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Characterizing Sobolev spaces of vector-valued functions ☆

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ABSTRACT

We are concerned here with Sobolev-type spaces of vector-valued functions. For an open subset $\Omega \subset \mathbb{R}^N$ and a Banach space V, we characterize the functions in the Sobolev-Reshetnyak space $R^{1,p}(\Omega, V)$, where $1 \leq p \leq \infty$, in terms of the existence of partial metric derivatives or partial w^*-derivatives with suitable integrability properties. In the case $p = \infty$ the Sobolev-Reshetnyak space $R^{1,\infty}(\Omega, V)$ is characterized in terms of a uniform local Lipschitz property. We also consider the special case of the space $V = \ell^\infty$.

0. Introduction

This work deals with Sobolev-type functions defined on an open set $\Omega \subset \mathbb{R}^N$ and taking values in a Banach space V. More precisely, we will focus on the Sobolev-Reshetnyak space $R^{1,p}(\Omega, V)$, introduced by Reshetnyak for metric-valued functions in [18], which has been considered in [12] and extensively studied in [13]. The space $R^{1,p}(\Omega, V)$ is a Banach space which contains the classical first order Sobolev space $W^{1,p}(\Omega, V)$ as a closed subspace. The question about the coincidence of both spaces has been recently studied in [5], where it is shown that $R^{1,p}(\Omega, V) = W^{1,p}(\Omega, V)$ if, and only if, the space V enjoys the Radon-Nikodým Property. This result has been extended in [10] to the more general setting of vector-valued Sobolev spaces based on Banach function spaces.

Recall that, much in the same way as in the scalar-valued case, the space $W^{1,p}(\Omega, V)$ is defined as the set of all (classes of) functions in $L^p(\Omega, V)$ that admit partial weak (or distributional) derivatives, which also belong to $L^p(\Omega, V)$. Here the space $L^p(\Omega, V)$ is understood in the sense of Bochner integral. It is known that the classical Beppo Levi characterization of Sobolev functions still holds in the vector-valued setting.

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That is, $W^{1,p}(\Omega, V)$ consists of all (classes of) functions in $L^p(\Omega, V)$ that admit a representative which is absolutely continuous and a.e. differentiable along almost every line parallel to a coordinate axis, and such that the classical partial derivatives belong to $L^p(\Omega, V)$ (see Theorem 4.16 in [16] or Theorem 3.2 in [3]).

On the other hand, the space $R^{1,p}(\Omega, V)$ is defined by means of a *scalarization* procedure, as those functions $f : \Omega \to V$ such that its compositions $\langle v^*, f\rangle$ with continuous linear functionals v^* in the dual space V^* belong to the scalar-valued space $W^{1,p}(\Omega)$, and with a suitable uniform bound for the gradient. See the precise definition in Section 1 below.

Our main purpose in this paper is to characterize the functions of the space $R^{1,p}(\Omega, V)$ in terms of their directional properties along lines, and also in terms of their differentiability properties. To this end, we have to deal with weaker notions of differentiability, such as metric differentiability or, in the case of V being the dual of a separable space, w^*-differentiability.

The contents of the paper are as follows. A short review about vector-valued Sobolev functions is given in Section 1. In Section 2 we examine the properties of vector-valued functions which are absolutely continuous on almost all lines parallel to coordinate axes, regarding the almost everywhere existence of partial metric derivatives and partial w^*-derivatives, and the connections between them (see Theorem 2.8). The main results of the paper are given in Section 3. In Theorem 3.1 we characterize the space $R^{1,p}(\Omega, V)$ as the space of (classes of) functions in $L^p(\Omega, V)$ that admit a representative which is absolutely continuous and a.e. metrically differentiable along almost every line parallel to a coordinate axis, and such that the partial metric derivatives belong to $L^p(\Omega)$. In the case where V is the dual of a separable space, the analogous result for partial w^*-derivatives is contained in Theorem 3.5. This Theorem provides also other characterizations of functions in $R^{1,p}(\Omega, V)$ using several variants of w^*-scalarizations, by composing the function only with elements of the *predual* of V in a suitable uniform way. For the case $p = \infty$, in Theorem 3.3 we characterize functions of $R^{1,\infty}(\Omega, V)$ in terms of a uniform local Lipschitz property, a result which parallels the scalar-valued case. Finally, as a relevant case, we consider the space $R^{1,p}(\Omega, \ell^\infty)$, since $V = \ell^\infty$ is a canonical example of dual of a separable space that does not satisfy the Radon-Nikodým Property.

When the research of this paper was completed, we were informed by Creutz and Evseev about their preprint [6], where very similar questions are studied. In fact, two of the characterizations of our Theorem 3.5 have been independently obtained in [6].

1. Preliminaries on vector-valued Sobolev spaces

In this paper, Ω will denote an open subset of $\mathbb{R}^N$, where we consider the Lebesgue measure $\mathcal{L}^N$ and the euclidean norm $|\cdot|$, and V will denote a Banach space. Recall that a function $s : \Omega \to V$ is said to be a *measurable simple function* if there exist vectors $v_1, \dots, v_m \in V$ and disjoint measurable subsets $E_1, \dots, E_m$ of Ω such that

$$s = \sum_{i=1}^{m} v_i \chi_{E_i}.$$

A function $f : \Omega \to V$ is said to be *measurable* if there exists a sequence of measurable simple functions $\{s_n : \Omega \to V\}_{n=1}^{\infty}$ that converges to f almost everywhere on Ω. The Pettis measurability theorem gives the following characterization of measurable functions (see e.g. [7] or [14]):

Theorem 1.1 *(Pettis). Let Ω be an open subset of $\mathbb{R}^N$ and V a Banach space. A function $f : \Omega \to V$ is measurable if and only if it satisfies the following two conditions:*

(1) f is weakly measurable, i.e., for each $v^* \in V^*$, we have that $\langle v^*, f\rangle : \Omega \to \mathbb{R}$ is measurable.
(2) f is essentially separably valued, i.e., there exists $Z \subset \Omega$ with $\mathcal{L}^N(Z) = 0$ such that $f(\Omega \setminus Z)$ is a separable subset of V.

For measurable Banach-valued functions, the Bochner integral is defined as follows. Suppose first that $s = \sum_{i=1}^{m} v_i \chi_{E_i}$ is a measurable simple function as before, where $E_1, \ldots, E_m$ are measurable, pairwise disjoint, and furthermore $\mathcal{L}^N(E_i) < \infty$ for each $i \in \{1, \ldots, m\}$. We say then that s is *integrable* and we define the integral of s by

$$\int_\Omega s(x)\, dx := \sum_{i=1}^{m} \mathcal{L}^N(E_i) v_i.$$

Now consider an arbitrary measurable function $f : \Omega \to V$, and let $\|\cdot\|$ denote the norm of V. We say that f is *integrable* if there exists a sequence $\{s_n\}_{n=1}^{\infty}$ of integrable simple functions such that

$$\lim_{n\to\infty} \int_\Omega \|s_n(x) - f(x)\|\, dx = 0.$$

In this case, the *Bochner integral* of f is defined as:

$$\int_\Omega f(x)\, dx := \lim_{n\to\infty} \int_\Omega s_n(x)\, dx.$$

We have the following characterization of Bochner integrability (see e.g. Proposition 3.2.7 in [14]):

Proposition 1.2. *Let Ω be an open subset of $\mathbb{R}^N$ and V a Banach space. A function $f : \Omega \to V$ is Bochner-integrable if, and only if, f is measurable and $\int_\Omega \|f(x)\|\, dx < \infty$. Furthermore, if $f : \Omega \to V$ is integrable, then for each $v^* \in V^*$ we have that $\langle v^*, f\rangle : \Omega \to \mathbb{R}$ is also integrable, and*

$$\left\langle v^*, \int_\Omega f \right\rangle = \int_\Omega \langle v^*, f\rangle.$$

For $1 \le p < \infty$, the space $L^p(\Omega, V)$ is defined as the space of all equivalence classes of measurable functions $f : \Omega \to V$ for which

$$\int_\Omega \|f\|^p = \int_\Omega \|f(x)\|^p\, dx < \infty.$$

Here, two measurable functions $f, g : \Omega \to V$ are *equivalent* if they coincide almost everywhere. We endow $L^p(\Omega, V)$ with its natural Banach space norm:

$$\|f\|_p := \left(\int_\Omega \|f\|^p \right)^{\frac{1}{p}} = \left(\int_\Omega \|f(x)\|^p\, dx \right)^{\frac{1}{p}}.$$

On the other hand, $L^\infty(\Omega, V)$ is defined as the space of all equivalence classes of essentially bounded measurable functions $f : \Omega \to V$, endowed with the natural Banach space norm:

$$\|f\|_\infty := \operatorname{ess\,sup}_{x\in\Omega} \|f(x)\|.$$

Moreover, for each $1 \le p \le \infty$ we consider $L^p_{\mathrm{loc}}(\Omega, V)$ to be, as usual, the set of all classes of functions $f : \Omega \to V$ such that for each $x \in \Omega$ there is a neighborhood U_x of x for which $f \in L^p(U_x, V)$.

Now let denote by $C_0^\infty(\Omega)$ the space of all real-valued functions that are infinitely differentiable and have compact support contained in Ω. This class of functions allows us to apply the integration by parts formula against functions in $L^p(\Omega, V)$, in order to define the classical Sobolev space $W^{1,p}(\Omega, V)$ as follows. Given $f \in L^1_{\text{loc}}(\Omega, V)$ and $i \in \{1, \ldots, N\}$, a function $f_i \in L^1_{\text{loc}}(\Omega, V)$ is said to be the *i-th weak (or distributional) partial derivative* of f if

$$\int_\Omega \frac{\partial \varphi}{\partial x_i} f = -\int_\Omega \varphi f_i$$

for every $\varphi \in C_0^\infty(\Omega)$. Partial derivatives are unique, and we denote f_i by $\partial f/\partial x_i$. If f admits all weak partial derivatives, we define its *weak gradient* as the N-vector $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N})$, and the *length* of the gradient is

$$|\nabla f| := \left(\sum_{i=1}^N \left\| \frac{\partial f}{\partial x_i} \right\|^2 \right)^{\frac{1}{2}}.$$

Now the space $W^{1,p}(\Omega, V)$ is defined as the set of all classes of functions $f \in L^p(\Omega, V)$ that admit a weak gradient satisfying $\partial f/\partial x_i \in L^p(\Omega, V)$ for all $i \in \{1, \ldots, N\}$. For $1 \leq p < \infty$, this space is equipped with the natural Banach space norm

$$\|f\|_{W^{1,p}} := \left(\int_\Omega \|f\|^p \right)^{\frac{1}{p}} + \left(\int_\Omega |\nabla f|^p \right)^{\frac{1}{p}}.$$

For $p = \infty$, the natural Banach space norm on $W^{1,\infty}(\Omega, V)$ is

$$\|f\|_{W^{1,\infty}} := \operatorname{ess\,sup} \|f\| + \operatorname{ess\,sup} |\nabla f|.$$

As customary, when $V = \mathbb{R}$, we denote $W^{1,p}(\Omega) := W^{1,p}(\Omega, \mathbb{R})$.

A different notion of Sobolev space was introduced by Reshetnyak in [18] for functions defined in an open subset of $\mathbb{R}^N$ and taking values in a metric space. Here we will consider only the case of functions with values in a Banach space. These Sobolev-Reshetnyak spaces have been considered in [13] and [12]. We give a definition taken from [12], which is slightly different, but equivalent, to the original definition in [18].

Definition 1.3. Let Ω be an open subset of $\mathbb{R}^N$ and let V be a Banach space. Given $1 \leq p \leq \infty$, the Sobolev-Reshetnyak space $R^{1,p}(\Omega, V)$ is defined as the space of all classes of functions $f \in L^p(\Omega, V)$ satisfying

(1) for every $v^* \in V^*$ such that $\|v^*\| \leq 1$, $\langle v^*, f \rangle \in W^{1,p}(\Omega)$;
(2) there is a nonnegative function $g \in L^p(\Omega)$ such that the inequality $|\nabla \langle v^*, f \rangle| \leq g$ holds almost everywhere, for all $v^* \in V^*$ satisfying $\|v^*\| \leq 1$.

We now define the norm

$$\|f\|_{R^{1,p}} := \|f\|_p + \inf_{g \in \mathcal{R}(f)} \|g\|_p,$$

where $\mathcal{R}(f)$ denotes the family of all nonnegative functions $g \in L^p(\Omega)$ satisfying (2).

As it is pointed out in page 692 of [12], it can be checked, using standard techniques, that the space $R^{1,p}(\Omega, V)$, endowed with the norm $\|\cdot\|_{R^{1,p}}$, is a Banach space.

For $1 \leq p < \infty$, it can be seen in Theorem 4.3 of [5] that $W^{1,p}(\Omega, V)$ is a closed subspace of $R^{1,p}(\Omega, V)$. Furthermore, from Theorem 4.6 of [5] we have that these spaces coincide if, and only if, V has the Radon-Nikodým Property. It can be seen that this holds also for the case $p = \infty$ (see Corollary 3.4 below).

2. Absolute continuity and differentiability

The notion of absolute continuity is closely related to Sobolev spaces. Recall that, if V is a Banach space, a function $f : [a, b] \longrightarrow V$ is said to be *absolutely continuous* if, for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^{m} |b_i - a_i| < \delta \Longrightarrow \sum_{i=1}^{m} \|f(b_i) - f(a_i)\| < \varepsilon,$$

for every pairwise disjoint intervals $[a_1, b_1], \dots, [a_m, b_m] \subset [a, b]$.

In the case of functions $f : \Omega \to V$ defined on an open subset of $\mathbb{R}^N$, we will consider the property of being absolutely continuous when restricted to each compact subinterval of $\ell \cap \Omega$, for almost every line ℓ parallel to a coordinate axis. The precise definition is as follows. Here, as usual, $\{e_1, \dots, e_N\}$ denotes the unit vector basis of $\mathbb{R}^N$.

Definition 2.1. Let $\Omega \subset \mathbb{R}^N$ be an open set and V a Banach space. We say that $f : \Omega \to V$ is *absolutely continuous on a.e. line parallel to coordinate axes* (in short ACL) if, for each $i = 1, \dots, N$ there exists an $\mathcal{L}^{N-1}$-null subset Z_i of the hyperplane $H_i = \{(x_1, \dots, x_N) : x_i = 0\}$ such that for every $x \in H_i \setminus Z_i$, the function $t \mapsto f(x + te_i)$ is absolutely continuous on every compact interval $[a, b]$ such that $x + te_i \in \Omega$ for $a \leq t \leq b$.

In this section we will be concerned with differentiability properties of this kind of vector-valued functions. In particular, we will discuss in our context the notions of metric and weak* differentiability, as well as the measurability of these derivatives.

We recall the definition of (partial) metric derivative, a concept which was considered by Kirchheim in [15] (see also [8]).

Definition 2.2. Let $\Omega \subset \mathbb{R}^N$ be an open set and V a Banach space. For each $i = 1, \dots, N$, we define the *i-th partial metric derivative* of a function $f : \Omega \to V$ at $x \in \Omega$ as the following limit, if it exists:

$$m\partial_i f(x) := \lim_{h \to 0} \frac{\|f(x + he_i) - f(x)\|}{|h|}.$$

In the case that $N = 1$, we denote by $md\,f(x)$ its *metric derivative*, that is, the only partial derivative.

The following result is due to Ambrosio [1] (see also Theorem 4.4.8 of [14] and Theorem 2.1 (ii) of [8]).

Theorem 2.3. *Let V be a Banach space and $f : [a, b] \to V$ be an absolutely continuous function. Then f is metrically differentiable almost everywhere.*

The following lemma extends this previous result to functions with N-dimensional domain $\Omega \subset \mathbb{R}^N$. Note that if $f : \Omega \to V$ is absolutely continuous on a.e. line parallel to coordinate axes, then given a direction e_i with $1 \leq i \leq N$, for almost every line ℓ parallel to e_i we have from the previous theorem that the set E_ℓ of all points in $\ell \cap \Omega$ for which the i-th partial metric derivative of f does not exist is $\mathcal{L}^1$-null. In order to see that the set E of all points in Ω for which the i-th partial metric derivative of f does not exist is $\mathcal{L}^N$-null, we apply Fubini's theorem, and this requires to show first that the set E is measurable in $\mathbb{R}^N$. In fact, measurability is the key point of the lemma, where we follow the ideas of Corollary 3.4 in [9].

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^N$ be an open set and V a Banach space. If $f : \Omega \to V$ is measurable and absolutely continuous on a.e. line parallel to coordinate axes, then it admits partial metric derivatives almost everywhere in Ω.*

Proof. The proof is independent for each direction $i = 1, \dots, N$, so without loss of generality we show it for the last coordinate x_N. Consider $D \subset \Omega$ the set of all points $x \in \Omega$ such that $m\partial_N f(x)$ exists. Hence if D is measurable then Fubini's theorem yields that $\Omega \backslash D$ is a null set. As we just need to prove that D is measurable and Ω can be expressed as a countable union of open N-dimensional cubes, we can assume that Ω is a cartesian product of the form

$$\Omega = I_1 \times \cdots \times I_N,$$

where each I_j is an open interval, for $1 \leq j \leq N$. Consider also the set

$$\widetilde{\Omega} := \{x = u + te_N \in \Omega \, : \, u \in I_1 \times \cdots \times I_{N-1}, t \mapsto f(u + te_N) \text{ is continuous on } I_N\}.$$

By the ACL property of f, we have that $\Omega \backslash \widetilde{\Omega}$ is an $\mathcal{L}^N$-null set. Then, it will be sufficient to prove the measurability of the set

$$\widetilde{D} := \{x \in \widetilde{\Omega} : m\partial_N f(x) \text{ exists } \}.$$

Now fix $x \in \widetilde{\Omega}$. Note that $\alpha = m\partial_N f(x)$ if, and only if, for every $\varepsilon > 0$ there exists some $\delta > 0$ such that, for every $t \in \mathbb{R}$ with $0 < |t| < \delta$, we have that

$$\left| \frac{\|f(x + te_N) - f(x)\|}{|t|} - \alpha \right| \leq \varepsilon. \tag{1}$$

Using the continuity of the map $t \mapsto f(x + te_N)$, it will be sufficient to have that the above inequality (1) holds for every $t \in \mathbb{Q}$ with $0 < |t| < \delta$.

From here, we see that $\alpha = m\partial_N f(x)$ if, and only if, for every null sequence (r_k) of nonzero rational numbers, we have that

$$\alpha = \lim_{k \to \infty} \frac{\|f(x + r_k e_N) - f(x)\|}{|r_k|}.$$

Therefore, it is easy to check that $x \in \widetilde{D}$ if, and only if, for every null sequence (r_k) of nonzero rational numbers, the sequence

$$\left\{ \frac{\|f(x + r_k e_N) - f(x)\|}{|r_k|} \right\}$$

is a Cauchy sequence in $\mathbb{R}$.

Now denote $\mathbb{Q}^* = \mathbb{Q} \backslash \{0\}$ and define, for $r, s \neq 0$ and $\varepsilon > 0$:

$$D(r, s, \varepsilon) := \left\{ x \in \widetilde{\Omega} : \left| \frac{\|f(x + re_N) - f(x)\|}{|r|} - \frac{\|f(x + se_N) - f(x)\|}{|s|} \right| \leq \varepsilon \right\}.$$

As a consequence of our previous observations, we obtain that

$$\widetilde{D} = \bigcap_{\varepsilon \in \mathbb{Q}^+} \bigcup_{\delta \in \mathbb{Q}^+} \bigcap_{r,s \in (-\delta,\delta) \cap \mathbb{Q}^*} D(r, s, \varepsilon).$$

Since the sets $D(r, s, \varepsilon)$ are measurable, this gives the measurability of $\widetilde{D}$. □

When the target space is the dual of a separable Banach space, the notion of partial w^*-derivatives turns out to be specially appropriate for the characterizations that we will address in Section 3. The partial w^*-derivatives were considered by Hajłasz and Tyson in [12]. We also refer to [2], where w^*-differentiability is developed in the context of dual Banach space valued Lipschitz functions. We proceed with the same scheme followed for the metric derivatives.

Definition 2.5. Let $\Omega \subset \mathbb{R}^N$ be an open set, $V = Y^*$ be the dual of a separable Banach space Y, and $f : \Omega \to V$. For each $i = 1, \dots, N$, we define the *i-th partial w^*-derivative* of f at a point $x \in \Omega$ as the following weak*-limit, if it exists:

$$w^*\partial_i f(x) = \text{weak}^* - \lim_{h\to 0} \frac{f(x + he_i) - f(x)}{h}$$

In the case that $N = 1$, we denote $w^*df(x)$ the corresponding *w^*-derivative.*

The next result and its proof are contained in Lemma 2.8 of [12].

Lemma 2.6. *Let $V = Y^*$ be the dual of a separable Banach space and let $f : [a, b] \to V$ be an absolutely continuous function. Then f is w^*-differentiable almost everywhere in $[a, b]$. Furthermore, if $F \subset Y$ is a dense and countable vector space over $\mathbb{Q}$, we have that f is w^*-differentiable at a point x if, and only if, for each $y \in F$ the following limit exists:*

$$\lim_{h\to 0} \left\langle y, \frac{f(x + h) - f(x)}{h} \right\rangle.$$

We now give an analog to Lemma 2.4, concerning the almost everywhere existence of partial w^*-derivatives. Recall that, if $V = Y^*$ is a dual space, a function $f : \Omega \to V$ is said to be *w^*-measurable* if $\langle y, f\rangle$ is measurable for each $y \in Y$.

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^N$ be an open set, $V = Y^*$ the dual of a separable Banach space. If $f : \Omega \to V$ is w^*-measurable and absolutely continuous on a.e. line parallel to coordinate axes, then f admits partial w^*-derivatives almost everywhere in Ω.*

Proof. The proof follows the same lines of Lemma 2.4. As in that case, we consider the direction e_N, and it will suffice to prove that the set D of all points $x \in \Omega$ such that $w^*\partial_N f(x)$ exists is measurable. We may also assume that $\Omega = I_1 \times \cdots \times I_N$ is an open cube, and we consider the set

$$\widetilde{\Omega} := \{x = u + te_N \in \Omega \,:\, u \in I_1 \times \cdots \times I_{N-1}, t \mapsto f(u + te_N) \text{ is continuous on } I_N\}.$$

By the ACL property of f, we have that $\Omega\backslash\widetilde{\Omega}$ is an $\mathcal{L}^N$-null set. Then, it will be sufficient to prove the measurability of the set

$$\widetilde{D} := \{x \in \widetilde{\Omega} : w^*\partial_N f(x) \text{ exists }\}.$$

Let $F \subset Y$ be a dense and countable vector space over $\mathbb{Q}$. Given $x \in \widetilde{\Omega}$, by Lemma 2.6 we have that $x \in \widetilde{D}$ if, and only if, for each $y \in F$ the following limit exists:

$$\lim_{h\to 0} \left\langle y, \frac{f(x + he_N) - f(x)}{h} \right\rangle.$$

Now for each $y \in F$, each $r, s \neq 0$ and each $\varepsilon > 0$, denote

$$D(y,r,s,\varepsilon) := \left\{ x \in \widetilde{\Omega} : \left| \left\langle y,\ \frac{f(x+re_N)-f(x)}{r} - \frac{f(x+se_N)-f(x)}{s} \right\rangle \right| \leq \varepsilon \right\}.$$

Then reasoning as in Lemma 2.4 we see that

$$\widetilde{D} = \bigcap_{y\in F} \bigcap_{\varepsilon\in\mathbb{Q}^+} \bigcup_{\delta\in\mathbb{Q}^+} \bigcap_{r,s\in(-\delta,\delta)\cap\mathbb{Q}^*} D(y,r,s,\varepsilon).$$

Since the sets $D(y,r,s,\varepsilon)$ are measurable, the result follows. □

The following theorem collects the information above and relates both types of partial derivatives as the main result of this section.

Theorem 2.8. *Let Ω be an open subset of $\mathbb{R}^N$ and $V = Y^*$ the dual of a separable Banach space. If $f : \Omega \to V$ is measurable and absolutely continuous on a.e. line parallel to coordinate axes, then f admits partial metric derivatives and partial w^*-derivatives almost everywhere. For each $i = 1, \dots, N$ the functions $m\partial_i f$ and $\|w^*\partial_i f\|$ are measurable, and*

$$m\partial_i f(x) = \|w^*\partial_i f(x)\|$$

at almost every $x \in \Omega$.

Proof. Fix $i \in \{1, \dots, N\}$. Lemma 2.4 and Lemma 2.7 yield, respectively the existence of $m\partial_i f$ and $w^*\partial_i f$ almost everywhere in Ω. Moreover $m\partial_i f$ is measurable, since f is measurable and $m\partial_i f$ is the limit a.e. of the sequence of measurable functions:

$$m\partial_i f(x) = \lim_{k\to\infty} \left\{ k \left\| f\left(x + \frac{1}{k}e_i\right) - f(x) \right\| \right\}.$$

In the same way, for each $y \in Y$ we have that the function $\langle y, w^*\partial_i f\rangle$ is measurable since

$$\langle y, w^*\partial_i f(x)\rangle = \lim_{k\to\infty} \left\langle y,\ k\left(f(x + \frac{1}{k}e_i) - f(x) \right) \right\rangle.$$

In addition, $\|w^*\partial_i f\|$ is also measurable, since

$$\|w^*\partial_i f(x)\| = \sup_{y\in D} |\langle y, w^*\partial_i f(x)\rangle|,$$

where D is a countable dense subset of the unit ball of Y.

Note that, for every $x \in \Omega$ and every $y \in Y$, we have that

$$\left| \left\langle y,\ \frac{f(x+he_i)-f(x)}{h} \right\rangle \right| \leq \|y\| \cdot \left\| \frac{f(x+he_i)-f(x)}{h} \right\|.$$

Taking limits we obtain that, for almost every $x \in \Omega$ and every $y \in Y$ with $\|y\| \leq 1$,

$$|\langle y, w^*\partial_i f(x)\rangle| \leq m\partial_i f(x).$$

Then

$$\|w^* \partial_i f(x)\| = \sup_{\|y\| \le 1} |\langle y, w^* \partial_i f(x)| \le m\partial_i f(x)$$

for almost every $x \in \Omega$.

On the other hand, f is absolutely continuous on almost every compact segment parallel to a coordinate axis and contained in Ω. Let $\sigma : [a, b] \to \Omega$ one of such segments on which f is absolutely continuous, and suppose that $\sigma(t) = x + te_i$ for $a \le t \le b$, where $x \in \Omega$ and $i \in \{1, \dots, N\}$. Thus for each $y \in Y$ with $\|y\| \le 1$ we have that $\langle y, f \circ \sigma \rangle$ is also absolutely continuous on $[a, b]$. Therefore, using Proposition 1.2, for every $a \le s < t \le b$:

$$|\langle y, f(x + te_i) - f(x + se_i) \rangle| \le \int_s^t |\langle y, w^* \partial_i f(x + \tau e_i) \rangle| \, d\tau \le \int_s^t \|w^* \partial_i f(x + \tau e_i)\| \, d\tau.$$

As a consequence we obtain that

$$\|f(x + te_i) - f(x + se_i)\| = \sup_{\|y\| \le 1} |\langle y, f(x + te_i) - f(x + se_i) \rangle| \le \int_s^t \|w^* \partial_i f(x + \tau e_i)\| \, d\tau.$$

Again by the absolute continuity of $f \circ \sigma$ on $[a, b]$, we know from Lemma 2.7 of [12] that the function $\tau \mapsto m\partial_i f(x + \tau e_i)$ belongs to $L^1([a, b])$, and then the function $\tau \mapsto \|w^* \partial_i f(x + \tau e_i)\|$ also belongs to $L^1([a, b])$. Therefore, using the Lebesgue differentiation theorem we conclude that

$$m\partial_i f(z) = \lim_{h \to 0^+} \frac{\|f(z + he_i) - f(z)\|}{|h|} \le \lim_{h \to 0^+} \frac{1}{h} \int_0^h \|w^* \partial_i f(z + \tau e_i)\| \, d\tau = \|w^* \partial_i f(z)\|$$

holds for $\mathcal{L}^1$-almost every point z of almost every line parallel to the i-th coordinate axis. In this way, using Fubini's theorem we see that $m\partial_i f \le \|w^* \partial_i f\|$ almost everywhere in Ω. □

To finish this Section we give the analog to Lemma 2.4 concerning the existence of classical partial derivatives. Recall that a Banach space V has the *Radon-Nikodým Property* if every Lipschitz function $f : [a, b] \to V$ is differentiable almost everywhere or, equivalently, if every absolutely continuous function $f : [a, b] \to V$ is differentiable almost everywhere (see e.g. Theorem 5.21 in [4]). We refer to [7] for further information about the Radon-Nikodým Property of Banach spaces.

Lemma 2.9. *Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $f : \Omega \to V$ be measurable and absolutely continuous on a.e. line parallel to coordinate axes. Then, the set of all points in Ω where f admits classical partial derivatives is measurable. Furthermore, if V has the Radon-Nikodým Property, then f admits classical partial derivatives almost everywhere in Ω.*

Proof. The proof can be carried out as in Lemma 2.4. We may assume that $\Omega = I_1 \times \cdots \times I_N$ is an open cube, we fix the direction e_N, and we consider the set

$$\widetilde{\Omega} := \{x = u + te_N \in \Omega \,:\, u \in I_1 \times \cdots \times I_{N-1}, t \mapsto f(u + te_N) \text{ is continuous on } I_N\}.$$

Since $\mathcal{L}^N(\Omega \backslash \widetilde{\Omega}) = 0$ by the ACL property of f, it is sufficient to prove the measurability of the set $\widetilde{D}$ of all points $x \in \widetilde{\Omega}$ such that the classical partial derivative $\partial f(x)/\partial x_N$ exists. Reasoning as in Lemma 2.4, this follows from the fact that

$$\widetilde{D} = \bigcap_{\varepsilon \in \mathbb{Q}^+} \bigcup_{\delta \in \mathbb{Q}^+} \bigcap_{r,s \in (-\delta,\delta) \cap \mathbb{Q}^*} \left\{ x \in \widetilde{\Omega} : \left\| \frac{f(x+re_N)-f(x)}{r} - \frac{f(x+se_N)-f(x)}{s} \right\| \leq \varepsilon \right\}.$$

If, in addition, V has the Radon-Nikodým Property, we obtain that f admits classical partial derivatives at $\mathcal{L}^1$-almost every point of almost every line parallel to a coordinate axis. Then Fubini's theorem gives the result. □

3. Characterizations of $R^{1,p}(\Omega, V)$

The main purpose in this Section is to characterize the functions in Sobolev-Reshetnyak spaces $R^{1,p}(\Omega, V)$ in terms of the existence of a representative which admits a.e. a suitable kind of partial derivatives, which satisfy the corresponding integrability properties. In the case of a general Banach space V, we will consider the notion of partial metric derivatives, and in the case where $V = Y^*$ is the dual of a separable space, we will consider the notion of partial w^*-derivatives. Furthermore, for $p = \infty$, we will also characterize the functions in $R^{1,\infty}(\Omega, V)$ in terms of the existence of a uniformly locally Lipschitz representative.

Theorem 3.1. *Let V be a Banach space and $\Omega \subset \mathbb{R}^N$ open. Given $1 \leq p \leq \infty$ and $f \in L^p(\Omega, V)$ the following are equivalent:*

(i) $f \in R^{1,p}(\Omega, V)$.
(ii) *f admits a representative that is absolutely continuous on a.e. line parallel to coordinate axes, and admits metric partial derivatives almost everywhere, satisfying that $m\partial_i f \in L^p(\Omega)$ for all $i = 1, \ldots, N$.*

Proof. $(i) \Rightarrow (ii)$ Consider first the case $1 \leq p < \infty$. Let $f \in R^{1,p}(\Omega, V)$ and consider $g \in \mathcal{R}(f)$. In order to obtain the ACL property, the key point is to show that f admits a representative f_0 such that for almost every compact segment $\sigma : [a, b] \to \Omega$ parallel to a coordinate axis, that is, of the form $\sigma(t) = x + te_i$, the following upper gradient inequality holds:

$$\|f_0(x + be_i) - f_0(x + ae_i)\| \leq \int_a^b g(\sigma(\tau))\, d\tau.$$

This can be combined with the result that the path integral of $g \in L^p(\Omega)$ on almost every segment parallel to the coordinate axes is finite, and the use of the absolutely continuity of integrals. The details can be seen, e.g., in the proof of Lemma 2.13 in [12]. We can also refer to the proof of $(v) \Rightarrow (ii)$ in Theorem 3.5 below. In our case, instead of the countable subset of the predual Y of V considered in the proof of Theorem 3.5, one would need to choose a countable dense subset W of the separable set of differences $f(\Omega \setminus E_0) - f(\Omega \setminus E_0)$, where $E_0 \subset \Omega$ is a null set such that $f(\Omega \setminus E_0)$ is separable; and then, for each $w \in W$, we select a unit vector $w^* \in V^*$ such that $|\langle w^*, w\rangle| = \|w\|$.

By Lemma 2.4, the ACL property of f_0 yields the existence of all partial metric derivatives at almost every $x \in \Omega$, and moreover for each $i \in \{1, \ldots, N\}$ the partial metric derivative $m\partial_i f_0$ is measurable. From our previous comments, we see that

$$m\partial_i f_0(z) = \lim_{h \to 0^+} \frac{\|f_0(z + he_i) - f_0(z)\|}{h} \leq \lim_{h \to 0^+} \frac{1}{h} \int_0^h g(z + \tau e_i)\, d\tau = g(z)$$

holds for $\mathcal{L}^1$-almost every point z of almost every line parallel to the i-th coordinate axis. Thus we obtain that $m\partial_i f_0(x) \leq g(x)$ at almost every $x \in \Omega$. In this way, we conclude that $m\partial_i f_0 \in L^p(\Omega)$ for each $i = 1, \ldots, N$.

On the other hand, the case $p = \infty$ follows combining the characterization given in Theorem 3.3 below with the almost everywhere metric differentiability of Lipschitz functions obtained by Kirchheim in [15].

$(ii) \Rightarrow (i)$ Let $1 \leq p \leq \infty$ and $f \in L^p(\Omega, V)$ as in (ii). Keep denoting by f the given representative which is absolutely continuous on a.e. line parallel to coordinate axes. Then for every $v^* \in V^*$ we obtain that $\langle v^*, f\rangle$ is also absolutely continuous on a.e. line parallel to coordinate axes. In addition, for each $i = 1, \dots, N$ the corresponding partial derivative satisfies

$$\left|\frac{\partial\langle v^*, f\rangle}{\partial x_i}(x)\right| = \lim_{h\to 0}\left|\left\langle v^*, \frac{f(x+he_i)-f(x)}{h}\right\rangle\right| \leq \|v^*\| \cdot m\partial_i f(x),$$

for almost every $x \in \Omega$. Since $m\partial_i f$ is a nonnegative function in $L^p(\Omega)$, we have that

$$\frac{\partial\langle v^*, f\rangle}{\partial x_i} \in L^p(\Omega).$$

In this way, from the Beppo Levi characterization of Sobolev functions (see, e.g., Theorem 6.1.13 in [14] or Theorem 1.41 in [17]) it follows that $\langle v^*, f\rangle \in W^{1,p}(\Omega)$. Consider now

$$g := \sum_{i=1}^{N} m\partial_i f \in L^p(\Omega).$$

Then we obtain that, for every $v^* \in V^*$ with $\|v^*\| \leq 1$, and almost everywhere on Ω:

$$|\nabla\langle v^*, f\rangle| = \left(\sum_{i=1}^{N}\left|\frac{\partial\langle v^*, f\rangle}{\partial x_i}\right|^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{N}(m\partial_i f)^2\right)^{\frac{1}{2}} \leq g\,.$$

This shows that $g \in \mathcal{R}(f)$, and $f \in R^{1,p}(\Omega, V)$. $\square$

Remark 3.2. Let $f \in R^{1,p}(\Omega, V)$ as before. By the previous theorem, we can choose a suitable representative satisfying (ii). The above proof gives that, for this representative:

$$\|f\|_p + \sup_{1\leq i\leq N}\|m\partial_i f\|_p \leq \|f\|_{R^{1,p}} \leq \|f\|_p + \sum_{i=1}^{N}\|m\partial_i f\|_p.$$

In the case $p = \infty$, we give a more precise characterization of $R^{1,\infty}(\Omega, V)$ in terms of uniformly locally Lipschitz functions, which parallels the case of the real-valued Sobolev space $W^{1,\infty}(\Omega)$. Recall that f is said to be *uniformly locally Lipschitz* on Ω if there exists some $K \geq 0$ such that f is locally K-Lipschitz, that is, every point $x \in \Omega$ admits a neighborhood $U_x \subset \Omega$ such that

$$\|f(p) - f(q)\| \leq K|p-q| \quad \text{for all} \quad p, q \in U_x.$$

Moreover, using the convexity of the balls, it is easy to see that f is uniformly locally Lipschitz on Ω if and only if, for some constant $K \geq 0$, it is K-Lipschitz on each open ball contained in Ω.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^N$ be an open set, V a Banach space and $f \in L^\infty(\Omega, V)$. Then $f \in R^{1,\infty}(\Omega, V)$ if and only if f has a representative which is locally K-Lipschitz, for some $K \geq 0$. Furthermore, in this case, the optimal local Lipschitz constant is*

$$K = \inf_{g\in\mathcal{R}(f)} \|g\|_\infty.$$

Proof. Suppose first that $f \in R^{1,\infty}(\Omega, V)$. Since Ω can be covered by a sequence of balls, it will suffice to show that, for each open ball $B \subset \Omega$ and each $g \in \mathcal{R}(f)$, there exists a representative f_0 of f on B such that $\|f_0(p) - f_0(q)\| \leq \|g\|_\infty |p - q|$, for all $p, q \in B$.

Note that, for each ball $B \subset \Omega$ and $1 < p < \infty$, we have that $f \in R^{1,p}(B, V)$. Therefore, as in the proof of Theorem 3.1 and Theorem 3.5, we have that f admits a representative f_0 on B such that, for almost every compact segment $\sigma : [a, b] \to B$ parallel to a coordinate axis, of the form $\sigma(t) = x + te_i$, the following inequality holds

$$\|f_0(x + be_i) - f_0(x + ae_i)\| \leq \int_a^b g(\sigma(\tau))\, d\tau; \tag{2}$$

and furthermore, if we consider the null set $Z = \{x \in \Omega : g(x) > \|g\|_\infty\}$, we also have that the length of σ in Z is zero. In this way we obtain a null set $E \subset B$ such that, for each $p, q \in B \setminus E$, there exists a polygonal path γ in B, with sides parallel to the coordinate axes, joining p and q, and such that the above inequality (2) holds for each piece of γ. Therefore,

$$\|f_0(q) - f_0(p)\| \leq \|g\|_\infty \,\text{lenght}(\gamma).$$

This implies that f_0 is Lipschitz on $B \setminus E$ and, since $B \setminus E$ is dense in B, f_0 admits a unique Lipschitz extension to B. In order to obtain the optimal Lipschitz constant, we proceed as follows. Given a pair of different points $p, q \in B$, consider a unit vector u in the direction of $p - q$. Again, we know that f_0 admits a representative satisfying the above inequality (2) for almost all compact segments parallel to u contained in B. Then we can select two sequences (p_k) and (q_k), convergent respectively to p and q, such that each segment $\sigma_k = [p_k, q_k]$ is parallel to u, and satisfies that

$$\|f_0(q_k) - f_0(p_k)\| \leq \int_{\sigma_k} g \leq \|g\|_\infty |p_k - q_k|.$$

Using the continuity of f_0 we obtain that, in fact, $\|f_0(p) - f_0(q)\| \leq \|g\|_\infty |p - q|$, for all $p, q \in B$. This shows that the local Lipschitz constant K of the representative f_0 satisfies

$$K \leq \inf_{g \in \mathcal{R}(f)} \|g\|_\infty.$$

Conversely, suppose now that f is locally K-Lipschitz. Then for each $v^* \in V^*$ with $\|v^*\| \leq 1$ we have that $\langle v^*, f\rangle$ is also locally K-Lipschitz. Thus $\langle v^*, f\rangle \in W^{1,\infty}(\Omega)$ and $|\nabla\langle v^*, f\rangle| \leq K$ almost everywhere on Ω. Therefore, the constant function $g = K$ belongs to $\mathcal{R}(f)$ and $f \in R^{1,\infty}(\Omega, V)$. In addition, we have that

$$\inf_{g \in \mathcal{R}(f)} \|g\|_\infty \leq K. \quad \square$$

In our next corollary we see that the coincidence of the Sobolev-Reshetnyak space $R^{1,\infty}(\Omega, V)$ with the classical Sobolev space $W^{1,\infty}(\Omega, V)$ characterizes the Radon-Nikodým Property of the space V. This extends Theorem 4.6 of [5] to the case $p = \infty$.

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^N$ be an open set and V a Banach space. Then $R^{1,\infty}(\Omega, V) = W^{1,\infty}(\Omega, V)$ if, and only if V has the Radon-Nikodým Property.*

Proof. The proof can be carried out much in the same way as in Theorem 4.6 of [5]. Suppose first that V has the Radon-Nikodým Property, and let $f \in R^{1,\infty}(\Omega, V)$. From Theorem 3.3 we can assume that f

is locally K-Lipschitz, with $K = \inf_{g\in\mathcal{R}(f)} \|g\|_\infty$. By the Radon-Nikodým Property of V, we know that f admits classical partial derivatives almost everywhere on Ω (see, Lemma 2.9). For every $i \in \{1, \ldots, N\}$, the classical partial derivative $\partial f/\partial x_i$ is a measurable function, with $\|\partial f/\partial x_i\|_\infty \leq K$. Composing with functionals $v^* \in V^*$, it is easy to see that, in fact, the classical partial derivative $\partial f/\partial x_i$ is the i-th weak (or distributional) derivative of f. In this way we see that $f \in W^{1,\infty}(\Omega, V)$.

Conversely, if V does not have the Radon-Nikodým Property, there exists a Lipschitz function $h : [a, b] \to V$ which is not differentiable almost everywhere. We may assume that $[a, b] \times R_0 = R$ is an N-dimensional rectangle contained in Ω, where R_0 is an $(N-1)$-dimensional rectangle. The function $f : [a, b] \times R_0 \to V$ defined by $f(x_1, \ldots, x_N) = h(x_1)$ is Lipschitz, so it admits a Lipschitz extension $\tilde{f} : \Omega \to V$ (see, e.g. Theorem 4.1.21 in [14]). Then $\tilde{f} \in R^{1,\infty}(\Omega, V)$. Nevertheless, $\tilde{f}$ is not almost everywhere differentiable along any horizontal segment contained in $[a, b] \times R_0 = R$, so from Theorem 3.2 in [3] we obtain that $\tilde{f} \notin W^{1,\infty}(\Omega, V)$. □

We now consider the case where $V = Y^*$ is the dual of a separable Banach space Y, in order to obtain a characterization of $R^{1,p}(\Omega, V)$ involving the partial w^*-derivatives. Note that a countable subset $\{y_k\}_{k=1}^\infty$ of the unit ball of Y is said to be *norming for* V if, for each $v \in V$, we have that

$$\|v\| = \sup_{k\in\mathbb{N}} |\langle y_k, v\rangle|.$$

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $V = Y^*$ be the dual of a separable Banach space. Given $1 \leq p \leq \infty$ and $f \in L^p(\Omega, V)$, the following conditions are equivalent:*

- *(i)* $f \in R^{1,p}(\Omega, V)$.
- *(ii) There exists a representative of f that is absolutely continuous on a.e. line parallel to coordinate axes and admits partial w^*-derivatives almost everywhere, satisfying that $\|w^*\partial_i f\| \in L^p(\Omega)$ for all $i = 1, \ldots, N$.*
- *(iii) For each $i = 1, \ldots, N$ there exists a w^*-measurable function $g_i : \Omega \to V$, with $\|g_i\| \in L^p(\Omega)$, and such that for every $y \in Y$:*

$$\int_\Omega \langle y, f(x)\rangle \frac{\partial\varphi(x)}{\partial x_i}\, dx = -\int_\Omega \langle y, g_i(x)\rangle\varphi(x)\, dx \quad \textit{for all} \quad \varphi \in C_0^\infty(\Omega)\,.$$

- *(iv) There exists a nonnegative function $g \in L^p(\Omega)$ such that, for every $y \in Y$ with $\|y\| \leq 1$ we have that $\langle y, f\rangle \in W^{1,p}(\Omega)$ and $|\nabla\langle y, f\rangle| \leq g$ almost everywhere on Ω.*
- *(v) There exist a countable set $\{y_k\}_{k=1}^\infty \subset Y$ which is norming for V, and a nonnegative function $g \in L^p(\Omega)$ such that, for every $k \in \mathbb{N}$ we have that $\langle y_k, f\rangle \in W^{1,p}(\Omega)$ and $|\nabla\langle y_k, f\rangle| \leq g$ almost everywhere on Ω.*

Proof. $(ii) \Leftrightarrow (i)$ This follows at once from Theorem 2.8 and Theorem 3.1.

$(ii) \Rightarrow (iii)$ Consider a representative f as in (ii) and for each $i = 1, \ldots, N$ define $g_i = w^*\partial_i f$. As we have noted in the proof of Theorem 2.8, each $g_i : \Omega \to V$ is w^*-measurable and $\|g_i\| \in L^p(\Omega)$. Also, for each $y \in Y$, the function $\langle y, f\rangle$ is absolutely continuous on a.e. line parallel to coordinate axes, and therefore its i-th classical partial derivative exists almost everywhere on Ω and coincides with its i-th weak (or distributional) derivative. That is, for each $i = 1, \ldots, N$ we have that

$$\langle y, w^*\partial_i f(x)\rangle = \frac{\partial\langle y, f\rangle}{\partial x_i}(x) \quad \text{a.e. } x \in \Omega.$$

Then (iii) follows.

(iii) ⇒ *(iv)* For each $y \in Y$ with $\|y\| \leq 1$ it is clear that the function $\langle y, f\rangle \in L^p(\Omega)$ and also for each $i = 1, \dots, N$ the function $\langle y, g_i\rangle \in L^p(\Omega)$. The integral equality in the statement of *(iii)* gives that $\langle y, g_i\rangle$ is the i-th weak (or distributional) partial derivative of $\langle y, f\rangle$. In this way, by the Beppo Levi characterization of Sobolev functions, we see that $\langle y, f\rangle \in W^{1,p}(\Omega)$. Additionally, if we define

$$g = \sum_{i=1}^{N} \|g_i\| \in L^p(\Omega)$$

we obtain that

$$|\nabla\langle y, f\rangle| \leq g \quad \text{a.e. for all} \quad y \in Y \quad \text{such that} \quad \|y\| \leq 1.$$

(iv) ⇒ *(v)* Let $\{y_k\}_{k=1}^{\infty}$ be a dense countable subset of the unit ball of Y. It is clear that $\{y_k\}_{k=1}^{\infty}$ is norming for V, so the implication follows.

(v) ⇒ *(ii)* Let $\{y_k\}_{k=1}^{\infty}$ be a countable set in the unit ball of Y, which is norming for V. For each $k \in \mathbb{N}$ we have that $\langle y_k, f\rangle \in W^{1,p}(\Omega)$, and therefore it admits a representative f_k that is absolutely continuous on a.e. line parallel to coordinate axes. Let E_k denote the set where f_k differs from $\langle y_k, f\rangle$ and define $\Omega_0 = \bigcup_k E_k$, which is a null set.

Consider the family Σ of all compact segments $\sigma : [a, b] \to \Omega$ which are parallel to a coordinate axis, and satisfy the following properties:

(1) the composition $g \circ \sigma$ is integrable on $[a, b]$;
(2) the length of σ in Ω_0 is zero, that is, $\mathcal{L}^1(\{t \in [a, b] : \sigma(t) \in \Omega_0\}) = 0$;
(3) for each $k \in \mathbb{N}$, the composition $f_k \circ \sigma$ is absolutely continuous on $[a, b]$.

We claim that the family Σ represents almost all compact segments in Ω which are parallel to a coordinate axis, that is, for each $i = 1, \dots, N$ there exists an $\mathcal{L}^{N-1}$-null set Z_i in $\{(x_1, \dots, x_N) \in \Omega : x_i = 0\}$ such that all compact segments $t \in [a, b] \mapsto x_0 + te_i \in \Omega$ with $x_0 \in \Omega \setminus Z_i$ belong to Σ. This follows taking into account for property (3) that each f_k is absolutely continuous on a.e. line parallel to coordinate axes, and on the other hand using Fubini's theorem for properties (1) and (2). Note that every segment $\sigma : [a, b] \to \Omega$ in Σ is of the form $\sigma(t) = x + te$ for $a \leq t \leq b$, where $x \in \Omega$ and e is a unit vector $e \in \{\pm e_1, \dots, \pm e_N\}$. Also, if $\sigma : [a, b] \to \Omega$ is a segment in Σ, it is clear that for all $a \leq s < t \leq b$ the restriction $\sigma|_{[s,t]}$ also belongs to Σ. Now we distinguish two cases:

Suppose first that $\sigma : [a, b] \to \Omega$ is a segment in Σ whose endpoints satisfy $\sigma(a), \sigma(b) \notin \Omega_0$. Then for each $k \in \mathbb{N}$ we have that

$$|\langle y_k, f(\sigma(b))\rangle - \langle y_k, f(\sigma(a))\rangle| = |f_k(\sigma(b)) - f_k(\sigma(a))| \leq \int_a^b |\nabla\langle y_k, f(\sigma(\tau))\rangle| \, d\tau \leq \int_a^b g(\sigma(\tau)) \, d\tau.$$

As a consequence,

$$\|f(\sigma(b)) - f(\sigma(a))\| = \sup_{k \in \mathbb{N}} |\langle y_k, f(\sigma(b)) - f(\sigma(a))\rangle| \leq \int_a^b g(\sigma(\tau)) \, d\tau.$$

Suppose now that $\sigma : [a, b] \to \Omega$ is a segment in Σ with at least one endpoint in Ω_0. In fact, we can suppose that $\sigma(a) \in \Omega_0$. By property (2), we can choose a sequence $\{t_k\}_{k=1}^{\infty} \subset [a, b]$ converging to a and such that $\sigma(t_k) \notin \Omega_0$. Then by the previous case

$$\|f(\sigma(t_k)) - f(\sigma(t_l))\| \leq \int_{t_k}^{t_l} g(\sigma(\tau))\, d\tau$$

for any $k, l \in \mathbb{N}$, and hence, as $g \circ \sigma$ is integrable on $[a, b]$, we see that $\{f(\sigma(t_k))\}_{k=1}^{\infty}$ is convergent. Suppose now that $\gamma : [c, d] \to \Omega$ is another segment in Σ satisfying $\sigma(a) = \gamma(c)$, and let $\{s_m\}_{m=1}^{\infty} \subset [c, d]$ be a sequence converging to c such that $\gamma(s_m) \notin \Omega_0$ for every $m \in \mathbb{N}$. Then

$$\|f(\sigma(t_k)) - f(\gamma(s_m))\| \leq \int_a^{t_k} g(\sigma(\tau))\, d\tau + \int_c^{s_m} g(\gamma(\tau))\, d\tau \overset{k,m\to\infty}{\longrightarrow} 0.$$

This proves that the limit of $f(\sigma(t_k))$ as $k \to \infty$ is independent of the choice of the segment σ and the sequence $\{t_k\}_{k=1}^{\infty}$.

Now we define a representative f_0 of f in the following way:

(1) If $x \in \Omega \setminus \Omega_0$ we set $f_0(x) = f(x)$.
(2) If $x \in \Omega_0$ and there exists a segment $\sigma : [a, b] \to \Omega$ in Σ such that $\sigma(a) = x$, we set $f_0(x) = \lim_{k\to\infty} f(\sigma(t_k))$ where $\{t_k\}_{k=1}^{\infty} \subset [a, b]$ is a sequence converging to a such that $\sigma(t_k) \notin \Omega_0$ for each k.
(3) Otherwise, we set $f_0(x) = 0$.

By definition, $f_0 = f$ almost everywhere and, for every segment $\sigma : [a, b] \to \Omega$ in Σ and every $a \leq s < t \leq b$, we have:

$$\|f_0(\sigma(t)) - f_0(\sigma(s))\| \leq \int_s^t g(\sigma(\tau))\, d\tau.$$

Since $g \circ \sigma \in L^1([a, b])$, this implies that f_0 is absolutely continuous on a.e. line parallel to coordinate axes. Then using Theorem 2.8 and the Lebesgue differentiation theorem we deduce that

$$\|w^*\partial_i f(z)\| = \lim_{h\to 0^+} \frac{\|f(z + he_i) - f(z)\|}{h} \leq \lim_{h\to 0^+} \frac{1}{h} \int_0^h g(z + \tau e_i)\, d\tau = g(z),$$

holds for $\mathcal{L}^1$-almost every point z of almost every line parallel to the i-th coordinate axis. In this way we see that $\|w^*\partial_i f\| \in L^p(\Omega)$ for all $i = 1, \dots, N$. □

Remark 3.6. Equivalences $(i) \Leftrightarrow (iii)$ and $(i) \Leftrightarrow (iv)$ of the above Theorem have been independently obtained by Creutz and Evseev in [6]. In fact, it is shown in [6] that the equivalence $(i) \Leftrightarrow (iv)$ holds for every dual space $V = Y^*$ and it does not require the separability of Y.

The following result provides a simple criterion for deciding whether a member of $R^{1,p}(\Omega, V)$ belongs to $W^{1,p}(\Omega, V)$ or not, involving only the measurability of the partial w^*-derivatives.

Proposition 3.7. *Suppose that $\Omega \subset \mathbb{R}^N$ is open, $V = Y^*$ is the dual of a separable Banach space Y, $1 \leq p \leq \infty$, and $f \in R^{1,p}(\Omega, V)$. Then, $f \in W^{1,p}(\Omega, V)$ if and only if $w^*\partial_i f$ is measurable for all $i = 1, \dots, N$.*

Proof. The direct implication is clear. For the converse, let $i \in \{1, \dots, N\}$ and suppose that $w^*\partial_i f$ is measurable. Since by Theorem 3.5 we know that $\|w^*\partial_i f\| \in L^p(\Omega)$, we obtain that $w^*\partial_i f \in L^p(\Omega, V)$. We are going to check that, in fact, $w^*\partial_i f$ is the i-th weak (or distributional) partial derivative of f. To this end, fix $\varphi \in C_0^\infty(\Omega)$. Note that, for each $y \in Y$ with $\|y\| \leq 1$, by the proof of Theorem 3.5, together with the measurability of $w^*\partial_i f$ and Proposition 1.2, we see that

$$\left\langle y, \int_\Omega f(x) \frac{\partial \varphi(x)}{\partial x_i}\, dx \right\rangle = \int_\Omega \langle y, f(x)\rangle \frac{\partial \varphi(x)}{\partial x_i}\, dx =$$

$$= - \int_\Omega \langle y, w^*\partial_i f(x)\rangle \varphi(x)\, dx = - \left\langle y, \int_\Omega w^*\partial_i f(x)\, \varphi(x)\, dx \right\rangle.$$

As a consequence,

$$\int_\Omega f(x) \frac{\partial \varphi(x)}{\partial x_i}\, dx = - \int_\Omega w^*\partial_i f(x) \varphi(x)\, dx.$$

Therefore, $f \in W^{1,p}(\Omega, V)$. □

In the final part of the paper we concentrate on the special case of $V = \ell^\infty$, the space of all bounded real sequences, endowed with the usual sup-norm. Note that $\ell^\infty = (\ell^1)^*$ is the dual of the separable space ℓ^1 of absolutely summable sequences. In addition, the unit vector basis $\{e_k\}_{k=1}^\infty$ of ℓ^1 is clearly a norming set for ℓ^∞. For an open set $\Omega \subset \mathbb{R}^N$, a function $f : \Omega \to \ell^\infty$ has components $f = (f_k)_{k=1}^\infty$, where $f_k = \langle e_k, f\rangle$ are real-valued functions and for each $x \in \Omega$, $\sup_{k \geq 1} |f_k(x)| < \infty$.

Then a direct application of Theorem 3.5 gives the following:

Corollary 3.8. *Let $\Omega \subset \mathbb{R}^N$ be an open set, $1 \leq p \leq \infty$, and $f \in L^p(\Omega, \ell^\infty)$, with components $f = (f_k)_{k=1}^\infty$. Then $f \in R^{1,p}(\Omega, \ell^\infty)$ if and only the following two conditions are satisfied:*

(i) $f_k \in W^{1,p}(\Omega)$ for all $k \in \mathbb{N}$.
(ii) There exists a nonnegative function $g \in L^p(\Omega)$ such that $|\nabla f_k| \leq g$ a.e. for all $k \in \mathbb{N}$.

Finally, we use the previous results to discuss a well-known example. Recall that ℓ^∞ lacks the Radon-Nikodým Property, and therefore, for any open set $\Omega \subset \mathbb{R}^N$ the classical Sobolev space $W^{1,p}(\Omega, \ell^\infty)$ is a proper subspace of $R^{1,p}(\Omega, \ell^\infty)$. The function f in our example below belongs to $R^{1,p}((0,1), \ell^\infty) \setminus W^{1,p}((0,1), \ell^\infty)$. We justify that f cannot be in $W^{1,p}((0,1), \ell^\infty)$ by proving that its w^*-derivative is not essentially separably valued.

Example 3.9. Consider the function $f : (0,1) \to \ell^\infty$ given by

$$f(t) = \left(\frac{\sin(2\pi n t)}{2\pi n} \right)_{n=1}^\infty.$$

Then f belongs to $R^{1,p}((0,1), \ell^\infty)$ and $g(t) := (\cos(2\pi n t))_{n=1}^\infty$ is the w^*-derivative of f. Furthermore, g is not essentially separably valued and so f does not belong to $W^{1,p}((0,1), \ell^\infty)$.

Proof. By Corollary 3.8, f belongs to $R^{1,p}((0,1), V)$. It is easy to see that g is the w^*-derivative of f. We use Proposition 3.7 to prove that f does not belong to $W^{1,p}(\Omega, V)$. We claim that g is not measurable. By Pettis theorem, it is enough to see that g is not essentially separably valued. To this end, given a null set $Z \subset (0,1)$, we will find a non-countable set $B \subset (0,1) \setminus Z$ such that for any pair of different elements $x, y \in B$ the inequality $\|g(x) - g(y)\|_\infty > \frac{1}{3}$ holds. Notice that the function $h(x) = \cos(2\pi x)$ is uniformly continuous on $\mathbb{R}$, so there exists $\delta > 0$ such that $|\cos(2\pi x) - \cos(2\pi y)| < \frac{1}{3}$ for each $x, y \in \mathbb{R}$ such that $|x - y| < \delta$. It is known (see e.g. [19], page 108) that there exists a non-measurable Hamel basis of $\mathbb{R}$ as a $\mathbb{Q}$-vector space, and hence we can find a non-measurable set $A \subset (0,1) \setminus \mathbb{Q}$ such that if $x \neq y$ in A, then the set $\{1, x, y\}$ is $\mathbb{Q}$-linearly independent. Given any null set $Z \subset (0,1)$, consider the set $B = A \backslash Z$, that is not measurable, and hence non-countable. Take $x \neq y$ in B. Since $\{1, x, y\}$ are $\mathbb{Q}$-linearly independent, by the Kronecker's Theorem on Diophantic Approximation (see [11], page 507) there exist $n, p, q \in \mathbb{Z}$ such that

$$\left| p + nx - \frac{1}{4} \right| < \delta \quad \text{and} \quad |q + ny| < \delta$$

If $n \in \mathbb{N}$,

$$\begin{aligned} |g_n(x) - g_n(y)| &= |\cos(2\pi nx) - \cos(2\pi ny)| = |\cos(2\pi(p + nx)) - \cos(2\pi(q + ny))| \\ &\geq |\cos(2\pi 0) - \cos\left(2\pi \frac{1}{4}\right)| - |\cos(2\pi(p + nx)) - \cos\left(2\pi \frac{1}{4}\right)| \\ &- |\cos(2\pi(q + ny)) - \cos(2\pi 0)| \\ &> 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}. \end{aligned}$$

For $n \leq -1$, consider the component g_m with $m = -n$ and the proof is analogous. Hence,

$$\|g(x) - g(y)\|_\infty > \frac{1}{3} \text{ for all different } x, y \in B$$

and there exists a non-countable set in $g((0,1)\backslash Z)$ with that property, proving $g((0,1)\backslash Z)$ is not separable. □

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