ADDITIVE RECURSIVE RULES

J. Montero, A. del Amo
Faculty of Mathematics
Complutense University
Madrid, Spain

E. Molina
Faculty of Applied Sciences
Miguel Hernández University
Elche, Spain

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Abstract

Recursiveness is a generalization of associativity, initially introduced in order to explain what an *Ordered Weighted Averaging (OWA)* rule is. In this paper, additive recursive rules are presented, showing the relevance of some particular OWA recursive rules.

Key words: Fuzzy Connective Rules, Recursiveness, Associativeness

1 Introduction.

A *rule* here is a *consistent* family of connectives

$$\{\phi_n : [0,1]^n \rightarrow [0,1]\}_{n>1}$$

being each operator ϕ_n the one to be applied when the numer of items is n. By definition we also assume that all these connectives are continuous and non-decreasing in each coordinate.

Classical approach assumes the existence of a unique commutative binary operator general approach . But as pointed out in [4], key aggregation operators can not be explained by means of a sequential application of a unique binary operator. Then it happens that we may find serious difficulties in understanding the rule to which a given operator belongs to, i.e., we get confused about how to proceed when we get more items to be aggregated (see also [5]).

2 Recursiveness

First of all, we realize that many aggregation procedures require a previous re-arrangement of data. For example, most facts need to be analyzed according time.

Definition 2.1. Let us denote

$$\pi_n(a_1, a_2, \dots, a_n) = (a_{\pi_n(1)}, a_{\pi_n(2)}, \dots, a_{\pi_n(n)})$$

An *ordering rule* π is a *consistent* family of permutations $\{\pi_n\}_{n>1}$ such that for any possible finite collections of numbers, each extra item a_{n+1} is allocated keeping relative positions of items, i.e.,

$$\pi_{n+1}(a_1, a_2, \dots, a_n, a_{n+1}) = (a_{\pi_n(1)}, \dots, a_{\pi_n(j-1)}, a_{\pi_{n+1}(j)}, a_{\pi_n(j+1)} \dots, a_{\pi_n(n)}),$$

for some $j \in \{1, \dots, n+1\}$.

The following definition was then proposed in [5].

Definition 2.2. A left-recursive connective rule is a family of connective operators

$$\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$$

such that there exists a sequence of binary operators

$$\{L_n : [0,1]^2 \to [0,1]\}_{n>1}$$

verifying

$$\begin{aligned} \phi_2(a_1, a_2) &= L_2(a_{\pi(1)}, a_{\pi(2)}) \\ \phi_n(a_1, \dots, a_n) &= L_n(\phi_{n-1}(a_{\pi(1)}, \dots, a_{\pi(n-1)}), a_{\pi(n)}) \quad \forall n > 2 \end{aligned}$$

for some ordering rule π.

Right recursiveness can be analogouly defined, and then we talk about a *recursive rule* when both left and right representations hold for the same ordering rule. We talk about *standard* recursive rules when they are based upon the identity ordering rule.

In this way recursiveness generalizes the property of associativity, still assuring an operational constructive consistency, but allowing to explain key rules in practice, like the *mean rule*. Alternative approaches can be found in [6, 7, 9] and [10].

3 Main result

The main result of this paper follows from the fact that a connective rule $\{\phi_n\}_{n>1}$ is recursive if and only if a set of general associativity equations (in the sense of Mak [8]) hold for each n, once the ordering rule π has been already applied:

$$\phi_n(a_1,\ldots,a_n) = R_n(a_{\pi(1)}, \phi_{n-1}(a_{\pi(2)},\ldots,a_{\pi(n)})) = L_n(\phi_{n-1}(a_{\pi(1)},\ldots,a_{\pi(n-1)}), a_{\pi(n)})$$

must hold for all n. We shall consider here only *regular recursive connective rules*, those families of connective operators

$$\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$$

such that there exists a sequence of binary operators

$$\{L_n : [0,1]^2 \to [0,1]\}_{n>1}$$

and

$$\{R_n : [0,1]^2 \to [0,1]\}_{n>1}$$

which are surjective continuous mappings verifying the Mak´s conditions [8].

Theorem 3.1. *Let*

$$\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$$

be a regular standard recursive rule. If ϕ_n is strictly increasing in each coordinate for all $n > 1$, then there exists:

(a) $p : [0,1] \to I\!R_+$, continuous and strictly increasing function

(b) $\{\delta_n : [0,1] \to I\!R_+\}_{n>1}$, family of continuous and strictly increasing functions, and

(c) $\{c_n\}_{n\geq 1}$, sequence of positive real numbers

in such a way that

$$\phi_n(a_1,\ldots,a_n) = \delta_n^{-1}\left(\prod_{j=2}^{n-2} c_j \sum_{k=1}^{n} c_1^{k-1} p(a_k)\right), \quad \forall\ (a_1,\ldots,a_n) \in [0,1]^n, \quad \forall\ n \geq 2 \tag{3.1}$$

where $\prod_{j=2}^{\ell} c_j$ is taken as 1 whenever $\ell \leq 2$.

4 Final comments

This paper shows how usual additive aggregation rules can be explained in terms of a recursive approach, and which ones of those additive rules are in fact recursive.

Moreover, it is shown how easily we fall into a quite intuitive practical observation: we have very few *consistent* choices for the rule as soon as the aggregation operator of the first two items has been defined.

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