\{\frac{a_i}{4} + \frac{N}{4p_i}\}, 1 - \max_i\{\frac{b_i}{4} + \frac{N}{4p_i}\})$, which is nonempty because $p_i > \frac{N}{4-k_i}$. To see this note that the latter condition gives $\frac{a_i}{4} + \frac{N}{4p_i} < 1 - \frac{b_i}{4} < 1$ and $\frac{b_i}{4} + \frac{N}{4p_i} < 1 - \frac{a_i}{4} < 1$. □

Remark 2.13. Note that now, since $p_i > \frac{N}{4-k_i}$, $I(q,a,b) \supset [-\min\{\frac{b_i}{4}\}, \min\{\frac{a_i}{4}\}]$, thus all the comments on Remark 2.11 hold for $\min\{b_i\}, \min\{a_i\}$ instead of b, a.

Remark 2.14. In some cases the condition (2.17) can be simplified and simpler description can be given.

i) If there is only one perturbation, then (2.17) is equivalent to $p > \frac{N}{4-k}$ as in Theorem 2.10.

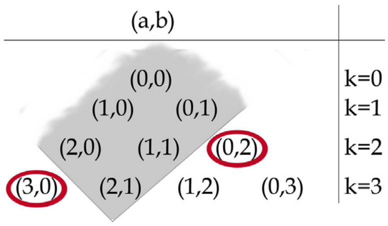

Fig. 2. Combining perturbations.

ii) If $a_i = a$ and $b_i = b$ (thus $k_i = k$) for all i, then

$$P = \sum_i D^b\big(d_i(x)D^a\big) = D^b\big(d(x)D^a\big) \quad \text{where } d := \sum_i d_i$$

can be considered as a perturbation with $d \in L^p_U(\mathbb{R}^N)$ for $p = \min_i\{p_i\}$. Then (2.17) holds if and only if $p > \frac{N}{4-k}$ as in Theorem 2.10.

iii) Assume now $p_i = p$ for all i. Then (2.17) is equivalent to

$$\max_i\{a_i\} + \max_i\{b_i\} < 4 - \left(\frac{N}{p} - \frac{N}{q}\right)_+ - \left(\frac{N}{p} - \frac{N}{q'}\right)_+. \tag{2.18}$$

Hence, if we denote $k := \max\{a_i\} + \max\{b_i\}$, then (2.18) is satisfied provided $p > \frac{N}{4-k}$, which resembles the condition in Theorem 2.10. Note that k can be regarded as the order of the perturbation $P = \sum_i P_i$.
In particular, if

$$k := \max_i\{a_i\} + \max_i\{b_i\} < 4 \quad \text{and} \quad p > \frac{N}{4-k}$$

are satisfied, then Proposition 2.12 applies with an interval for P given by

$$I(q,P) = \left(-1 + \frac{\max_i\{a_i\}}{4} + \frac{N}{4}\left(\frac{1}{p} - \frac{1}{q'}\right)_+, 1 - \frac{\max_i\{b_i\}}{4} - \frac{N}{4}\left(\frac{1}{p} - \frac{1}{q}\right)_+\right).$$

Compare it with $I(q,a,b)$ in Theorem 2.10 to see the resemblance.

iv) We now describe how to determine if two perturbations as in iii) can be combined.
For example, if we fix a perturbation $P_{a,b}$ with $k = 3$, then any perturbation $P_{c,d}$ with $c \leqslant a$ and $d \leqslant b$ can be combined with it, and the interval is $I(q,P) = I(q,a,b)$, $P = P_{a,b} + P_{c,d}$.
Also, a perturbation $P_{2,1}$ can be combined with all the ones included in the shaded area in Fig. 2 with interval $I(q,2,1)$.
However, the encircled perturbations $P_{3,0}$ and $P_{0,2}$ cannot be combined together.
If we fix a perturbation $P_{a,b}$ with $k = 2$ then, all perturbations $P_{c,d}$ with $c \leqslant a$ and $d \leqslant b$ can be combined with it, and also those with $c - 1 \leqslant a$ or $d - 1 \leqslant b$, but not both at the same time.
The same happens for $P_{a,b}$ with $k = 1$, all perturbations $P_{c,d}$ with $k \leqslant 1$ can be combined with it.

v) There are 127 possible combinations for pairs of perturbations as in iv).

Observe that perturbations in (2.9) can be handled as above because we could determine the spaces of the Bessel scale between which a perturbation $P_{a,b}$ is a well behaved linear operator; see Proposition 2.9. However the fact that a, b are integer derivatives is not really essential. Therefore, this class of perturbations can be extended to the following one, where derivatives are replaced by fractional powers of the Laplacian as long as this one is well defined in our scale. For example $-\Delta + cI$, with $c > 0$ can be used in this way, because the operator $(-\Delta + cI)^{r/2}$, $r > 0$, satisfies for any $s \in \mathbb{R}$,

$$(-\Delta + cI)^{r/2} \in \mathcal{L}\big(H^{s,q}(\mathbb{R}^N), H^{s-r,q}(\mathbb{R}^N)\big), \qquad \big\|(-\Delta + cI)^{r/2}\big\|_{\mathcal{L}(H^{s,q}(\mathbb{R}^N), H^{s-r,q}(\mathbb{R}^N))} \leqslant C$$

for some C independent of s, r, q. Note that this estimate is analogous to (2.5) for a non-integer r.

Thus, the perturbations

$$P_{a,b}u = (-\Delta + cI)^{b/2}\big(d(x)(-\Delta + cI)^{a/2}u\big), \quad a, b \geqslant 0,$$

for any $0 \leqslant a, b \in \mathbb{R}$, in the sense that for any smooth enough φ

$$\langle P_{a,b}u, \varphi\rangle = \int_{\mathbb{R}^N} d(x)(-\Delta + cI)^{a/2}u(-\Delta + cI)^{b/2}\varphi, \tag{2.19}$$

with $d \in L^p_U(\mathbb{R}^N)$, satisfy the statement in Proposition 2.9.

Then proceeding exactly as in Theorem 2.10, we recover the same results for this kind of perturbations, with the only difference that now $k = a + b$ is a real number smaller than 4.

Theorem 2.15. *Let $a, b, k \geqslant 0$ be real numbers such that $k = a + b < 4$ and $P_{a,b}$ be as in (2.19). Assume that $\|d\|_{L^p_U(\mathbb{R}^N)} \leqslant R_0$ with $p > \frac{N}{4-k}$, then for any $1 < q < \infty$ and such $P_{a,b}$ there exists an interval $I(q,a,b) \subset (-1 + \frac{a}{4}, 1 - \frac{b}{4})$ containing $(-1 + \frac{a}{4} + \frac{N}{4p}, 1 - \frac{b}{4} - \frac{N}{4p})$, such that for any $\gamma \in I(q,a,b)$, we have a strongly continuous, analytic semigroup, $S_{P_{a,b}}(t)$, in the space $H^{4\gamma,q}(\mathbb{R}^N)$, $1 < q < \infty$, for the problem*

$$\begin{cases} u_t + \Delta^2 u + P_{a,b} u = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimates

$$\left\| S_{P_{a,b}}(t) u_0 \right\|_{H^{4\gamma',q}(\mathbb{R}^N)} \leqslant \frac{M_{\gamma',\gamma} e^{\mu t}}{t^{\gamma'-\gamma}} \|u_0\|_{H^{4\gamma,q}(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in H^{4\gamma,q}\left(\mathbb{R}^N\right),$$

for every $\gamma, \gamma' \in I(q,a,b)$ with $\gamma' \geqslant \gamma$, and

$$\left\| S_{P_{a,b}}(t) u_0 \right\|_{L^r(\mathbb{R}^N)} \leqslant \frac{M_{q,r} e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{L^q(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in L^q\left(\mathbb{R}^N\right),$$

with $1 < q \leqslant r \leqslant \infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q,a,b)$ is given by

$$I(q,a,b) = \left(-1 + \frac{a}{4} + \frac{N}{4}\left(\frac{1}{p} - \frac{1}{q'}\right)_+, 1 - \frac{b}{4} - \frac{N}{4}\left(\frac{1}{p} - \frac{1}{q}\right)_+ \right).$$

Finally, if, as $\varepsilon \to 0$,

$$d_\varepsilon \to d \quad \text{in } L^p_U\left(\mathbb{R}^N\right),\ p > \frac{N}{4-k},$$

then for every $1 < q \leqslant r \leqslant \infty$ and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\left\| S_{P_\varepsilon}(t) - S_P(t) \right\|_{\mathcal{L}(H^{4\gamma,q}(\mathbb{R}^N), H^{4\gamma',q}(\mathbb{R}^N))} \leqslant \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \quad \forall 0 < t \leqslant T,$$

for all $\gamma, \gamma' \in I(q,a,b)$ with $\gamma' > \gamma$ and for any $1 < q \leqslant r \leqslant \infty$

$$\left\| S_{P_\varepsilon}(t) - S_P(t) \right\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \leqslant \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \quad \forall 0 < t \leqslant T.$$

Note that Remark 2.3 and Remark 2.11, Proposition 2.12 and Remark 2.14 apply here as well.

3. Fourth order equations in the uniform Bessel–Lebesgue spaces in $\mathbb{R}^N$

The heat equation (2.2) and therefore the bi-Laplacian equation (2.4) can be also considered in much larger spaces than the Bessel spaces above, by taking the initial data in locally uniform spaces.

For this, consider the locally uniform space $L^q_U(\mathbb{R}^N)$ for $1 \leqslant q \leqslant \infty$ defined as in (2.10) and denote by $\dot{L}^q_U(\mathbb{R}^N)$ the closed subspace of $L^q_U(\mathbb{R}^N)$ consisting of all elements which are translation continuous with respect to $\|\cdot\|_{L^q_U(\mathbb{R}^N)}$, that is

$$\|\tau_y \phi - \phi\|_{L^q_U(\mathbb{R}^N)} \to 0 \quad \text{as } |y| \to 0,$$

where $\{\tau_y,\ y \in \mathbb{R}^N\}$ denotes the group of translations. Note that $L^q(\mathbb{R}^N) \subset \dot{L}^q_U(\mathbb{R}^N)$ for $1 \leqslant q < \infty$ and for $q = \infty$ we get $L^\infty_U(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$ and $\dot{L}^\infty_U(\mathbb{R}^N) = BUC(\mathbb{R}^N)$.

Thus we introduce the *uniform Bessel–Sobolev spaces* $H^{k,q}_U(\mathbb{R}^N)$, with $k \in \mathbb{N}$, as the set of functions $\phi \in H^{k,q}_{loc}(\mathbb{R}^N)$ such that

$$\|\phi\|_{H^{k,q}_U(\mathbb{R}^N)} = \sup_{x \in \mathbb{R}^N} \|\phi\|_{H^{k,q}(B(x,1))} < \infty$$

for $k \in \mathbb{N}$. Then denote by $\dot{H}^{k,q}_U(\mathbb{R}^N)$ a subspace of $H^{k,q}_U(\mathbb{R}^N)$ consisting of all elements which are translation continuous with respect to $\|\cdot\|_{H^{k,q}_U(\mathbb{R}^N)}$, that is

$$\|\tau_y \phi - \phi\|_{H^{k,q}_U(\mathbb{R}^N)} \to 0 \quad \text{as } |y| \to 0$$

where $\{\tau_y,\ y \in \mathbb{R}^N\}$ denotes the group of translations.

Consider the complex interpolation functor denoted by $[\cdot,\cdot]_\theta$, for $\theta \in (0,1)$, see [18] for details. Then for $1 \leqslant q < \infty$, $k \in \mathbb{N} \cup \{0\}$ and $s \in (k, k+1)$ we define $\theta \in (0,1)$ such that $s = \theta(1+k) + (1-\theta)k$, that is $\theta = s - k$. Then one can define the intermediate spaces by interpolation as

$$H_U^{s,q}(\mathbb{R}^N) = \left[H_U^{k+1,q}(\mathbb{R}^N), H_U^{k,q}(\mathbb{R}^N)\right]_\theta,$$

and

$$\dot{H}_U^{s,q}(\mathbb{R}^N) = \left[\dot{H}_U^{k+1,q}(\mathbb{R}^N), \dot{H}_U^{k,q}(\mathbb{R}^N)\right]_\theta.$$

For details on the construction of the interpolation scale, see [2] and Appendix B.

Using Proposition 4.2 in [4] it is easy to see that the sharp embeddings of Bessel spaces translate into

$$\dot{H}_U^{s,q}(\mathbb{R}^N) \subset \begin{cases} \dot{L}_U^r(\mathbb{R}^N),\ s - \frac{N}{q} \geqslant -\frac{N}{r},\ 1 \leqslant r < \infty, & \text{if } s - \frac{N}{q} < 0, \\ \dot{L}_U^r(\mathbb{R}^N),\ 1 \leqslant r < \infty, & \text{if } s - \frac{N}{q} = 0, \\ C_b^\eta(\mathbb{R}^N) & \text{if } s - \frac{N}{q} > \eta \geqslant 0. \end{cases} \tag{3.1}$$

In [4], the Laplace operator was considered in the scale of spaces $H_U^{s,q}(\mathbb{R}^N)$, $s \geqslant 0$, and $\dot{H}_U^{s,q}(\mathbb{R}^N)$, and it was proved that $-\Delta$ defines an analytic semigroup. However in the "undotted" spaces the semigroup generated by $-\Delta$ is analytic but not strongly continuous. These spaces are less convenient to use because smooth functions are not dense in them; see [4].

It was moreover proved in [4, Theorem 5.3, p. 290], that $-\Delta$ has bounded imaginary powers, and therefore this scale coincides with the fractional power one; see [2, V.1.5.13, p. 283] (see Remark B.4). Note that from the results in [4] we have in particular that $\dot{H}_U^{1,q}(\mathbb{R}^N) = [\dot{H}_U^{2,q}(\mathbb{R}^N), \dot{L}_U^q(\mathbb{R}^N)]_{1/2}$; see Remark 5.7, page 291 in that reference. From this reiterations properties of interpolation gives that $\dot{H}_U^{2\theta,q}(\mathbb{R}^N) = [\dot{H}_U^{2,q}(\mathbb{R}^N), \dot{L}_U^q(\mathbb{R}^N)]_\theta$ for $\theta \in [0,1]$.

The scale above can be extended to negative indexes by a general extrapolation procedure as in [2], see Appendix B and Remark 2.5. In this way one can define the extrapolated space $\dot{H}_U^{-k,q}(\mathbb{R}^N)$ as the completion of $\dot{L}_U^q(\mathbb{R}^N)$ with the norm $\|(-\Delta + I)^{-k/2} u\|_{\dot{L}_U^q(\mathbb{R}^N)}$. Again, by complex interpolation, for $0 < s < k$, $k \in \mathbb{N}$, the intermediate spaces are given by

$$\dot{H}_U^{-s,q}(\mathbb{R}^N) = \left[\dot{L}_U^q(\mathbb{R}^N), \dot{H}_U^{-k,q}(\mathbb{R}^N)\right]_\theta, \quad \text{with } \theta = \frac{s}{k}.$$

Note that because of the reiteration property of the complex interpolation (see (2.8.4) in page 31 in [2] and Theorem 1.5.4 in [2]) this definition of $\dot{H}_U^{-s,q}(\mathbb{R}^N)$ does not depend on k. Also the operator $-\Delta$ and the analytic semigroup it generates, extends to the spaces with negative index above.

However, since the uniform Sobolev spaces are not reflexive, even for $q = 2$, we do not get the description of the negative part of the scale in terms of the dual spaces as in (2.1), see Appendix B.

Therefore, we start with some description of the negative spaces which complements the results in [4].

Proposition 3.1. *We have that*

$$\dot{L}_U^p(\mathbb{R}^N) \hookrightarrow \dot{H}_U^{-s,q}(\mathbb{R}^N) \quad \text{if } s - \frac{N}{q'} \geqslant -\frac{N}{p'},\ s > 0.$$

Proof. We first assume that $0 \leqslant s \leqslant 2$.

i) First note that $\dot{H}_U^{-s,q}(\mathbb{R}^N)$ is the completion of $\dot{H}_U^{2-s,q}(\mathbb{R}^N)$ with the norm $\|(-\Delta + I)^{-1} \cdot\|_{\dot{H}_U^{2-s,q}(\mathbb{R}^N)}$ (see Appendix B). This means that $f \in \dot{H}_U^{-s,q}(\mathbb{R}^N)$ if and only if there exists an approximating sequence $\{f_n\} \in \dot{H}_U^{2-s,q}(\mathbb{R}^N)$ that converges to f in $\dot{H}^{-s,q}(\mathbb{R}^N)$.

Since $(-\Delta + I)^{-1}$ is an isometry from $\dot{H}_U^{2-s,q}(\mathbb{R}^N)$ to $\dot{H}_U^{-s,q}(\mathbb{R}^N)$, see Appendix B, this is equivalent to

$$(-\Delta + I)^{-1} f_n \to (-\Delta + I)^{-1} f \quad \text{in } \dot{H}^{2-s,q}(\mathbb{R}^N),$$

and observe that since $f_n \in \dot{H}_U^{2-s,q}(\mathbb{R}^N)$ then $(-\Delta + I)^{-1} f_n \in \dot{H}_U^{4-s,q}(\mathbb{R}^N)$. Thus, we get that $f \in \dot{H}_U^{-s,q}(\mathbb{R}^N)$ if and only if there exists $\{u_n\} \in \dot{H}_U^{4-s,q}(\mathbb{R}^N)$ such that $u_n \to (-\Delta + I)^{-1} f$ in $\dot{H}_U^{2-s,q}(\mathbb{R}^N)$.

ii) Now, take $f \in \dot{L}_U^p(\mathbb{R}^N)$, then from the results in [4] we have $u = (-\Delta + I)^{-1} f \in \dot{H}_U^{2,p}(\mathbb{R}^N)$ and since $s - \frac{N}{q'} \geqslant -\frac{N}{p'}$ holds by assumption, we have $\dot{H}_U^{2,p}(\mathbb{R}^N) \hookrightarrow \dot{H}_U^{2-s,q}(\mathbb{R}^N)$, and $2 - s \geqslant 0$. Therefore $u \in \dot{H}_U^{2-s,q}(\mathbb{R}^N)$.

Since $\dot{H}_U^{4-s,q}(\mathbb{R}^N)$ is dense in $\dot{H}_U^{2-s,q}(\mathbb{R}^N)$, there exist $u_n \in \dot{H}_U^{4-s,q}(\mathbb{R}^N)$ such that $\|u_n - u\|_{\dot{H}_U^{2-s,q}(\mathbb{R}^N)} \xrightarrow{n\to\infty} 0$ and therefore by i), $f \in \dot{H}_U^{-s,q}(\mathbb{R}^N)$. Note that the inclusion is continuous, since $(-\Delta + I)^{-1}$ is an isometry on the scale and then

$$\|f\|_{\dot{H}_U^{-s,q}(\mathbb{R}^N)} = \left\|(-\Delta + I)^{-1} f\right\|_{\dot{H}_U^{2-s,q}(\mathbb{R}^N)} \leqslant C \left\|(-\Delta + I)^{-1} f\right\|_{\dot{H}_U^{2,p}(\mathbb{R}^N)} = C\|f\|_{\dot{L}_U^p(\mathbb{R}^N)}.$$

In order to prove the result for $s \geqslant 0$, we can repeat the whole argument above, using $(-\Delta + I)^{-n}$, which is an isometry on the scale, for a suitable n. If $2 \leqslant s \leqslant 4$ we use $n = 2$, thus in part i) we obtain that $f \in \dot{H}_U^{-s,q}(\mathbb{R}^N)$ if there exists a sequence $\{u_n\} \in \dot{H}_U^{6-s,q}(\mathbb{R}^N)$ converging to $u = (-\Delta + I)^{-2} f$ in $\dot{H}_U^{4-s,q}(\mathbb{R}^N)$. In part ii) we now have $u \in \dot{H}_U^{4,p}(\mathbb{R}^N) \hookrightarrow \dot{H}_U^{4-s,q}(\mathbb{R}^N)$ since now $4 - s \geqslant 0$ and the result follows as before.

In the same way, for $2(k-1) \leqslant s \leqslant 2k$, we use $n = k$ and repeat the argument above. □

Remark 3.2. Note that the embedding in Proposition 3.1 is precisely the one that one could expect from (3.1) if the spaces were reflexive. Also this is the embedding that holds for the standard Bessel scale as in Section 2. Needless to say the conditions for the embeddings read also $s \geqslant \frac{N}{p} - \frac{N}{q}$.

Using the spaces above and the convolution with the heat kernel, it was proved in Proposition 2.1, Theorem 2.1 and Theorem 5.3 in [4] that the heat equation defines an order preserving analytic semigroup in $L_U^q(\mathbb{R}^N)$ and, for $1 \leqslant q < \infty$, which is strongly continuous in $\dot{L}_U^q(\mathbb{R}^N)$ and in $E_\alpha := \dot{H}_U^{2\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$. Moreover, this semigroup satisfies the smoothing estimates

$$\|S_{-\Delta}(t)u_0\|_{\dot{L}_U^r(\mathbb{R}^N)} \leqslant \frac{M_{r,q} e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{\dot{L}_U^q(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{L}_U^q(\mathbb{R}^N),$$

for $1 \leqslant q \leqslant r \leqslant \infty$ for $\mu > 0$ arbitrary, and

$$\|S_{-\Delta}(t)u_0\|_{\dot{H}_U^{2\alpha,q}(\mathbb{R}^N)} \leqslant \frac{M_{\alpha,\beta} e^{\mu t}}{t^{\alpha-\beta}} \|u_0\|_{\dot{H}_U^{2\beta,q}(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{H}_U^{2\beta,q}(\mathbb{R}^N),$$

with $\mu > 0$ arbitrary, for any $\alpha, \beta \in \mathbb{R}$, $\alpha \geqslant \beta$.

It was also proved in [4] using a parabolic argument that type$(-\Delta) = 0$ in the $\dot{L}_U^q(\mathbb{R}^N)$ spaces (and thus in $\dot{H}_U^{\alpha,q}(\mathbb{R}^N)$), which explains why $\mu > 0$ above is arbitrary.

We now show some relevant information on the spectrum and resolvent of $-\Delta$ and Δ^2 in the uniform spaces which is analogous to Lemma 2.1.

Proposition 3.3.

i) *For $1 < q < \infty$, in the space $E_0 := \dot{L}_U^q(\mathbb{R}^N)$ the operator $-\Delta$, with domain $E_1 := D(-\Delta) = \dot{H}_U^{2,q}(\mathbb{R}^N)$, satisfies the estimate*

$$\|(-\Delta - \lambda)^{-1}\|_{\mathcal{L}(E_0)} \leqslant M|\lambda|^{-1}$$

for all λ in a sector $S_{0,\phi}$ as in (2.3) for $\phi > 0$ arbitrarily small.
Furthermore, $\sigma(-\Delta) = [0, \infty)$, and thus, type$(-\Delta) = 0$.

ii) *For $1 < q < \infty$, in the space $E_0 := \dot{L}_U^q(\mathbb{R}^N)$ the operator Δ^2, with domain $E_2 := D(\Delta^2) = \dot{H}_U^{4,q}(\mathbb{R}^N)$, satisfies the estimate*

$$\|(\Delta^2 - \lambda)^{-1}\|_{\mathcal{L}(E_0)} \leqslant M|\lambda|^{-1}$$

for all λ in a sector $S_{0,2\phi}$ as in (2.3) for $\phi > 0$ arbitrarily small.
Furthermore, $\sigma(\Delta^2) = [0, \infty)$, and thus, type$(\Delta^2) = 0$.

Proof. First recall that from Theorem 2.1 in [4] we have that the domain of the Laplacian operator in $\dot{L}_U^q(\mathbb{R}^N)$ is given by $D(-\Delta) = \dot{H}_U^{2,q}(\mathbb{R}^N)$. To prove part i), observe that, as in pages 32–33 in [14], we can obtain an expression for the operator $(-\Delta + \mu I)^{-1}$, provided $\mathrm{Re}(\sqrt{\mu}) > 0$, as a convolution operator. The expression is

$$u = (-\Delta + \mu)^{-1} f = \Gamma_\mu * f, \quad \mathrm{Re}(\sqrt{\mu}) > 0,$$

with

$$\Gamma_\mu(x) = \sqrt{\mu}^{N-2} G_2(\sqrt{\mu} x), \quad x \in \mathbb{R}^N,\ \mathrm{Re}(\sqrt{\mu}) > 0,$$

where G_2 is as in page 132 in [17] or page 33 in [14], that is

$$G_2(x) = \frac{1}{(4\pi)^{N/2}} \int_0^\infty t^{-N/2} e^{-t - \frac{x\cdot x}{4t}}\, dt = \frac{\xi^{1-N/2}}{(4\pi)^{N/2}} \int_0^\infty s^{-N/2} e^{-\xi(s + \frac{1}{4s})}\, ds, \quad x \in \mathbb{R}^N,$$

with $\xi = \sqrt{x \cdot x} > 0$. This definition can be extended to complex variables as

$$G_2(z) = \frac{\xi^{1-N/2}}{(4\pi)^{N/2}} \int_0^\infty s^{-N/2} e^{-\xi(s + \frac{1}{4s})}\, ds, \quad z \in \mathbb{C}^N,\ \xi = \sqrt{z \cdot z},\ \mathrm{Re}(\xi) > 0.$$

According to [14], we have for $z \in \mathbb{C}^N$ with $\mathrm{Re}(\xi) > 0$, if $N > 2$

$$\left|G_2(z)\right| \leqslant C|\xi|^{(2-N)/2}(\mathrm{Re}\,\xi)^{(2-N)/2}e^{-\frac{1}{2}\mathrm{Re}\,\xi}, \quad \xi = \sqrt{z \cdot z}, \tag{3.2}$$

and if $N = 2$,

$$\left|G_2(z)\right| \leqslant C \max\left\{\ln\frac{1}{\mathrm{Re}\,\xi}, 1\right\}e^{-\frac{1}{2}\mathrm{Re}\,\xi}, \quad \xi = \sqrt{z \cdot z}. \tag{3.3}$$

Now observe that if $\lambda \in S_{0,\phi}$ with $\phi > 0$ then for $\mu = -\lambda \in \mathbb{C} \setminus (-\infty, 0]$ we can choose $\mathrm{Re}(\sqrt{\mu}) > 0$. For such λ and similarly to Lemma 2.1 we are going to check that for $f \in \dot{L}^q_U(\mathbb{R}^N)$ we have the following estimate for $u = \Gamma_\mu * f$,

$$\|u\|_{L^q_U(\mathbb{R}^N)} \leqslant C\frac{1}{|\lambda|}\|f\|_{L^q_U(\mathbb{R}^N)}, \quad \lambda \in S_{0,\phi},\ \phi > 0.$$

Let $\{Q_i\}$, $i \in \mathbb{Z}^N$, be a partition of $\mathbb{R}^N$ in open disjoint cubes centered in $i \in \mathbb{Z}^N$ with edges of length 1, parallel to the axes. Thus $Q_i \cap Q_j = \emptyset$ for $i \neq j$ and $\mathbb{R}^N = \bigcup_i \overline{Q_i}$.

Then we fix $i \in \mathbb{Z}^N$ and decompose $f \in \dot{L}^q_U(\mathbb{R}^N)$ in a *far* and a *near* region as in Proposition 2.1 in [4]. For this we denote by $N(i)$ the set for indices j such that $\overline{Q_i} \cap \overline{Q_j} \neq 0$. That is, the set for which

$$d_{ij} := \inf\left\{\mathrm{dist}(x, y),\ x \in Q_i,\ y \in Q_j\right\}$$

satisfies that $d_{ij} = 0$. Thus we can define, for each $i \in \mathbb{Z}^N$ fixed

$$Q_i^{near} = \bigcup_{j \in N(i)} Q_j \quad \text{and} \quad Q_i^{far} = \mathbb{R}^N \setminus Q_i^{near}.$$

Hence, we decompose $f := f_i^{near} + f_i^{far} := f\chi_{Q_i^{near}} + f\chi_{Q_i^{far}}$, where χ denotes the characteristic function and $u := u_i^{near} + u_i^{far}$ with

$$u_i^{near} := \Gamma_\mu * f_i^{near}, \qquad u_i^{far} := \Gamma_\mu * f_i^{far}.$$

The resolvent estimate will follow from the following estimates of the two terms of the decomposition. For λ as above, we have first,

$$\left|u_i^{near}\right|_{L^q(Q_i)} \leqslant \frac{C}{|\lambda|}\|f\|_{L^q(Q_i^{near})}, \quad \lambda \in S_{0,\phi}, \tag{3.4}$$

and, second,

$$\left\|u_i^{far}\right\|_{L^\infty(Q_i)} \leqslant \frac{C}{|\lambda|}\|f\|_{L^1_U(Q_i^{far})}, \quad \lambda \in S_{0,\phi}, \tag{3.5}$$

for some C independent if $i \in \mathbb{Z}^N$.

Using (3.4) and (3.5), since the constants for the embedding $L^\infty(Q_i) \hookrightarrow L^q(Q_i)$ and the restrictions $L^q_U(\mathbb{R}^N) \hookrightarrow L^q(Q_i^{near})$, $L^q_U(\mathbb{R}^N) \hookrightarrow L^1_U(Q_i^{near})$ depend on N but can be chosen independent of p, q and i, (3.4) and (3.5) imply

$$\|u\|_{L^q(Q_i)} \leqslant \frac{C}{|\lambda|}\|f\|_{L^q_U(\mathbb{R}^N)}, \quad \lambda \in S_{0,\phi},$$

for each $i \in \mathbb{Z}^N$ with C independent of i and $\lambda \in S_{0,\phi}$, which gives the result.

Hence, we first prove (3.4). As a consequence of Lemma 2.1, we get for all $\lambda \in S_{0,\phi}$

$$\left\|u_i^{near}\right\|_{L^q(Q_i)} \leqslant \left\|u_i^{near}\right\|_{L^q(\mathbb{R}^N)} \leqslant \frac{C}{|\lambda|}\left\|f_i^{near}\right\|_{L^q(\mathbb{R}^N)} = \frac{C(N)}{|\lambda|}\|f\|_{L^q(Q_i^{near})}.$$

We show now (3.5) for $N > 2$. Observe that $f_i^{far} = f\chi_{Q_i^{far}} = \sum_{j \in \mathbb{Z}^N \setminus N(i)} f\chi_{Q_j}$. Hence, because of (3.2) with $z = \sqrt{\mu}x$, $\mathrm{Re}(\sqrt{\mu}) > 0$, $x \in \mathbb{R}^N$, $\mu = -\lambda$ and $\lambda \in S_{0,\phi}$, we have for all $x \in Q_i$

$$\begin{aligned}
\left|u_i^{far}(x)\right| &= \sum_{j \notin N(i)} \left|(\Gamma_\mu * f\chi_{Q_j})(x)\right| \\
&\leqslant \sum_{j \notin N(i)} C \sup_{y \in Q_j} \left|\sqrt{\mu}^{N-2} \cdot \left(\sqrt{\mu}|x-y|\right)^{1-N/2} \mathrm{Re}\left(\sqrt{\mu}|x-y|\right)^{1-N/2} e^{-\frac{1}{2}\mathrm{Re}\sqrt{\mu}|x-y|}\right| \|f\|_{L^1(Q_j)} \\
&\leqslant C\|f\|_{L^1_U(Q_i^{far})} \sqrt{|\lambda|}^{N/2-1} \mathrm{Re}(\sqrt{\mu})^{1-N/2} \sum_{j \notin N(i)} \sup_{y \in Q_i} |x-y|^{2-N} e^{-\frac{1}{2}|x-y|\,\mathrm{Re}\sqrt{\mu}}.
\end{aligned}$$

Note that for all $x \in Q_i$ and $y \in Q_j$ it holds $|x-y| \geqslant d_{ij}$, thus

$$\left|u_i^{far}(x)\right| \leqslant C\|f\|_{L^1_U(Q_i^{far})}\left(\frac{\sqrt{|\lambda|}}{\operatorname{Re}(\sqrt{\mu})}\right)^{N/2-1}\sum_{j\notin N(i)} d_{ij}^{2-N}e^{-\frac{1}{2}d_{ij}\operatorname{Re}\sqrt{\mu}}.$$

Hence

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leqslant C\|f\|_{L^1_U(Q_i^{far})}\left(\frac{\sqrt{|\lambda|}}{\operatorname{Re}(\sqrt{\mu})}\right)^{N/2-1}\sum_{j\notin N(i)} d_{ij}^{2-N}e^{-\frac{1}{2}d_{ij}\operatorname{Re}\sqrt{\mu}}.$$

Now, using that $\sharp\{j\in\mathbb{Z},\ d_{ij}=k\}\leqslant Ck^{N-1}$ we obtain

$$\begin{aligned}\|u_i^{far}\|_{L^\infty(Q_i)} &\leqslant C\|f\|_{L^1_U(Q_i^{far})}\left(\frac{\sqrt{|\lambda|}}{\operatorname{Re}(\sqrt{\mu})}\right)^{N/2-1}\sum_{k=1}^{\infty} ke^{-\frac{1}{2}k\operatorname{Re}\sqrt{\mu}}\\ &\leqslant C\|f\|_{L^1_U(Q_i^{far})}\left(\frac{\sqrt{|\lambda|}}{\operatorname{Re}(\sqrt{\mu})}\right)^{N/2-1}\int_1^{\infty} se^{-\frac{1}{2}s\operatorname{Re}\sqrt{\mu}}\,ds.\end{aligned}$$

Finally, changing variables in the integral above as $r=\operatorname{Re}(\sqrt{\mu})s$, we obtain

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leqslant C\left(\frac{\sqrt{|\lambda|}}{\operatorname{Re}(\sqrt{\mu})}\right)^{N/2-1}\frac{1}{\operatorname{Re}(\sqrt{\mu})^2}\|f\|_{L^1_U(Q_i^{far})}$$

which can be arranged as

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leqslant \left(\frac{\sqrt{|\lambda|}}{\operatorname{Re}(\sqrt{\mu})}\right)^{N/2+1}\frac{C}{|\lambda|}\|f\|_{L^1_U(Q_i^{far})}.$$

To conclude, observe that for all $\lambda\in S_{0,\phi}$ we find

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leqslant \frac{C}{\cos(\phi/2)^{N/2+1}}\frac{1}{|\lambda|}\|f\|_{L^1_U(Q_i^{far})}.$$

Thus, (3.5) is proved for $N>2$.

We show now (3.5) for $N=2$. Proceeding as above and using (3.3) we get

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leqslant C\|f\|_{L^1_U(Q_i^{far})}\sum_{j\notin N(i)}\max\left\{\ln\frac{1}{d_{ij}\operatorname{Re}(\sqrt{\mu})},1\right\}e^{-\frac{1}{2}d_{ij}\operatorname{Re}\sqrt{\mu}}.$$

Using again that $\sharp\{j\in\mathbb{Z},\ d_{ij}=k\}\leqslant Ck^{N-1}$ we get

$$\begin{aligned}\|u_i^{far}\|_{L^\infty(Q_i)} &\leqslant C\|f\|_{L^1_U(Q_i^{far})}\sum_{k=1}^{\infty}k\max\left\{\ln\frac{1}{k\operatorname{Re}(\sqrt{\mu})},1\right\}e^{-\frac{1}{2}k\operatorname{Re}\sqrt{\mu}}\\ &\leqslant C\|f\|_{L^1_U(Q_i^{far})}\int_0^{\infty}s\max\left\{\ln\frac{1}{s\operatorname{Re}(\sqrt{\mu})},1\right\}e^{-\frac{1}{2}s\operatorname{Re}\sqrt{\mu}}\,ds\end{aligned}$$

and with the change of variables $r=\operatorname{Re}(\sqrt{\mu})s$ we obtain

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leqslant \|f\|_{L^1_U(Q_i^{far})}\frac{C}{\operatorname{Re}(\sqrt{\mu})^2} = \left(\frac{\sqrt{|\lambda|}}{\operatorname{Re}(\sqrt{\mu})}\right)^{2}\frac{C}{|\lambda|}\|f\|_{L^1_U(Q_i^{far})}.$$

Thus for all $\lambda\in S_{0,\phi}$ we find

$$\|u_i^{far}\|_{L^\infty(Q_i)} \leqslant \frac{C}{\cos(\phi/2)^2}\frac{1}{|\lambda|}\|f\|_{L^1_U(Q_i^{far})}$$

and the result is proved.

In particular since $\phi > 0$ is arbitrary, $\sigma(-\Delta) \subset [0,\infty)$. For the opposite inclusion, note that $u(x) = e^{i\omega x}$, $\omega \in \mathbb{R}^N$, satisfies $u \in \dot{L}^p_U(\mathbb{R}^N)$ and

$$-\Delta u = \lambda u$$

for $\lambda = |\omega|^2 \subset [0,\infty)$, and thus $[0,\infty) \subset \sigma(-\Delta)$.

For part ii), since $-\Delta$ is sectorial with sector $S_{0,\phi}$ with $\phi < \pi/4$ and we have the estimate $\|(-\Delta - \lambda)^{-1}\| \leqslant \frac{C}{|\lambda|}$ for $\lambda \in S_{0,\phi}$, we apply [15, 10.5] (see Proposition C.1). Therefore, we get that Δ^2 is sectorial with sector $S_{0,2\phi}$. Note that $\sigma(\Delta^2) \subset [0,\infty)$ because $\phi > 0$ is arbitrarily small. Also, note again that $u(x) = e^{i\omega x}$, $\omega \in \mathbb{R}^N$, satisfies $u \in \dot{L}^q_U(\mathbb{R}^N)$ and

$$\Delta^2 u = \lambda u$$

for $\lambda = |\omega|^4 \subset [0,\infty)$. □

Now, using Proposition C.2 and an argument as in Lemma 2.2 we get the next result.

Lemma 3.4. *Consider the problem*

$$\begin{cases} u_t + \Delta^2 u = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases} \tag{3.6}$$

i) *Then for each $1 < q < \infty$, (3.6) defines an analytic semigroup, $S_{\Delta^2}(t)$, in the scale $X_\alpha := E_{2\alpha} = \dot{H}^{4\alpha,q}_U(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, such that for any $\mu_0 > 0$ there exists C such that*

$$\|S_{\Delta^2}(t)u_0\|_{\dot{H}^{4\alpha,q}_U(\mathbb{R}^N)} \leqslant \frac{M_{\alpha,\beta}e^{\mu t}}{t^{\alpha-\beta}}\|u_0\|_{\dot{H}^{4\beta,q}_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{H}^{4\beta,q}_U(\mathbb{R}^N),$$

with $\alpha, \beta \in \mathbb{R}$, $\alpha \geqslant \beta$.

ii) *The analytic semigroup $S_{\Delta^2}(t)$, in $\dot{L}^q_U(\mathbb{R}^N)$, $1 < q < \infty$, satisfies*

$$\|S_{\Delta^2}(t)u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \leqslant \frac{M_{q,r}e^{\mu_0 t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{L}^q_U(\mathbb{R}^N),$$

for any $\mu_0 > 0$ and $1 < q \leqslant r \leqslant \infty$ and some $M_{q,r} > 0$.

For a similar estimate with $q = 1 < r \leqslant \infty$, we can proceed as in Remark 2.3.

We can now adapt the arguments for Bessel and Lebesgue spaces in Section 2 to the uniform Bessel spaces to perturb Eq. (3.6) as follows. First, as in [16, Lemma 26, p. 43] we have

Lemma 3.5.

i) *Assume that $m \in L^p_U(\mathbb{R}^N)$, then the multiplication operator*

$$Pu(x) = m(x)u(x)$$

satisfies, for $r \geqslant p'$ and $\frac{1}{s} = \frac{1}{r} + \frac{1}{p}$, that

$$P \in \mathcal{L}\big(L^r_U(\mathbb{R}^N), L^s_U(\mathbb{R}^N)\big), \qquad \|P\|_{\mathcal{L}(L^r_U(\mathbb{R}^N), L^s_U(\mathbb{R}^N))} \leqslant C\|m\|_{L^p_U(\mathbb{R}^N)}.$$

ii) *If moreover $m \in \dot{L}^p_U(\mathbb{R}^N)$ we have for $r \geqslant p'$ and $\frac{1}{s} = \frac{1}{r} + \frac{1}{p}$, that*

$$P \in \mathcal{L}\big(\dot{L}^r_U(\mathbb{R}^N), \dot{L}^s_U(\mathbb{R}^N)\big), \qquad \|P\|_{\mathcal{L}(\dot{L}^r_U(\mathbb{R}^N), \dot{L}^s_U(\mathbb{R}^N))} \leqslant C\|m\|_{L^p_U(\mathbb{R}^N)}.$$

Now, we consider perturbations similar to the perturbations in (2.9) with $b = 0$, that is,

$$P_a u = d(x)D^a u \tag{3.7}$$

with $d \in \dot{L}^p_U(\mathbb{R}^N)$ and $a \in \mathbb{N}$. Note that since the uniform Bessel spaces are not reflexive (even for $q = 2$), the negative spaces cannot be described as dual spaces, and thus, the approach in Proposition 2.9 cannot be carried out for $b \neq 0$ in uniform spaces. We will use Proposition 3.1 instead.

Proposition 3.6. *Let $P_a u = d(x)D^a u$ with $d \in \dot{L}^p_U(\mathbb{R}^N)$, $a \in \{0,1,2,3\}$ and let $s \geqslant a$, $\sigma \geqslant 0$. Then for $1 < q < \infty$, if*

$$\left(s - a - \frac{N}{q}\right)_- + \left(\sigma - \frac{N}{q'}\right)_- \geqslant -\frac{N}{p'} \tag{3.8}$$

we have

$$P_a \in \mathcal{L}\big(\dot{H}^{s,q}_U(\mathbb{R}^N), \dot{H}^{-\sigma,q}_U(\mathbb{R}^N)\big), \qquad \|P_a\|_{\mathcal{L}(\dot{H}^{s,q}_U(\mathbb{R}^N), \dot{H}^{-\sigma,q}_U(\mathbb{R}^N))} \leqslant C\|d\|_{\dot{L}^p_U(\mathbb{R}^N)}.$$

Proof. First note that $u \in \dot{H}^{s,q}_U(\mathbb{R}^N)$, thus $D^a u \in \dot{H}^{s-a,q}_U(\mathbb{R}^N)$. Because of (3.8) we can choose $r, \rho \geqslant 1$ such that $(s-a-\frac{N}{q})_- \geqslant -\frac{N}{r}$ and $(\sigma - \frac{N}{q'})_- \geqslant -\frac{N}{\rho'}$ with $\frac{1}{\rho} = \frac{1}{r} + \frac{1}{p}$ (and so $r \geqslant p'$).

Therefore we can use the inclusion $\dot{H}^{s-a,q}_U(\mathbb{R}^N) \hookrightarrow \dot{L}^r_U(\mathbb{R}^N)$ and then part ii) in Lemma 3.5 gives $P_a u \in \dot{L}^\rho_U(\mathbb{R}^N)$ and finally, because of Proposition 3.1, we use the inclusion $\dot{L}^\rho_U(\mathbb{R}^N) \hookrightarrow \dot{H}^{-\sigma,q}_U(\mathbb{R}^N)$ and we get the result. □

With this, we can obtain the main result for perturbations of (3.6).

Theorem 3.7. *Let $a \in \{0,1,2,3\}$, $d \in \dot{L}^p_U(\mathbb{R}^N)$ be such that $\|d\|_{\dot{L}^p_U(\mathbb{R}^N)} \leqslant R_0$ with $p > \frac{N}{4-a}$. Then for any $1 < q < \infty$ and any P_a as in (3.7) there exists an interval $I(q,a) \subset (-1+\frac{a}{4}, 1)$ containing $(-1+\frac{a}{4}+\frac{N}{4p}, 1-\frac{N}{4p})$, such that for any $\gamma \in I(q,a)$, we have a continuous, analytic semigroup, $S_{P_a}(t)$, in the space $\dot{H}^{4\gamma,q}_U(\mathbb{R}^N)$, for the problem*

$$\begin{cases} u_t + \Delta^2 u + d(x)D^a u = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimate

$$\big\|S_{P_a}(t)u_0\big\|_{\dot{H}^{4\gamma',q}_U(\mathbb{R}^N)} \leqslant \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{\dot{H}^{4\gamma,q}_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{H}^{4\gamma}_U(\mathbb{R}^N),$$

for every $\gamma, \gamma' \in I(q,a)$ with $\gamma' \geqslant \gamma$, and

$$\big\|S_{P_a}(t)u_0\big\|_{\dot{L}^r_U(\mathbb{R}^N)} \leqslant \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{L}^q_U(\mathbb{R}^N),$$

for $1 < q \leqslant r \leqslant \infty$ with some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

For each P_a, the interval $I(q,a)$ is given by

$$I(q,a) = \left(-1 + \frac{a}{4} + \frac{N}{4}\left(\frac{1}{p} - \frac{1}{q'}\right)_+, 1 - \frac{N}{4}\left(\frac{1}{p} - \frac{1}{q}\right)_+\right) \subset \left(-1 + \frac{a}{4}, 1\right).$$

Finally, if, as $\varepsilon \to 0$

$$d_\varepsilon \to d \quad \text{in } \dot{L}^p_U(\mathbb{R}^N),\ p > \frac{N}{4-k},$$

then for every $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\big\|S_{P_\varepsilon}(t) - S_P(t)\big\|_{\mathcal{L}(\dot{H}^{4\gamma,q}_U(\mathbb{R}^N), \dot{H}^{4\gamma',q}_U(\mathbb{R}^N))} \leqslant \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \quad \forall 0 < t \leqslant T,$$

for all $\gamma, \gamma' \in I(q,a,b)$, $\gamma' \geqslant \gamma$ and for all $1 < q \leqslant r \leqslant \infty$,

$$\big\|S_{P_\varepsilon}(t) - S_P(t)\big\|_{\mathcal{L}(\dot{L}^q_U(\mathbb{R}^N), \dot{L}^r_U(\mathbb{R}^N))} \leqslant \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \quad \forall 0 < t \leqslant T.$$

Proof. The proof is as the proof of Theorem 2.10 but using Proposition 3.6 instead of Proposition 2.9. The analyticity comes again from [16, Theorem 12] (see Theorem A.3). □

Note that Remark 2.11, Proposition 2.12 and Remark 2.14 apply here as well. Also, we can replace D^a in (3.7) by $(-\Delta + cI)^{a/2}$ with $0 \leqslant a < 4$ as in Theorem 2.15.

4. Some other higher order equations

In this section we show that all the results in Sections 2 and 3 above also hold true for other natural powers of suitable operators, and in particular, for any power of the Laplacian, $(-\Delta)^m$, with $m \in \mathbb{N}$. The proofs below have barely no changes with respect to the ones above, and we now detail the main points for them.

Lemma 4.1. *For $1 < q < \infty$, in $E_0 = L^q(\mathbb{R}^N)$ the operator $(-\Delta)^m$, with domain $E_m = D(-\Delta^m) = H^{2m,q}(\mathbb{R}^N)$, satisfies the estimate*

$$\left\| \left((-\Delta)^m - \lambda\right)^{-1} \right\|_{L^q(\mathbb{R}^N)} \leqslant M|\lambda|^{-1} \quad \text{for all } \lambda \in S_{0,m\phi},$$

where $\phi > 0$ is arbitrarily small. Furthermore $\sigma((-\Delta)^m) = [0, \infty)$ and therefore

$$\operatorname{type}\left((-\Delta)^m\right) = 0.$$

The proof is exactly as the one of Lemma 2.1, using again Proposition 10.5 in [15] (see also Proposition C.1). Also, Proposition C.2 can be adapted to any m. These two pieces of information together lead to

Lemma 4.2. *Consider the problem*

$$\begin{cases} u_t + (-\Delta)^m u = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{4.1}$$

with $m \in \mathbb{N}$.

i) *Then for $1 < q < \infty$, (4.1) defines an analytic semigroup, $S_{(-\Delta)^m}(t)$, in the scale $X_\alpha = E_{m\alpha} = H^{2m\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, such that for any $\mu_0 > 0$ there exists $C(\alpha - \beta)$ such that*

$$\left\| S_{(-\Delta)^m}(t) \right\|_{\mathcal{L}(H^{2m\beta,q}(\mathbb{R}^N), H^{2m\alpha,q}(\mathbb{R}^N))} \leqslant \frac{C(\alpha - \beta)}{t^{\alpha-\beta}} e^{\mu_0 t}, \quad t > 0,\ \alpha, \beta \in \mathbb{R},\ \alpha \geqslant \beta.$$

ii) *The analytic semigroup, $S_{(-\Delta)^m}(t)$, in $L^q(\mathbb{R}^N)$, $1 < q < \infty$, satisfies that for any $\mu_0 > 0$ there exists $M_{q,r}$ such that*

$$\left\| S_{(-\Delta)^m}(t) \right\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \leqslant \frac{M_{q,r}}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}} e^{\mu_0 t}, \quad t > 0,$$

for $1 < q \leqslant r \leqslant \infty$.

Note that the proof of Lemma 2.2 can be carried out now taking $(-\Delta)^m$ instead of Δ^2 in the scale of spaces.

Also note that the solution of problem (4.1) can also be described as the convolution of the initial data with the fundamental kernel for the m-Laplacian operator, which satisfies suitable Gaussian bounds; see e.g. [11,7].

We can now add the perturbations to (4.1), as in Theorem 2.10.

Theorem 4.3. *Let $a, b \in \mathbb{N}$ with $k = a + b \leqslant 2m - 1$ and $P_{a,b}$ be as in (2.9). Assume that $\|d\|_{L^p_U(\mathbb{R}^N)} \leqslant R_0$ with $p > \frac{N}{2m-k}$. Then for any $1 < q < \infty$ and such $P_{a,b}$ there exists an interval $I(q, a, b) \subset (-1 + \frac{a}{2m}, 1 - \frac{b}{2m})$ containing $(-1 + \frac{a}{2m} + \frac{N}{2mp}, 1 - \frac{b}{2m} - \frac{N}{2mp})$, such that for any $\gamma \in I(q, a, b)$, we have a strongly continuous, analytic semigroup, $S_{P_{a,b}}(t)$, in the space $H^{2m\gamma,q}(\mathbb{R}^N)$, for the problem*

$$\begin{cases} u_t + (-\Delta)^m u + D^b\left(d(x) D^a u\right) = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimates

$$\left\| S_{P_{a,b}}(t) u_0 \right\|_{H^{2m\gamma',q}(\mathbb{R}^N)} \leqslant \frac{M_{\gamma',\gamma} e^{\mu t}}{t^{\gamma'-\gamma}} \|u_0\|_{H^{2m\gamma,q}(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in H^{2m\gamma,q}\left(\mathbb{R}^N\right),$$

for every $\gamma, \gamma' \in I(q, a, b)$ with $\gamma' \geqslant \gamma$, and

$$\left\| S_{P_{a,b}}(t) u_0 \right\|_{L^r(\mathbb{R}^N)} \leqslant \frac{M_{q,r} e^{\mu t}}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{L^q(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in L^q\left(\mathbb{R}^N\right),$$

with $1 < q \leqslant r \leqslant \infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q, a, b)$ is given by

$$I(q, a, b) = \left(-1 + \frac{a}{2m} + \frac{N}{2m}\left(\frac{1}{p} - \frac{1}{q'}\right)_+, 1 - \frac{b}{2m} - \frac{N}{2m}\left(\frac{1}{p} - \frac{1}{q}\right)_+\right).$$

Finally, if as $\varepsilon \to 0$

$$d_\varepsilon \to d \quad \text{in } L^p_U(\mathbb{R}^N),\ p > \frac{N}{2m-k},$$

then for every $1 < q < \infty$ and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\left\| S_{P_\varepsilon}(t) - S_P(t) \right\|_{\mathcal{L}(H^{2m\gamma,q}(\mathbb{R}^N), H^{2m\gamma',q}(\mathbb{R}^N))} \leqslant \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \quad \forall 0 < t \leqslant T,$$

for all $\gamma, \gamma' \in I(q,a,b)$, with $\gamma' \geqslant \gamma$ and

$$\left\| S_{P_\varepsilon}(t) - S_P(t) \right\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \leqslant \frac{C(\varepsilon)}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}}, \quad \forall 0 < t \leqslant T,$$

for all $1 < q \leqslant r \leqslant \infty$.

Note that now, the amount of possible combinations of perturbations becomes enormous, however, they can be combined just as explained in Proposition 2.12 and Remark 2.14. Also, Remark 2.11 still holds.

We finally turn into the uniform spaces $\dot{L}^q_U(\mathbb{R}^N)$. First of all, we check the information about the spectrum and resolvent set for $(-\Delta)^m$ in $\dot{L}^q_U(\mathbb{R}^N)$, with the same ideas as in Proposition 3.3 and using again Proposition C.1.

Lemma 4.4. *For $1 < q < \infty$, the operator $(-\Delta)^m$, in the space $E_0 = \dot{L}^q_U(\mathbb{R}^N)$ with domain $E_m = D((-\Delta)^m) = \dot{H}^{2m,q}_U(\mathbb{R}^N)$, satisfies the estimate*

$$\left\| \left((-\Delta)^m - \lambda \right)^{-1} \right\|_{\dot{L}^q_U(\mathbb{R}^N)} \leqslant M|\lambda|^{-1}$$

for all λ in a sector $S_{0,m\phi}$ as in (2.3) for $\phi > 0$ arbitrarily small.

Furthermore, $\sigma((-\Delta)^m) = [0,\infty)$, and thus, $\operatorname{type}((-\Delta)^m) = 0$.

Again, this leads to

Lemma 4.5. *Consider the problem*

$$\begin{cases} u_t + (-\Delta)^m u = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases} \tag{4.2}$$

i) *Then for each $1 < q < \infty$, (4.2) defines an analytic semigroup, $S_{(-\Delta)^m}(t)$, in the scale $X_\alpha := E_{m\alpha} = \dot{H}^{2m\alpha,q}_U(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, such that for any $\mu_0 > 0$ there exists C such that*

$$\left\| S_{(-\Delta)^m}(t) u_0 \right\|_{\dot{H}^{2m\alpha,q}_U(\mathbb{R}^N)} \leqslant \frac{M_{\alpha,\beta} e^{\mu_0 t}}{t^{\alpha-\beta}} \|u_0\|_{\dot{H}^{4\beta,q}_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{H}^{4\beta,q}_U(\mathbb{R}^N),$$

with $\alpha, \beta \in \mathbb{R}$, $\alpha \geqslant \beta$.

ii) *The analytic semigroup $S_{(-\Delta)^m}(t)$, in $\dot{L}^q_U(\mathbb{R}^N)$, $1 < q < \infty$, satisfies*

$$\left\| S_{(-\Delta)^m}(t) u_0 \right\|_{\dot{L}^r_U(\mathbb{R}^N)} \leqslant \frac{M_{q,r} e^{\mu t}}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{L}^q_U(\mathbb{R}^N),$$

for any $1 < q \leqslant r \leqslant \infty$ and μ_0 and some $M_{q,r} > 0$.

Then adding perturbations as above, we have

Theorem 4.6. *Let $a \in \mathbb{N}$, $a \leqslant 2m-1$ and $\|d\|_{\dot{L}^p_U(\mathbb{R}^N)} \leqslant R_0$ with $p > \frac{N}{2m-a}$, then for any $1 < q < \infty$ and any P_a as in (3.7) there exists an interval $I(q,a) \subset (-1 + \frac{a}{2m}, 1)$ containing $(-1 + \frac{a}{2m} + \frac{N}{2mp}, 1 - \frac{N}{2mp})$, such that for any $\gamma \in I(q,a)$, we have a continuous, analytic semigroup, $S_{P_a}(t)$, in the space $\dot{H}^{2m\gamma,q}_U(\mathbb{R}^N)$, for the problem*

$$\begin{cases} u_t + (-\Delta)^m u + d(x) D^a u = 0, & x \in \mathbb{R}^N,\ t > 0, \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimate

$$\left\| S_{P_a}(t) u_0 \right\|_{\dot{H}^{2m\gamma',q}_U(\mathbb{R}^N)} \leqslant \frac{M_{\gamma',\gamma} e^{\mu t}}{t^{\gamma'-\gamma}} \|u_0\|_{\dot{H}^{2m\gamma,q}_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{H}^{2m\gamma,q}_U(\mathbb{R}^N),$$

for every $\gamma, \gamma' \in I(q,a)$ with $\gamma' \geqslant \gamma$, and

$$\left\| S_{P_a}(t)u_0 \right\|_{\dot{L}^r_U(\mathbb{R}^N)} \leqslant \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \quad t>0,\ u_0 \in \dot{L}^q_U\big(\mathbb{R}^N\big),$$

with $1<q \leqslant r \leqslant \infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

For each P_a, the interval $I(q,a)$ is given by

$$I(q,a) = \left(-1+\frac{a}{2m}+\frac{N}{2m}\left(\frac{1}{p}-\frac{1}{q'}\right)_+, 1-\frac{N}{2m}\left(\frac{1}{p}-\frac{1}{q}\right)_+\right) \subset \left(-1+\frac{a}{2m}, 1\right).$$

Finally, if as $\varepsilon \to 0$

$$d_\varepsilon \to d \quad \text{in } \dot{L}^p_U\big(\mathbb{R}^N\big),\ p>\frac{N}{2m-k},$$

then for every $1<q<\infty$ and $T>0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\left\| S_{P_\varepsilon}(t) - S_P(t) \right\|_{\mathcal{L}(\dot{H}^{2m\gamma,q}_U(\mathbb{R}^N), \dot{H}^{2m\gamma',q}_U(\mathbb{R}^N))} \leqslant \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \quad \forall 0<t \leqslant T,$$

for all $\gamma, \gamma' \in I(q,a,b)$, $\gamma' \geqslant \gamma$ and

$$\left\| S_{P_\varepsilon}(t) - S_P(t) \right\|_{\mathcal{L}(\dot{L}^q_U(\mathbb{R}^N), \dot{L}^r_U(\mathbb{R}^N))} \leqslant \frac{C(\varepsilon)}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}}, \quad \forall 0<t \leqslant T,$$

for all $1<q \leqslant r \leqslant \infty$.

The proofs of both Lemma 4.5 and Theorem 4.6 follow the proofs of Lemma 3.4 and Theorem 3.7, just replacing Δ^2 by $(-\Delta)^m$ as the order of the operator involved.

Appendix A. Some previous results

We recall some results from [16] that are needed for this article. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of Banach spaces, with α in an interval I, endowed with a norm $\|\cdot\|_\alpha$. Let $S(t)$ be a semigroup on a scale $\{X_\alpha\}_{\alpha \in I}$, such that

$$\left\| S(t) \right\|_{\beta,\alpha} := \left\| S(t) \right\|_{\mathcal{L}(X_\beta, X_\alpha)} \leqslant \frac{M_0(\beta,\alpha)}{t^{\alpha-\beta}}, \quad \forall 0<t \leqslant 1, \tag{A.1}$$

for all $\alpha, \beta \in I$, $\alpha \geqslant \beta$ for some constant $M_0(\beta,\alpha)>0$.

Now, assume that for some fixed $\alpha \geqslant \beta$, with $0 \leqslant \alpha-\beta<1$ we have a linear perturbation satisfying

$$P \in \mathcal{L}(X_\alpha, X_\beta), \quad 0 \leqslant \alpha-\beta<1. \tag{A.2}$$

Sometimes, "nested" spaces are used, that is, for all $\alpha, \beta \in I$ with $\alpha \geqslant \beta$ we have

$$X_\alpha \subset X_\beta \tag{A.3}$$

with continuous inclusion and the norm of the inclusion will be denoted $\|i\|_{\alpha,\beta}$.

Consider the perturbed problem

$$u(t;u_0) = S(t)u_0 + \int_0^t S(t-\tau)Pu(\tau;u_0)\,d\tau, \quad t>0, \tag{A.4}$$

which corresponds to solving the problem $u_t + Au = Pu$, where $-A$ is the infinitesimal generator of the semigroup $S(t)$.

The following result is taken from [16, Proposition 10] and states the existence of a perturbed semigroup defined by (A.4).

Theorem A.1. *Assume (A.1) and (A.2). Then for every $R_0>0$ and every*

$$P \in \mathcal{L}(X_\alpha, X_\beta) \quad \text{with } \|P\|_{\mathcal{L}(X_\alpha, X_\beta)} \leqslant R_0$$

and for every $\gamma, \gamma' \in I$ such that

$$\gamma \in E(\alpha) = (\alpha-1, \alpha] \cap I, \qquad \gamma' \in R(\beta) = [\beta, \beta+1) \cap I, \qquad \gamma' \geqslant \gamma,$$

there exist constants $\omega = \omega(\gamma, \gamma', R_0) \geqslant 0$ and $M_0 = M_0(\gamma, \gamma', R_0)$ such that, for $t > 0$, there exists a unique solution of (A.4), which defines a mapping from X_γ into $X_{\gamma'}$ as

$$S_P(t)u_0 := u(t; u_0) \quad \textit{for all } t > 0$$

such that

$$\left\| S_P(t)u_0 \right\|_{\gamma'} \leqslant M_0 e^{\omega t} t^{-(\gamma'-\gamma)} \|u_0\|_\gamma, \quad \gamma' \geqslant \gamma.$$

In particular for any $\gamma \in [\beta, \alpha]$, $S_P(t) \in \mathcal{L}(X_\gamma)$ and it is a semigroup of linear continuous operators in X_γ.
The same is true for any $\gamma \in E(\alpha)$, if the scale is nested.

Now we turn into the continuity of the perturbed semigroup with respect to the perturbation. With the setting above, assume that we have two perturbations

$$P_i \in \mathcal{L}(X_\alpha, X_\beta), \qquad \|P_i\|_{\mathcal{L}(X_\alpha, X_\beta)} \leqslant R_0, \quad i = 1, 2,\ 0 \leqslant \alpha - \beta < 1,$$

for some $R_0 > 0$.

Consider then an initial data $u_0 \in X_\gamma$, and the corresponding solutions of the perturbed problem

$$u^i(t; u_0) = S_{P_i}(t)u_0 = S(t)u_0 + \int_0^t S(t-\tau)P_i u^i(\tau; u_0)\, d\tau, \quad t > 0.$$

Then we have the following continuity result, see [16, Theorem 14].

Theorem A.2. *With the notations above, for any $R_0 > 0$, there exists a sufficiently small T_0 such that for all perturbations P_i, $i = 1, 2$, such that $\|P_i\|_{\mathcal{L}(X_\alpha, X_\beta)} \leqslant R_0$,*

$$\left\| S_{P_1}(t) - S_{P_2}(t) \right\|_{\mathcal{L}(X_\gamma, X_{\gamma'})} \leqslant \frac{L(T_0, R_0)}{t^{\gamma'-\gamma}} \|P_1 - P_2\|_{\mathcal{L}(X_\alpha, X_\beta)} \quad \textit{for all } 0 < t \leqslant T_0,$$

and for every $T > T_0$

$$\left\| S_{P_1}(t) - S_{P_2}(t) \right\|_{\mathcal{L}(X_\gamma, X_{\gamma'})} \leqslant L(T, T_0, R_0) \|P_1 - P_2\|_{\mathcal{L}(X_\alpha, X_\beta)} \quad \textit{for all } T_0 < t \leqslant T,$$

with

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha] \cap I, \qquad \gamma' \in R(\beta) = [\beta, \beta + 1) \cap I, \qquad \gamma' \geqslant \gamma.$$

Finally we will also need the following result about the analyticity of the semigroup defined by (A.4). Note that the first part of the theorem below is taken from [16, Theorem 12], while the second part is easy to prove.

Theorem A.3. *Assume that the scale is nested, that is, (A.3), and that for any $\gamma \in I$, if $-A_\gamma$ denotes the infinitesimal generator of $S(t)$ in X_γ, then its domain is given by $D(A_\gamma) = X_{\gamma+1}$.*

Also assume that the scale satisfies either one of the following interpolation properties:

i) *If Y is a Banach space and $T \in \mathcal{L}(X_\gamma, Y)$ and $T \in \mathcal{L}(X_{\gamma'}, Y)$ then $T \in \mathcal{L}(X_{\theta\gamma+(1-\theta)\gamma'}, Y)$ for $\theta \in [0, 1]$ and*

$$\|T\|_{\mathcal{L}(X_{\theta\gamma+(1-\theta)\gamma'}, Y)} \leqslant \|T\|^\theta_{\mathcal{L}(X_\gamma, Y)} \|T\|^{1-\theta}_{\mathcal{L}(X_{\gamma'}, Y)}.$$

ii) *For any $\gamma, \gamma' \in I$ and $0 < \theta < 1$*

$$\|u\|_{X_{\theta\gamma+(1-\theta)\gamma}} \leqslant C \|u\|^\theta_{X_\gamma} \|u\|^{1-\theta}_{X_{\gamma'}}.$$

Finally, as in Theorem A.1, assume that for some fixed $\alpha \geqslant \beta$, with $0 \leqslant \alpha - \beta < 1$ we have a linear perturbation satisfying

$$P \in \mathcal{L}(X_\alpha, X_\beta) \quad \textit{with } \|P\|_{\mathcal{L}(X_\alpha, X_\beta)} \leqslant R_0.$$

Then, there exists some $0 < \omega_0 = \omega_0(R_0)$ such that for any $\mathrm{Re}(\lambda) \geqslant \omega_0$ and any $\gamma \in (\alpha - 1, \beta)$ the operator $A_\gamma + \lambda I - P$, between $X_{\gamma+1}$ and X_γ, is invertible and

$$\left\| (A_\gamma + \lambda I - P)^{-1} \right\|_{\mathcal{L}(X_\gamma, X_\gamma)} \leqslant \frac{C}{|\lambda|}, \qquad \mathrm{Re}(\lambda) \geqslant \omega_0,$$

and

$$\left\|(A_\gamma+\lambda I-P)^{-1}\right\|_{\mathcal{L}(X_\gamma,X_{\gamma+1})}\leqslant C,\qquad \mathrm{Re}(\lambda)\geqslant\omega_0$$

where C is independent of P and λ.

In particular, for every $\gamma\in(\alpha-1,\beta)$, the semigroup $S_P(t)$ in X_γ in Theorem A.1 is analytic.

Remark A.4. Using this we get that $u(t)=S_P(t)u_0$, with $S_P(t)$ as above, satisfies the equation

$$u_t+A_\gamma u=Pu,\quad t>0,\ \text{in } X^\beta,$$

see also Remark 6 in [16].

Appendix B. Scales of spaces for sectorial operators

Here, we review the construction of suitable scales of spaces for sectorial operators in Banach spaces in [2] and, in view of the applications in this paper, we particularize for the scales of complex interpolation–extrapolation spaces and the scale of fractional power spaces.

Following [2], let E_0, E_1 be Banach spaces with continuous inclusion $E_1\subset E_0$ and consider the class $\mathcal{H}(E_1,E_0)$ of linear operators in E_0, with dense domain E_1 such that if $A_0\in\mathcal{H}(E_1,E_0)$, then $-A_0$ generates a strongly continuous analytic semigroup in E_0, $\{e^{-A_0t};\ t\geqslant 0\}$. In other words, A_0 is sectorial as defined in [14].

Note that for $A_0\in\mathcal{H}(E_1,E_0)$, we define $\mathrm{type}(A_0)=-\inf\{\mathrm{Re}(\sigma(A_0))\}$.

In what follows we will momentarily assume that

$$0\in\rho(A_0). \tag{B.1}$$

With this it can be proved that the norm $\|\cdot\|_{E_1}$ is equivalent $\|A_0\cdot\|_{E_0}$, and we can start a recurring construction as follows.

Consider $E_2:=D(A_1)=\{u\in E_1,\ A_1u\in E_1\}$ where $A_1:E_2\hookrightarrow E_1$ is the realization (and also the closure) of A_0 in E_1 and endowed with the norm $\|\cdot\|_{E_2}=\|A_1\cdot\|_{E_1}$.

We can iterate this process to get a discrete scale of Banach spaces $\{E_n,\ n\in\mathbb{N}\}$ and the realizations of A_0 in E_n, which we denote by A_n, satisfy $A_n\in\mathcal{H}(E_{n+1},E_n)$ and are isometric isomorphisms from $E_{n+1}\to E_n$, see [2, V.1.2.1, p. 256].

For the construction of the negative side of the scale, define E_{-1} as the completion of E_0 relatively to the norm $\|\cdot\|_{E_{-1}}:=\|A_0^{-1}\cdot\|_{E_0}$, which is a Banach space such that $E_0\hookrightarrow E_{-1}$ densely and A_{-1} is the unique continuous extension of A_0, which is an isometric isomorphism from $E_0\to E_{-1}$. This extension is called again the realization of A_0 in E_{-1}.

Again, we iterate the process of completion with the norm generated by the new operator and we get a negative discrete scale $\{E_{-n},\ n\in\mathbb{N}\}$ and $A_{-n}\in\mathcal{H}(E_{-n+1},E_{-n})$, where A_{-n} denotes the realization of A_0, the closure of A_{-n+1} in E_{-n} and is an isometric isomorphism from $E_{-n+1}\to E_{-n}$ see [2, V.1.3.2, p. 263] and the comments on [2, p. 264].

So we have a two-sided discrete nested scale [2, V.1.3.4, p. 264]:

$$\{E_k,\ k\in\mathbb{Z}\},\qquad A_k\in\mathcal{H}(E_{k+1},E_k) \tag{B.2}$$

where A_k denotes the realization of A_0, the closure of A_{k+1} in E_k and is an isometric isomorphism from $E_{k+1}\to E_k$ which satisfies

$$\rho(A_k)=\rho(A_0),\quad k\in\mathbb{Z}. \tag{B.3}$$

Now we construct intermediate spaces between the discrete scale $\{E_k,\ k\in\mathbb{Z}\}$ following two different procedures.

B.1. Construction of the interpolation–extrapolation scale for A_0

Starting with the discrete scale (B.2) and taking the complex interpolation method, we proceed as in [2, V.1.5.1, p. 275] to obtain the spaces

$$E_\alpha:=E_{k+\theta}:=[E_{k+1},E_k]_\theta,\quad \theta\in(0,1),\ k\in\mathbb{Z},$$

and the operator A_α as the interpolation of A_{k+1} and A_k. Thus we obtain the continuous nested interpolation scale

$$\{E_\alpha,\ \alpha\in\mathbb{R}\},\qquad A_\alpha\in\mathcal{H}(E_{\alpha+1},E_\alpha)$$

and A_α is an isometry from $E_{\alpha+1}$ into E_α. Note that if $\alpha>\beta$, E_α is densely included in E_β and A_α is the realization of A_0 in E_α. Moreover, for every $\alpha\in\mathbb{R}$

$$\rho(A_\alpha)=\rho(A_0),$$

see [2, V.1.1.2.e), p. 252].

Now, since $A_\beta \in \mathcal{H}(E_{\beta+1}, E_\beta)$, $-A_\beta$ generates an analytic semigroup in E_β with the property [2, V.2.1.3, p. 289]:

$$\left\|e^{-A_\beta t}\right\|_{\mathcal{L}(E_\beta, E_\alpha)} \leqslant \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\sigma t}, \quad t>0,\ \alpha, \beta \in \mathbb{R},\ \alpha \geqslant \beta, \tag{B.4}$$

for any $\sigma > \operatorname{type}(A_0)$ and $C(\alpha-\beta)$ is bounded for α, β in bounded sets of $\mathbb{R}$.

If E_0 is reflexive, we can interpolate in the dual scale $\{E_n^\sharp,\ n \in \mathbb{Z}\}$ as well. We take again the complex interpolation $[\cdot,\cdot]_\theta$, and the negative intermediate spaces can be identified with the dual of the positive ones as

$$E_{-\alpha} = \left(E_\alpha^\sharp\right)' \quad \text{and} \quad A_{-\alpha} = \left(A_\alpha^\sharp\right)' \quad \text{for } \alpha > 0,$$

see [2, V.1.5.12, p. 282]. Also, the semigroup in the spaces of the negative side can be identified with the duals by [2, V.2.3.2, p. 298]:

$$e^{-A_{-\alpha} t} = \left(e^{-A_\alpha^\sharp t}\right)', \quad \alpha > 0.$$

Note that the semigroups in (B.4) are extensions or restrictions of each other one, that is, given $\alpha \geqslant \beta$, then

$$e^{-A_\beta t}\big|_{E_\alpha} = e^{-A_\alpha t}, \quad t \geqslant 0.$$

For details see Lemma [2, V.2.1.2]. Hence, we have the following.

Definition B.1. Under the assumptions above we say that the operator A_0 defines an analytic semigroup $S_{A_0}(t)$ in the interpolation scale $\{E_\alpha\}_{\alpha\in\mathbb{R}}$ in the sense that

$$S_{A_0}(t)|_{E_\alpha} = e^{-A_\alpha t}, \quad \forall \alpha \in \mathbb{R}.$$

Observe that

$$\left\|S_{A_0}(t)\right\|_{\mathcal{L}(E_\beta, E_\alpha)} \leqslant \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\sigma t}, \quad t>0,\ \alpha, \beta \in \mathbb{R},\ \alpha \geqslant \beta,$$

for any $\sigma > \operatorname{type}(A_0)$ and $C(\alpha-\beta)$ is bounded for α, β in bounded sets of $\mathbb{R}$.

Now we construct the interpolation scale and the semigroup in the scale, as in Definition B.1, without assuming (B.1).

Proposition B.2. *Let $A_0 \in \mathcal{H}(E_1, E_0)$ and take c such that $0 \in \rho(A_0 + cI)$.*

Then the scale $\{E_\alpha\}_{\alpha\in\mathbb{R}}$ generated by $A_0 + cI$, as above, is independent of c and for any $\alpha \in \mathbb{R}$, the realization of A_0 in E_α, denoted as A_α, satisfies

$$A_\alpha \in \mathcal{H}(E_{\alpha+1}, E_\alpha)$$

and for all $\alpha \in \mathbb{R}$, $\rho(A_\alpha) = \rho(A_0)$.

Hence we have an analytic semigroup $S_{A_0}(t)$ defined in the scale $\{E_\alpha\}_{\alpha\in\mathbb{R}}$ such that $S_{A_0}(t)|_{E_\alpha} = e^{-A_\alpha t}$, $\alpha \in \mathbb{R}$, and satisfies

$$\left\|S_{A_0}(t)\right\|_{\mathcal{L}(E_\beta, E_\alpha)} \leqslant \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\sigma t}, \quad t>0,\ \alpha \geqslant \beta \in \mathbb{R},$$

for any $\sigma > \operatorname{type}(A_0)$.

Furthermore if E_0 is reflexive, then $E_{-\alpha} = (E_\alpha^\sharp)'$, $A_{-\alpha} = (A_\alpha^\sharp)'$ for $\alpha > 0$, and

$$e^{-A_{-\alpha} t} = \left(e^{-A_\alpha^\sharp t}\right)'.$$

B.2. Construction of the fractional power scale for A_0

Now, starting again with the discrete scale (B.2), we construct a fractional power scale $\{F_\alpha\}_{\alpha\in\mathbb{R}}$ following [2]. See also [14] and [15]. For this we will also assume for a moment that

$$(-\infty, 0] \subset \rho(A_0). \tag{B.5}$$

Since the intermediate spaces between the integer scale (B.2) might be different to the ones in the previous section, see Remark B.4 below, we denote now

$$F_k = E_k \quad \text{for } k \in \mathbb{Z}.$$

For the negative scale, note that (B.5) together with (B.3) implies $(-\infty, 0] \subset \rho(A_n)$ for any $n \in \mathbb{Z}$. Fix now $N \in \mathbb{N}$ and take $A_{-N} \in \mathcal{H}(F_{-N+1}, F_{-N})$. Thus, we get the extrapolated fractional power scale of order N,

$$F_{\alpha-N} = D\big(A_{-N}^{\alpha}\big), \quad \alpha \geqslant 0,$$

see [2, V.1.3.8, p. 266] and [2, V.1.3.9, p. 267]. Then we have

$$\{F_\alpha,\ \alpha \geqslant -N\}, \qquad A_\alpha \in \mathcal{H}(F_{\alpha+1}, F_\alpha), \qquad \rho(A_\alpha) = \rho(A_0), \quad \alpha \geqslant -N,$$

and A_α is an isometry from $E_{\alpha+1}$ into E_α.

Now fix $A_\beta : F_{\beta+1} \to F_\beta$ for any $\beta \geqslant -N$. Renaming $F_\beta = Z$, $F_{\beta+1} = Z^1$ we have the following reiteration property (see [2, V.1.2.6, p. 260] or [15, Proposition 10.6])

$$Z^\varepsilon = D\big(A_\beta^\varepsilon\big) = F_{\beta+\varepsilon}$$

for $\varepsilon \in [0, 1]$, and A_β is sectorial in Z, thus we can apply [14, I.1.4.3, p. 26] to get

$$\big\|e^{-A_\beta t}\big\|_{\mathcal{L}(F_\beta, F_\alpha)} \leqslant \frac{C(\alpha-\beta)}{t^{\alpha-\beta}}, \quad t > 0,\ \alpha \geqslant \beta \geqslant -N,$$

for any $\sigma > \mathrm{type}(A_0)$.

As above, if E_0 is reflexive, we can identify the negative side of the scale with some dual spaces by means of [2, V.1.4.12, p. 274] getting

$$F_{-\alpha} = \big(F_\alpha^\sharp\big)' \quad \text{and} \quad A_{-\alpha} = \big(A_\alpha^\sharp\big)', \quad \alpha > 0,$$

with

$$e^{-A_{-\alpha} t} = \big(e^{-A_\alpha^\sharp t}\big)'.$$

Therefore analogously to Definition B.1 we say that A_0 defines an analytic semigroup $S_{A_0}(t)$ in the fractional power scale $\{F_\alpha\}_{\alpha \geqslant -N}$ in the sense that

$$S_{A_0}(t)|_{F_\alpha} = e^{-A_\alpha t}, \quad \forall \alpha \geqslant -N,$$

and

$$\big\|S_{A_0}(t)\big\|_{\mathcal{L}(F_\beta, F_\alpha)} \leqslant \frac{C(\alpha-\beta)}{t^{\alpha-\beta}}, \quad t > 0,\ \alpha \geqslant \beta \geqslant -N.$$

Now we construct the fractional power scale and the semigroup without assuming (B.5).

Proposition B.3. *Let $A_0 \in \mathcal{H}(E_1, E_0)$ and take c such that $(-\infty, 0] \in \rho(A_0 + cI)$.*

Then given $N \in \mathbb{N}$, the scale $\{F_\alpha\}_{\alpha \geqslant -N}$ generated by $A_0 + cI$, as above, is independent of c and the realizations of A_0 in F_α, denoted by A_α, satisfy

$$A_\alpha \in \mathcal{H}(F_{\alpha+1}, F_\alpha), \qquad \rho(A_\alpha) = \rho(A_0), \quad \alpha \geqslant -N.$$

Hence we have an analytic semigroup $S_{A_0}(t)$ defined in the scale $\{F_\alpha\}_{\alpha \geqslant -N}$ such that $S_{A_0}(t)|_{F_\alpha} = e^{-A_\alpha t}$, $\alpha \geqslant -N$, satisfies

$$\big\|S_{A_0}(t)\big\|_{\mathcal{L}(F_\beta, F_\alpha)} \leqslant \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\sigma t}, \quad t > 0,\ \alpha \geqslant \beta \geqslant -N,$$

for any $\sigma > \mathrm{type}(A_0)$.

Furthermore if E_0 is reflexive, then $F_{-\alpha} = (F_\alpha^\sharp)'$, $A_{-\alpha} = (A_\alpha^\sharp)'$ and $e^{-A_{-\alpha} t} = (e^{-A_\alpha^\sharp t})'$ for $0 < \alpha \leqslant N$.

Remark B.4. Note that after Propositions B.2 and B.3, for $A_0 \in \mathcal{H}(E_1, E_0)$ we have a discrete scale (B.2) and with the notations of these propositions, we have

$$F_k = E_k \quad \text{for } k \in \mathbb{Z},\ k \geqslant -N.$$

However, the intermediate spaces, F_α and E_α, for $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, $\alpha \geqslant -N$, do not need to coincide in general. But, if A_0 has bounded imaginary powers, that is, there exist $\varepsilon > 0$ and $M \geqslant 1$ such that

$$\big\|A_0^{it}\big\|_{\mathcal{L}(E_1, E_0)} \leqslant M \quad \text{for } t \in [-\varepsilon, \varepsilon], \tag{B.6}$$

then E_α and the scale of fractional powers F_α coincide, see [2, V.1.5.13, p. 283].

An important case when this happens is when E_0 is a Hilbert space and A_0 is self-adjoint.

Appendix C. The scales and semigroup for A_0^2

In this section we show how the scale of spaces constructed in Appendix B for A_0 can be used for the squared operator $A_0^2 := A_0 \circ A_1$. That is, our goal here is to relate the scales of the square of an operator, A_0^2, with the scale of the A_0. We will show that if we perform the constructions in Appendix B with A_0^2 we arrive to the same spaces as for A_0 with a suitable labeling.

Hence, we assume as in the previous section that

$$A_0 \in \mathcal{H}(E_1, E_0).$$

Observe that by Propositions B.2 and B.3 we can consider the associated interpolation scale $\{E_\alpha\}_{\alpha\in\mathbb{R}}$ or the fractional power scale $\{F_\alpha\}_{\alpha\geqslant -N}$, $N \in \mathbb{N}$, without assuming $0 \in \rho(A_0)$ or $(-\infty, 0] \in \rho(A_0)$, respectively. Also, note that with the notation of the previous section,

$$A_0^2 := A_0 \circ A_1 : E_2 \to E_0.$$

Hence, we will assume furthermore that

$$A_0^2 \in \mathcal{H}(E_2, E_0).$$

The following result, which is a particular case of [15, Proposition 10.5], gives a criteria for determining when A_0^2 is a sectorial operator.

Proposition C.1. *Let $A_0 \in \mathcal{H}(E_1, E_0)$ with $(-\infty, 0] \subset \rho(A_0)$ and satisfying $\|(A_0 - \lambda)^{-1}\| \leqslant \frac{K}{|\lambda|}$ for $\lambda \in S_{0,\phi}$ with $\phi \in (0, \frac{\pi}{4})$ where $S_{0,\phi}$ is a sector as in (2.3) with vertex $a = 0$.*

Then A_0^2 satisfies $S_{0,2\phi} \subset \rho(A_0^2)$ and

$$\left\|\left(A_0^2 - \lambda\right)^{-1}\right\|_{E_0} \leqslant \frac{K}{|\lambda|}$$

for $\lambda \in S_{0,2\phi}$, thus $A_0^2 \in \mathcal{H}(E_2, E_0)$.

So now we can construct both interpolation and fractional scales for A_0^2 following the procedures explained in Appendix B. In the next two results we will show that these scales coincide with the ones for A_0 after a suitable labeling.

Proposition C.2. *Let $A_0 \in \mathcal{H}(E_1, E_0)$ and assume $A_0^2 := A_0 \circ A_1 \in \mathcal{H}(E_2, E_0)$. Let $\{E_\alpha\}_{\alpha\in\mathbb{R}}$ be the interpolation scale for A_0 as in Proposition B.2. Then on the scale $X_\alpha = E_{2\alpha}$ with $\alpha \in \mathbb{R}$ we have $A_\alpha^2 := A_\alpha \circ A_{\alpha+1} \in \mathcal{H}(X_{\alpha+1}, X_\alpha)$ and A_0^2 defines a semigroup $S_{A_0^2}(t)$ in the scale $\{X_\alpha\}_{\alpha\in\mathbb{R}}$ that satisfies $S_{A_0^2}(t)|_{X_\alpha} = e^{-A_\alpha^2 t}$ and*

$$\left\|S_{A_0^2}(t)\right\|_{\mathcal{L}(X_\beta, X_\alpha)} \leqslant \frac{C(\alpha-\beta)}{t^{\beta-\alpha}} e^{\mu t}, \quad t > 0,\ \alpha, \beta \in \mathbb{R},\ \alpha \geqslant \beta,$$

for any $\mu > \mathrm{type}(A_0^2)$. The constant $C(\alpha-\beta)$ is bounded for α, β in bounded sets of $\mathbb{R}$.

If E_0 is reflexive, the negative side of the scale can be described as

$$X_{-\alpha} = \left(X_\alpha^\sharp\right)' \quad \text{and} \quad A_{-\alpha}^2 = \left(A_\alpha^{2\sharp}\right)', \quad \alpha > 0,$$

and it holds that $e^{-A_{-\alpha}^2 t} = (e^{-A_\alpha^{2\sharp} t})'$.

Furthermore, the problem

$$\begin{cases} u_t + A_\alpha^2 u = 0, & t > 0, \\ u(0) = u_0 \in X_\alpha, \end{cases}$$

for any $\alpha \in \mathbb{R}$ has a unique solution $u(t) = S_{A_0^2}(t)u_0 = e^{-A_\alpha^2 t}u_0$.

Now we turn to the fractional power scale to obtain

Proposition C.3. *Let $A_0 \in \mathcal{H}(E_1, E_0)$ and assume $A_0^2 := A_0 \circ A_1 \in \mathcal{H}(E_2, E_0)$. Let $N \in \mathbb{N}$ and $\{F_\alpha\}_{\alpha\geqslant -2N}$ be the fractional power scale for A_0 as in Proposition B.3. Then on the fractional power scale $Y_\alpha = F_{2\alpha}$ with $\alpha \geqslant -N$ we have $A_\alpha^2 := A_\alpha \circ A_{\alpha+1} \in \mathcal{H}(Y_{\alpha+1}, Y_\alpha)$ and A_0^2 defines a semigroup $S_{A_0^2}(t)$ in the scale $\{Y_\alpha\}_{\alpha\geqslant -N}$ that satisfies $S_{A_0^2}(t)|_{F_\alpha} = e^{-A_\alpha^2 t}$ and*

$$\left\|S_{A_0^2}(t)\right\|_{\mathcal{L}(Y_\beta, Y_\alpha)} \leqslant \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\mu t}, \quad t > 0,\ \alpha \geqslant \beta \geqslant -N,$$

for any $\mu > \mathrm{type}(A_0^2)$. The constant $C(\alpha-\beta)$ is bounded for α, β in bounded sets of $\mathbb{R}$.

If E_0 is reflexive, the negative side of the scale can be described as

$$Y_{-\alpha} = \left(Y_\alpha^\sharp\right)' \quad \text{and} \quad A^2_{-\alpha} = \left(A_\alpha^{\sharp 2}\right)', \quad \alpha > 0,$$

and it holds that $e^{-A^2_{-\alpha}t} = (e^{-A_\alpha^{2\sharp}t})'$.

Furthermore, the problem

$$\begin{cases} u_t + A^2_\alpha u = 0, & t > 0, \\ u(0) = u_0 \in Y_\alpha, \end{cases}$$

for any $\alpha \geqslant -N$ *has a unique solution* $u(t) = S_{A_0^2}(t)u_0 = e^{-A_\alpha^2 t}u_0$.

Remark C.4. According to Remark B.4 if A_0 has bounded imaginary powers, then A_0^2 does as well, see (B.6). In such a case both scales and semigroups in Propositions C.2 and C.3 coincide, that is, $X_\alpha = Y_\alpha$ for $\alpha \geqslant -N$, see [2, V.1.5.13, p. 283].

Appendix D. The scales and semigroup for A_0^m

We can extend these results to any other natural powers A_0^m, $m \in \mathbb{N}$, reviewing the abstract results above.

Proposition D.1. *Proposition C.1 remains true for* A_0^m, $m \in \mathbb{N}$, *as long as the sector* $S_{0,\phi}$ *for* A_0 *has an opening angle* $\phi < \frac{\pi}{2m}$.

In fact, this is the original result in Theorem 10.5 in [15].

Now, for the interpolation scale, similarly to Propositions C.2, we get

Proposition D.2. *Let* $A_0 \in \mathcal{H}(E_1, E_0)$ *and assume* $A_0^m := A_0 \circ \cdots \circ A_m \in \mathcal{H}(E_m, E_0)$, $m \in \mathbb{N}$. *Then on the interpolation scale* $X_\alpha = E_{m\alpha}$ *with* $\alpha \in \mathbb{R}$ *we have* $A_\alpha^m := A_\alpha \circ \cdots \circ A_{\alpha+m} \in \mathcal{H}(X_{\alpha+m}, X_\alpha)$ *and* A_0^m *defines a semigroup* $S_{A_0^m}(t)$ *in* $\{X_\alpha\}_{\alpha\in\mathbb{R}}$ *such that* $S_{A_0^m}(t)|_{X_\alpha} = e^{-A_\alpha^m t}$ *and*

$$\left\| S_{A_0^m}(t) \right\|_{\mathcal{L}(X_\beta, X_\alpha)} \leqslant \frac{C(\alpha-\beta)}{t^{\beta-\alpha}} e^{\mu t}, \quad t > 0,\ \alpha, \beta \in \mathbb{R},\ \alpha \geqslant \beta,$$

for any $\mu > \mathrm{type}(A_0^m)$. *The constant* $C(\alpha-\beta)$ *is bounded for* α, β *in bounded sets of* $\mathbb{R}$.

If E_0 is reflexive, the negative side of the scale can be described as

$$X_{-\alpha} = \left(X_\alpha^\sharp\right)' \quad \text{and} \quad A^m_{-\alpha} = \left(A_\alpha^{m\sharp}\right)', \quad \alpha > 0,$$

and it holds that $e^{-A^m_{-\alpha}t} = (e^{-A_\alpha^{m\sharp}})'$.

Furthermore, the problem

$$\begin{cases} u_t + A^m_\alpha u = 0, & t > 0, \\ u(0) = u_0 \in X_\alpha, \end{cases}$$

for $\alpha \in \mathbb{R}$ *has a unique solution* $u(t) = S_{A_0^m}(t) = e^{-A_\alpha^m t}u_0$.

On the other hand, for the fractional power scale, as in Proposition C.3, we get

Proposition D.3. *Let* $A_0 \in \mathcal{H}(E_1, E_0)$ *and assume* $A_0^m := A_0 \circ \cdots \circ A_m \in \mathcal{H}(E_m, E_0)$. *Also, fix* $N \in \mathbb{N}$. *Then on the fractional power scale* $Y_\alpha = F_{m\alpha}$ *with* $\alpha \geqslant -N$ *we have* $A_\alpha^m := A_\alpha \circ \cdots \circ A_{\alpha+m} \in \mathcal{H}(Y_{\alpha+m}, Y_\alpha)$ *and* A_0^m *defines a semigroup* $S_{A_0^m}(t)$ *in* $\{Y_\alpha\}_{\alpha\geqslant -N}$ *such that* $S_{A_0^m}(t)|_{Y_\alpha} = e^{-A_\alpha^m t}$ *and*

$$\left\| S_{A_0^m}(t) \right\|_{\mathcal{L}(Y_\beta, Y_\alpha)} \leqslant \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\mu t}, \quad t > 0,\ \alpha \geqslant \beta \geqslant -N,$$

for any $\mu > \mathrm{type}(A_0^m)$. *The constant* $C(\alpha-\beta)$ *is bounded for* α, β *in bounded sets of* $\mathbb{R}$.

If E_0 is reflexive, the negative side of the scale can be described as

$$Y_{-\alpha} = \left(Y_\alpha^\sharp\right)' \quad \text{and} \quad A^m_{-\alpha} = \left(A_\alpha^{m\sharp}\right)', \quad \alpha > 0,$$

and it holds that $e^{-A^m_{-\alpha}t} = (e^{-A_\alpha^{m\sharp}})'$.

Furthermore, the problem

$$\begin{cases} u_t + A^m_\alpha u = 0, & t > 0, \\ u(0) = u_0 \in Y_\alpha, \end{cases}$$

for $\alpha \geqslant -N$ *has a unique solution* $u(t) = S_{A_0^m}(t) = e^{-A_\alpha^m t}u_0$.

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