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On the relation between completely bounded and $(1, cb)$-summing maps with applications to quantum XOR games ☆

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In this work we show that, given a linear map from a general operator space into the dual of a C*-algebra, its completely bounded norm is upper bounded by a universal constant times its $(1, cb)$-summing norm. This problem is motivated by the study of quantum XOR games in the field of quantum information theory. In particular, our results imply that for such games entangled strategies cannot be arbitrarily better than those strategies using one-way classical communication.

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1. Introduction and main results

During the last years there have been many interactions between quantum information and the fields of operator algebras and operator spaces/systems. To mention a few examples, free probability has been successfully applied in the study of quantum channel capacities [3,4], operator systems and operator algebras techniques have been recently used to study synchronous games [13,20,24] and operator spaces have been key to solve several problems on nonlocal games and Bell inequalities [22]. In fact, these connections also go in the converse direction, as it is shown by the new proofs of Grothendieck's Theorem for operator spaces based on the use of the Embezzlement state [30], the proof of new embeddings between noncommutative L_p-spaces [17] and certain operator algebras [11] by using some classical protocols in quantum information and, probably the most notable example, the recent resolution of the famous Connes Embedding Problem by using techniques from quantum computer sciences [14].

The main goal of this paper is to study the relation between certain norms defined on linear maps from a general operator space X to the dual of a C*-algebra A^*. This problem has a clear mathematical motivation, since some fundamental results such as the noncommutative versions of Grothendieck's Theorem, can be read in similar terms. However, in the spirit of the previous paragraph, in the second part of the paper we explain that this problem, when restricted to the case where both X and A are matrix algebras, is equivalent to the study of certain values of the so-called quantum XOR games, hence stressing the close connection between pure mathematical problems and some questions motivated by quantum information theory.

In order to state our main results we need to introduce some elements. Let us recall that an operator space X is a closed subspace of $B(H)$ [8,28]. For any such subspace the operator norm on $B(H)$ automatically induces a sequence of matrix norms $\|\cdot\|_d$ on $M_d(X)$, $d \geq 1$, via the inclusions $M_d(X) \subseteq M_d(B(H)) \simeq B(H^{\oplus d})$. In this way, given a linear map $T : X \to Y$ between two operator spaces X and Y, we say that T is completely bounded if

$$\|T\|_{cb} := \sup_d \|\mathbb{1} \otimes T : M_d(X) \to M_d(Y)\| < \infty.$$

The study of operator spaces was initiated in [32] and can be understood as a noncommutative version of Banach space theory. Since then, an important line of research has been devoted to developing the "Grothendieck's program" for operator spaces (see for instance [7,10,16,29,35]). A crucial definition in the local theory of Banach spaces is that of absolutely p-summing maps, as those linear maps between Banach spaces $T : X \to Y$ such that

$$\pi_p(T) := \|\mathbb{1} \otimes T : \ell_p \otimes_\epsilon X \to \ell_p(Y)\| < \infty.$$

Motivated by the great relevance of these maps [6], in [27] Pisier introduced and studied a noncommutative analogue in the context of operator spaces. Given a linear map between operator spaces $T : X \to Y$, we say that T is completely p-summing if

$$\pi_p^0(T) := \|\mathbb{1} \otimes T : S_p \otimes_{min} X \to S_p(Y)\| < \infty.$$

Note that the previous definition requires the highly nontrivial concept of noncommutative vector-valued L_p-spaces, which was also developed in [27].

However, the noncommutative context admits some other generalizations of p-summing maps. Here, we will deal with the (p, cb)-summing maps, introduced by the first author in [15] (see also [19]) and which can be understood as an intermediate definition between the one for Banach spaces and the one for operator spaces above. Given an operator space X and a Banach space Y, a linear map $T : X \to Y$ is said to be (p, cb)-summing if

$$\pi_{p,\,cb}(T) := \|\mathbb{1} \otimes T : \ell_p \otimes_{min} X \to \ell_p(Y)\| < \infty.$$

It is clear from the previous definitions that for every linear map $T : X \to Y$ between operator spaces, the inequality $\max\{\|T\|_{cb},\, \pi_{p,\,cb}(T)\} \leq \pi_p^0(T)$ holds. However, there is no general relation between the quantities $\|T\|_{cb}$ and $\pi_{p,\,cb}(T)$. That is, one can find examples of linear maps and operator spaces for which $\|T\|_{cb} < \infty$ and $\pi_{p,\,cb}(T) = \infty$ and also for which $\|T\|_{cb} = \infty$ and $\pi_{p,\,cb}(T) < \infty$.

In this work we study the relation between $\|T\|_{cb}$ and $\pi_{1,\,cb}(T)$ for maps T defined from a general operator space X to the dual of a C*-algebra A^*. Our main result is as follows.

Theorem 1.1. *There exists a universal constant K such that for any linear map $T : X \to A^*$, where X is an operator space and A is a C*-algebra, we have*

$$\|T\|_{cb} \leq K\pi_{1,cb}(T).$$

In order to prove Theorem 1.1 we will need to study the quantity $\Gamma_{R\cap C}$ (the factorizable "norm" through the operator space $R \cap C$) and prove that it fits very well in our context. Once this is done, Theorem 1.1 will follow as an application of the noncommutative Little Grothendieck's Theorem. In fact, we will prove a stronger result than the one stated above, namely $\Gamma_{R\cap C}(T) \leq K\pi_{1,cb}(T)$, where the constant K can be taken equal to $2\sqrt{2}$. It is worth mentioning that one cannot expect to have a converse inequality in Theorem 1.1; not even in the commutative case. That is, there exist maps $T : \ell_\infty \to \ell_\infty^*$ for which $\|T\|_{cb} < \infty$ and $\pi_{1,\,cb}(T) = \infty$ (see Section 4 for details). Let us also mention that we do not know if the constant K in Theorem 1.1 can be taken equal to one. However, we stress that the techniques used in the present work lead irremediably to $K > 1$.

Theorem 1.1 can be read in the context of quantum XOR games [31] when $X = A = M_n$ is the C*-algebra of $n \times n$ complex matrices. Quantum XOR games are collaborative games where a referee asks some (quantum) questions to a couple of players, usually called Alice and Bob, who must answer with outputs $a, b \in \{\pm 1\}$. According to the questions and the parity of the answers, ab, the players win or lose the game. It turns out that these games can be identified with selfadjoint matrices $G \in M_{nm}$ such that $\|G\|_{S_1^{nm}} \leq 1$, where S_1^{nm} denotes the corresponding 1-Schatten class (see Section 4 for details). Moreover, the largest possible winning probability of the game depending on the type of strategies performed by the players can be expressed by means of some norms on $\hat{G} : M_n \to M_m$, where $\hat{G}$ is the linear map associated to the matrix $G \in M_{nm}$ according to the algebraic identification $M_{nm} = L(M_n \to M_m)$. In this context, if we denote by $\beta^*(G)$ the largest bias[1] of the game G when the players are allowed to perform entangled strategies, it is known that $\beta^*(G) = \|\hat{G} : M_n \to S_1^m\|_{cb}$ (see Section 4). On the other hand, if we denote by $\beta_{owc}(G)$ the largest bias of the game when the players are allowed to send one-way classical communication as part of their strategies, we will show in Section 4 that $\beta_{owc}(G) \approx \pi_{1,cb}(\hat{G} : M_n \to S_1^m)$, where $\approx$ means equivalence up to a universal constant. Hence, Theorem 1.1 above leads to the following consequence.

Corollary 1.2. *Let G be a quantum XOR games. Then,*

$$\beta^*(G) \leq K' \beta_{owc}(G)$$

for a certain universal constant K'.

One of the main goals of quantum information theory is to find scenarios where quantum entanglement is "much more powerful" than classical resources. When working with classical XOR games [2] (see also [22]), it is very easy to see that the one-way communication of classical information is as powerful as possible. That is, those games can always be won with probability one if the players can use classical communication as part of their strategy. On the contrary, as a consequence of the classical Grothendieck theorem, entanglement is a quite limited resource to play classical XOR games, providing only small advantages over classical strategies. The situation changes dramatically for *quantum* XOR games. Within this more general family of games there exist instances for which the use of entanglement allows to attain biases which are unboundedly larger than the best bias that players sharing only classical randomness can achieve [31]. On the other hand, since the questions now are quantum states, classical communication is not enough to win with certainty. In fact, there exist quantum XOR games for which sharing one-way classical communication does not provide any advantage at all (see Section 4 for further clarification on the previous statements). This new phenomenology motivates

[1] For some reasons that will become clear in Section 4, when working with XOR games one usually works with the bias $\beta = 2P_{win} - 1$ rather than with the winning probability P_{win}.

us to ask whether there exist quantum XOR games for which quantum entanglement allows Alice and Bob to answer much more successfully than using one-way classical communication. Corollary 1.2 says that this is not the case. Hence, one needs to consider more involved tasks than winning quantum XOR games in order to find examples for which quantum entanglement is much better than sending classical information.

The structure of the paper is as follows. In Section 2 we introduce some notation and basic results that will be used along the paper. Section 3 will be devoted to proving Theorem 1.1. Finally, in Section 4 we will introduce quantum XOR games and we will explain and prove how different values of these games can be written in terms of norms on linear maps between some operator spaces. As a consequence of this mathematical formulation for the different values of quantum XOR games we will see how Corollary 1.2 can be obtained from Theorem 1.1.

2. Preliminaries and some basic results

In this section we introduce some tools and well known results that we will use later. We assume the reader to be familiar with the basic elements of Banach spaces [33] and operator spaces [28].

2.1. Absolutely p-summing maps and completely p-summing maps

Given a linear map $T : X \to Y$ between two Banach spaces and $1 \leq p < \infty$, we say that T is *absolutely p-summing* if

$$\pi_p(T) := \|id \otimes T : \ell_p \otimes_\epsilon X \to \ell_p(X)\| < \infty, \tag{2.1}$$

where here $\ell_p \otimes_\epsilon X$ denotes the (complete) injective tensor product and $\ell_p(X)$ is the corresponding vector valued L_p-space. It is not difficult to see that π_p is a norm on the set of all absolutely p-summing maps. The *factorization theorem* for these maps [6, Theorem 2.13] states that $T : X \to Y$ is absolutely p-summing if and only if there exist a regular Borel probability measure μ on the unit ball of the dual space of X^*, B_{X^*}, a closed subspace $E_p \subseteq L_p(B_{X^*}, \mu)$ and a linear map $u : E_p \to Y$ with $\|u\| = \pi_p(T)$ such that the following diagram commutes:

$$\begin{tikzcd}
C(B_{X^*}) \arrow[r, "i"] & L_p(\mu) \\
\cup & \cup \\
j(X) \arrow[r, "i|_{j(X)}"] & E_p \arrow[d, "u"] \\
X \arrow[u, "j"] \arrow[r, "T"] & Y
\end{tikzcd}$$

Here, $j : X \hookrightarrow C(B_{X^*})$ is the canonical embedding and $i : C(B_{X^*}) \to L_p(B_{X^*}, \mu)$ is the identity map.

Motivated by the great relevance of absolutely p-summing maps in the local theory of Banach spaces, in [27] Pisier developed the theory of completely p-summing maps in the context of operator spaces. Given a linear map $T : X \to Y$ between two operator spaces and $1 \leq p < \infty$, we say that T is *completely p-summing* if

$$\pi_p^0(T) := \|id \otimes T : S_p \otimes_{min} X \to S_p(X)\| < \infty,$$

where here $S_p \otimes_{min} X$ denotes the minimal tensor product in the category of operator spaces and $S_p(X)$ is the corresponding non-commutative vector valued L_p-space.

It is interesting to note that completely p-summing maps satisfy a factorization theorem analogous to the one for absolutely p-summing maps. However, in order to explain that result we need to recall some definitions about ultraproducts of Banach spaces and operator spaces. We refer to [12] for a detailed exposition on ultraproducts of Banach spaces and to [28, Section 2.8] for the operator space case. Given a family of Banach spaces $(X_i)_{i\in I}$ and a nontrivial ultrafilter $\mathcal{U}$ on the set I, denote by ℓ the set of elements $(x_i)_{i\in I}$ with $x_i \in X_i$ for every i and such that $\sup_i \|x_i\| < \infty$. We equip this space with the norm $\|x\| = \sup_i \|x_i\|$. Let us now denote by $\nu_{\mathcal{U}}$ the subspace of ℓ given by the elements x such that $\lim_{\mathcal{U}} \|x_i\| = 0$. The quotient $\ell/\nu_{\mathcal{U}}$ is a Banach space called ultraproduct of the family $(X_i)_{i\in I}$ and denoted by $\prod X_i/\mathcal{U}$. Note that if $[x]$ is the equivalence class associated to an element $(x_i)_i$, then $\|[x]\| = \lim_{\mathcal{U}} \|x_i\|$. If in addition X_i is endowed with an operator space structure for every i, we can endow the space $\prod X_i/\mathcal{U}$ with a natural operator space structure by defining $M_n(\prod X_i/\mathcal{U}) = \prod M_n(X_i)/\mathcal{U}$ for every $n \in \mathbb{N}$. It can be seen that given a family of bounded (resp. completely bounded) maps $T_i : X_i \to Y_i$ for every i, one can define a linear map $\hat{T} : \prod X_i/\mathcal{U} \to \prod Y_i/\mathcal{U}$ by $\hat{T}([(x_i)_i]) = [(T_i(x_i))_i]$ which satisfies $\|\hat{T}\| \leq \sup_i \|T_i\|$ (resp. $\|\hat{T}\|_{cb} \leq \sup_i \|T_i\|_{cb}$). It is also interesting to mention that ultraproducts respect isometries and quotients both in the Banach space category and in the operator space category.

The *factorization theorem* for completely p-summing maps [27, Remark 5.7] states that given a linear map $T : X \to Y$ between operator spaces, such that $X \subset B(H)$, there exist an ultrafilter $\mathcal{U}$ over an index set I, families $(a_i)_i$, $(b_i)_i$ in the unit sphere of $S_{2p}(H)$, a closed (operator) space $E_p \subseteq \prod S_p/\mathcal{U}$ and a linear map $u : E_p \to Y$ with $\|u\|_{cb} = \pi_p^0(T)$ such that the following diagram commutes:

$$\begin{tikzcd}
\prod B(H)/\mathcal{U} \arrow[r, "\mathcal{M}"] & \prod S_p/\mathcal{U} \\
\cup & \cup \\
j(X) \arrow[r, "\mathcal{M}|_{j(X)}"] & E_p \arrow[d, "u"] \\
X \arrow[u, "j"] \arrow[r, "T"] & Y
\end{tikzcd}$$

Here, $j : X \hookrightarrow \prod B(H)/\mathcal{U}$ is the complete isometry defined as $j(x) = [(x)_{i\in I}]$ and $\mathcal{M} : \prod B(H)/\mathcal{U} \to \prod S_p/\mathcal{U}$ is the linear map defined by the family $(M_i)_i$, where M_i :

$B(H) \to S_p(H)$ is defined as $M_i(x) = a_i x b_i$ for every $i \in I$. In the previous picture, $E_p = \overline{\mathcal{M}(j(X))}$.

As mentioned in the Introduction, one can define an intermediate notion between absolutely p-summing and completely p-summing maps. Indeed, given an operator spaces X and a Banach space Y, a linear map $T : X \to Y$ is said to be (p, cb)*-summing* (see [15], [19]) if

$$\pi_{p,\,cb}(T) := \|\mathbb{1} \otimes T : \ell_p \otimes_{min} X \to \ell_p(Y)\| < \infty. \tag{2.2}$$

Remark 2.1. It was observed by Pisier [27, Remark 5.11] that $T : X \to Y$ is (p, cb)-summing if and only if it verifies a similar factorization theorem to the one for completely p-summing maps but where, in this case, $\|u\| = \pi_{p,\,cb}(T)$.

Given a complex Hilbert space H, the operator space structures defined by the isometric identifications

$$H \simeq B(H, \mathbb{C}) \qquad \text{and} \qquad H \simeq B(\mathbb{C}, H) \tag{2.3}$$

are the row and column operator space structures on H, denoted by R_H and C_H, respectively. When the underlying Hilbert space is $H = \ell_2$, we use the simpler notation R and C. Moreover, we can also define the $R_H \cap C_H$ operator space structure on H by means of the embedding

$$j : R_H \cap C_H \to R_H \oplus_\infty C_H, \tag{2.4}$$

defined as $j(x) = (x, x)$. Finally, the $R_H + C_H$ operator space structure on H can be defined so that $R_H + C_H = (R_H \cap C_H)^*$ completely isometrically. The following stability properties under ultraproducts will play a role later on:

Remark 2.2. It is well known that, at the Banach space level, the ultraproduct of a family of Hilbert spaces $(H_i)_i$, $\hat{H} = \prod H_i/\mathcal{U}$, is a Hilbert space. Furthermore, according to the definition of R_H and C_H and the comments above, it is not difficult to see that

$$\prod R_{H_i}/\mathcal{U} = R_{\hat{H}} \qquad \text{and} \qquad \prod C_{H_i}/\mathcal{U} = C_{\hat{H}}.$$

Moreover, the properties of the ultraproducts together with the definition of $R_H \cap C_H$ via the embedding (2.4) ensure that

$$\prod (R_{H_i} \cap C_{H_i})/\mathcal{U} = R_{\hat{H}} \cap C_{\hat{H}}.$$

In fact, this stability under ultraproducts is a property of any homogeneous Hilbertian operator space (as, e.g., R_H, C_H or $R_H \cap C_H$). See [26, Lemma 3.1 and remarks in page 82].

2.2. Little Grothendieck's theorem

Although most maps between Banach spaces are not p-summing for any p, a famous result, called *little Grothendieck's theorem*, asserts that every linear map $T : C(K) \to L_2(\mu)$, where K is a compact space and μ is any measure verifies that $\pi_2(T) \leq K_{LG}\|T\|$. Here, $K_{LG} = \sqrt{\pi/2}$ in the real case and $K_{LG} = 2/\sqrt{\pi}$ in the complex case.

There is also a noncommutative version of this result, which was first proved in [25] and later in [9] (with an improvement in the constant). This result is usually referred to as *non-commutative little Grothendieck's theorem.*[2]

Theorem 2.1. *Let A be a C^*-algebra and H be a Hilbert space. Then, for any bounded linear map $T : A \to H$ there exist states f_1 and f_2 on A such that*

$$\|T(x)\| \leq \|T\|\Big(f_1(x^*x) + f_2(xx^*)\Big)^{\frac{1}{2}} \tag{2.5}$$

for every $x \in A$.

Although the following corollary is folklore, we have not found any proof in the literature. Since it will be crucial for us, we give some hints about its proof.

Corollary 2.2. *Let A be a C^*-algebra and H be a Hilbert space. Then, any bounded linear map $T : A \to H$ satisfies that*

$$\|T : A \to R_H + C_H\|_{cb} \leq 2\|T\|.$$

Proof. The key idea is to use the non-commutative little Grothendieck theorem to decompose T into two maps $T = T_1 + T_2$ such that

$$\max\{\|T_1 : A \to R_H\|_{cb}, \|T_2 : A \to C_H\|_{cb}\} \leq \|T : A \to H\|. \tag{2.6}$$

With this at hand, the statement is obtained noticing that:

$$\|T : A \to R_H + C_H\|_{cb} \leq \|T_1\|_{cb} + \|T_2\|_{cb} \leq 2\|T\|.$$

Therefore, the main part of the proof consists on constructing T_1 and T_2 with the claimed properties. For that, observe that the states f_1 and f_2 from Theorem 2.1 define pre-inner products $\langle x, y\rangle_1 := f_1(xy^*)$, $\langle x, y\rangle_2 := f_2(x^*y)$, for any $x, y \in A$, which naturally induce Hilbert spaces H_1, H_2 in the standard way ($H_i = \overline{A/N_i}$, where $N_i = \{x \in A : \langle x, x\rangle_i = 0\}$).

[2] In order to see that this result generalizes the classical little Grothendieck's theorem one must use the *domination theorem* for 2-summing maps [6, Theorem 2.13]. We omit this result here because we will not use it.

Given that, we consider the Hilbert space $H_1 \oplus_2 H_2$ and the injections:

$$\begin{array}{cccc} j_1: & A & \longrightarrow & H_1 \oplus_2 H_2 \\ & x & \mapsto & [x] \oplus 0 \end{array}, \qquad \begin{array}{cccc} j_2: & A & \longrightarrow & H_1 \oplus_2 H_2 \\ & x & \mapsto & 0 \oplus [x] \end{array}.$$

Let us now consider the Hilbert space V defined by the closure of the linear subspace $E = \{[x] \oplus [x] \,:\, x \in A\} \subset H_1 \oplus_2 H_2$ and let us define the map

$$\begin{array}{cccc} \tilde{T}: & E & \longrightarrow & H \\ & [x] \oplus [x] & \mapsto & T(x) \end{array}.$$

Using Equation (2.5) one can see that $\tilde{T}$ is a well defined linear map satisfying $\|\tilde{T}\| \leq \|T\|$. In particular, it can be extended to V with the same norm.

Now, by defining the linear maps

$$T_i : A \xrightarrow{j_i} H_1 \oplus_2 H_2 \xrightarrow{p} V \xrightarrow{\tilde{T}} H \qquad \text{for } i = 1, 2,$$

where $p : H_1 \oplus_2 H_2 \to V$ is the orthogonal projection onto the subspace V, one can easily check that $T(x) = T_1(x) + T_2(x)$ and, moreover, $\|T_1(x)\|_H \leq f_1(xx^*)^{\frac{1}{2}}$ and $\|T_2(x)\|_H \leq f_2(x^*x)^{\frac{1}{2}}$ for every $x \in A$. These last estimates imply Equation (2.6). $\square$

3. Main result

In this section we will prove our main result, that we state again for convenience.

Theorem 3.1. *There exists a universal constant K such that for any linear map $T : X \to A^*$, where X is an operator space and A is a C*-algebra, we have*

$$\|T\|_{cb} \leq K\pi_{1,cb}(T).$$

Before proving the result, let us make some comments.

It follows from the proof of Theorem 3.1 that the constant K can be taken equal $2\sqrt{2}$. We did not attempt any optimization in terms of this constant. However, our proof inevitably leads to a constant strictly larger than one. Whether one can get $K = 1$ in the previous statement seems an interesting problem (see Section 4 for a related problem in quantum information).

The fact that the image of T is in the dual of a C*-algebra is crucial in Theorem 3.1, since one can find examples of (operator) spaces X, Y for which $\|T : X \to Y\|_{cb}$ can be arbitrary larger than $\pi_{1,cb}(T : X \to Y)$. Indeed, this can be shown, for instance, by considering $X = CL_n$, the operator space associated to the Clifford algebra with n generators [28, Section 9.3], and $Y = \max(\ell_2^n)$. With this choice, we have $\pi_{1,cb}(id : CL_n \to \ell_2^n) \leq 2$ [15, Proposition 4.3.2] and $\|id : CL_n \to \max(\ell_2^n)\|_{cb} \geq \sqrt{n}$ [28, Theorem 10.4].

Finally, recall that while it is known [27, Corollary 5.5] that

$$\pi_1^0(T) = \|id \otimes T : S_1 \otimes_{min} X \to S_1(Y)\| = \|id \otimes T : S_1 \otimes_{min} X \to S_1(Y)\|_{cb},$$

$\pi_{1,cb}(T) = \|id \otimes T : \ell_1 \otimes_{min} X \to \ell_1(Y)\|$ does not coincide in general with $\|id \otimes T : \ell_1 \otimes_{min} X \to \ell_1(Y)\|_{cb}$. Indeed, it follows from [17] that $\|id \otimes T : \ell_1 \otimes_{min} X \to \ell_1(X)\|_{cb} = \pi_1^0(T)$ and there are known examples showing that $\pi_1^0(T)$ can be much larger than $\pi_{1,cb}(T)$ for maps $T : B(H) \to S_1(H)$.

In order to prove Theorem 3.1, let us first define, for a given linear map $T : X \to Y$ between operator spaces, the quantity

$$\Gamma_{R\cap C}(T) := \inf\{\|a : X \to R \cap C\|_{cb}\|b : R \cap C \to Y\|_{cb} : T = b \circ a\}, \tag{3.1}$$

where we understand $\Gamma_{R\cap C}(T) = \infty$ if there does not exist any such factorization of T. This quantity has been previously studied by different authors (see for instance [26, Section 8] and [15, Section 4.2]). While it is not clear whether Equation (3.1) defines a norm, it is known to be equivalent to a norm.

The following result will be key in the proof of Theorem 3.1.

Proposition 3.2. *Let H be a complex Hilbert space, a and b be elements in the unit ball of $S_2(H)$ and $M_{a,b} : B(H) \to S_1(H)$ be the linear map defined as $M_{a,b}(x) = axb$ for every $x \in B(H)$. Then, $\Gamma_{R\cap C}(M_{a,b}) \leq K$ for a universal constant K. Moreover, K can be taken $2\sqrt{2}$.*

We thank the reviewer very much for providing us with the following proof, which simplified considerably (and slightly improved the constant K) a previous one by the authors, based on the theory of weights introduced by Pisier in [26].

Proof. Let us first notice that, for every $x \in B(H)$, we have

$$\|M_{a,b}(x)\|_{S_1(H)} \leq \min\{\|ax\|_{S_2(H)}, \|xb\|_{S_2(H)}\}.$$

Hence, for any decomposition $x = x_1 + x_2$, x_1, $x_2 \in B(H)$, we have

$$\|M_{a,b}(x)\|_{S_1(H)} \leq \|M_{a,b}(x_1)\|_{S_1(H)} + \|M_{a,b}(x_2)\|_{S_1(H)} \leq \|ax_1\|_{S_2(H)} + \|x_2 b\|_{S_2(H)},$$

from where one easily obtains

$$\|M_{a,b}(x)\|_{S_1(H)} \leq \sqrt{2} \inf_{x=x_1+x_2} \left(\|ax_1\|^2_{S_2(H)} + \|x_2 b\|^2_{S_2(H)}\right)^{\frac{1}{2}}.$$

Now, in order to give an explicit factorization of the operator $M_{a,b}$ through a Hilbert space, let us consider the linear subspace $V = \{(ay, -yb) : y \in B(H)\}$ of $S_2(H) \oplus_2 S_2(H)$ and the Hilbert space obtained by considering the corresponding quotient space:

$$H_0 = S_2(H) \oplus_2 S_2(H)/\overline{V}.$$

Then, note that for every $x \in B(H)$ the norm of the equivalence class $[(ax, 0)]$ is given by:

$$\begin{aligned} \|[(ax,0)]\|_{H_0} &= \inf_{z \in \overline{V}} \|(ax,0) - z\|_{S_2(H)\oplus_2 S_2(H)} = \inf_{y \in B(H)} \|(ax,0) - (ay,-yb)\|_{S_2(H)\oplus_2 S_2(H)} \\ &= \inf_{y \in B(H)} \|(a(x-y), yb)\|_{S_2(H)\oplus_2 S_2(H)} \\ &= \inf_{x = x_1 + x_2} \left(\|ax_1\|^2_{S_2(H)} + \|x_2 b\|^2_{S_2(H)} \right)^{\frac{1}{2}}, \end{aligned}$$

where the last equality follows trivially by identifying $x_1 = x - y$, $x_2 = y$.

Therefore, we conclude that

$$\|M_{a,b}(x)\|_{S_1(H)} \leq \sqrt{2} \|[(ax,0)]\|$$

for every $x \in B(H)$.

Let us now define the map $T_1 : B(H) \to H_0$ given by

$$T_1(x) = [(ax,0)] \quad \text{for every } x \in B(H).$$

It is clear that T_1 is a well defined linear map (note that T_1 is nothing else than the map $x \mapsto (ax, 0)$, composed with the quotient map) satisfying that

$$\|M_{a,b}(x)\|_{S_1(H)} \leq \sqrt{2} \|T_1(x)\|_{H_0} \quad \text{for every } x \in B(H). \tag{3.2}$$

In order to obtain a second map, we consider the linear subspace $\tilde{H}_0 = \{T_1(x) : x \in B(H)\}$ of H_0 and define $T_2 : \tilde{H}_0 \to S_1(H)$ by

$$T_2(T_1(x)) = M_{a,b}(x) \quad \text{for every } x \in B(H).$$

Now, Equation (3.2) guarantees that T_2 is well defined and, moreover, one easily checks that it is a linear map. Furthermore, by using again Equation (3.2) one easily concludes that $\|T_2\| \leq \sqrt{2}$. This allows us to extend T_2 to another linear map (that we will call T_2 again) from the closure of the space $\tilde{H}_0$, that we denote $\mathcal{H}$, to $S_1(H)$ such that $\|T_2\| \leq \sqrt{2}$ and $M_{a,b} = T_2 \circ T_1$. Hence, we have found an explicit factorization of $M_{a,b}$ through a Hilbert space $\mathcal{H}$. In order to conclude the proof, we need to study the completely bounded norm of T_1 and T_2.

We will start proving that $\|T_1 : B(H) \to R_{\mathcal{H}} \cap C_{\mathcal{H}}\|_{cb} \leq 1$. To this end, let us first recall the well known result ([26, Proposition 5.11])

$$\begin{aligned} \|T_1 : B(H) \to R_{\mathcal{H}}\|_{cb} &= \|id \otimes T_1 : R \otimes_{min} B(H) \to \ell_2(\mathcal{H})\|, \\ \|T_1 : B(H) \to C_{\mathcal{H}}\|_{cb} &= \|id \otimes T_1 : C \otimes_{min} B(H) \to \ell_2(\mathcal{H})\|. \end{aligned}$$

Now, given $z = \sum_i e_i \otimes x_i \in R \otimes_{min} B(H)$, we have

$$\|(id \otimes T_1)(z)\|_{\ell_2(\mathcal{H})} = \Big(\sum_i \|T_1(x_i)\|_{\mathcal{H}}^2\Big)^{\frac{1}{2}} = \Big(\sum_i \|[(ax_i, 0)]\|_{H_0}^2\Big)^{\frac{1}{2}} \leq \Big(\sum_i \|ax_i\|_{S_2(H)}^2\Big)^{\frac{1}{2}}$$
$$= \Big(tr\Big[a\Big(\sum_i x_i x_i^*\Big)a^*\Big]\Big)^{\frac{1}{2}} \leq \Big\|\sum_i x_i x_i^*\Big\|^{\frac{1}{2}},$$

from where we conclude that $\|T_1 : B(H) \to R_{\mathcal{H}}\|_{cb} \leq 1$.

Using that $\|[(ax, 0)]\|_{H_0} \leq \|xb\|_{S_2(H)}$ for every $x \in B(H)$, we can analogously prove that $\|T_1 : B(H) \to C_{\mathcal{H}}\|_{cb} \leq 1$. According to the definition of $R_{\mathcal{H}} \cap C_{\mathcal{H}}$ (see Equation (2.4)), we obtain that $\|T_1 : B(H) \to R_{\mathcal{H}} \cap C_{\mathcal{H}}\|_{cb} \leq 1$.

Finally, Corollary 2.2 in its dual form, guarantees that

$$\|T_2 : R_{\mathcal{H}} \cap C_{\mathcal{H}} \to S_1(H)\|_{cb} \leq 2\|T_2 : \mathcal{H} \to S_1(H)\| \leq 2\sqrt{2}.$$

This concludes the proof. □

Remark 3.1. The proof of Proposition 3.2 shows, in particular, that given a complex Hilbert space H, a and b elements in the unit ball of $S_2(H)$ and the linear map $M_{a,b} : B(H) \to S_1(H)$ defined as $M_{a,b}(x) = axb$ for every $x \in B(H)$, there exists a Hilbert space $\mathcal{H}$ and linear maps $T_1 : B(H) \to \mathcal{H}$ and $T_2 : \mathcal{H} \to S_1(H)$ such that $M_{a,b} = T_2 \circ T_1$, $\|T_1 : B(H) \to R_{\mathcal{H}} \cap C_{\mathcal{H}}\|_{cb} \leq 1$ and $\|T_2 : \mathcal{H} \to S_1(H)\| \leq \sqrt{2}$.

We are now ready to prove our main result.

Proof of Theorem 3.1. By homogeneity it suffices to show that for every linear map $T : X \to A^*$ such that $\pi_{1,cb}(T) \leq 1$, we have $\|T\|_{cb} \leq K$. To this end, assume that $\iota : X \hookrightarrow B(H)$ is a complete isometry and let us invoke the factorization theorem for $(1, cb)$-summing maps (see Remark 2.1) to conclude the existence of an ultrafilter $\mathcal{U}$ over an index set I, families $(a_i)_i$, $(b_i)_i$ in the unit ball of $S_2(H)$, a closed (operator) space $E_1 \subseteq \prod S_1/\mathcal{U}$ and a linear map $u : E_1 \to A^*$ with $\|u\| = \pi_{1,cb}(T) \leq 1$ such that the following diagram commutes:

$$\begin{tikzcd}
\prod B(H)/\mathcal{U} \arrow[r, "\mathcal{M}"] & \prod S_1/\mathcal{U} \\
\cup & \cup \\
j(X) \arrow[r, "\mathcal{M}|_{j(X)}"] & E_1 \arrow[d, "u"] \\
X \arrow[u, "j"] \arrow[r, "T"] & A^*
\end{tikzcd}$$

Here, $j : X \hookrightarrow \prod B(H)/\mathcal{U}$ is a complete isometry and $\mathcal{M} : \prod B(H)/\mathcal{U} \to \prod S_1/\mathcal{U}$ is the linear map defined by the family $(M_i)_i$, where $M_i : B(H) \to S_1(H)$ is given by

$M_i(x) = a_i x b_i$ for every $i \in I$. In order to simplify notation let us denote $\hat{B} = \prod B(H)/\mathcal{U}$, $\hat{S}_1 = \prod S_1/\mathcal{U}$ and $\tilde{\mathcal{M}} = \mathcal{M}|_{j(X)}$. We will show that the previous factorization implies that T factorizes through $R \cap C$ with completely bounded maps. Hence, T must be completely bounded.

In first place, according to Remark 3.1, for every $i \in I$ there exists a Hilbert space $\mathcal{H}_i$ and linear maps $T_{1,i} : B(H) \to \mathcal{H}_i$ and $T_{2,i} : \mathcal{H}_i \to S_1(H)$ such that $M_{a_i,b_i} = T_{2,i} \circ T_{1,i}$, $\|T_{1,i} : B(H) \to R_{\mathcal{H}_i} \cap C_{\mathcal{H}_i}\|_{cb} \leq 1$ and $\|T_{2,i} : \mathcal{H}_i \to S_1(H)\| \leq \sqrt{2}$. Then, one can deduce from Remark 2.2 and the properties about ultraproducts explained in Section 2.1 the existence of another Hilbert space $\hat{\mathcal{H}}$ and linear maps $\alpha : \hat{B} \to \hat{\mathcal{H}}$ and $\beta : \hat{\mathcal{H}} \to \hat{S}_1$ such that $\mathcal{M} = \beta \circ \alpha$, $\|\alpha : \hat{B} \to R_{\hat{\mathcal{H}}} \cap C_{\hat{\mathcal{H}}}\|_{cb} \leq 1$, and $\|\beta : \hat{\mathcal{H}} \to \hat{S}_1\| \leq \sqrt{2}$. Moreover, a similar factorization can be obtained for $\tilde{\mathcal{M}}$. Indeed, to see this let us define $\tilde{\mathcal{H}} = \overline{\alpha(j(X))} \subset \hat{\mathcal{H}}$. In virtue of the homogeneity of $R_{\hat{\mathcal{H}}} \cap C_{\hat{\mathcal{H}}}$, $\tilde{\mathcal{H}}$ inherits the same $R \cap C$ operator space structure. Then, by denoting $\tilde{\alpha} = \alpha|_{j(X)} : j(X) \to \tilde{\mathcal{H}}$ and $\tilde{\beta} = \beta|_{\tilde{\mathcal{H}}} : \tilde{\mathcal{H}} \to E_1$, it is clear that $\tilde{\mathcal{M}} = \tilde{\beta} \circ \tilde{\alpha}$, $\|\tilde{\alpha} : j(X) \to R_{\tilde{\mathcal{H}}} \cap C_{\tilde{\mathcal{H}}}\|_{cb} \leq 1$ and $\|\tilde{\beta} : \tilde{\mathcal{H}} \to E_1\| \leq \sqrt{2}$.

Therefore, we obtain a decomposition $T = (u \circ \tilde{\beta}) \circ (\tilde{\alpha} \circ j)$ such that

$$\begin{aligned}\|T : X \to A^*\|_{cb} &\leq \|\tilde{\alpha} \circ j : X \to R_{\tilde{\mathcal{H}}} \cap C_{\tilde{\mathcal{H}}}\|_{cb} \|u \circ \tilde{\beta} : R_{\tilde{\mathcal{H}}} \cap C_{\tilde{\mathcal{H}}} \to A^*\|_{cb} \\ &\leq 2\|\tilde{\alpha} : j(X) \to R_{\tilde{\mathcal{H}}} \cap C_{\tilde{\mathcal{H}}}\|_{cb} \|\tilde{\beta} : \tilde{\mathcal{H}} \to E_1\| \|u : E_1 \to A^*\| \\ &\leq 2\sqrt{2},\end{aligned}$$

where in the second inequality we have used that $\|j : X \to j(X)\|_{cb} \leq 1$ and Corollary 2.2 (in its dual form) to write

$$\|u \circ \tilde{\beta} : R_{\tilde{\mathcal{H}}} \cap C_{\tilde{\mathcal{H}}} \to A^*\|_{cb} \leq 2\|u \circ \tilde{\beta} : \tilde{\mathcal{H}} \to A^*\| \leq 2\|\tilde{\beta} : \tilde{\mathcal{H}} \to E_1\| \, \|u : E_1 \to A^*\|,$$

and in the third inequality we have used that u is a contraction.

This concludes the proof. □

Remark 3.2. Note that we have actually proved that any linear map $T : X \to A^*$, where X is an operator space and A is a C*-algebra, satisfies

$$\|T\|_{cb} \leq \Gamma_{R \cap C}(T) \leq 2\sqrt{2}\pi_{1,cb}(T).$$

4. Quantum XOR games via tensor norms

A *bipartite quantum XOR game* is described by means of a family of bipartite quantum states $(\rho_x)_{x=1}^N$, a family of signs $c = (c_x)_{x=1}^N \in \{-1, 1\}^N$ and a probability distribution $p = (p_x)_x$ on $\{1, \cdots, N\}$. Here, a bipartite quantum state ρ is just a positive semidefinite operator acting on the tensor product of two finite dimensional complex Hilbert spaces, $H_A \otimes H_B$, with trace one. Note that ρ is an element of the unit ball of $S_1(H_A \otimes H_B)$.

In order to understand the game, we can think of two (spatially separated) players, Alice and Bob, and a referee. The game starts with the referee choosing one of the states

ρ_x according to the probability distribution p. Then, the referee sends register H_A to Alice and register H_B to Bob (this can be understood as some quantum questions). After receiving the states, Alice and Bob must answer an output, $a = \pm 1$ in the case of Alice and $b = \pm 1$ in the case of Bob. Then, the players win the game if $ab = c_x$. These games were first considered in [31] as a natural generalization of classical XOR games, which have a great relevance in both quantum information and computer science. As we will see below, the relevant information of the game is encoded in the operator

$$G = \sum_{x=1}^{N} c_x p_x \rho_x, \tag{4.1}$$

which is a selfadjoint operator acting on $H_A \otimes H_B$ such that $\|G\|_{S_1(H_A \otimes H_B)} \leq 1$.

In the following we will denote by M_k (resp. M_k^{sa}) the complex (real) vector space of $k \times k$ (selfadjoint) matrices. This space, endowed with the trace and operator norms will be denoted by S_1^k ($S_1^{k,\mathrm{sa}}$) and S_∞^k ($S_\infty^{k,\mathrm{sa}}$), respectively. In the rest of this section we will identify $H_A = \mathbb{C}^n$ and $H_B = \mathbb{C}^m$. In this case, according to the previous paragraph, a quantum XOR game G can be identified with an element in $B_{S_1^{nm,\mathrm{sa}}}$, the unit ball of $S_1^{nm,\mathrm{sa}}$.

When playing a quantum XOR game, Alice and Bob generate their answers by means of some operation (a quantum channel, see e.g. [21]) on the system received from the referee. We call such an operation a *strategy*. Formally, a strategy for Alice and Bob can be expressed by a linear map $\mathcal{P} : M_{nm}^{\mathrm{sa}} \to \mathbb{R}^4$ such that, for any given state ρ, it assigns a probability distribution over the set of possible answers:

$$\mathcal{P}(\rho) = P(a, b|\rho)_{a,b=\pm 1}.$$

Note that, for a fixed strategy, it is very easy to write the probability of winning the game:

$$\begin{aligned}\mathbf{P}_{win}(G; \mathcal{P}) = \sum_{x: c_x = 1} & p_x \Big(P(1, 1|\rho_x) + P(-1, -1|\rho_x) \Big) \\ & + \sum_{x: c_x = -1} p_x \Big(P(1, -1|\rho_x) + P(-1, 1|\rho_x) \Big).\end{aligned}$$

It is also easy to see that if Alice and Bob answer randomly (somehow the most naive strategy), that is, $P(a, b|\rho_x) = \frac{1}{4}$ for every $a, b = \pm 1$ and every ρ_x, then $\mathbf{P}_{win}(G; \mathcal{P}) = \frac{1}{2}$. Hence, when working with XOR games, it is very common to study the so-called *bias* of the game, $\beta(G; \mathcal{P}) = 2(\mathbf{P}_{win}(G; \mathcal{P}) - 1/2)$ or, equivalently,

$$\mathbf{P}_{win}(G; \mathcal{P}) - \mathbf{P}_{lose}(G; \mathcal{P}) = \sum_{x=1}^{N} p_x c_x \sum_{a,b=\pm 1} ab P(a, b|\rho_x).$$

We see that, in order to compute the bias, the only relevant part of the strategies are the correlations. That is, given a strategy $\mathcal{P}$ and a state ρ, if we define the correlation

$$\gamma_{\mathcal{P}}(\rho) = \sum_{a,b=\pm 1} abP(a,b|\rho),$$

we have

$$\beta(G;\mathcal{P}) = \sum_{x=1}^{N} p_x c_x \gamma_{\mathcal{P}}(\rho_x).$$

As the reader may guess, the winning probability of the game (and so its bias) will strongly depend on the form of the strategies under consideration. The strategies considered in a given context will be determined by the *resources* allowed to Alice and Bob to play the game. One extreme case is that where the players are allowed to perform any global quantum measurement. This case can be understood as if both players were located at the same place so that they can act as a single person with access to both registers H_A and H_B. In this case, a strategy will be given by a family of positive semidefinite operators $(E_{a,b})_{a,b=-1,1}$ acting on $\mathbb{C}^n \otimes \mathbb{C}^m$ verifying that $\sum_{a,b=-1,1} E_{a,b} = \mathbb{1}_{M_{nm}}$ and such that

$$P(a,b|\rho) = tr(E_{a,b}\rho) \text{ for every } a,b = \pm 1.$$

It is very easy to see that the supremum of the bias of the game $G \in S_1^{nm,\mathrm{sa}}$ when the players are restricted to these kinds of strategies is given by

$$\beta_{owq}(G) = \sup\{tr(XG) : X \in B_{S_\infty^{nm,\mathrm{sa}}}\} = \|G\|_{S_1^{nm}},$$

where G was defined in Equation (4.1). The sub-index owq stands for *one-way quantum communication.* This is justified by the observation that the quantity above coincides with the bias achieved by strategies in which Alice is allowed to send quantum information to Bob (or the other way around). Indeed, in that case Alice can send her question to Bob so that he has access to the whole bipartite state.

In this section we will be interested in the identification between the elements $G \in S_1^{n,\mathrm{sa}} \otimes S_1^{m,\mathrm{sa}} \subset S_1^n \otimes S_1^m$ and the linear maps $\hat{G} : S_\infty^n \to S_1^m$, where we recall that, given G, we define $\hat{G}(x) = (tr \otimes \mathbb{1}_{M_m})\big(G(x^T \otimes \mathbb{1}_{M_m})\big)$. Note that we must see G as an element in the complex space $S_1^n \otimes S_1^m$ in order to work with operator spaces. With this identification in mind, it is well known that

$$\|G\|_{S_1^{nm}} = \pi_1^o(\hat{G} : S_\infty^n \to S_1^m).$$

Indeed, for every linear map $\hat{G} : S_\infty^n \to S_1^m$ the completely 1-summing norm coincides with the completely nuclear norm [7, Corollary 15.5.4] and the fact that the operator spaces are finite dimensional guarantees that the nuclear norm of $\hat{G}$ is exactly the same as $\|G\|_{S_1^{nm}}$.

Another extreme set of strategies (somehow, at the opposite side, because they are the most limited ones) are those where Alice and Bob must answer independently. These strategies are usually called *product* or *unentangled strategies* [31]. In this case there exist operators E_a acting on $\mathbb{C}^n$ and F_b acting on $\mathbb{C}^m$, for $a, b = \pm 1$ such that they are positive semidefinite, verify $E_1 + E_{-1} = \mathbb{1}_{M_n}$, $F_1 + F_{-1} = \mathbb{1}_{M_m}$ and

$$P(a, b|\rho) = tr(E_a \otimes F_b \rho) \text{ for any } a, b = \pm 1 \text{ and } \rho.$$

It is easy to see that the supremum of the bias of the game G when the players are restricted to these kinds of strategies is given by

$$\beta(G) = \sup\{tr(A \otimes BG) : \ A \in B_{S_\infty^{n,\mathrm{sa}}}, B \in B_{S_\infty^{m,\mathrm{sa}}}\}.$$

In particular, the "norm expression" of this quantity has the form

$$\beta(G) = \|G\|_{S_1^{n,\mathrm{sa}} \otimes_\epsilon S_1^{m,\mathrm{sa}}}.$$

One can also show [31, Claim 4.7] that

$$\beta(G) \leq \|G\|_{S_1^n \otimes_\epsilon S_1^m} = \|\hat{G} : S_\infty^n \to S_1^m\| \leq \sqrt{2}\beta(G).$$

There are many more possible strategies one can consider in the study of quantum XOR games. A very important family of strategies are the so-called *entangled strategies*, in which the players are allowed to use a bipartite quantum state. This situation has been deeply studied and it leads to the expression

$$\beta^*(G) = \sup\{tr\big((A \otimes B)(G \otimes \rho_{A'B'})\big)\},$$

where in this case the supremum runs over all possible complex Hilbert spaces $H_{A'}$, $H_{B'}$, bipartite quantum states $\rho_{A'B'}$ acting on $H_{A'} \otimes H_{B'}$ and selfadjoint contractive operators A and B acting on $\mathbb{C}^n \otimes H_{A'}$ and $\mathbb{C}^m \otimes H_{B'}$, respectively. In this case, the norm to be considered in $S_1^n \otimes S_1^m$ is the minimal norm (in the category of operator spaces) and one can show [31, Claim 4.14] that

$$\beta^*(G) = \|G\|_{S_1^n \otimes_{min} S_1^m} = \|\hat{G} : S_\infty^n \to S_1^m\|_{cb}. \tag{4.2}$$

Notice that, in contrast with the case of the unentangled bias $\beta(G)$, what [31, Claim 4.14] proves is that there is no need for any selfadjoint restriction on the norm. This follows from G being self adjoint by means of a standard expansion trick.

In light of the previous paragraphs, we see that the bias of the game G according to different type of strategies can be expressed by means of different norms of G as a linear map from S_1^n to S_1^m. This is the way in which we aim to understand the bias of G when the players are restricted to sending *classical communication* from Alice to Bob. The study of this set of strategies is the main goal of this section.

Denoting by $\beta_{owc}(G)$ the bias of G when the players are restricted to the use of one-way classical communication (from Alice to Bob), we will show:

Proposition 4.1. *Given a quantum XOR game $G \in S_1^{n,\mathrm{sa}} \otimes S_1^{m,\mathrm{sa}} \subset S_1^n \otimes S_1^m$, we have*

$$\beta_{owc}(G) \leq \pi_{1,cb}(\hat{G} : S_\infty^n \to S_1^m) \leq 4\beta_{owc}(G).$$

In order to prove the previous proposition we must study the correlations obtained from the strategies we are considering. Let us assume that Alice can send c bits of classical information (so, 2^c classical messages) to Bob as a part of their strategy. Hence, after receiving her part of the system from the referee, Alice will have to produce two different data: the message to be sent to Bob and the output a to be sent to the referee. This can be modelled by a family of positive semidefinite operators $E_{a,k}$ acting on $\mathbb{C}^n$, where $a = \pm 1$, $k = 1, \cdots, 2^c$, and such that $\sum_{a,k} E_{a,k} = \mathbb{1}_{M_n}$. Indeed, given a state ρ acting on $\mathbb{C}^n$, the probability that Alice outputs the pair (a, k) upon the reception of ρ is given by $tr(E_{a,k}\rho)$. On the other hand, after this first stage Bob will have access to his part of the state ρ_x as well as the message received from Alice, and he will have to output $b = \pm 1$. Hence, Bob's action is modelled by a family of positive semidefinite operators $F_{b,k}$ acting on $\mathbb{C}^m$, where $b = \pm 1$, $k = 1, \cdots, 2^c$, and such that $\sum_b F_{b,k} = \mathbb{1}_{M_m}$ for every k (that is, Bob can perform a measurement according to the message received from Alice). In this way, the strategy will be given by

$$P(a, b|\rho) = \sum_{k=1}^{2^c} tr(E_{a,k} \otimes F_{b,k}\rho) \quad \text{for any } a, b = \pm 1 \text{ and } \rho.$$

It can be seen that the supremum of the bias of G over all possible strategies of this form is given by

$$\beta_{owc}(G) = \sup \left\{ \sum_{k=1}^{2^c} tr\left((A_k \otimes B_k)G\right) : A_k = E_{1,k} - E_{-1,k},\, B_k = F_{1,k} - F_{-1,k} \right\}, \quad (4.3)$$

where here the supremum is taken over families of operators $\{E_{a,k}\}_{a,k}$ and $\{F_{b,k}\}_{b,k}$ as above.

In Proposition 4.1 we will relate the bias $\beta_{owc}(G)$ with the $\pi_{1,cb}$-norm (defined in Equation (2.2)) of the corresponding map $\hat{G}$. It is easy to see that this norm can be equivalently written as

$$\pi_{1,cb}(\hat{G} : S_\infty^n \to S_1^m) = \sup_d \left\| \mathbb{1} \otimes \hat{G} : \ell_1^d \otimes_{min} S_\infty^n \to \ell_1^d(S_1^m) \right\|. \quad (4.4)$$

Let us write this norm in more detail. For each natural number d in the above supremum we have:

$$
\begin{aligned}
&\Big\| \mathbb{1} \otimes \hat{G} : \ell_1^d \otimes_{min} S_\infty^n \to \ell_1^d(S_1^m) \Big\| \\
&= \sup \left\{ \Big\| (\mathbb{1} \otimes \hat{G}) \Big(\sum_{k=1}^d e_k \otimes A_k \Big) \Big\|_{\ell_1^d(S_1^m)} : \sum_{k=1}^d e_k \otimes A_k \in B_{\ell_1^d \otimes_{min} S_\infty^n} \right\} \\
&= \sup \left\{ \sum_{k=1}^d tr\big(G(A_k^T \otimes B_k) \big) : \begin{array}{l} \sum_{k=1}^d e_k \otimes A_k \in B_{\ell_1^d \otimes_{min} S_\infty^n}, \\ \sum_{i=1}^d e_k \otimes B_k \in B_{\ell_\infty^d(S_\infty^m)} \end{array} \right\}. \qquad (4.5)
\end{aligned}
$$

Note that the transpose in A_k doesn't play any role in Equation (4.5) because the supremum is taken over all $x = \sum_{k=1}^d e_k \otimes A_k$ with $\|x\|_{\ell_1^d \otimes_{min} S_\infty^m} \leq 1$. Replacing A_k by A_k^T doesn't change the norm of x, so we can ignore it.

As a final preamble before proving Proposition 4.1, we need to recall the following decomposition theorem due to Wittstock [34]. We use the statement appearing in [28, Corollary 1.9].

Theorem 4.2. *Let A be a C^*-algebra and H be a Hilbert space. Then, for any completely bounded map $u : A \to B(H)$ there exist completely positive maps $u_k : A \to B(H)$ for $k = 1, \dots, 4$, such that $u = (u_1 - u_2) + i(u_3 - u_4)$ and*[3]

$$\max\{\|u_1 + u_2\|_{cb}, \|u_3 + u_4\|_{cb}\} \leq \|u\|_{cb}.$$

In particular, if $\|u\|_{cb} \leq 1$, $(u_1 + u_2)(\mathbb{1}_A) \leq \mathbb{1}_{B(H)}$ and $(u_3 + u_4)(\mathbb{1}_A) \leq \mathbb{1}_{B(H)}$.

Proof of Proposition 4.1. Let us first show that $\beta_{owc}(G) \leq \pi_{1,cb}(\hat{G} : S_\infty^n \to S_1^m)$. To do so, let us consider operators $A_k = E_{1,k} - E_{-1,k}$ and $B_k = F_{1,k} - F_{-1,k}$ for every k such that the E's and the F's are positive semidefinite, $\sum_{a,k} E_{a,k} = \mathbb{1}_{M_n}$ and $\sum_b F_{b,k} = \mathbb{1}_{M_m}$ for every k. Let us show that

$$\Big\| \sum_k e_k \otimes A_k \Big\|_{\ell_1^{2^c} \otimes_{min} S_\infty^n} \leq 1 \quad \text{and} \quad \|B_k\|_{S_\infty^m} \leq 1 \text{ for every } k. \qquad (4.6)$$

The second bound in Equation (4.6) is very easy from the definition of B_k and the fact that $F_{1,k}$ and $F_{-1,k}$ are positive semidefinite verifying $F_{1,k} + F_{-1,k} = \mathbb{1}_{M_m}$ for every k. In order to see the first bound in (4.6), note that

$$\Big\| \sum_k e_k \otimes A_k \Big\|_{\ell_1^{2^c} \otimes_{min} S_\infty^n} = \|\hat{A} : \ell_\infty^{2^c} \to S_\infty^n\|_{cb},$$

[3] For this part of the statement, see the last line in the proof of [28, Corollary 1.9].

where $\hat{A}$ is the linear map defined by $\hat{A}(e_k) = A_k$ for every k. Now, if we consider the linear maps $\hat{u}^{\pm} : \ell_\infty^{2^c} \to S_\infty^n$ defined by $\hat{u}^{\pm}(e_k) = E_{\pm 1,k}$, respectively, they verify that $\hat{A} = \hat{u}^+ - \hat{u}^-$, both maps $\hat{u}^+$ and $\hat{u}^-$ are completely positive[4] and $\hat{u}^+ + \hat{u}^-$ is a unital map. Then, using Stinespring's dilation theorem [23, Theorem 4.1] on the maps $\hat{u}^+$ and $\hat{u}^-$ one can check that $\hat{A}$ is indeed completely contractive. This proves the desired implication.

Let us now show that $\pi_{1,cb}(\hat{G} : S_\infty^n \to S_1^m) \le 4\beta_{owc}(G)$. According to the equations (4.4) and (4.5), given $\epsilon > 0$ there exist $d \in \mathbb{N}$, $x = \sum_{k=1}^d e_i \otimes A_k$ with $\|x\|_{\ell_1^d \otimes_{min} S_\infty^n} \le 1$ and $\|B_k\|_{S_\infty^m} \le 1$ for every k such that

$$\pi_{1,cb}(\hat{G}) \le \sum_{k=1}^d tr(G(A_k \otimes B_k)) + \epsilon.$$

Next, we construct a strategy from these elements in order to bound $\beta_{owc}(G)$. On the one hand, we write $B_k = B_k^1 + iB_k^2$, with $B_k^j \in S_\infty^{m,\mathrm{sa}}$, and $\|B_k^j\| \le 1$ for $j = 1, 2$. On the other hand, if we realize x as a completely contractive map $\hat{x} : \ell_\infty^d \to S_\infty^n$, we can apply Theorem 4.2 to obtain completely positive maps $u_i : \ell_\infty^d \to S_\infty^n$ such that $\hat{x} = (u_1 - u_2) + i(u_3 - u_4)$ and

$$\max\{\|u_1 + u_2\|_{cb}, \|u_3 + u_4\|_{cb}\} \le \|\hat{x}\|_{cb} \le 1.$$

Moreover, $(u_1+u_2)(\mathbb{1}_{\ell_\infty^d}) \le \mathbb{1}_{M_n}$ and $(u_3+u_4)(\mathbb{1}_{\ell_\infty^d}) \le \mathbb{1}_{M_n}$. Let us define $E_{1,k} = u_1(e_k)$, $E_{-1,k} = u_2(e_k)$, $\tilde{E}_{1,k} = u_3(e_k)$ and $\tilde{E}_{-1,k} = u_4(e_k)$ for every $k = 1, \cdots, d$. Note that, $\sum_{a,k} E_{a,k} \le \mathbb{1}_{M_n}$ and $\sum_{a,k} \tilde{E}_{a,k} \le \mathbb{1}_{M_n}$. In order to sum up to one, we artificially define $E_{1,0} = \mathbb{1}_{M_n} - \sum_{a,k} E_{a,k}$, $E_{-1,0} = 0$, $\tilde{E}_{1,0} = \mathbb{1}_{M_n} - \sum_{a,k} \tilde{E}_{a,k}$, $\tilde{E}_{-1,0} = 0$. Then, if we set $C_k = E_{1,k} - E_{-1,k}$ and $\tilde{C}_k = \tilde{E}_{1,k} - \tilde{E}_{-1,k}$ for $k = 0, 1, \cdots, d$, we obtain a couple of families $\{C_k\}_k$ and $\{\tilde{C}_k\}_k$ as in the Equation (4.3). Notice that, by construction, $A_k = C_k + i\tilde{C}_k$ for $k = 1, \cdots, d$.

Hence, we can write

$$\begin{aligned}
\Big|\sum_{k=1}^d tr(G(A_k \otimes B_k))\Big| &\le 2 \sup\left\{\Big|\sum_{k=1}^d tr(G(A_k \otimes D_k))\Big| : D_k \in B_{S_\infty^{m,\mathrm{sa}}}\right\} \\
&\le 2 \sup\left\{\Big|\sum_{k=0}^d tr(G(C_k \otimes D_k))\Big| : D_k \in B_{S_\infty^{m,\mathrm{sa}}}\right\} \\
&\quad + 2 \sup\left\{\Big|\sum_{k=0}^d tr(G(\tilde{C}_k \otimes D_k))\Big| : D_k \in B_{S_\infty^{m,\mathrm{sa}}}\right\} \\
&\le 4\beta_{owc}(G).
\end{aligned}$$

[4] Since $A = \ell_\infty^{2^c}$ is a commutative C*-algebra, positive maps are automatically completely positive maps.

With this, we have proved that $\pi_{1,cb}(\hat{G}) \leq 4\beta_{owc}(G) + \epsilon$ for every $\epsilon > 0$, from where we immediately conclude that $\pi_{1,cb}(\hat{G}) \leq 4\beta_{owc}(G)$, as we wanted. $\square$

Proposition 4.1 complements the clean connection between the different values of quantum XOR games and certain norms on the corresponding linear maps associated to these games. This connection, implicitly initiated in [31] (see also [18], where the 1-summing norm was used to study classical XOR games with communication and [5], where the authors analyzed some properties of rank-one quantum games by studying some tensor norms on $S_1^n \otimes S_1^m$), allows us to reformulate the chain of inequalities

$$\beta(G) \leq \left\{ \begin{array}{c} \beta^*(G) \\ \beta_{owc}(G) \end{array} \right\} \leq \beta_{owq}(G),$$

which is trivial from a physical point of view, as

$$\|\hat{G} : S_\infty^n \to S_1^m\| \leq \left\{ \begin{array}{c} \|\hat{G} : S_\infty^n \to S_1^m\|_{cb} \\ \pi_{1,cb}(\hat{G} : S_\infty^n \to S_1^m) \end{array} \right\} \leq \pi_1^o(\hat{G} : S_\infty^n \to S_1^m).$$

This establishes a clear hierarchy on the relative power of different resources when playing quantum XOR games. However, this hierarchy does not say anything about the comparison between players sharing entanglement (but no communication) and players with one-way classical communication (but no entanglement). That is, the comparison between the norms $\|\cdot\|_{cb}$ and $\pi_{1,cb}(\cdot)$.

As a first approach to understand the previous relation, we can restrict to operators acting on the diagonals of S_∞^n and S_1^m; that is, $\hat{G} : \ell_\infty^n \to \ell_1^m$ (or equivalently $G \in \ell_1^n \otimes \ell_1^m$). We have[5]

$$\pi_{1,cb}(\hat{G} : \ell_\infty^n \to \ell_1^m) = \pi_1^o(\hat{G} : \ell_\infty^n \to \ell_1^m). \tag{4.7}$$

Read in the context of quantum XOR games, the previous equation says that one-way classical communication allows the players to achieve the same bias as if they were performing a global measurement. This observation easily implies that for these kinds of maps

$$\|\hat{G} : \ell_\infty^n \to \ell_1^m\|_{cb} \leq \pi_{1,cb}(\hat{G} : \ell_\infty^n \to \ell_1^m). \tag{4.8}$$

Moreover, there exist maps for which

$$\frac{\pi_{1,cb}(\hat{G} : \ell_\infty^n \to \ell_1^m)}{\|\hat{G} : \ell_\infty^n \to \ell_1^m\|_{cb}} \geq C\sqrt{\min\{n, m\}} \tag{4.9}$$

[5] It is well-known that for these kinds of maps $\pi_1(\hat{G}) = \pi_{1,cb}(\hat{G}) = \pi_1^o(\hat{G})$. That is, the three notions of 1-summing maps coincide.

for a universal constant C.

This last inequality is not surprising once we know that the classical Grothendieck's Theorem implies

$$\|\hat{G}:\ell_\infty^n \to \ell_1^m\|_{cb} \le K_G \|\hat{G}:\ell_\infty^n \to \ell_1^m\|. \tag{4.10}$$

Hence, Equation (4.9) follows from the well known estimate $\|id : \ell_1^n \otimes_\epsilon \ell_1^m \to \ell_1^{nm}\| \geq C\sqrt{\min\{n,m\}}$.

In fact, restricting to real tensors $G \in \ell_1^n \otimes \ell_1^m$ (that is, selfadjoint operators) corresponds to considering classical XOR games [22]. In this sense, the previous comments are not new at all. Equation (4.8) means that for classical XOR games strategies using classical communication are always better than entangled strategies and, in some cases, can actually be much better, cf. Equation (4.9). Moreover, Equation (4.10) tells us that for classical XOR games entangled strategies are very limited (in fact comparable to product strategies), something we already mentioned in the introduction.

One could wonder if something similar happens for general quantum XOR games or, on the contrary, in the setting of quantum XOR games one can find examples for which quantum entanglement is much more useful than classical information. Note that Equation (4.9) immediately implies the existence of maps $\hat{G} : S_\infty^n \to S_1^m$ for which $\pi_{1,cb}(\hat{G})/\|\hat{G}\|_{cb} \geq C\sqrt{\min\{n,m\}}$. However, Equation (4.7) does not extend from ℓ_1 to S_1 and, therefore, Equation (4.8) might not hold in this more general case. In fact, such an extension of Equation (4.7) is manifestly false. A very simple counterexample is provided by the transpose map $\tau : S_\infty^n \to S_1^n$, for which $\pi_1^o(\tau)/\pi_{1,cb}(\tau) = n$. Indeed, it is very easy to see that $\pi_1^o(\tau) = n^2$ while $\pi_{1,cb}(\tau) = \|\tau\| = n$. Furthermore, the equality $\pi_{1,cb}(\tau) = \|\tau\|$ can be reinterpreted in terms of quantum XOR games as an example for which classical one-way communication does not provide any advantage at all over product strategies. Together with the result that there exist quantum XOR games for which entangled strategies attain a bias unboundedly larger than the one achieved by product strategies [31, Theorem 1.2], this points out to the possibility that games G for which $\|\hat{G}\|_{cb}/\pi_{1,cb}(\hat{G})$ is arbitrarily large might exist. Contrary to this intuition, Theorem 3.1 applied to $X = S_\infty^n$ and $A = S_\infty^m$ implies that this is not the case.

Corollary 4.3. *There exists a universal constant C such that for every quantum XOR game G*

$$\beta^*(G) \le C\beta_{owc}(G).$$

Proof. According to Equation (4.2) and Proposition 4.1,

$$\frac{\beta^*(G)}{\beta_{owc}(G)} \le 4\frac{\|\hat{G}:S_\infty^n \to S_1^m\|_{cb}}{\pi_{1,cb}(\hat{G}:S_\infty^n \to S_1^m)} \le 4K,$$

where K is the constant appearing in Theorem 3.1. □

Let us mention here that we do not know if C can be taken equal to one in Corollary 4.3. Hence, it could still happen that quantum entanglement is strictly better than sending classical information in some instances.

To finish we make a comment about strategies that mix entanglement and one-way classical communication. From the quantum information point of view, it is well known that the access to both entanglement and one-way classical communication allows Alice to send one-way quantum communication to Bob (thanks to the quantum teleportation protocol [1]). So we recover the value $\beta_{owq}(G)$. From the mathematical point of view, this argument can be understood by showing that the corresponding bias of the game coincides, up to a constant, with the norm

$$\|\hat{G} : \ell_1 \otimes_{min} S_\infty^n \to \ell_1(S_1^m)\|_{cb}.$$

As we explained in the comments right below Theorem 3.1, this norm equals $\pi_1^o(\hat{G}) = \|\hat{G}\|_{S_1^{nm}}$.

Data availability

No data was used for the research described in the article.

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