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Self-similarity in homogeneous stationary and evolution problems

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Abstract. We analyse self-similarity properties related to linear elliptic and evolutionary problems involving homogeneous operators in several spaces including measures. We employ these techniques to analyse in particular $2m$th-order diffusion equations and the associated fractional problems.

1. Introduction

In this paper, motivated by the study of $2m$-order parabolic equations ($m \in \mathbb{N}$) and fractional diffusion problems, we consider general evolution problems defined by homogeneous operators and address general properties related to self-similarity as we now describe.

Homogenous operators are those that interact in a special form with the dilations of functions in $\mathbb{R}^N$, defined by

$$\phi_R(x) = \phi(Rx), \quad x \in \mathbb{R}^N, \qquad R > 0.$$

For example, a differential operator L that involves only derivatives of order $m \in \mathbb{N}$,

$$L = \sum_{|\beta|=m} a_\beta \partial^\beta, \quad \text{with constant coefficients } \ a_\beta \in \mathbb{C},$$

satisfies for a sufficiently smooth function ϕ,

$$L(\phi_R)(x) = \sum_{|\beta|=m} a_\beta \partial^\beta (\phi_R)(x) = \sum_{|\beta|=m} R^m a_\beta \partial^\beta \phi(Rx) = R^m L\phi(Rx), \quad x \in \mathbb{R}^N.$$

In general, an operator L with domain $D(L)$ in a Banach space X of functions or distributions which is invariant by rescaling is homogeneous of degree $\sigma \in \mathbb{R}$ if

$$L\big(\phi_R\big) = R^\sigma \left(L\phi\right)_R, \quad \phi \in D(L), \ R > 0.$$

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The semigroup of solutions to

$$u_t + Lu = 0, \qquad x \in \mathbb{R}^N, \ t > 0, \tag{1.1}$$

$\{S(t)\}_{t>0}$, turns out to be homogenous of degree σ in the sense that for $t > 0$, $R > 0$ and $\phi \in X$

$$S(t)\phi_R = \big(S(R^\sigma t)\phi\big)_R \qquad \text{in } \mathbb{R}^N.$$

It was proved in [7,8] that if the space X is moreover homogeneous of degree $\nu \in \mathbb{R}$, that is, its norm satisfies

$$\|\phi_R\|_X = R^\nu \|\phi\|_X, \quad \phi \in X, \ R > 0, \tag{1.2}$$

then both semigroups and resolvent operators must satisfy some sharp estimates derived only from homogeneity.

In this paper, we further analyse these types of semigroups and resolvent operators along the following lines. In Sect. 2 we show that under quite general conditions, $2m$-order parabolic equations

$$u_t + \sum_{|\alpha|\le 2m} a_\alpha(x)(-i)^{|\alpha|}\partial^\alpha u = 0, \ t > 0, \ x \in \mathbb{R}^N, \qquad u(0,x) = \phi(x), \ x \in \mathbb{R}^N \tag{1.3}$$

have an associated kernel with suitable Gaussian bounds, that is,

$$|k(t,x,y)| \le c\frac{e^{\mathrm{at}}}{t^{\frac{N}{2m}}} \exp\left(\frac{-|x-y|^{\frac{2m}{2m-1}}}{4t^{\frac{1}{2m-1}}}\right), \quad t > 0, \ x, y \in \mathbb{R}^N$$

that allows to represent solutions as

$$S(t)\phi(x) = \int_{\mathbb{R}^N} k(t,x,y)\phi(y)\,\mathrm{d}y, \qquad \phi \in X, \quad t > 0, \ x \in \mathbb{R}^N$$

in large spaces of initial data X which include Radon uniform measures, that is $\phi = \mu$ such that

$$\|\mu\|_{\mathcal{M}_U(\mathbb{R}^N)} = \sup_{x_0 \in \mathbb{R}^{\mathbb{N}}} |\mu|(B(x_0,1)) < \infty, \tag{1.4}$$

see Sect. 3.

Then, in Sect. 4, we prove that from the kernel for $S(t)$, the resolvent operator $\mathcal{R}(\lambda, L)$ inherits a Green's function so that

$$\mathcal{R}(\lambda, L)\phi(x) = (L - \lambda I)^{-1}\phi(x) = \int_{\mathbb{R}^N} G_\lambda(x,y)\phi(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^N,$$

see Proposition 4.1. If moreover the semigroup commutes with spatial translations, then the representation by the kernel is actually a convolution, that is $k(t,x,y) =$

$K(t, x-y)$ and the same is true for the Green function, $G_\lambda(x-y)$, see Proposition 4.3. This holds for parabolic problems as in (1.3) with constant coefficients. On the other hand, for a homogeneous semigroup of degree $\sigma \neq 0$, we prove the kernel is self-similar in the sense that

$$k(t, x, y) = \frac{1}{t^{\frac{N}{\sigma}}} k_0\left(\frac{x}{t^{\frac{1}{\sigma}}}, \frac{y}{t^{\frac{1}{\sigma}}}\right), \quad t > 0,\ x, y \in \mathbb{R}^N,$$

and the Green's function is also self-similar, see Proposition 4.5. With these pieces of information, similar results are then proved for the associated fractional semigroups and operators, see Sect. 4.4, where moreover we obtain bounds on the fractional kernel and the fractional Green's function from the semigroup kernel alone.

In Sect. 5, we study self-similar solutions for (1.1) and prove the existence of suitable naturally associated self-similar variables, which provide an alternative form of describing the semigroup. First, in Theorem 5.2 we prove that homogeneous distributions in X of degree $-\beta$ give rise to β-self-similar solutions of (1.1) and characterise all possible β-self-similar solutions, which must have the form

$$u(t, x) = \frac{1}{t^{\frac{\beta}{\sigma}}} \Phi\left(\frac{x}{t^{\frac{1}{\sigma}}}\right), \quad t > 0,\ x \in \mathbb{R}^N,$$

where the self-similar profile Φ satisfies a suitable *elliptic* problem. Indeed, we show that β must be an eigenvalue of this *elliptic* operator and the profile Φ must be an associated eigenfunction. The existence of self-similar variables is then shown in Theorem 5.6 where we show that, in self-similar variables, the semigroup is equivalent to an Ornstein–Uhlenbeck-type semigroup.

Finally, in Sect. 6, we employ the previous general tools to analyse the heat as well as higher-order diffusion equations and their corresponding fractional problems. We also include some discussion on the heat equation with a Hardy potential.

The final Appendix contains a discussion involving dilations and homogeneous distributions, which play an important role in the results of Sect. 5. In particular we characterise the homogenous elements of several function spaces appearing in the paper.

2. Elliptic operators and semigroups in uniform spaces

In this section, we show that under quite general conditions a linear $2m$-order parabolic equations (1.3) have an associated kernel with Gaussian bounds, see Theorem 2.2.

We denote by ∂_j the partial derivative with respect to the variable x_j, where $j = 1, \ldots, N$ and $x = (x_1, \ldots, x_N)$ is a point in $\mathbb{R}^N$. Following [1, p. 635] we let $D_j := -i\partial_j$, and given a multi-index $\alpha \in \mathbb{N}^N$, we also set $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$.

Using this notation, we consider a $2m$th-order operator of the form

$$A = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha \tag{2.1}$$

with higher-order coefficients $a_\alpha \in BUC^\mu(\mathbb{R}^N)$ with some $\mu \in (0,1)$, for $|\alpha| = 2m$, and lower-order coefficients $a_\alpha \in L^\infty(\mathbb{R}^N)$ when $|\alpha| < 2m$.

Definition 2.1. The differential operator A in (2.1) is $(M, \theta_0)-$uniformly elliptic if the principal symbol $A_0(x,\xi) = \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha$, $x, \xi \in \mathbb{R}^N$ satisfies

$$\inf_{x,\xi\in\mathbb{R}^N, |\xi|=1} \mathrm{Re}(A_0(x,\xi)) \geq \frac{1}{M}, \qquad \theta_0 \geq \sup_{x\in\mathbb{R}^N, |\xi|=1} |\mathrm{Arg}(A_0(x,\xi))|,$$

with $\theta_0 < \frac{\pi}{2}$ and $\max_{|\alpha|<2m} \|a_\alpha\|_{L^\infty(\mathbb{R}^N)} \leq M$.

Consider the locally uniform spaces $L^p_U(\mathbb{R}^N)$, $1 \leq p \leq \infty$, composed of $\phi \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for which

$$\|\phi\|_{L^p_U(\mathbb{R}^N)} = \sup_{x_0\in\mathbb{R}^N} \|\phi\|_{L^p(B(x_0,1))} < \infty.$$

In particular, $L^\infty_U(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N) \subset L^p_U(\mathbb{R}^N) \subset L^1_U(\mathbb{R}^N)$ for $1 < p < \infty$. By considering the translations $\tau_y\phi(x) = \phi(x-y)$, $x \in \mathbb{R}^N$, the dotted uniform spaces, $\dot{L}^p_U(\mathbb{R}^N)$, are defined as the subspace of $L^p_U(\mathbb{R}^N)$ such that $\tau_y\phi - \phi \to 0$ as $y \to 0$ in the norm of $L^p_U(\mathbb{R}^N)$. In particular, $\dot{L}^\infty(\mathbb{R}^N) = \mathrm{BUC}(\mathbb{R}^N)$. Sobolev type spaces of functions with derivatives in uniform spaces are denoted $W^{k,p}_U(\mathbb{R}^N)$ or $\dot{W}^{k,p}_U(\mathbb{R}^N)$, respectively.

It was proved in [6] that if A is as above, with nondense domain $W^{2m,p}_U(\mathbb{R}^N)$ in $L^p_U(\mathbb{R}^N)$, $1 < p < \infty$, then (1.3) defines an analytic (but not strongly continuous) semigroup of solutions $\{S(t)\}_{t\geq 0}$. The solutions are smooth for $t > 0$ and enter $W^{2m,q}_U(\mathbb{R}^N)$ for any $p \leq q < \infty$. Also the semigroup extends to initial data in $L^1_U(\mathbb{R}^N)$. Furthermore, if the lower-order terms satisfy $a_\alpha \in \mathrm{BUC}(\mathbb{R}^N)$, $|\alpha| < 2m$, then the semigroup is analytic and strongly continuous in $\dot{L}^p_U(\mathbb{R}^N)$.

Then, we have the following result.

Theorem 2.2. *The semigroup $\{S(t)\}_{t\geq 0}$ above has a kernel function such that for $u_0 \in L^1_U(\mathbb{R}^N)$*

$$S(t)u_0(x) = \int_{\mathbb{R}^N} k(t,x,y)u_0(y)\,\mathrm{d}y \quad x \in \mathbb{R}^N,\ t > 0 \tag{2.2}$$

and satisfies, for certain constants $a \geq 0$, $b, c > 0$,

$$|k(t,x,y)| \leq ce^{\mathrm{at}} G_{\mathrm{bt}}(x-y), \quad t > 0,\ x, y \in \mathbb{R}^N \tag{2.3}$$

where

$$G_t(x) := \frac{c_{mN}^{-1}}{t^{\frac{N}{2m}}} \exp\left(\frac{-|x|^{\frac{2m}{2m-1}}}{4t^{\frac{1}{2m-1}}}\right) \quad x \in \mathbb{R}^N, \quad t > 0 \tag{2.4}$$

and $c_{mN} := \|\exp\big(\frac{-|\cdot|^{\frac{2m}{2m-1}}}{4}\big)\|_{L^1(\mathbb{R}^N)}$, $\int_{\mathbb{R}^N} G_t(x)\,dx = 1$ for all $t > 0$.

Proof. From the proof of [6, Theorem 3.1], for the weight function $\rho(x) = (1 + |x|^2)^{-\nu}$, with some $\nu > \frac{N}{2}$, there exists a uniformly elliptic operator Λ that defines a semigroup $\{S_\Lambda(t)\}_{t\geq 0}$ such that $S(t) = \rho^{-1} S_\Lambda(t)\rho$. Also, $S_\Lambda(t)$ satisfies the estimates in [6, Corollary 2.5], in particular $S_\Lambda(t)$ takes $L^1(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$. Hence, [2, Theorem 1.3] ensures that $S_\Lambda(t)$ in $L^1(\mathbb{R}^N)$ has a kernel, $k_\Lambda(t, x, y)$, which in turn implies that $\rho^{-1}(x)k_\Lambda(t, x, y)\rho(y) =: k(t, x, y)$ is a kernel for $S(t)$ in $L^1_\rho(\mathbb{R}^N)$. In particular, (2.2) holds for any $u_0 \in L^1_U(\mathbb{R}^N) \subset L^1_\rho(\mathbb{R}^N)$.

On the other hand, by [13, Example 2.6 (B), pp. 151-152] (see also [6, Proposition 2.3 iv)]), the semigroup $\{S(t)\}_{t\geq 0}$ in $L^1(\mathbb{R}^N)$ satisfies $|S(t)u_0|(x) \leq ce^{\mathrm{at}}\big(G_{\mathrm{bt}} * |u_0|\big)(x)$ in $(0, \infty) \times \mathbb{R}^N$ for every $u_0 \in C^\infty_c(\mathbb{R}^N)$.

Hence, for any $t > 0$ and $x \in \mathbb{R}^N$ and all $0 \leq u_0 \in C^\infty_c(\mathbb{R}^N)$, we get

$$\int_{R^N} \big(ce^{\mathrm{at}} G_{\mathrm{bt}}(x-y) \pm k(t, x, y)\big)u_0(y)\, \mathrm{d}y \geq 0$$

which yields (2.3). □

3. Diffusion semigroups in the space of uniform measures

Consider now the space of uniform measures $\mathcal{M}_U(\mathbb{R}^N)$ as in (1.4). Clearly, $L^1_U(\mathbb{R}^N) \subset \mathcal{M}_U(\mathbb{R}^N)$ isometrically. Then, the semigroup in $L^1_U(\mathbb{R}^N)$ in Theorem 2.2 can be further extended to the space of uniform measures, by defining for $\mu \in \mathcal{M}_U(\mathbb{R}^N)$

$$S(t)\mu(x) = \int_{\mathbb{R}^N} k(t, x, y)\, \mathrm{d}\mu(y), \qquad t > 0. \tag{3.1}$$

Then, we have the following result. Observe that this result, in particular, improves the estimates obtained in [6].

Theorem 3.1. *For $\mu \in \mathcal{M}_U(\mathbb{R}^N)$, (3.1) defines a function such that for $1 \leq p \leq \infty$*

$$\|S(t)\mu\|_{L^p_U(\mathbb{R}^N)} \leq ce^{\mathrm{at}}(1 + \frac{1}{t^{\frac{N}{2m}(1-\frac{1}{p})}})\|\mu\|_{\mathcal{M}_U(\mathbb{R}^N)}, \quad t > 0 \tag{3.2}$$

and solves (1.3) for $t > 0$. In particular, $S(t)\mu$ has all the regularity obtained in [6].

Additionally, if $\mu = u_0 \in L^p_U(\mathbb{R}^N)$, $1 \leq p \leq q \leq \infty$, then

$$\|S(t)u_0\|_{L^q_U(\mathbb{R}^N)} \leq ce^{\mathrm{at}}(1 + \frac{1}{t^{\frac{N}{2m}(\frac{1}{p}-\frac{1}{q})}})\|u_0\|_{L^p_U(\mathbb{R}^N)}, \quad t > 0. \tag{3.3}$$

Proof. *Step 1* For $x \in \mathbb{R}^N$, $|S(t)\mu|(x) \leq ce^{\mathrm{at}} \int_{\mathbb{R}^N} G_{\mathrm{bt}}(x-y)\mathrm{d}|\mu(y)|$. Define

$$g^\delta_t(z) := \frac{1}{t^{\frac{1}{2m}}} \exp\left(\frac{-|z|^{\frac{2m}{2m-1}}}{\delta t^{\frac{1}{2m-1}}}\right) \quad z \in \mathbb{R}, \quad t > 0$$

and then $\int_{\mathbb{R}} g_t^\delta(z)\,\mathrm{d}z = c(\delta)$ and there exists δ such that $G_{\mathrm{bt}}(x) \le \prod_{j=1}^N g_t^\delta(x_j)$ for $x \in \mathbb{R}^N$, $t > 0$. Let now $k = (k_1, \dots, k_N) \in \mathbb{Z}^N$ and denote $Q_k = \prod_{i=1}^N [k_i - 1, k_i]$. Therefore,

$$J = \int_{\mathbb{R}^N} G_{\mathrm{bt}}(x-y)\mathrm{d}|\mu(y)| \le \sum_{k\in\mathbb{Z}^N} \int_{Q_k} \prod_{j=1}^N g_t^\delta(x_j - y_j)\mathrm{d}|\mu(y)|$$

hence

$$\begin{aligned} J &\le \sum_{k\in\mathbb{Z}^N} \sup_{y\in Q_k} \prod_{j=1}^N g_t^\delta(x_j - y_j)|\mu(Q_k)| \le c \sum_{k\in\mathbb{Z}^N} \prod_{j=1}^N \sup_{s\in[k_j-1,k_j]} g_t^\delta(x_j - s)\|\mu\|_{\mathcal{M}_U(\mathbb{R}^N)} \\ &= c\Big(\prod_{i=1}^N \sum_{k_i=-\infty}^{\infty} \sup_{s\in[k_i-1,k_i]} g_t^\delta(x_i - s)\Big)\|\mu\|_{\mathcal{M}_U(\mathbb{R}^N)}. \end{aligned}$$

Now for $z \in \mathbb{R}$ there exists at most two $k \in \mathbb{Z}$ such that $z \in [k-1, k]$. Removing these intervals and using the fact that g_t^δ is an even function, decaying and with a maximum $g_t^\delta(0) = \frac{1}{t^{\frac{1}{2m}}}$,

$$\sum_{k=-\infty}^{\infty} \sup_{s\in[k-1,k]} g_t^\delta(z-s) \le c\left(\frac{1}{t^{\frac{1}{2m}}} + \int_0^\infty g_t^\delta(z)\,\mathrm{d}z\right) \le c(\delta)\left(1 + \frac{1}{t^{\frac{1}{2m}}}\right)$$

and then $|S(t)\mu|(x) \le ce^{\mathrm{at}}(1 + \frac{1}{t^{\frac{N}{2m}}})\|\mu\|_{\mathcal{M}_U(\mathbb{R}^N)}$, that is

$$\|S(t)\mu\|_{L^\infty(\mathbb{R}^N)} \le ce^{\mathrm{at}}\left(1 + \frac{1}{t^{\frac{N}{2m}}}\right)\|\mu\|_{\mathcal{M}_U(\mathbb{R}^N)}.$$

Step 2 Using (2.3) and Fubini for every $z \in \mathbb{R}^N$,

$$\begin{aligned} J = \|S(t)\mu\|_{L^1(B(z,1))} &\le ce^{\mathrm{at}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G_{\mathrm{bt}}(x-y)\mathcal{X}_{B(z,1)}(x)\,\mathrm{d}|\mu(y)|\,\mathrm{d}x \\ &= ce^{\mathrm{at}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G_{\mathrm{bt}}(x-y)\mathcal{X}_{B(z,1)}(x)\,\mathrm{d}x\,\mathrm{d}|\mu(y)| \end{aligned}$$

and changing variables and Fubini again

$$\begin{aligned} &= ce^{\mathrm{at}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G_{\mathrm{bt}}(w)\mathcal{X}_{B(z-y,1)}(w)\,\mathrm{d}w\,\mathrm{d}|\mu(y)| \\ &= ce^{\mathrm{at}} \int_{\mathbb{R}^N} G_{\mathrm{bt}}(w) \int_{\mathbb{R}^N} \mathcal{X}_{B(z-y,1)}(w)\,\mathrm{d}|\mu(y)|\,\mathrm{d}w. \end{aligned}$$

Now for fixed $w \in \mathbb{R}^N$ we have, as a function of $y \in \mathbb{R}^N$, $\mathcal{X}_{B(z-y,1)}(w) = \mathcal{X}_{B(z-w,1)}(y)$ and

$$\begin{aligned} J &\le ce^{\mathrm{at}} \int_{\mathbb{R}^N} G_{\mathrm{bt}}(w) \int_{\mathbb{R}^N} \mathcal{X}_{B(z-w,1)}(y)\,\mathrm{d}|\mu(y)|\,\mathrm{d}w \\ &\le ce^{\mathrm{at}} \int_{\mathbb{R}^N} G_{\mathrm{bt}}(w)|\mu|(B(z-w,1))\,\mathrm{d}w \end{aligned}$$

and then $J \leq Ce^{\mathrm{at}}\|\mu\|_{\mathcal{M}_U(\mathbb{R}^N)} \int_{\mathbb{R}^N} G_{\mathrm{bt}}(w)\,\mathrm{d}w = ce^{\mathrm{at}}\|\mu\|_{\mathcal{M}_U(\mathbb{R}^N)}$, that is

$$\|S(t)\mu\|_{L^1_U(\mathbb{R}^N)} \leq ce^{\mathrm{at}}\|\mu\|_{\mathcal{M}_U(\mathbb{R}^N)}.$$

Step 3 Now, by interpolation, for any $z \in \mathbb{R}^N$ and $1 \leq p \leq \infty$ we get

$$\begin{aligned}\|S(t)\mu\|_{L^p(B(z,1))} &\leq \|S(t)\mu\|^{1-\frac{1}{p}}_{L^\infty(B(z,1))}\|S(t)\mu\|^{\frac{1}{p}}_{L^1(B(z,1))}\\ &\leq \|S(t)\mu\|^{1-\frac{1}{p}}_{L^\infty(\mathbb{R}^N)}\|S(t)\mu\|^{\frac{1}{p}}_{L^1_U(\mathbb{R}^N)}\end{aligned}$$

which leads to (3.2).

Step 4 Now we prove, for $1 \leq p < \infty$ and $u_0 \in L^p_U(\mathbb{R}^N)$,

$$\|S(t)u_0\|_{L^\infty(\mathbb{R}^N)} \leq c_p e^{\mathrm{at}}\left(1 + \frac{1}{t^{\frac{N}{2mp}}}\right)\|u_0\|_{L^p_U(\mathbb{R}^N)}, \quad t > 0. \tag{3.4}$$

In fact from (2.3) we get for each $1 \leq p < \infty$ and $x \in \mathbb{R}^N$, $t > 0$,

$$\begin{aligned}|S(t)u_0|(x) &\leq ce^{\mathrm{at}}\int_{\mathbb{R}^N} G_{\mathrm{bt}}(z)|u_0(x-z)|\mathrm{d}z\\ &= ce^{\mathrm{at}}\int_{\mathbb{R}^N}(G_{\mathrm{bt}}(z))^{1-\frac{1}{p}}(G_{\mathrm{bt}}(z))^{\frac{1}{p}}|u_0(x-z)|\,\mathrm{d}z\\ &\leq ce^{\mathrm{at}}\left(\int_{\mathbb{R}^N} G_{\mathrm{bt}}(z)\,\mathrm{d}z\right)^{1-\frac{1}{p}}(\int_{\mathbb{R}^N} G_{\mathrm{bt}}(z)|u_0(x-z)|^p\,\mathrm{d}z)^{\frac{1}{p}}\\ &= c_p e^{\mathrm{at}}\left(\int_{\mathbb{R}^N} G_{\mathrm{bt}}(z)|u_0(x-z)|^p\,\mathrm{d}z\right)^{\frac{1}{p}}.\end{aligned}$$

Now we proceed as in Step 1 in the proof of Theorem 4.3 in [6] and split the integral as the sum of integrals in the cubes $Q_k = \prod_{i=1}^N [k_i - 1, k_i]$ and use properties of the function $G_{\mathrm{bt}}(z)$ as in Step 1 above to get

$$|S(t)u_0|(x) \leq c^0_p e^{\mathrm{at}}\left(1 + \frac{1}{t^{\frac{N}{2mp}}}\right)\|u\|_{L^p_U(\mathbb{R}^N)}, \quad x \in \mathbb{R}^N, \quad t > 0$$

which proves (3.4).

Step 5 Now we show that for each $1 \leq p < \infty$ and $u_0 \in L^p_U(\mathbb{R}^N)$ we have

$$\|S(t)u_0\|_{L^p_U(\mathbb{R}^N)} \leq c^1_p e^{\mathrm{at}}\|u_0\|_{L^p_U(\mathbb{R}^N)} \quad t > 0. \tag{3.5}$$

Indeed, again the Gaussian bounds give for $y \in \mathbb{R}^N$, $t > 0$,

$$\begin{aligned}\|S(t)u_0\|_{L^p(B(y,1))} &\le ce^{\text{at}}\left(\int_{B(y,1)}\left(\int_{\mathbb{R}^N} G_{\text{bt}}(z)|u_0(x-z)|\mathrm{d}z\right)^p \mathrm{d}x\right)^{\frac{1}{p}}\\ &\le ce^{\text{at}}\int_{\mathbb{R}^N} G_{\text{bt}}(z)\left(\int_{B(y,1)} |u_0(x-z)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\mathrm{d}z\\ &= ce^{\text{at}}\int_{\mathbb{R}^N} G_{\text{bt}}(z)\left(\int_{B(y-z,1)} |u_0(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\mathrm{d}z\\ &\le ce^{\text{at}}\left(\int_{\mathbb{R}^N} G_{\text{bt}}(z)\,\mathrm{d}z\right)\|u_0\|_{L^p_U(\mathbb{R}^N)} = c_p^1 e^{\text{at}}\|u_0\|_{L^p_U(\mathbb{R}^N)}\end{aligned}$$

where we have used the generalised Minkowski's inequality (see [20, §18.1, formula (4)]). This proves (3.5).

Step 6 Now, by interpolation, for any $y \in \mathbb{R}^N$ and $1 \le p \le q \le \infty$

$$\begin{aligned}\|S(t)u_0\|_{L^q(B(y,1))} &\le \|S(t)u_0\|^{1-\frac{p}{q}}_{L^\infty(B(y,1))}\|S(t)u_0\|^{\frac{p}{q}}_{L^p(B(y,1))}\\ &\le \|S(t)u_0\|^{1-\frac{p}{q}}_{L^\infty(\mathbb{R}^N)}\|S(t)u_0\|^{\frac{p}{q}}_{L^p_U(\mathbb{R}^N)}\end{aligned}$$

and using (3.4) and (3.5) we get (3.3). □

Restricting to Lebesgue spaces and to the space of measures of bounded total variation, $\mathcal{M}_{\text{BTV}}(\mathbb{R}^N)$ that is, satisfying $\|\mu\|_{\text{BTV}} := |\mu|(\mathbb{R}^N) < \infty$, we get the following result.

Proposition 3.2. *For $\mu \in \mathcal{M}_{\text{BTV}}(\mathbb{R}^N)$, (3.1) defines a function such that for $1 \le p \le \infty$*

$$\|S(t)\mu\|_{L^p(\mathbb{R}^N)} \le \frac{ce^{at}}{t^{\frac{N}{2m}(1-\frac{1}{p})}}\|\mu\|_{\mathcal{M}_{\text{BTV}}(\mathbb{R}^N)}, \quad t > 0. \tag{3.6}$$

Additionally, if $\mu = u_0 \in L^p(\mathbb{R}^N)$, $1 \le p \le q \le \infty$, then

$$\|S(t)u_0\|_{L^q(\mathbb{R}^N)} \le \frac{ce^{at}}{t^{\frac{N}{2m}(\frac{1}{p}-\frac{1}{q})}}\|u_0\|_{L^p(\mathbb{R}^N)}, \quad t > 0.$$

Proof. For $x \in \mathbb{R}^N$, $|S(t)\mu|(x) \le ce^{\text{at}}\int_{\mathbb{R}^N} G_{\text{bt}}(x-y)\mathrm{d}|\mu(y)|$ and then, using Fubini,

$$\|S(t)\mu\|_{L^1(\mathbb{R}^N)} \le ce^{\text{at}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G_{\text{bt}}(x-y)\mathrm{d}|\mu(y)|\,\mathrm{d}x = ce^{\text{at}}\|\mu\|_{\mathcal{M}_{\text{BTV}}(\mathbb{R}^N)}.$$

Also, $\|S(t)\mu\|_{L^\infty(\mathbb{R}^N)} \le \frac{ce^{\text{at}}}{t^{\frac{N}{2m}}}\|\mu\|_{\mathcal{M}_{\text{BTV}}(\mathbb{R}^N)}$ and then (3.6) follows by interpolation.

On the other hand, from properties of convolution, we have for $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$ that $\|S(t)\mu\|_{L^q(\mathbb{R}^N)} \le ce^{\text{at}}\|G_{\text{bt}}\|_{L^r(\mathbb{R}^N)}\|u_0\|_{L^p(\mathbb{R}^N)} = \frac{ce^{\text{at}}}{t^{\frac{N}{2m}(1-\frac{1}{r})}}\|u_0\|_{L^p(\mathbb{R}^N)}$
$= \frac{ce^{\text{at}}}{t^{\frac{N}{2m}(\frac{1}{p}-\frac{1}{q})}}\|u_0\|_{L^p(\mathbb{R}^N)}$. □

Now, we show the semigroup in Theorem 3.1 attains the initial data in the sense of measures, if the kernel is symmetric. This will hold in the example in Sect. 6.

Proposition 3.3. *Assume that $k(t,x,y)$ in (3.1) is symmetric in $x, y \in \mathbb{R}^N$.*

Then, the semigroup $\{S(t)\}_{t\geq 0}$ in $\mathcal{M}_U(\mathbb{R}^N)$ defined in Theorem 3.1 is continuous at $t = 0^+$ in the sense that given any $\mu \in \mathcal{M}_U(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} \phi S(t)\mu \, \mathrm{d}y \to \int_{\mathbb{R}^N} \phi \, d\mu \ \text{ as } \ t \to 0^+ \ \text{ for each } \ \phi \in C_c(\mathbb{R}^N). \tag{3.7}$$

Proof. From (3.1), we have

$$\int_{\mathbb{R}^N} \phi(x) S(t)\mu(x)\,\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(x)k(t,x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}x, \quad t>0, \ \phi \in C_c(\mathbb{R}^N).$$

From the bound (2.3), Step 1 in the proof of Theorem 3.1 and Lemma 3.4 we can change the order of integration and using the symmetry of the kernel we get

$$\int_{\mathbb{R}^N} \phi(x) S(t)\mu(x)\,\mathrm{d}x = \int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N} k(t,y,x)\phi(x)\,\mathrm{d}x\Big)\,\mathrm{d}\mu(y) = \int_{\mathbb{R}^N} S(t)\phi(y)\,\mathrm{d}\mu(y).$$

Now, since $\phi \in C_c(\mathbb{R}^N)$ we have $S(t)\phi \to \phi$ uniformly in $\mathbb{R}^N$, see [6, (4.6)], and using Lemma 3.4 and Lebesgue's dominated convergence theorem, we get the result. □

Now we prove the lemma used above.

Lemma 3.4. *Given $\phi \in C_c(\mathbb{R}^N)$ and $T > 0$ there are constants $C, \gamma > 0$ such that*

$$|S(t)\phi(x)| \leq Ce^{-\gamma |x|^{\frac{2m}{2m-1}}} \tag{3.8}$$

for all $x \in \mathbb{R}^N$, $0 \leq t \leq T$.

Proof. If $1 < p \leq 2$, $x, y \in \mathbb{R}^N$ are fixed and $x \neq 0$, then the function

$$\ell(\delta) = (1-\delta^{p-1})|x|^p - (\delta^{1-p}-1)|y|^p \text{ for } \delta > 0,$$

attains at $\delta = \left(\frac{|y|}{|x|}\right)^{\frac{p}{2(p-1)}}$ its maximum $\ell_{\max} = \big||x|^{\frac{p}{2}} - |y|^{\frac{p}{2}}\big|^2 \leq |x-y|^p$ and letting $p = \frac{2m}{2m-1}$, $m \in \mathbb{N}$, we get

$$(1-\delta^{\frac{1}{2m-1}})|x|^{\frac{2m}{2m-1}} - (\delta^{-\frac{1}{2m-1}}-1)|y|^{\frac{2m}{2m-1}} \leq |x-y|^{\frac{2m}{2m-1}}, \ x,y \in \mathbb{R}^N, \ \delta > 0.$$

We next consider a ball B_R of radius R around zero such that supp$\phi \subset B_R$ and using the inequality above with arbitrarily fixed $0 < \delta < 1$ we obtain

$$\begin{aligned}
|S(t)\phi|(x) &\leq ce^{\mathrm{at}}\Big(G_{\mathrm{bt}} * |\phi|\Big)(x) = ce^{\mathrm{at}} \int_{\mathbb{R}^N} \frac{c_{mN}^{-1}}{t^{\frac{N}{2m}}} \exp\left(\frac{-|x-y|^{\frac{2m}{2m-1}}}{4t^{\frac{1}{2m-1}}}\right) |\phi|(y)\mathrm{d}y \\
&\leq ce^{\mathrm{at}} \int_{B_R} \frac{c_{mN}^{-1}}{t^{\frac{N}{2m}}} \exp\left(\frac{-(1-\delta^{\frac{1}{2m-1}})|x|^{\frac{2m}{2m-1}} + (\delta^{-\frac{1}{2m-1}}-1)|y|^{\frac{2m}{2m-1}}}{4t^{\frac{1}{2m-1}}}\right) |\phi|(y)\mathrm{d}y \\
&\leq \frac{\tilde{c}}{t^{\frac{N}{2m}}} e^{\mathrm{at}} e^{-\frac{1}{4}t^{-\frac{1}{2m-1}}(1-\delta^{\frac{1}{2m-1}})\left(|x|^{\frac{2m}{2m-1}} - \delta^{-\frac{1}{2m-1}} R^{\frac{2m}{2m-1}}\right)}, \quad x \in \mathbb{R}^N, \ t > 0,
\end{aligned}$$

where $\tilde{c} = cc_{mN}^{-1}\|\phi\|_{L^1(B_R)}$. Since for $|x| \ge 2^{\frac{2m-1}{2m}} R\delta^{-\frac{1}{m}}$ we have

$$|x|^{\frac{2m}{2m-1}} - \delta^{-\frac{1}{2m-1}} R^{\frac{2m}{2m-1}} \ge (1-\delta^{\frac{1}{2m-1}})|x|^{\frac{2m}{2m-1}} + \delta^{-\frac{1}{2m-1}} R^{\frac{2m}{2m-1}},$$

letting

$$M(t) = \tilde{c} t^{-\frac{N}{2m}} e^{\mathrm{at}} e^{-\frac{1}{4}t^{-\frac{1}{2m-1}}(\delta^{-\frac{1}{2m-1}}-1)R^{\frac{2m}{2m-1}}}$$

we get

$$|S(t)\phi(x)| \le M(t) e^{-\frac{1}{4}t^{-\frac{1}{2m-1}}(1-\delta^{\frac{1}{2m-1}})^2|x|^{\frac{2m}{2m-1}}}, \quad |x| \ge 2^{\frac{2m-1}{2m}} R\delta^{-\frac{1}{m}},\ t > 0,$$

and we get the estimate (3.8) with $C = \sup_{t\in(0,T]} M(t)$ and $\gamma = \frac{1}{4}T^{-\frac{1}{2m-1}}(1-\delta^{\frac{1}{2m-1}})^2$.

For $|x| \le 2^{\frac{2m-1}{2m}} R\delta^{-\frac{1}{m}}$ from (3.3) with $p = q = \infty$, we get

$$|S(t)\phi(x)| \le 2ce^{\mathrm{at}}\|\phi\|_{L^\infty(B_R)}, \quad t > 0,\ |x| \le 2^{\frac{2m-1}{2m}} R\delta^{-\frac{1}{m}},$$

and we get (3.8) for any $\gamma > 0$ with $C = 2ce^{\mathrm{at}}\|\phi\|_{L^\infty(B_R)} e^{2\gamma R^{\frac{2m}{2m-1}}\delta^{-\frac{2}{2m-1}}}$. □

4. Kernels, Green functions and self-similarity

We now analyse semigroup kernels and Green functions.

4.1. Semigroup kernels and Green functions

Motivated by (2.2) and the results in Sect. 3, in this section we consider semigroups $\{S(t)\}_{t\ge0}$ defined in a linear space of functions in $\mathbb{R}^N$, X, which have a kernel $k(t,x,y)$, that is, a function defined in $(0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$ such that

$$S(t)\phi(x) = \int_{\mathbb{R}^N} k(t,x,y)\phi(y)\,\mathrm{d}y \quad \text{for} \quad \phi \in X, \quad x \in \mathbb{R}^N. \tag{4.1}$$

Observe that if $\mathscr{D}(\mathbb{R}^N) \subset X$, then the kernel is uniquely determined. Also from the results in [15,21] we can assume

$$\|S(t)\|_{\mathcal{L}(X)} \le Me^{\omega t}, \quad t \ge 0$$

for some $M > 0$ and $\omega \in \mathbb{R}$ and the semigroup curves solve

$$u_t + Lu = 0, \qquad t > 0, \quad u(0) \in X$$

where $-L$ is the generator of $\{S(t)\}_{t\ge0}$, and has a domain $D(L) \subset X$.

Proposition 4.1. *Under the assumptions above,* $\{\lambda,\ \mathrm{Re}(\lambda) < -\omega\} \subset \rho(L)$ *and if*

$$\mathrm{Re}(\lambda) < -\omega,$$

the resolvent operator has a Green function such that for $\phi \in \overline{D(L)}$

$$\mathcal{R}(\lambda, L)\phi(x) = (L - \lambda I)^{-1}\phi(x) = \int_{\mathbb{R}^N} G_\lambda(x, y)\phi(y)\, dy, \qquad x \in \mathbb{R}^N,$$

given by

$$G_\lambda(x, y) = \int_0^\infty e^{\lambda t} k(t, x, y)\, dt, \quad x, y \in \mathbb{R}^N. \tag{4.2}$$

Proof. For $\mathrm{Re}(\lambda) < -\omega$, we know, see, e.g. [9, Theorem 1.10, Chapter II], that $\mathcal{R}(\lambda, L) = (L - \lambda)^{-1} = \int_0^\infty e^{\lambda t} S(t)\, \mathrm{d}t$ on $\overline{D(L)}$ which implies that for all $\phi \in \overline{D(L)}$ we have $\mathcal{R}(\lambda, L)\phi(x) = \int_{\mathbb{R}^N} \big(\int_0^\infty e^{\lambda t} k(t, x, y)\, \mathrm{d}t\big)\phi(y)\, \mathrm{d}y$, $x \in \mathbb{R}^N$. □

In view of Sect. 2, we assume now that

$$|k(t, x, y)| \leq \frac{e^{\mathrm{at}}}{t^{\frac{N}{\sigma}}} g\left(\frac{x - y}{t^{\frac{1}{\sigma}}}\right), \quad t > 0,\ x, y \in \mathbb{R}^N, \qquad \sigma > 1 \tag{4.3}$$

where $g(z) = ce^{-b|z|^{\frac{\sigma}{\sigma-1}}}$ for some constants $b, c > 0$. Observe that for the differential operators in Sect. 2 with $L = A = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$, from (2.3), the bound above holds with $\sigma = 2m$. Then, we get the following result.

Corollary 4.2. *Assuming the bound above, we have the following estimate for the Green's function for* $\mathrm{Re}(\lambda) < -a$*,*

$$|G_\lambda(x, y)| \quad \textit{decays exponentially as} \quad |x - y| \to \infty$$

and

$$|G_\lambda(x, y)| \begin{cases} \textit{bounded} & \textit{if } N < \sigma \\ \leq C|\ln(|x - y|)| & \textit{if } N = \sigma \\ \leq \frac{C}{|x-y|^{N-\sigma}} & \textit{if } N > \sigma \end{cases} \quad \textit{as} \quad |x - y| \to 0.$$

Proof. For $\mathrm{Re}(\lambda) < -a$

$$|G_\lambda(x, y)| \leq \int_0^\infty e^{(\mathrm{Re}(\lambda)+a)t} |k(t, x, y)|\, \mathrm{d}t \leq c \int_0^\infty \frac{e^{(\mathrm{Re}(\lambda)+a)t}}{t^{\frac{N}{\sigma}}} g\left(\frac{|x - y|}{t^{\frac{1}{\sigma}}}\right) \mathrm{d}t$$
$$= H_\lambda(x - y)$$

and splitting the integral according to whether $\frac{|x-y|}{t^{\frac{1}{\sigma}}}$ is larger or smaller than 1, we get

$$H_\lambda(x - y) \leq \frac{C}{|x - y|^m} \int_0^{|x-y|^\sigma} t^{\frac{m-N}{\sigma}} e^{(\mathrm{Re}(\lambda)+a)t}\, \mathrm{d}t + C \int_{|x-y|^\sigma}^\infty t^{-\frac{N}{\sigma}} e^{(\mathrm{Re}(\lambda)+a)t}\, \mathrm{d}t$$

where we used the fact that $g(z) \le cz^{-m}$ for any $m \ge N$ and $|z| \ge 1$ and g is bounded for $|z| \le 1$.

Then, as $|x-y| \to 0$, taking $m = N$ the first term is of order $\frac{C}{|x-y|^{N-\sigma}}$ while for the second: (a) If $\frac{N}{\sigma} < 1$, it is bounded. (b) If $\frac{N}{\sigma} = 1$, it is of order $C|\ln(|x-y|)|$. (c) If $\frac{N}{\sigma} > 1$, it is of order $\frac{C}{|x-y|^{N-\sigma}}$.

For $|x-y| \to \infty$, notice that for $|z| \ge 1$ and $m \ge N$, $z^m g(z) \le ce^{-b'|z|^{\frac{\sigma}{\sigma-1}}}$ for any $0 < b' < b$ and that for any $0 < \alpha < 1$ to be chosen below, $z = \frac{|x-y|}{t^{\frac{1}{\sigma}}} \ge |x-y|^{\alpha} > 1$ iff $t \le |x-y|^{(1-\alpha)\sigma}$ and in such a case $z^m g(z) \le c\exp(-b'|x-y|^{\frac{\alpha\sigma}{\sigma-1}})$. Hence, we split the integral for $H_\lambda(x-y)$ according to whether $\frac{|x-y|}{t^{\frac{1}{\sigma}}}$ is larger or smaller than $|x-y|^{\alpha}$ to get

$$H_\lambda(x-y) \le \frac{Ce^{-b'|x-y|^{\frac{\alpha\sigma}{\sigma-1}}}}{|x-y|^m}\int_0^{|x-y|^{(1-\alpha)\sigma}} t^{\frac{m-N}{\sigma}} e^{(\mathrm{Re}(\lambda)+a)t}\,\mathrm{d}t$$
$$+\, C\int_{|x-y|^{(1-\alpha)\sigma}}^{\infty} t^{-\frac{N}{\sigma}} e^{(\mathrm{Re}(\lambda)+a)t}\,\mathrm{d}t.$$

Then, as $|x-y| \to \infty$, the first term is of order $\frac{C\exp(-b'|x-y|^{\frac{\alpha\sigma}{\sigma-1}})}{|x-y|^m}$ while the second is of order $C|x-y|^{-N(1-\alpha)}e^{(\mathrm{Re}(\lambda)+a)|x-y|^{(1-\alpha)\sigma}}$. By comparing $\frac{\alpha\sigma}{\sigma-1}$ and $(1-\alpha)\sigma$ the fastest exponential decay in both terms is obtained when $\frac{\alpha\sigma}{\sigma-1} = (1-\alpha)\sigma$ which gives $\alpha = \frac{\sigma-1}{\sigma} = \frac{1}{\sigma'}$ and then $\frac{\alpha\sigma}{\sigma-1} = 1$ and we get the result. □

4.2. Invariance under translations: convolution kernels

Now we prove the following result that applies to (1.3) if the operator has constant coefficients.

Proposition 4.3. *Assume X is a Banach space of functions or distributions in $\mathbb{R}^N$ which is invariant under translations that is, for every $\phi \in X$ and $y \in \mathbb{R}$, $\tau_y\phi \in X$ and $\phi \mapsto \tau_y\phi$ are continuous.*

Assume $\{S(t)\}_{t\ge0}$ is a semigroup with $\|S(t)\|_{\mathcal{L}(X)} \le Me^{\omega t}$, $t > 0$, and $-L$ is the generator with domain $D(L)$. Assume furthermore that the semigroup commutes with translations, that is for $t > 0$,

$$\tau_y S(t) = S(t)\tau_y \quad \textit{in } X.$$

(i) Then, the domain $D(L)$ is invariant under translations and $\tau_y L = L\tau_y$ in $D(L)$. Conversely, if L commutes with translations as above, then $\tau_y S(t) = S(t)\tau_y$ in $\overline{D(L)}$.
(ii) Assume furthermore that the semigroup has a kernel $k(t,x,y)$. Then,

$$k(t,x,y) = k_0(t,x-y), \quad x,y \in \mathbb{R}^N \tag{4.4}$$

for some function $k_0(t,z)$. Hence, k is a convolution kernel.

iii) Finally, for $\mathrm{Re}(\lambda) < -\omega$*, the Green function of L satisfies*

$$G_\lambda(x, y) = G_{0,\lambda}(x - y) \quad x, y \in \mathbb{R}^N$$

for some function $G_{0,\lambda}(z)$*.*

Proof. (i) Observe that if $\phi \in D(L)$, then $-L\phi = \lim_{t\to 0} \frac{S(t)\phi-\phi}{t}$ in X and then

$$-L\tau_y\phi = \lim_{t\to 0} \frac{S(t)\tau_y\phi - \tau_y\phi}{t} = \lim_{t\to 0} \tau_y \frac{S(t)\phi - \phi}{t} = -\tau_y L\phi.$$

For the converse, by the uniqueness for the equation $u_t + Lu = 0$ with $u(0) \in D(L)$ we get $S(t)\tau_y u(0) = \tau_y S(t)u(0)$ and the result follows.
(ii) Now for $z \in \mathbb{R}^N$, $\tau_z S(t)\phi(x) = \int_{\mathbb{R}^N} k(t, x - z, y)\phi(y)\,\mathrm{d}y$ equals

$$S(t)(\tau_z\phi)(x) = \int_{\mathbb{R}^N} k(t, x, y)\phi(y - z)\,\mathrm{d}y, = \int_{\mathbb{R}^N} k(t, x, y + z)\phi(y)\,\mathrm{d}y,$$

for all $\phi \in X$. Hence, $k(t, x - z, y) = k(t, x, y + z)$, $x, y, z \in \mathbb{R}^N$ which implies

$$k(t, x, y) = k(t, x + z, y + z), \qquad x, y, z \in \mathbb{R}^N.$$

Taking $z = -x$ or $z = -y$ gives (4.4).
iii) The result for the Green function follows from (4.2). □

4.3. Homogeneous operators and semigroups: self-similar kernels

Definition 4.4. Let X be a linear space of functions or distributions in $\mathbb{R}^N$ which is invariant by rescaling, that is if $\phi \in X$ then the dilation $\phi_R(x) = \phi(Rx) \in X$ for $R > 0$.
(i) A linear operator L is homogeneous of degree $\sigma \in \mathbb{R}$ if $D(L) \subset X$ is invariant by rescaling and for $\phi \in D(L)$ and $R > 0$ we have

$$L\big(\phi_R\big) = R^\sigma \big(L\phi\big)_R.$$

Notice that for $\sigma = 0$ this implies that L commutes with dilations.
(ii) We say that a family $\{T(t)\}_{t>0}$ of linear mappings in X is a scaling family of degree (α, β) on X, with $\alpha, \beta \in \mathbb{R}$, if for every $\phi \in X$, $R > 0$ and $t > 0$ we have

$$T(t)\big(\phi_R\big) = R^{-\beta}\big(T(R^\alpha t)\phi\big)_R.$$

(iii) If a scaling family $\{T(t)\}_{t>0}$ is also a semigroup (which implies necessarily $\beta = 0$), then we say that $\{T(t)\}_{t>0}$ is a *homogeneous semigroup* of degree $\alpha \in \mathbb{R}$.

It was proved in [8] that homogenous semigroups of degree σ have homogenous generator of the same degree. If additionally X is a homogeneous space of degree $\nu \in \mathbb{R}$ as in (1.2), then

$$\|S(t)\|_{\mathcal{L}(X)} = M, \quad t > 0. \tag{4.5}$$

For homogenous operators, L, of degree σ it was proved in [7] that the resolvent set $\rho(L)$ is invariant by multiplication by positive numbers, that is $s\rho(L) = \{s\lambda,\ \lambda \in \rho(L)\} = \rho(L)$ for all $s > 0$, and the resolvent operator $\mathcal{R}(\lambda, L)\phi = (L - \lambda I)^{-1}$ satisfies for $\lambda \in \rho(L)$

$$\mathcal{R}(\lambda, L)(\phi_R) = \frac{1}{R^\sigma}\Big(\mathcal{R}(\frac{\lambda}{R^\sigma}, L)\phi\Big)_R \tag{4.6}$$

that is, the resolvent $\{\mathcal{R}(\lambda, L)\}_{\lambda\in\rho(L)}$ is a scaling family of degree $(-\sigma, \sigma)$.

For kernels and Green functions, we get the following self-similarity result.

Proposition 4.5. *If the semigroup (4.1) is homogeneous semigroup of degree $\sigma \in \mathbb{R}$, then*
(i) If $\sigma \neq 0$, then the kernel is N-self-similar, that is

$$k(t, x, y) = \frac{1}{t^{\frac{N}{\sigma}}} k_0\left(\frac{x}{t^{\frac{1}{\sigma}}}, \frac{y}{t^{\frac{1}{\sigma}}}\right), \quad t > 0,\ x, y \in \mathbb{R}^N. \tag{4.7}$$

If moreover (4.3) holds, then we can assume $a = 0$.

If $\sigma = 0$, then

$$k(t, Rx, Ry) = \frac{1}{R^N} k(t, x, y), \quad t > 0,\ x, y \in \mathbb{R}^N,\ R > 0, \tag{4.8}$$

that is, the kernel is homogeneous of degree $-N$.
(ii) If $-L$ is the generator of the semigroup, the Green function of L is self-similar, that is, for $\mathrm{Re}(\lambda) < -\omega \max\{1, R^\sigma\}$

$$G_{\frac{\lambda}{R^\sigma}}(Rx, Ry) = \frac{1}{R^{N-\sigma}} G_\lambda(x, y) \quad x, y \in \mathbb{R}^N. \tag{4.9}$$

In particular, if $\sigma = 0$ then the Green function is homogeneous of degree $-N$.

Proof. (i) From Definition 4.4 we have that

$$S(t)\phi_R(x) = \int_{\mathbb{R}^N} k(t, x, y)\phi(Ry)\,\mathrm{d}y = \frac{1}{R^N}\int_{\mathbb{R}^N} k\left(t, x, \frac{z}{R}\right)\phi(z)\,\mathrm{d}z$$

equals $S(R^\sigma t)\phi(Rx) = \displaystyle\int_{\mathbb{R}^N} k(R^\sigma t, Rx, y)\phi(y)\,\mathrm{d}y$ for all $\phi \in X$ which gives

$$\frac{1}{R^N} k\left(t, x, \frac{z}{R}\right) = k(R^\sigma t, Rx, z), \quad x, z \in \mathbb{R}^N.$$

Setting $y = \frac{z}{R}$ gives $\frac{1}{R^N} k(t, x, y) = k(R^\sigma t, Rx, Ry)$. Now, if $\sigma \neq 0$, choosing R such that $R^\sigma t = 1$ leads to (4.7). If $\sigma = 0$ we get (4.8).

If $\sigma > 0$ and (4.3) holds, then taking $t = 1$ we get $|k_0(x, y)| \leq g(x - y)$ and from (4.7) we get (4.3) with $a = 0$.

(ii) Using now the Green function as in Proposition 4.3, the self-similarity of the resolvent (4.6) implies that

$$\mathcal{R}(\lambda, L)(\phi_R)(x) = \int_{\mathbb{R}^N} G_\lambda(x, y)\phi(Ry)\,\mathrm{d}y = \frac{1}{R^N}\int_{\mathbb{R}^N} G_\lambda\left(x, \frac{z}{R}\right)\phi(z)\,\mathrm{d}z$$

equals $\frac{1}{R^\sigma}\mathcal{R}(\frac{\lambda}{R^\sigma}, L)(\phi)(Rx) = \frac{1}{R^\sigma}\int_{\mathbb{R}^N} G_{\frac{\lambda}{R^\sigma}}(Rx, y)\phi(y)\,\mathrm{d}y$ for all $\phi \in X$ which implies

$$\frac{1}{R^N}G_\lambda\left(x, \frac{z}{R}\right) = \frac{1}{R^\sigma}G_{\frac{\lambda}{R^\sigma}}(Rx, z), \quad x, z \in \mathbb{R}^N.$$

Setting $y = \frac{z}{R}$ leads to $\frac{1}{R^N}G_\lambda(x, y) = \frac{1}{R^\sigma}G_{\frac{\lambda}{R^\sigma}}(Rx, Ry)$, $x, y \in \mathbb{R}^N$. Hence, we get (4.9). □

4.4. Fractional diffusion

In this section, we show the results above apply in particular to the case of fractional diffusion. Our goal is to obtain as much information as possible on the fractional diffusion problem from information on the original semigroup. So we consider $\{S(t)\}_{t\geq 0}$ a bounded semigroup in a Banach space X of functions in $\mathbb{R}^N$, that is,

$$\|S(t)\|_{\mathcal{L}(X)} \leq M, \quad t \geq 0. \tag{4.10}$$

Denote by $-L$ the generator of the semigroup with domain $D(L)$ in X.

From the general results compiled in the Appendix in [8], we have the fractional semigroup $\{S_\alpha(t)\}_{t>0}$ associated with the fractional evolution equation

$$u_t + L^\alpha u = 0, \qquad \alpha \in (0, 1).$$

This applies in particular for the case $L = (-\Delta)^m$, $m \in \mathbb{N}$, in $X = L^p(\mathbb{R}^N)$, $1 < p < \infty$, since the corresponding semigroup of solutions of

$$u_t + (-\Delta)^m u = 0, \qquad x \in \mathbb{R}^N,\ t > 0,$$

satisfies (4.5). For uniform spaces, see Proposition 6.1 and Sect. 6.2.

Then, we show below that the fractional semigroup also has a kernel and there exists a Green function for L^α.

Corollary 4.6. *Assume X, $\{S(t)\}_{t\geq 0}$ and L as in (4.10), and the semigroup has a kernel. Then, for $\alpha \in (0, 1)$ the fractional semigroup $\{S_\alpha(t)\}_{t\geq 0}$ has kernel, called* fractional kernel, *$k_\alpha(t, x, y)$.*

Also, the resolvent of L^α contains the set $\{\lambda,\ \mathrm{Re}(\lambda) < 0\}$ and has the Green's function, called fractional Green function,

$$G_{\alpha,\lambda}(x, y) = \int_0^\infty e^{\lambda t} k_\alpha(t, x, y)\,\mathrm{d}t, \quad \mathrm{Re}(\lambda) < 0. \tag{4.11}$$

Proof. Following [24, p. 259] for $\alpha \in (0,1)$ and $t>0$ let $f_{t,\alpha}(\lambda) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{z\lambda - tz^{\alpha}}\,\mathrm{d}z$ for $\lambda \geq 0$ and zero for $\lambda < 0$ where $\sigma > 0$ and the branch for z^{α} is chosen such that $\mathrm{Re}(z^{\alpha}) > 0$ if $\mathrm{Re}(z) > 0$. From the results collected in [8, Appendix], the fractional semigroup is given by

$$S_{\alpha}(t) = \int_0^{\infty} f_{t,\alpha}(s)S(s)\,\mathrm{d}s, \quad t>0, \tag{4.12}$$

that is $S_{\alpha}(t)\phi(x) = \int_{\mathbb{R}^N}\big(\int_0^{\infty} f_{t,\alpha}(s)k(s,x,y)\,\mathrm{d}s\big)\phi(y)\,\mathrm{d}y$. Hence, $S_{\alpha}(t)$ has the kernel given by

$$k_{\alpha}(t,x,y) = \int_0^{\infty} f_{t,\alpha}(s)k(s,x,y)\,\mathrm{d}s. \tag{4.13}$$

Then, (4.11) follows from (4.2). □

For semigroups that commute with translations, we get the following result.

Corollary 4.7. *In addition to (4.10), assume as in Proposition 4.3 that the semigroup commutes with translations and translations are continuous in X.*

Then, for $\alpha \in (0,1)$, L^{α} is invariant under translations. If the semigroup has a kernel, then the fractional semigroup $\{S_{\alpha}(t)\}_{t\geq 0}$ has a fractional convolution kernel

$$k_{\alpha}(t,x,y) = k_{0,\alpha}(t,x-y)$$

for some function $k_{0,\alpha}(t,z)$. Also, for $\mathrm{Re}(\lambda) < 0$ the fractional Green function satisfies

$$G_{\alpha,\lambda}(x,y) = G_{0,\alpha,\lambda}(x-y) \quad x,y \in \mathbb{R}^N$$

for some function $G_{0,\alpha,\lambda}(z)$.

Proof. It suffices to show that L^{α} is invariant under translations and apply Proposition 4.3. For this, notice
$L^{\alpha}\tau_y\phi(x) = \frac{1}{\Gamma(-\alpha)}\int_0^{\infty}\frac{S(t)\tau_y\phi(x)-\tau_y\phi(x)}{t^{1+\alpha}}\,\mathrm{d}t = \frac{1}{\Gamma(-\alpha)}\tau_y\Big(\int_0^{\infty}\frac{S(t)\phi(x)-\phi(x)}{t^{1+\alpha}}\,\mathrm{d}t\Big)$
$= \tau_y\Big(L^{\alpha}\phi(x)\Big)$. □

In particular, for homogenous semigroups, we get the following result.

Corollary 4.8. *In addition to (4.10) assume that $S(t)$ is homogenous of degree $\sigma \in \mathbb{R}$ as in Definition 4.4.*

Then, for any $\alpha \in (0,1)$ the fractional operator L^{α} is homogeneous of degree $\sigma\alpha$. In particular the fractional semigroup $\{S_{\alpha}(t)\}_{t>0}$ is a homogeneous semigroup of degree $\sigma\alpha$, the resolvent set $\rho(L^{\alpha})$ satisfies $s\rho(L^{\alpha}) = \rho(L^{\alpha})$ for all $s>0$ and the fractional resolvent operator satisfies

$$\mathcal{R}(\lambda, L^{\alpha})(\phi_R) = \frac{1}{R^{\sigma\alpha}}\Big(\mathcal{R}(\frac{\lambda}{R^{\sigma\alpha}}, L^{\alpha})(\phi)\Big)_R,$$

for $\mathrm{Re}(\lambda) < 0$ *and for all* $\lambda \in \rho(L^\alpha)$*, if the dilations are continuous. Therefore, the resolvent* $\{\mathcal{R}(\lambda, L^\alpha)\}_{\lambda\in\rho(L^\alpha)}$ *is a scaling family of degree* $(-\sigma\alpha, \sigma\alpha)$*.*

If the semigroup $S(t)$ *has a kernel* $k(t, x, y)$*, then for* $\alpha \in (0, 1)$

(i) The fractional kernel is N*-self-similar, that is*

$$k_\alpha(t, x, y) = \frac{1}{t^{\frac{N}{\sigma\alpha}}} k_{0,\alpha}\left(\frac{x}{t^{\frac{1}{\sigma\alpha}}}, \frac{y}{t^{\frac{1}{\sigma\alpha}}}\right), \quad x, y \in \mathbb{R}^N, \ t > 0.$$

(ii) The fractional Green function is self-similar, that is, for $\mathrm{Re}(\lambda) < 0$

$$G_{\alpha, \frac{\lambda}{R^\sigma}}(Rx, Ry) = \frac{1}{R^{N-\sigma\alpha}} G_{\alpha,\lambda}(x, y), \quad x, y \in \mathbb{R}^N, \ R > 0.$$

Now, we exploit (4.13) to obtain upper bounds on the fractional kernel from those on the semigroup kernel. For this, in view of the examples, Sect. 2 and Proposition 4.5, we will assume (4.3) with $a = 0$. Then, we have the following result.

Proposition 4.9. *Assume* $\{S(t)\}_{t\geq 0}$ *is as in (4.10) in a Banach space* X *of functions in* $\mathbb{R}^N$ *which has a kernel satisfying (4.3) with* $a = 0$*.*

If either $\alpha = \frac{1}{2}$ *or* $\alpha \in (0, \frac{1}{2})$ *and* $N < \sigma$*, then the kernel* k_α *of the fractional semigroup* $\{S_\alpha(t)\}_{t\geq 0}$ *satisfies the self-similar upper bound*

$$|k_\alpha(t, x, y)| \leq C \min\{t^{-\frac{N}{\sigma\alpha}}, t|x - y|^{-N-\sigma\alpha}\} = \frac{1}{t^{\frac{N}{\sigma\alpha}}} H_\alpha\left(\frac{x - y}{t^{\frac{1}{\sigma\alpha}}}\right),$$
$$t > 0, \ x, y \in \mathbb{R}^N,$$

where $H_\alpha(z) = C \min\{1, \frac{1}{|z|^{N+\sigma\alpha}}\}$*,* $C > 0$*.*

Proof. *Step 1* We first prove that for $\alpha \in (0, \frac{1}{2}]$ there exists $C > 0$ such that

$$|k_\alpha(t, x, y)| \leq Ct|x - y|^{-N-\sigma\alpha}, \quad t > 0, \ x, y \in \mathbb{R}^N.$$

Indeed, from (4.13), using $f_{t,\alpha} \geq 0$, (4.3) and Lemma 4.10, we have

$$|k_\alpha(t, x, y)| \leq \int_0^\infty f_{t,\alpha}(s)|k(s, x, y)| \,\mathrm{d}s \leq \int_0^\infty c_\alpha t s^{-1-\alpha-\frac{N}{\sigma}} g\left(\frac{x - y}{s^{\frac{1}{\sigma}}}\right) \mathrm{d}s.$$

Taking the change of variable $s = \xi^{1-\sigma}$ and for $\tilde{c}_\alpha$ a multiple of c_α, we get

$$|k_\alpha(t, x, y)| \leq (\sigma - 1)\tilde{c}_\alpha t \int_0^\infty \xi^{(\sigma-1)\alpha+\frac{(\sigma-1)N}{\sigma}-1} e^{-b|x-y|^{\frac{\sigma}{\sigma-1}}\xi} \,\mathrm{d}\xi.$$

Now, the integral above is a multiple of $t|x - y|^{-N-\sigma\alpha}$ (see (4.15)) and we get the result.

Step 2 If $\frac{N}{\sigma} < 1$, then for $\alpha \in (0, 1)$ there exist $C > 0$ such that

$$|k_\alpha(t, x, y)| \leq Ct^{-\frac{N}{\sigma\alpha}}, \quad t > 0, \ x, y \in \mathbb{R}^N.$$

If $\alpha = \frac{1}{2}$, then this also holds when $\frac{N}{\sigma} \geq 1$.

Indeed from (4.13), and using $f_{t,\alpha} \geq 0$, (4.3) and (4.16) with $\theta_\alpha = \frac{\pi}{1+\alpha}$ we get

$$|k_\alpha(t,x,y)| \leq \int_0^\infty f_{t,\alpha}(s)|k(s,x,y)|\,\mathrm{d}s$$
$$\leq \frac{1}{\pi}\int_0^\infty \left(\int_0^\infty e^{-|\cos\theta_\alpha|(sr+tr^\alpha)} \sin\left((sr-tr^\alpha)\sin\theta_\alpha+\theta_\alpha\right)\mathrm{d}r\right) \frac{g\left(\frac{x-y}{s^{\frac{1}{\sigma}}}\right)}{s^{\frac{N}{\sigma}}}\,\mathrm{d}s.$$

Changing variables as $\tilde{s} = st^{-\frac{1}{\alpha}}$, $\tilde{r} = rt^{\frac{1}{\alpha}}$ and omitting back the 'tilde', we get

$$|k_\alpha(t,x,y)| \leq \frac{1}{\pi}\int_0^\infty \left(\int_0^\infty e^{-|\cos\theta_\alpha|(sr+r^\alpha)} \sin\left((sr-r^\alpha)\sin\theta_\alpha+\theta_\alpha\right)\mathrm{d}r\right) \frac{g\left(\frac{x-y}{(st^{\frac{1}{\alpha}})^{\frac{1}{\sigma}}}\right)}{(st^{\frac{1}{\alpha}})^{\frac{N}{\sigma}}}\,\mathrm{d}s$$

and the integral in r above is precisely $f_{1,\alpha}(s)$, see (4.16). Now since $g(z) \leq c$ we obtain

$$|k_\alpha(t,x,y)| \leq \int_0^\infty f_{1,\alpha}(s)\frac{g\left(\frac{x-y}{(st^{\frac{1}{\alpha}})^{\frac{1}{\sigma}}}\right)}{(st^{\frac{1}{\alpha}})^{\frac{N}{\sigma}}}\,\mathrm{d}s \leq ct^{-\frac{N}{\sigma\alpha}}\int_0^\infty s^{-\frac{N}{\sigma}} f_{1,\alpha}(s)\,\mathrm{d}s$$

and the integral is finite by Lemma 4.11. If $\alpha = \frac{1}{2}$, then the integral is finite by Lemma 4.12 even if $\frac{N}{\sigma} \geq 1$. Hence, we proved Step 2.

The rest of the proof is immediate. □

We include here the three lemmas used in the proof above.

Lemma 4.10. *Given $\alpha \in (0, \frac{1}{2}]$ there is $c_\alpha > 0$ such that $f_{t,\alpha}(s) \leq c_\alpha t s^{-1-\alpha}$ for $s, t > 0$.*

Proof. We remark that from [24, (17), p. 263] and [24, (8) p. 261 and (14), p. 262]

$$f_{t,\alpha}(s) = \frac{1}{\pi}\int_0^\infty e^{sr\cos\theta - tr^\alpha\cos\alpha\theta}\sin\left(sr\sin\theta - tr^\alpha\sin\alpha\theta+\theta\right)\mathrm{d}r,\ \ \theta\in[\frac{\pi}{2},\pi],\ s>0,$$
$$f_{t,\alpha}\geq 0 \ \text{ and } \int_0^\infty f_{t,\alpha}(s)\mathrm{d}s = 1. \tag{4.14}$$

Choosing in (4.14) $\theta = \pi$ we have $f_{t,\alpha}(s) = \frac{1}{\pi}\int_0^\infty e^{-sr}e^{-tr^\alpha\cos\alpha\pi}\sin\left(tr^\alpha\sin\alpha\pi\right)\mathrm{d}r$ for $s > 0$. Since $|\sin\left(tr^\alpha\sin\alpha\pi\right)| \leq tr^\alpha\sin\alpha\pi$ for $r, t > 0$, and $e^{-r^\alpha\cos\alpha\pi} \leq 1$ when $\alpha \in (0, \frac{1}{2}]$, we get $f_{t,\alpha}(s) \leq \dfrac{\sin\alpha\pi}{\pi}t\displaystyle\int_0^\infty r^\alpha e^{-sr}\,\mathrm{d}r$ which gives the result with $c_\alpha = \frac{\sin\alpha\pi}{\pi}\Gamma(1+\alpha)$ using

$$\int_0^\infty r^z e^{-\kappa r}\,\mathrm{d}r = \Gamma(1+z)\kappa^{-1-z},\quad z>-1,\ \kappa>0. \tag{4.15}$$

□

Lemma 4.11. *Given any* $\alpha, \omega \in (0,1)$ *there is* $c_{\alpha,\omega} > 0$ *such that*

$$\int_0^\infty s^{-\omega} f_{t,\alpha}(s)\,\mathrm{d}s \le c_{\alpha,\omega} t^{-\frac{\omega}{\alpha}}, \quad t>0.$$

Proof. Choosing in (4.14) $\theta = \frac{\pi}{1+\alpha} =: \theta_\alpha$ we have

$$f_{t,\alpha}(s) = \frac{1}{\pi}\int_0^\infty e^{-|\cos\theta_\alpha|(sr+tr^\alpha)} \sin\left((sr-tr^\alpha)\sin\theta_\alpha + \theta_\alpha\right)\mathrm{d}r,\ s,t>0. \tag{4.16}$$

Hence, we get

$$\int_0^\infty s^{-\omega} f_{t,\alpha}(s)\,\mathrm{d}s \le \frac{1}{\pi}\int_0^\infty \left(\int_0^\infty s^{-\omega} e^{-|\cos\theta_\alpha| sr} e^{-|\cos\theta_\alpha| tr^\alpha}\,\mathrm{d}r\right)\mathrm{d}s$$

$$= \frac{1}{\pi}\int_0^\infty e^{-|\cos\theta_\alpha| tr^\alpha}\left(\int_0^\infty s^{-\omega} e^{-|\cos\theta_\alpha| rs}\,\mathrm{d}s\right)\mathrm{d}r$$
$$= \frac{1}{\pi}\int_0^\infty e^{-|\cos\theta_\alpha| tr^\alpha}\Gamma(1-\omega)(|\cos\theta_\alpha| r)^{-1+\omega}\mathrm{d}r,$$

which after the change of variable $r = \xi^{\frac{1}{\alpha}}$ gives

$$\int_0^\infty s^{-\omega} f_{t,\alpha}(s)\,\mathrm{d}s \le \frac{\Gamma(1-\omega)}{\alpha\pi|\cos\theta_\alpha|^{1-\omega}}\int_0^\infty \xi^{\frac{\omega}{\alpha}-1} e^{-|\cos\theta_\alpha| t\xi}\,\mathrm{d}\xi.$$

Due to (4.15), we get the result with $c_{\alpha,\omega} = \frac{\Gamma(1-\omega)\Gamma(\frac{\omega}{\alpha})}{\alpha\pi|\cos\theta_\alpha|^{1-\omega}|\cos\theta_\alpha|^{\frac{\omega}{\alpha}}}$. □

Lemma 4.12. *If* $\alpha = \frac{1}{2}$ *and* $\omega > 0$, *then* $\int_0^\infty s^{-\omega} f_{t,\frac{1}{2}}(s)\,ds = \frac{4^{\omega-4}\Gamma\left(\omega+\frac{1}{2}\right)}{\sqrt{\pi}} t^{-2\omega}$ *for* $t>0$.

Proof. From [24, (32), p. 268], we have $f_{t,\frac{1}{2}}(s) = (4^9\pi)^{-\frac{1}{2}} ts^{-\frac{3}{2}} e^{-\frac{t^2}{4s}}$, $s,t>0$, and then changing variables to $s = \xi^{-1}$, we get

$$\int_0^\infty s^{-\omega} f_{t,\frac{1}{2}}(s)\,\mathrm{d}s = (4^9\pi)^{-\frac{1}{2}} t\int_0^\infty s^{-\omega-\frac{3}{2}} e^{-\frac{t^2}{4s}}\,\mathrm{d}s = (4^9\pi)^{-\frac{1}{2}} t\int_0^\infty \xi^{\omega-\frac{1}{2}} e^{-\frac{t^2}{4}\xi}\,\mathrm{d}\xi$$

and using (4.15) we get the result. □

Now, for the fractional Green's function we get the following bounds.

Corollary 4.13. *Under the assumption of Proposition 4.9, assume either* $\alpha = \frac{1}{2}$ *or* $\alpha \in (0,\frac{1}{2})$ *and* $N < \sigma$. *Then, the fractional Green's function satisfies, for* $\mathrm{Re}(\lambda) < 0$,

$$|G_{\alpha,\lambda}(x,y)| \le \frac{C}{|x-y|^{N+\sigma\alpha}}, \quad |x-y|\to\infty$$

and $|G_{\alpha,\lambda}(x,y)| \begin{cases} \textit{bounded} & \textit{if } N<\sigma\alpha \\ \le C|\ln(|x-y|)| & \textit{if } N=\sigma\alpha, \\ \le \frac{C}{|x-y|^{N-\sigma\alpha}} & \textit{if } N>\sigma\alpha \end{cases}$ $\quad |x-y|\to 0.$

Proof. From the bounds in Proposition 4.9 on the fractional semigroup kernel

$$|G_{\alpha,\lambda}(x,y)|\leq \int_0^\infty e^{\mathrm{Re}(\lambda)t}|k_\alpha(t,x,y)|\,\mathrm{d}t\leq \int_0^\infty \frac{e^{\mathrm{Re}(\lambda)t}}{t^{\frac{N}{\sigma\alpha}}}H_\alpha\left(\frac{x-y}{t^{\frac{1}{\sigma\alpha}}}\right)\mathrm{d}t=I_{\alpha,\lambda}(x-y)$$

and splitting the integral according to whether $\frac{|x-y|}{t^{\frac{1}{\sigma\alpha}}}$ is larger or smaller than 1, we get

$$I_{\alpha,\lambda}(x-y)=\frac{C}{|x-y|^{N+\sigma\alpha}}\int_0^{|x-y|^{\sigma\alpha}} te^{\mathrm{Re}(\lambda)t}\,\mathrm{d}t + C\int_{|x-y|^{\sigma\alpha}}^\infty t^{-\frac{N}{\sigma\alpha}}e^{\mathrm{Re}(\lambda)t}\,\mathrm{d}t.$$

Then, as $|x-y|\to\infty$ the first term is of order $\frac{C}{|x-y|^{N+\sigma\alpha}}$, while the second is of order $C|x-y|^{-N}e^{\mathrm{Re}(\lambda)|x-y|^{\sigma\alpha}}$.

On the other hand, as $|x-y|\to 0$ then the first term is of order $\frac{C}{|x-y|^{N-\sigma\alpha}}$, while for the second: (a) If $\frac{N}{\sigma\alpha}<1$ it is bounded. (b) If $\frac{N}{\sigma\alpha}=1$ it is of order $C|\ln(|x-y|)|$. (c) If $\frac{N}{\sigma\alpha}>1$ it is of order $\frac{C}{|x-y|^{N-\sigma\alpha}}$. □

Finally, we show that if the fractional semigroup $\{S_\alpha(t)\}_{t\geq 0}$ is given as in (4.12), where the semigroup $\{S(t)\}_{t\geq 0}$ is the one in Sect. 3, then the fractional semigroup is continuous at $t=0$ in the sense of measures if the kernel in (3.1) is symmetric.

Proposition 4.14. *Assume that $k(t,x,y)$ in (3.1) is symmetric in $x,y\in\mathbb{R}^N$ and $a=0$ in (2.3).*

Then, the semigroup $\{S_\alpha(t)\}_{t\geq 0}$ is continuous at $t=0^+$ in the sense that given any $\mu\in\mathcal{M}_U(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N}\phi S_\alpha(t)\mu\,\mathrm{d}y\to\int_{\mathbb{R}^N}\phi\,d\mu \quad as \quad t\to 0^+ \quad for\ each \quad \phi\in C_c(\mathbb{R}^N)$$

and it is analytic and satisfies $\|S_\alpha(t)\|_{\mathcal{L}(\mathcal{M}_U(\mathbb{R}^N),\dot{L}^1_U(\mathbb{R}^N))}\leq c$ for all $t>0$.

Proof. Observe that for $\phi\in C_c(\mathbb{R}^N)$, using [24, (20'), p. 264],

$$\int_{\mathbb{R}^N}\phi S_\alpha(t)\mu\,\mathrm{d}y=\int_{\mathbb{R}^N}\int_0^\infty \phi f_{t,\alpha}(s)S(s)\mu\,\mathrm{d}s\,\mathrm{d}y=\int_{\mathbb{R}^N}\int_0^\infty \phi f_{1,\alpha}(\xi)S(\xi t^{\frac{1}{\alpha}})\mu\,\mathrm{d}\xi\,\mathrm{d}y$$
$$=\int_0^\infty f_{1,\alpha}(\xi)\int_{\mathbb{R}^N}\phi S(\xi t^{\frac{1}{\alpha}})\mu\,\mathrm{d}y\,\mathrm{d}\xi,\quad t>0. \tag{4.17}$$

Now if B_R is a ball in $\mathbb{R}^N$ containing the support of $\phi\in C_c(\mathbb{R}^N)$ then, using (3.2),

$$\left|\int_{\mathbb{R}^N}\phi S(\xi t^{\frac{1}{\alpha}})\mu\,\mathrm{d}y\right|\leq\|\phi\|_{L^\infty(B_R)}\|S(\xi t^{\frac{1}{\alpha}})\mu\|_{L^1(B_R)}\leq c_{R,N}\|\phi\|_{L^\infty(B_R)}\|S(\xi t^{\frac{1}{\alpha}})\mu\|_{L^1_U(\mathbb{R}^N)}$$
$$\leq 2cc_{R,N}\|\phi\|_{L^\infty(B_R)}\|\mu\|_{\mathcal{M}_U(\mathbb{R}^N)},$$

where $c_{R,N}$ is a number of balls in $\mathbb{R}^N$ of radius 1 necessary to cover B_R.

Since from (3.7) $\lim_{t\to 0^+}\int_{\mathbb{R}^N}\phi S(\xi t^{\frac{1}{\alpha}})\mu\,dy=\int_{\mathbb{R}^N}\phi\,d\mu$ for every $\xi>0$, using (4.14) and Lebesgue's dominated convergence theorem from (4.17), we get the result.

To prove that $\{S_\alpha(t)\}_{t\geq 0}$ is analytic, observe that uniform estimates on the semigroups and the time derivative can be obtained using (4.12) as in [24, proof of Theorem 1, pp. 263-264]. Then, [15, Proposition 2.1.9] concludes the result, by observing that the weak convergence to the initial data proved above is enough to reproduce the proof. Finally, the estimate on $\|S_\alpha(t)\|_{\mathcal{L}(\mathcal{M}_U(\mathbb{R}^N),\dot{L}^1_U(\mathbb{R}^N))}$ follows from [8, Lemma 4.4]. □

5. Self-similar solutions and self-similar variables

Now we pay attention to self-similar solutions of homogeneous semigroups. Our main goal below is to determine those $\beta \in \mathbb{R}$ for which β-self-similar solutions exist.

Definition 5.1. Assume $S(t)$ is a homogeneous semigroup of degree $\sigma \in \mathbb{R}$, with generator $-L$, in a Banach space of functions or distributions in $\mathbb{R}^N$, X, which is invariant by rescaling as in Definition 4.4.
(i) A β-self-similar solution of the semigroup is a function $(0,\infty) \ni t \mapsto u(t) \in X$, such that for each $t > s > 0$ we have $u(t) = S(t-s)u(s)$ and

$$R^{\beta}\big(u(R^{\sigma}t)\big)_R = u(t), \qquad t>0, \quad R>0.$$

(ii) A strong β-self-similar solution of the semigroup is a β-self-similar solution of the semigroup that is a strong solution of

$$u_t + Lu = 0, \quad t > 0.$$

That is, $u(t) \in D(L)$, u is differentiable in X for every $t > 0$ and satisfies the differential equation above.

Notice that for $\sigma = 0$ a β-self-similar solution is a homogeneous function of degree $-\beta$ for each $t > 0$.

Then, we prove the following result concerning self-similar solutions.

Theorem 5.2. *Self-similar solutions. Let X be a Banach space of functions or distributions in $\mathbb{R}^N$ which is invariant by rescaling. Assume also $S(t)$ is a homogeneous semigroup of degree $\sigma \neq 0$. Then,*
(i) If there exists $\phi \in X$ which is homogeneous of degree $-\beta$, that is $\phi_R = R^{-\beta}\phi$ for $R > 0$, then $u(t) = S(t)\phi$ is a β-self-similar solution of the semigroup and satisfies (5.1) with $\Phi = S(1)\phi$.

Conversely, if the semigroup is injective and $u(t)=S(t)\phi$ is a β-self-similar solution of the semigroup, then ϕ is homogeneous of degree $-\beta$.
(ii) A function $(0,\infty) \ni t \mapsto u(t) \in X$ is a β-self-similar solution of the semigroup if and only if it satisfies

$$u(t) = \frac{1}{t^{\frac{\beta}{\sigma}}}(\Phi)_{\frac{1}{t^{\frac{1}{\sigma}}}}, \quad t > 0 \tag{5.1}$$

for some $\Phi \in X$ *such that*

$$S(\tau)\Phi = \frac{1}{(1+\tau)^{\frac{\beta}{\sigma}}}(\Phi)_{\frac{1}{(1+\tau)^{\frac{1}{\sigma}}}}, \quad \tau > 0. \tag{5.2}$$

(iii) If u is β*-self-similar solution of the semigroup as in (5.1) and (5.2) and* $\Phi \in D(L)$ *then u is a strong* β*-self-similar solution,* $R \mapsto \Phi_R$ *is differentiable in X,* $x\nabla\Phi \in X$ *and*

$$L\Phi = \frac{1}{\sigma}x\nabla\Phi + \frac{\beta}{\sigma}\Phi \quad in \quad X, \tag{5.3}$$

that is, $\frac{\beta}{\sigma}$ *is an eigenvalue of the operator* $L - \frac{1}{\sigma}x\nabla$ *and* Φ *is a corresponding eigenfunction.*

This happens for any semigroup with a smoothing effect.

(iv) Conversely, assume there exists $\Phi \in D(L)$ *such that* $R \mapsto \Phi_R$ *is differentiable in X,* $x\nabla\Phi \in X$ *and satisfies (5.3). Then, u defined in (5.1) is a strong* β*-self-similar solution of the semigroup.*

Proof. (i) Notice that for each $\phi \in X$ and $\beta \in \mathbb{R}$

$$R^\beta\big(S(R^\sigma t)\phi\big)_R = R^\beta S(t)\phi_R = S(t)(R^\beta\phi_R).$$

Thus, if $R^\beta\phi_R = \phi$, then $u(t) = S(t)\phi$ is a β-self-similar solution. The converse is also true if the semigroup is injective. That u satisfies (5.1) follows by the argument below.

(ii) If $u(t)$ is a β-self-similar solution, then we take $R^\sigma t = 1$ in $R^\beta\big(u(R^\sigma t)\big)_R = u(t)$ and then, denoting $\Phi = u(1)$ we have (5.1).

On the other hand, notice that for a function as in (5.1),

$$S(t-s)u(s) = \frac{1}{s^{\frac{\beta}{\sigma}}}\left(S\left(\frac{t}{s}-1\right)\Phi\right)_{\frac{1}{s^{\frac{1}{\sigma}}}}$$

and setting $\tau = \frac{t}{s} - 1 > 0$, we get $u(t) = S(t-s)u(s)$ for $t > s > 0$ is equivalent to (5.2).

Conversely, (5.2) is equivalent to $u(t)=S(t-s)u(s)$ and by (5.1) we get $R^\beta(u(R^\sigma t))_R = \frac{1}{t^{\frac{\beta}{\sigma}}}((\Phi)_{\frac{1}{Rt^{\frac{1}{\sigma}}}})_R = \frac{1}{t^{\frac{\beta}{\sigma}}}(\Phi)_{\frac{1}{t^{\frac{1}{\sigma}}}} = u(t)$.

(iii) If $\Phi \in D(L)$, then $\tau \mapsto S(\tau)\Phi$ is differentiable in X and then $R \mapsto \Phi_R$ is differentiable in X, which by Proposition A.1 yields $x\nabla\Phi \in X$. Also, from (5.1), using Proposition A.1

$$\partial_t u(t) = \frac{-\beta/\sigma}{t^{\frac{\beta}{\sigma}+1}}(\Phi)_{\frac{1}{t^{\frac{1}{\sigma}}}} + \frac{1}{t^{\frac{\beta}{\sigma}}}x\big(\nabla\Phi\big)_{\frac{1}{t^{\frac{1}{\sigma}}}}\frac{-1/\sigma}{t^{\frac{1}{\sigma}+1}} = \frac{-\beta/\sigma}{t^{\frac{\beta}{\sigma}+1}}\big(\Phi\big)_{\frac{1}{t^{\frac{1}{\sigma}}}} - \frac{1/\sigma}{t^{\frac{\beta}{\sigma}+1}}\big(x\nabla\Phi\big)_{\frac{1}{t^{\frac{1}{\sigma}}}}$$

and $Lu(t) = \frac{1}{t^{\frac{\beta}{\sigma}+1}}\big(L(\Phi)\big)_{\frac{1}{t^{\frac{1}{\sigma}}}}$, hence Φ must satisfy (5.3).

For a semigroup with a smoothing effect, since $S(\tau)\Phi \in D(L)$ for $\tau > 0$, (5.2) implies that $\Phi \in D(L)$.
(iv) From the assumptions on Φ we get that u in (5.1) satisfies $u(t) \in D(L)$ for $t > 0$, it is differentiable in X for $t > 0$ and, from the computation in part (iii) above, is a strong self-similar solution of the semigroup. □

Remark 5.3. (i) Observe that from the semigroup property, either $S(t)$ is injective for all $t > 0$ or not injective for any $t > 0$.
(ii) For strongly continuous semigroups such that the curves $t \mapsto S(t)\phi$ are analytic (in particular for analytic semigroups), we obtain that $S(t)$ is injective for all $t > 0$.

The next result shows that, in general, in homogeneous spaces self-similar solutions either have constant norm or decay to zero.

Corollary 5.4. *Assume X is a homogeneous space of degree $\nu \in \mathbb{R}$ and $\{S(t)\}_{t\geq 0}$ is a homogeneous semigroup of degree $\sigma \neq 0$ in X.*
(i) Assume the semigroup is injective. A β-self-similar solution of the type $u(t) = S(t)\phi$ with $\phi \in X$, can only exist for $\beta = -\nu$. In such a case, the norm of $u(t)$ is constant in time.
(ii) If there is a β-self-similar solution of the semigroup as in Definition 5.1, then for $t > 0$

$$\|u(t)\| = \frac{1}{t^{\frac{\beta+\nu}{\sigma}}}\|\Phi\|, \qquad \text{with } \frac{\beta+\nu}{\sigma} \geq 0. \tag{5.4}$$

Hence, if $\beta = -\nu$ then all β-self-similar solutions have constant norm. Otherwise, the β-self-similar solution converges to 0 in X as $t \to \infty$.

Furthermore, if the self-similar solution above is a strong one, then additionally

$$\|\partial_t u(t)\| = \frac{1}{t^{\frac{\beta+\nu}{\sigma}+1}}\|L\Phi\| \to 0, \quad \text{as } t \to \infty.$$

Proof. For part (i), from Theorem 5.2 we know that self-similar solutions $u(t) = S(t)\phi$ correspond to $\phi \in X$ homogeneous of degree $-\beta$, that is $\phi_R = R^{-\beta}\phi$. Since X is a homogeneous space, by Lemma A.2, we get $\beta = -\nu$. In such a case the norm of $u(t)$ does not change in time as we prove (5.4).

For part (ii), from Theorem 5.2 we know that self-similar solutions are of the form (5.1), which implies the first part in (5.4), since X is a homogeneous space.

Also, we have $u(t+s) = S(s)u(t)$ and then, from (4.5) $\|u(t+s)\| \leq M\|u(t)\|$ which leads to $\left(\frac{t}{t+s}\right)^{\frac{\beta+\nu}{\sigma}} \leq M$ for $t, s > 0$ which gives $\frac{\beta+\nu}{\sigma} \geq 0$.

If furthermore the self-similar solution is a strong one, from the proof of Theorem 5.2,

$$\partial_t u = -\frac{1}{t^{\frac{\beta}{\sigma}+1}}(L\Phi)_{\frac{1}{t^{\frac{1}{\sigma}}}}, \quad t > 0$$

and we get the result since X is a homogeneous space. □

Observe that the results above relate the existence of some self-similar solutions to X containing some homogeneous functions. So we search for homogenous functions in some particular functions spaces. Notice that below we use the Morrey spaces $M^{p,\ell}(\mathbb{R}^N) \subset L^p_U(\mathbb{R}^N)$, $1 \le p \le \infty$, $\ell \in [0, N]$, of functions $\phi \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$ such that

$$\|\phi\|_{M^{p,\ell}(\mathbb{R}^N)} = \sup_{x_0 \in \mathbb{R}^N, R>0} R^{\frac{\ell}{p}-\frac{N}{p}} \|\phi\|_{L^p(B(x_0,R))} < \infty.$$

Dotted Morrey spaces, $\dot{M}^{p,\ell}(\mathbb{R}^N)$, are defined as the subspace of $M^{p,\ell}(\mathbb{R}^N)$ such that $\tau_y\phi - \phi \to 0$ as $y \to 0$ in the norm of $M^{p,\ell}(\mathbb{R}^N)$. These are homogenous spaces of degree $-\frac{\ell}{p}$. Also, Morrey measures, $\mathcal{M}^\ell(\mathbb{R}^N)$ with $\ell \in [0, N]$, are measures satisfying

$$\|\mu\|_{\mathcal{M}^\ell(\mathbb{R}^N)} = \sup_{x_0 \in \mathbb{R}^N, R>0} R^{\ell-N} |\mu|(B(x_0, R)) < \infty.$$

Also $M^{1,\ell}(\mathbb{R}^N) \subset \mathcal{M}^\ell(\mathbb{R}^N)$ isometrically and $\mathcal{M}^\ell(\mathbb{R}^N) \subset \mathcal{M}_U(\mathbb{R}^N)$. These are homogenous spaces of degree $-\ell$.

In particular, we get the following result.

Corollary 5.5. *(i) For $1 \le p < \infty$ and $0 \le \beta < \frac{N}{p}$, there exist homogeneous functions of degree $-\beta$ in $L^p_U(\mathbb{R}^N)$, for example $\phi(x) = \frac{1}{|x|^\beta}$. Thus, they give rise to β-self-similar solutions for any homogeneous semigroup in $L^p_U(\mathbb{R}^N)$.*
(ii) In the Morrey space $M^{p,\ell}(\mathbb{R}^N)$ there exist homogeneous functions only of degree $-\frac{\ell}{p}$, for example $\phi(x) = \frac{1}{|x|^{\frac{\ell}{p}}}$, which give rise to $\frac{\ell}{p}$-self-similar solutions of constant norm for any homogeneous semigroup in $M^{p,\ell}(\mathbb{R}^N)$.
(iii) For $0 \le \beta \le N$ there exist homogeneous measures of degree $-\beta$ in $\mathcal{M}_U(\mathbb{R}^N)$. Thus, they give rise to β-self-similar solutions for any homogeneous semigroup in $\mathcal{M}_U(\mathbb{R}^N)$.

In particular, for $\beta = N$, the measure is a multiple of $\delta \in \mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^N) \subset \mathcal{M}_U(\mathbb{R}^N)$ and gives rise to the only (except for multiples) N-self-similar solution if the semigroup is injective. Moreover, it has constant norm, for any homogeneous semigroup in $\mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^N)$.
(iv) In the Morrey space $\mathcal{M}^\ell(\mathbb{R}^N)$, there exist homogeneous measures only of degree $-\ell$. Thus, they give rise to ℓ-self-similar solutions of constant norm for any homogeneous semigroup in $\mathcal{M}^\ell(\mathbb{R}^N)$.

Proof. The existence of self-similar solutions in parts (i)–(iv) comes from Lemmas A.2, A.4 and A.5 and part (i) in Theorem 5.2.

That in (ii)–(iv) the self-similar solution has constant norm comes from Corollary 5.4. □

Besides self-similar solutions as in Theorem 5.2, we show below that homogeneous semigroups have naturally associated self-similar variables, which provide an alternative form of describing the semigroup. For this, observe that for the self-similar solutions in Theorem 5.2, from (5.2), we have that the self-similar profile satisfies

$$\Phi = (1+\tau)^{\frac{\beta}{\sigma}} \big(S(\tau)\Phi\big)_{(1+\tau)^{\frac{1}{\sigma}}}, \quad \tau > 0$$

and if $\Phi \in D(L)$ then it satisfies the *stationary* equation (5.3). Then, setting $s = \log(1+\tau)$, i.e. $e^s = 1+\tau$, we have the following.

Theorem 5.6. *Self-similar variables. Let X be a Banach space of functions or distributions in $\mathbb{R}^N$ which is invariant by rescaling. Assume also $S(t)$ is a homogeneous semigroup of degree $\sigma \neq 0$. Then,*
(i) For $\phi \in X$ and $\beta \in \mathbb{R}$

$$T_\beta(s)\phi = e^{\frac{\beta}{\sigma}s}\big(S(e^s-1)\phi\big)_{e^{\frac{s}{\sigma}}}, \quad s > 0$$

defines a semigroup in X that we denote the β-Ornstein–Uhlenbeck semigroup associated with $S(t)$.
(ii) Assume $X \subset \mathscr{D}'(\mathbb{R}^N)$. If $\phi \in D(L)$, then $v(s) = T_\beta(s)\phi$ is a solution of the Ornstein–Uhlenbeck equation

$$v_s + Lv = \frac{1}{\sigma} x\nabla v + \frac{\beta}{\sigma} v, \quad s > 0.$$

The same occurs for all $\phi \in X$ if $S(t)$ is a semigroup with a smoothing effect.

In particular, β-self-similar solutions of $S(t)$ correspond to stationary solutions of the semigroup $T_\beta(s)$. More generally, if $v(s,x)$ is a solution of the Ornstein–Uhlenbeck semigroup above, then

$$u(t,x) = \frac{1}{(1+t)^{\frac{\beta}{\sigma}}} v(\log(1+t), \frac{x}{(1+t)^{\frac{1}{\sigma}}}), \quad t > 0,\ x \in \mathbb{R}^N$$

is a solution of the semigroup $S(t)$.
(iii) If $X \subset \mathscr{D}'(\mathbb{R}^N)$ is a homogeneous space of degree $\nu \in \mathbb{R}$, then

$$\|u(t)\| = \frac{1}{(1+t)^{\frac{\beta+\nu}{\sigma}}} \|v(\log(1+t)\|$$

and if u is a strong solution then $\|u_t(t)\| = \frac{1}{(1+t)^{\frac{\beta+\nu}{\sigma}+1}} \|Lv(s)\|$.

Proof. That $T_\beta(0)\phi = \phi$ and $T_\beta(s_1) \circ T_\beta(s_2) = T_\beta(s_1+s_2)$ follows from immediate computations. Thus, (i) is proved.

For (ii) observe that if $\phi \in D(L)$ (or $\phi \in X$ if $S(t)$ has a smoothing effect), we have that $\tau \mapsto S(\tau)\phi \in X$ is differentiable and therefore $s \mapsto v(s)_{e^{-\frac{s}{\sigma}}} = e^{\frac{\beta}{\sigma}s} S(e^s-1)\phi \in X$ is differentiable. Differentiating both sides we have

$$(\partial_s v)_{e^{-\frac{s}{\sigma}}} + x\big(\nabla v\big)_{e^{-\frac{s}{\sigma}}} \left(\frac{-1}{\sigma}\right) e^{-\frac{s}{\sigma}} = \left(\frac{\beta}{\sigma}\right) e^{\frac{\beta}{\sigma}} S(e^s-1)\phi - e^{\frac{\beta}{\sigma}s} LS(e^s-1)\phi e^s.$$

Thus, we get

$$\partial_s v - \frac{1}{\sigma} x \nabla v = \left(\frac{\beta}{\sigma}\right) v - e^{\frac{\beta}{\sigma}s}\big(LS(e^s-1)\phi\big)_{e^{\frac{s}{\sigma}}} e^s = \left(\frac{\beta}{\sigma}\right) v - Lv.$$

The rest follows easily.

For (iii) the estimate of the norm of $u(t)$ is immediate. Then, with $s = \log(1+t)$,

$$\begin{aligned}\partial_t u(t) &= \frac{-\beta/\sigma}{(1+t)^{\frac{\beta}{\sigma}+1}}\big(v(s)\big)_{\frac{1}{(1+t)^{\frac{1}{\sigma}}}} + \frac{1}{(1+t)^{\frac{\beta}{\sigma}}}\left(\big(v_s(s)\big)_{\frac{1}{(1+t)^{\frac{1}{\sigma}}}}\frac{1}{1+t} + x\big(\nabla v(s)\big)_{\frac{1}{(1+t)^{\frac{1}{\sigma}}}}\frac{-1/\sigma}{(1+t)^{\frac{1}{\sigma}+1}}\right)\\ &= \frac{1}{(1+t)^{\frac{\beta}{\sigma}+1}}\left(-\frac{\beta}{\sigma}v(s) + v_s(s) - \frac{x}{\sigma}\nabla v(s)\right)_{\frac{1}{(1+t)^{\frac{1}{\sigma}}}} = \frac{1}{(1+t)^{\frac{\beta}{\sigma}+1}}\big(Lv(s)\big)_{\frac{1}{(1+t)^{\frac{1}{\sigma}}}}\end{aligned}$$

which leads to the estimate of the norm of the derivative. □

6. Some examples

We now apply the results to some relevant homogeneous operators.

6.1. Higher-order diffusion

Consider $m \in \mathbb{N}$ and a homogenous of degree $2m$ elliptic operator with constant real coefficients, $A_0 = \sum_{|\alpha|=2m} a_\alpha D^\alpha$ as in Sect. 2 and the associated parabolic equation

$$u_t + A_0 u = 0, \quad t > 0, \quad x \in \mathbb{R}^N. \tag{6.1}$$

Proposition 6.1. *Equation (6.1) defines a homogenous semigroup of degree $2m$ in $\mathcal{M}_U(\mathbb{R}^N)$, $S(t)$, that has a convolution kernel that satisfies*

$$k(t,z) = \frac{1}{t^{\frac{N}{2m}}} k_0\left(\frac{z}{t^{\frac{1}{2m}}}\right), \quad |k_0(z)| \le cG_b(z) = c\exp\left(\frac{-|z|^{\frac{2m}{2m-1}}}{4b^{\frac{1}{2m-1}}}\right), \quad t > 0,\ z \in \mathbb{R}^N,$$

for some $b > 0$ and estimates (2.3) and those in Theorem 3.1 and Proposition 3.2 with $a = 0$. Also k_0 is even and hence the kernel is symmetric.

(i) Except for multiples, $k(t,x)$ is the only N-self-similar solution of (6.1) originating in $\mathcal{M}_U(\mathbb{R}^N)$, and has constant $L^1(\mathbb{R}^N)$ norm,

$$k(t, x-y) = S(t)\delta_y(x), \quad x, y \in \mathbb{R}^N, \quad t > 0$$

and

$$A_0 k_0 = \frac{x}{2m}\nabla k_0 + \frac{N}{2m}k_0.$$

(ii) For $0 \le \beta < N$ there exist homogeneous measures of degree $-\beta$ in $\mathcal{M}_U(\mathbb{R}^N)$, which are actually in the Morrey spaces $\mathcal{M}^\beta(\mathbb{R}^N)$, that give rise to β-self-similar solutions as in (5.1), that is,

$$u(t) = \frac{1}{t^{\frac{\beta}{2m}}}(\Psi)_{\frac{1}{t^{\frac{1}{2m}}}}, \quad t > 0, \qquad A_0\Psi = \frac{x}{2m}\nabla\Psi + \frac{\beta}{2m}\Psi$$

of constant $\mathcal{M}^{\beta}(\mathbb{R}^N)$ norm.
(iii) For $\beta \in \mathbb{R}$, in self-similar variables, (6.1) is equivalent to the Ornstein–Uhlenbeck equation

$$v_s + A_0 v = \frac{1}{2m} x \nabla v + \frac{\beta}{2m} v, \quad s > 0$$

in such a way that

$$u(t,x) = \frac{1}{(1+t)^{\frac{\beta}{2m}}} v\left(\log(1+t), \frac{x}{(1+t)^{\frac{1}{2m}}}\right), \quad t > 0, \ x \in \mathbb{R}^N.$$

(iv) For $m = 1$ and $A_0 = -\Delta$, that is for the heat equation, $k_0(x) = (4\pi)^{-N/2} e^{-|x|^2/4}$ and for each $k \in \mathbb{N}$ and

$$\beta_k = N + k - 1, \qquad k = 1, 2, \ldots$$

there exists β_k-self-similar solutions of the form $u_k(t,x) = \frac{1}{t^{\frac{\beta_k}{2}}} \Psi(\frac{x}{\sqrt{t}})$ where Ψ belongs to the space spanned by $\{D^{\alpha} k_0, \ |\alpha| = k-1\}$.

Proof. The results in Sects. 2 and 3 apply to $L = A_0$, so we have a well-defined semigroup with a kernel and Gaussian bounds in $\mathcal{M}_U(\mathbb{R}^N)$. Hence, by Proposition 4.5 we get $a = 0$.

Now observe that if $\tilde{\phi}(x) = \phi(-x)$ then $(S(t)\tilde{\phi})(x) = (S(t)\phi)(-x)$ which easily gives that k_0 is even. Therefore, the kernel is symmetric. In particular, Proposition 3.3 applies and this and the analyticity of the semigroup curves in Theorem 3.1 and part (ii) in Remark 5.3 implies that the semigroup is injective in $\mathcal{M}_U(\mathbb{R}^N)$.

For part (i), according to Theorem 5.2 and Corollary 5.5, the initial data $\phi = \delta \in \mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^N) \subset \mathcal{M}_U(\mathbb{R}^N)$, which is homogeneous of degree $-N$, give rise to a N-self-similar solution as in (5.1), with constant $\mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^N)$ (or $L^1(\mathbb{R}^N)$) norm. Since the semigroup is injective, except for multiples, this is the only such N-self-similar solution. Since the kernel is N-self-similar and satisfies Eq. (6.1) pointwise, we get

$$k(t,x) = S(t)\delta(x) = \frac{1}{t^{\frac{N}{2m}}} \left(\Phi\right)_{\frac{1}{t^{\frac{1}{2m}}}}, \quad t > 0, \quad x \in \mathbb{R}^N$$

and $\Phi(x) = S(1)\delta = k_0(x)$. Also, using translations

$$S(t)\delta_y(x) = \tau_y S(t)\delta = k(t, x-y), \quad x, y \in \mathbb{R}^N, \quad t > 0$$

and we also get the elliptic equation for k_0 and part (i) is proved. Observe that an expression for k_0 in terms of Bessel functions can be found in [12].

Part (ii), follows from Corollary 5.5 and Theorem 5.2, while part (iii), in turn, follows from Theorem 5.6.

Finally, if $m=1$ and $A_0=-\Delta$, as $k(t,z)=\frac{1}{(4\pi t)^{\frac{N}{2}}}e^{-\frac{|z|^2}{4t}}$ we get the expression for $k_0(x)$. Also, the equation in self-similar variables can be written as

$$v_s-\frac{1}{\rho}\operatorname{div}(\rho\nabla v)=\frac{\beta}{2}v,\quad s>0.$$

with $\rho(x)=\exp(\frac{|x|^2}{4})$ so it can be naturally studied in $L^2_\rho(\mathbb{R}^N)$. Indeed, $A=-\frac{1}{\rho}\operatorname{div}(\rho\nabla\cdot)$ is self-adjoint in $L^2_\rho(\mathbb{R}^N)$ and since $H^1_\rho(\mathbb{R}^N)\subset L^2_\rho(\mathbb{R}^N)$ is compact, then it has an increasing sequence of eigenvalues which are explicitly given by $\mu_k=\frac{N+k-1}{2}$, $k=1,2,\ldots$, of which the first one is simple with positive eigenfunction $\Phi_1(x)=\rho^{-1}(x)=\exp(\frac{-|x|^2}{4})$ and the eigenspace of μ_k is spanned by $\{D^\alpha\Phi_1,\ |\alpha|=k-1\}$, see [10,11,14,23].

Therefore, β-self-similar solutions exist for $\beta_k=2\mu_k=N+k-1, k=1,2,\ldots$. □

6.2. Higher-order fractional diffusion

Given $m\in\mathbb{N}$ and $\alpha\in(0,1)$ such that $m\alpha$ is not an integer note that $L=(-\Delta)^{m\alpha}$ is homogenous of degree $2m\alpha$ and consider

$$u_t+(-\Delta)^{m\alpha}u=0,\quad t>0. \tag{6.2}$$

Notice the results below apply when $(-\Delta)^m$ is replaced by any other elliptic operator with constant real coefficients, $A_0=\sum_{|\alpha|=2m}a_\alpha D^\alpha$ as in Sect. 2.

Proposition 6.2. *Equation (6.2) defines a homogenous semigroup of degree $2m\alpha$ in $\mathcal{M}_U(\mathbb{R}^N)$, $S_\alpha(t)$, that has a convolution kernel that satisfies*

$$k_\alpha(t,z)=\frac{1}{t^{\frac{N}{2m\alpha}}}k_{0,\alpha}\left(\frac{z}{t^{\frac{1}{2m\alpha}}}\right),\quad z\in\mathbb{R}^N,\ t>0$$

for some even function $k_{0,\alpha}$ that satisfies

$$(-\Delta)^{m\alpha}\Phi=\frac{x}{2m\alpha}\nabla\Phi+\frac{N}{2m\alpha}\Phi$$

and $k_\alpha(t,x-y)=S_\alpha(t)\delta_y(x)=\tau_y S_\alpha(t)\delta$, $x,y\in\mathbb{R}^N$, $t>0$. In particular, the kernel is symmetric.

(i) Except for multiples, $k_\alpha(t,x)$ is the only N-self-similar solution of (6.2) originating in $\mathcal{M}_U(\mathbb{R}^N)$, and has constant $L^1(\mathbb{R}^N)$ norm.

(ii) For $0\le\beta<N$, there exist homogeneous measures of degree $-\beta$ in $\mathcal{M}_U(\mathbb{R}^N)$, which are actually in the Morrey spaces $\mathcal{M}^\beta(\mathbb{R}^N)$, that give rise to β-self-similar solutions as in (5.1), that is,

$$u(t)=\frac{1}{t^{\frac{\beta}{2m\alpha}}}(\Psi)_{\frac{1}{t^{\frac{1}{2m\alpha}}}},\quad t>0,\qquad(-\Delta)^{m\alpha}\Psi=\frac{x}{2m\alpha}\nabla\Psi+\frac{\beta}{2m\alpha}\Psi$$

of constant $\mathcal{M}^{\beta}(\mathbb{R}^N)$ norm.
(iii) For $\beta \in \mathbb{R}$, in self-similar variables, (6.2) is equivalent to the Ornstein–Uhlenbeck equation

$$v_s + (-\Delta)^{m\alpha} v = \frac{1}{2m\alpha} x\nabla v + \frac{\beta}{2m\alpha} v, \quad s > 0.$$

in such a way that

$$u(t,x) = \frac{1}{(1+t)^{\frac{\beta}{2m\alpha}}} v\left(\log(1+t), \frac{x}{(1+t)^{\frac{1}{2m\alpha}}}\right), \quad t > 0,\ x \in \mathbb{R}^N.$$

Proof. Observe that we proved in Proposition 6.1 that, for $\alpha = 1$, Theorem 3.1 and Proposition 3.2 hold with $a = 0$ and therefore we have (4.10) in those spaces, and we can apply the results in Sect. 4.4 to get all the stated properties of the kernel. Proposition 4.14 applies and, using part (ii) in Remark 5.3, the semigroup in the statement is analytic and injective in $\mathcal{M}_U(\mathbb{R}^N)$. The rest follows as in the proof of Proposition 6.1. □

In particular, for $m = 1, \alpha \in (0, 1)$ from the results in Sect. 4.4 we get the following result.

Theorem 6.3. *For $0 < \gamma < 1$, the fractional diffusion equation*

$$u_t + (-\Delta)^{\gamma} u = 0, \quad t > 0,\ x \in \mathbb{R}^N$$

has a nonnegative, convolution and N-self-similar kernel that satisfies

$$0 \le k_\gamma(t,x,y) = k_\gamma(t,x-y) \sim \frac{1}{t^{\frac{N}{2\gamma}}} H_\gamma\left(\frac{x-y}{t^{\frac{1}{2\gamma}}}\right),$$

where $H_\gamma(z) = C \min\{1, \frac{1}{|z|^{N+2\gamma}}\}$, $C > 0$.

Moreover, $k_{\frac{1}{2}}(t, x-y) = t^{-N} I_{\frac{1}{2}}(\frac{|x-y|}{t})$, where $I_{\frac{1}{2}}(z) = \dfrac{c_N}{(1+|z|^2)^{\frac{N+1}{2}}}$ and $c_N = \frac{\Gamma(\frac{N+1}{2})}{2^8 \pi^{\frac{N+1}{2}}}$.

Proof. First, that $k_\gamma(t,x,y) \ge 0$ follows from (4.13) since the heat kernel is positive and $f_{t,\alpha}(s) \ge 0$. Also that $k_\gamma(t,x,y) = k_\gamma(t,x-y)$ follows from Corollary 4.7. Finally, that the fractional kernel is self-similar follows from Corollary 4.8.

Fix $\gamma \in (0,1)$ and take $2 < m \in \mathbb{N}$ such that $N < 2m$. Now observe that the powers of the operator $-\Delta$ in, say, $L^p(\mathbb{R}^N)$, $1 < p < \infty$ satisfy , see [16, Theorem 5.4.3, p. 123],

$$\left((-\Delta)^m\right)^{\alpha} = (-\Delta)^{m\alpha}, \quad m \in \mathbb{N},\ \alpha > 0.$$

Now take the semigroup generated by $-(-\Delta)^m$. By Theorem 2.2, the kernel for this semigroup satisfies (2.3), that is $|k(t,x,y)| = |k(t,x-y)| \le c e^{\mathrm{at}} G_{\mathrm{bt}}(x-y)$, because of the invariance with respect to translations and Proposition 4.3. Since the semigroup is homogenous, by Proposition 4.5 we also have $k(t,z) = \frac{1}{t^{\frac{N}{2m}}} k_0(\frac{z}{t^{\frac{1}{2m}}})$. Therefore,

$$|k(1,z)| = |k_0(z)| \leq cG_b(x-y)$$

and self-similarity of the kernel gives in turn

$$|k(t,x,y)| = |k(t,x-y)| \leq \frac{c}{t^{\frac{N}{2m}}} G_b\left(\frac{x-y}{t^{\frac{1}{2m}}}\right),$$

i.e. the semigroup generated by $-(-\Delta)^m$ satisfies the assumptions in Proposition 4.9 with $\sigma = 2m$. Hence, we take $\alpha = \frac{\gamma}{m} < \frac{1}{2}$ and Proposition 4.9 gives the upper bound. The lower bounds can be seen in [4,5].

Now if $\alpha = \frac{1}{2}$ we have $f_{t,\frac{1}{2}}(s) = (4^9\pi)^{-\frac{1}{2}} t s^{-\frac{3}{2}} e^{-\frac{t^2}{4s}}$, see [24, (32), p. 268], and then (4.13) with $k(s,x,y) = (4\pi)^{-\frac{N}{2}} s^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4s}}$ gives

$$k_{\frac{1}{2}}(t,x-y) = 2^{-(N+9)} \pi^{-\frac{N+1}{2}} t \int_0^\infty s^{-\frac{N+3}{2}} e^{-\frac{t^2+|x-y|^2}{4s}} \, ds.$$

Changing variables as $s = \xi^{-1}$ gives, up to the constant,

$$k_{\frac{1}{2}}(t,x-y) = C \int_0^\infty \xi^{\frac{N-1}{2}} e^{-\frac{t^2+|x-y|^2}{4}\xi} \, d\xi$$

and by (4.15) we get the result. □

Remark 6.4. Observe that for $\gamma \in (0,1)$, $H_\gamma(z) \sim I_\gamma(z) = \frac{1}{(1+|z|^2)^{\frac{N}{2}+\gamma}}$.

Using estimates of the kernel as the ones in Theorem 6.3, it was proved in [5] that the semigroup can be extended to the optimal class of measures satisfying

$$\int_{\mathbb{R}^N} (1+|x|)^{-(N+2\alpha)} \, d|\mu|(x) < \infty$$

that contains $\mathcal{M}_U(\mathbb{R}^N)$. This space contains homogeneous functions $\phi(x) = c|x|^\gamma$ for $0 \leq \gamma < 2\alpha$ of degree γ, so they give rise to $(-\gamma)$-self-similar solutions, [5, Section 7.2].

As a consequence, we get the following result.

Corollary 6.5. *For $0 < \gamma < 1$ and $Re(\lambda) < 0$, the fractional Green's function of $(-\Delta)^\gamma$ is positive if $\lambda < 0$, of convolution type and self-similar and satisfies the estimates from above in Corollary 4.13 with $\sigma = 2$.*

Moreover, for $\gamma = \frac{1}{2}$ and $\lambda < 0$ we have the estimate from below of the form

$$G_{\frac{1}{2},\lambda}(x,y) \geq c_N e^{\lambda} \begin{cases} \ln \frac{\sqrt{1+|x-y|^2}}{|x-y|} & \text{if } N = 1, \\ \frac{1}{N-1}\left(\frac{1}{|x-y|^{N-1}} - \frac{1}{(1+|x-y|^2)^{\frac{N-1}{2}}}\right) & \text{if } N \geq 2. \end{cases} \tag{6.3}$$

Proof. That the fractional Green's function satisfies $G_{\gamma,\lambda}(x, y) = G_{\gamma,\lambda}(x-y)$ follows from Corollary 4.7 and that it is self-similar follows from Corollary 4.8.

From the bounds in Theorem 6.3, we can estimate the Green's function as in the proof of Corollary 4.13 and we get the result. Also, if $\lambda < 0$, from (4.11) we get $G_{\gamma,\lambda}(x, y) > 0$ since the fractional heat kernel is positive by Theorem 6.3.

Finally for $\gamma = \frac{1}{2}$, since $k_{\frac{1}{2}}(t, x, y) = \frac{c_N t}{(t^2+|x-y|^2)^{\frac{N+1}{2}}}$, from (4.11) we have for $\lambda < 0$

$$G_{\frac{1}{2},\lambda}(x, y) = c_N \int_0^1 e^{\lambda t} \frac{t}{(t^2 + |x - y|^2)^{\frac{N+1}{2}}} \,\mathrm{d}t + c_N \int_1^\infty e^{\lambda t} \frac{t}{(t^2 + |x - y|^2)^{\frac{N+1}{2}}} \,\mathrm{d}t = I_1 + I_2.$$

Observe that $e^\lambda \leq e^{\lambda t} \leq 1$ for $t \in [0, 1]$ and by direct integration

$$\int_0^1 \frac{t}{(t^2 + |x - y|^2)^{\frac{N+1}{2}}} \,dt = \begin{cases} \ln \frac{\sqrt{1+|x-y|^2}}{|x-y|} & \text{if } N = 1 \\ \frac{1}{N-1}\left(\frac{1}{|x-y|^{N-1}} - \frac{1}{(1+|x-y|^2)^{\frac{N-1}{2}}}\right) & \text{if } N \geq 2, \end{cases}$$

so the lower bound is (6.3). □

6.3. Heat equation with Hardy potential

Consider $L = (-\Delta) + \frac{\lambda}{|x|^2}$ with $\lambda \geq -\lambda_*$, $\lambda_* = \frac{(N-2)^2}{4}$ the critical Hardy constant, and the evolution equation

$$u_t - \Delta u + \frac{\lambda}{|x|^2} u = 0, \quad t > 0, \; x \in \mathbb{R}^N. \tag{6.4}$$

An associated number that plays an important role in the analysis is

$$\sigma = \sigma(\lambda) = \frac{N-2}{2} - \sqrt{\frac{(N-2)^2}{4} + \lambda}$$

which is the smallest root of $\sigma^2 - (N-2)\sigma - \lambda = 0$ and satisfies $\sigma(-\lambda_*) = \frac{N-2}{2}$, $0 \leq \sigma(\lambda) < \frac{N-2}{2}$ if $-\lambda_* < \lambda \leq 0$, $\sigma(0) = 0$, while $\sigma(\lambda) < 0$ for $\lambda > 0$, [3].

Then, we have the following result.

Proposition 6.6. *(i) For $\lambda \geq -\lambda_*$, equation (6.4) defines an analytic semigroup of contractions in $L^2(\mathbb{R}^N)$. It also defines a contraction semigroup in $L^p(\mathbb{R}^N)$ for $1 \leq p < \frac{N}{2}$ and $\lambda > -\tilde{\lambda}_*(p) \geq -\lambda_*$.*

(ii) The semigroup above has a self-similar kernel that satisfies the following estimates.

a) For $\lambda > -\lambda_*$

$$c_1 H_\sigma\left(\frac{|x|}{\sqrt{t}}\right) H_\sigma\left(\frac{|y|}{\sqrt{t}}\right) G_{b_1 t}(x-y) \leq k(t, x, y) \leq c_2 H_\sigma\left(\frac{|x|}{\sqrt{t}}\right) H_\sigma\left(\frac{|y|}{\sqrt{t}}\right) G_{b_2 t}(x-y)$$

where $G_{\mathrm{bt}}(x)$ is the heat kernel (2.4) with $m = 1$ and $H_\sigma(r)$ is a bounded function in $C^2((0,\infty))$ such that $H_\sigma(r) = \begin{cases} \frac{1}{r^\sigma} & r \leq 1 \\ 1 & r \geq 2. \end{cases}$

b) For $\lambda = -\lambda_*$, we have

$$k(t,x,y) \le \frac{c}{t^{\frac{N}{2}}} H_\sigma\left(\frac{|x|}{\sqrt{t}}\right) H_\sigma\left(\frac{|y|}{\sqrt{t}}\right).$$

(iii) For $\lambda \ge -\lambda_*$ and $\beta \in \mathbb{R}$, in self-similar variables, (6.4) is equivalent to the Ornstein–Uhlenbeck equation

$$v_s - \Delta v + \frac{\lambda}{|x|^2} v = \frac{1}{2} x \nabla v + \frac{\beta}{2} v, \quad s > 0.$$

in such a way that

$$u(t,x) = \frac{1}{(1+t)^{\frac{\beta}{2}}} v\left(\log(1+t), \frac{x}{(1+t)^{\frac{1}{2}}}\right), \quad t > 0,\ x \in \mathbb{R}^N$$

(iv) For $\lambda \ge \lambda_*$, there exists an increasing sequence of positive numbers $\{\beta_k(\lambda)\} \to \infty$, such that $\beta_k(\lambda)$ is increasing in λ for each $k \in \mathbb{N}$ and (6.4) has $\beta_k(\lambda)$-self-similar solutions of the form

$$u_k(t,x) = \frac{1}{t^{\frac{\beta_k}{2}}} \Psi\left(\frac{x}{\sqrt{t}}\right)$$

with Ψ in some finite dimensional space. Moreover for $-\lambda_* \le \lambda < 0$, $\beta_1(\lambda) = \frac{N+2+2\sqrt{\lambda_*+\lambda}}{2}$ so $\beta_1(-\lambda_*) = \frac{N+2}{2}$ and $\beta_1(0) = N$. For $\beta_1(\lambda)$ can take $\Psi(x) = |x|^{\sqrt{\lambda_*+\lambda}-\frac{N-2}{2}} \exp(-\frac{|x|^2}{4})$ so

$$u(x,t) = t^{-(1+\sqrt{\lambda_*+\lambda})} |x|^{\sqrt{\lambda_*+\lambda}-\frac{N-2}{2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

Proof. For the semigroup in $L^2(\mathbb{R}^N)$, see, e.g. [7,22,23]. For the semigroup in $L^p(\mathbb{R}^N)$, see [7, Theorem 8.11].

For part (ii), that the kernel is self-similar follows from Proposition 4.5, while the estimates on the kernel follow from [18,19,22] for $\lambda > -\lambda_*$ and from [19, Theorem 1] for $\lambda = -\lambda_*$.

Theorem 5.6 gives the expression in self-similar variables in part (iii). Observe that this equation can be written as $v_s - \frac{1}{\rho}\mathrm{div}(\rho \nabla v) + \frac{\lambda}{|x|^2} v = \frac{\beta}{2} v$, $s > 0$, with $\rho(x) = \exp(\frac{|x|^2}{4})$. Again, since $\lambda \ge -\lambda_*$, $A = -\frac{1}{\rho}\mathrm{div}(\rho \nabla \cdot) + \frac{\lambda}{|x|^2} I$ is self-adjoint in $L^2_\rho(\mathbb{R}^N)$ and since $H^1_\rho(\mathbb{R}^N) \subset L^2_\rho(\mathbb{R}^N)$ is compact, then it has an increasing sequence of positive eigenvalues $\{\mu_k\}$. Hence, β-self-similar solutions exist for $\beta_k = 2\mu_k$.

For the computation of $\beta_1(\lambda)$ and the corresponding Ψ, further details and a complete description of the spectrum above for $-\lambda_* \le \lambda < 0$, see [23, Section 9]. □

Remark 6.7. In [22, Proposition 3.1], it was shown that for $\lambda \ge -\lambda_*$, the equation has the following explicit solutions:

$$u_1(x,t) = t^{\sigma - \frac{N}{2}} |x|^{-\sigma} \exp\left(-\frac{|x|^2}{4t}\right), \quad u_2(x,t) = t^{\frac{N}{2}-\sigma-2} |x|^{2-N+\sigma} \exp\left(-\frac{|x|^2}{4t}\right)$$

which coincide for $\lambda = -\lambda_*$ with $u_0(x,t) = t^{-1}|x|^{-\frac{N-2}{2}}\exp(-\frac{|x|^2}{4t})$, see [22, Remark 3.2]. These are β-self-similar solutions as in (5.1) for $\beta_1 = N-\sigma$, $\Phi_1(x) = |x|^{-\sigma}\exp(-\frac{|x|^2}{4})$ and $\beta_2 = \sigma+2$, $\Phi_2(x) = |x|^{2-N+\sigma}\exp(-\frac{|x|^2}{4})$, respectively. Notice that u_0 is self-similar for $\beta = \beta_1(-\lambda_*) = \frac{N+2}{2}$ and is precisely the self-similar solution in part (iv) of Proposition 6.6, with $\lambda = -\lambda_*$, with $\Psi(x) = |x|^{-\frac{N-2}{2}}\exp(-\frac{|x|^2}{4})$.

Data availability Data sharing is not applicable to this article as no data sets were generated or analysed during the current study.

Appendix A. Dilations and homogeneous distributions

We address here various aspects involving dilations and homogeneous distributions. For functions in $\mathbb{R}^N$, the dilations are given by

$$D_R\phi(x) = \phi_R(x) = \phi(Rx), \quad x \in \mathbb{R}^N, \quad R > 0$$

and then the family of linear operators $\{D_R\}_{R>0}$ forms a multiplicative group, that is

$$D_R \circ D_S = D_{RS}, \ R, S > 0, \quad D_1 = I.$$

For distributions (either in $\mathscr{D}'(\mathbb{R}^N)$ or $\mathscr{S}'(\mathbb{R}^N)$) the dilation ϕ_R is defined by duality as

$$\langle \phi_R, \varphi\rangle = \left\langle \phi, \frac{1}{R^N}\varphi_{\frac{1}{R}}\right\rangle, \qquad \text{for all test functions } \varphi.$$

This applies, in particular, to Radon measures. Also, the group $\{D_R\}_{R>0}$ extends to distributions in $\mathcal{M}_{\text{loc}}(\mathbb{R}^N)$, $\mathscr{D}'(\mathbb{R}^N)$, or $\mathscr{S}'(\mathbb{R}^N)$.

Then, we have the following result.

Proposition A.1. *For any distributions $\phi \in \mathscr{D}'(\mathbb{R}^N)$, the dilation curve*

$$(0,\infty) \ni R \mapsto D_R(\phi) = \phi_R \in \mathscr{D}'(\mathbb{R}^N)$$

is differentiable in $\mathscr{D}'(\mathbb{R}^N)$ *with*

$$\partial_R \phi_R = x(\nabla\phi)_R = \frac{1}{R}(x\nabla\phi)_R = \frac{1}{R}x\nabla(\phi_R).$$

Proof. Step 1 We start by proving the result for test functions $\varphi \in \mathscr{D}(\mathbb{R}^N)$. That is we prove that $(0,\infty) \ni R \mapsto D_R(\varphi) = \phi_R \in \mathscr{D}(\mathbb{R}^N)$ is differentiable in $\mathscr{D}(\mathbb{R}^N)$ with

$$\partial_R \varphi_R = x(\nabla\varphi)_R = \frac{1}{R}(x\nabla\varphi)_R = \frac{1}{R}x\nabla(\varphi_R).$$

Define

$$\Delta_R\varphi(x,h) = \frac{1}{h}\big(\varphi_{R+h}(x) - \varphi_R(x)\big)$$

which converges, as $h \to 0$, to $\partial_R\varphi_R(x)$ pointwise for $x \in \mathbb{R}^N$. Also, by the mean value theorem, since φ has compact support, we get $\Delta_R\varphi(h) \to \partial_R\varphi_R$ uniformly in $\mathbb{R}^N$.

Now, we prove the convergence is also in $\mathscr{D}(\mathbb{R}^N)$. For this, for any multi-index α we have $\partial^\alpha\Delta_R\varphi(x,h) = \frac{1}{h}\big((R+h)^{|\alpha|}(\partial^\alpha\varphi)_{R+h}(x) - R^{|\alpha|}(\partial^\alpha\varphi)_R(x)\big)$ which we can write as

$$\frac{(R+h)^{|\alpha|} - R^{|\alpha|}}{h}(\partial^\alpha\varphi)_R(x) + (R+h)^{|\alpha|}\Delta_R\partial^\alpha\varphi(x,h).$$

Hence, from the argument above, this converges as $h \to 0$, uniformly in $\mathbb{R}^N$, to

$$|\alpha|R^{|\alpha|-1}(\partial^\alpha\varphi)_R(x) + R^{|\alpha|}x(\nabla\partial^\alpha\varphi)_R(x).$$

On the other hand, it is not difficult to show that

$$\partial^\alpha\partial_R\varphi_R(x) = R^{|\alpha|}x(\nabla\partial^\alpha\varphi)_R(x) + |\alpha|R^{|\alpha|-1}(\partial^\alpha\varphi)_R(x).$$

Therefore, $\partial^\alpha\Delta_R\varphi(h) \to \partial^\alpha\partial_R\varphi_R$ as $h \to 0$, uniformly in $\mathbb{R}^N$ and the claim is proved.
Step 2 Now it is standard to show that we can apply the chain rule to prove that

$$t \mapsto a(t)\varphi_{R(t)} \in \mathscr{D}(\mathbb{R}^N)$$

is differentiable in $\mathscr{D}(\mathbb{R}^N)$ with derivative $a'(t)\varphi_{R(t)} + a(t)(\partial_R\varphi)_{R(t)}R'(t)$, provided $a(t)$, $R(t)$ are differentiable.
Step 3 Now for a distribution $\phi \in \mathscr{D}'(\mathbb{R}^N)$ and any $\varphi \in \mathscr{D}(\mathbb{R}^N)$, $R > 0$ and $h \in \mathbb{R}$,

$$\frac{1}{h}\big[\langle\phi_{R+h} - \phi_R, \varphi\rangle\big] = \frac{1}{h}\left\langle\phi, \frac{1}{(R+h)^N}\varphi_{\frac{1}{R+h}} - \frac{1}{R^N}\varphi_{\frac{1}{R}}\right\rangle \longrightarrow \left\langle\phi, \frac{-N}{R^{N+1}}\varphi_{\frac{1}{R}} - \frac{1}{R^{N+2}}(\partial_R\varphi)_{\frac{1}{R}}\right\rangle$$

as $h \to 0$, where we used Step 2 for $R \mapsto \frac{1}{R^N}\varphi_{\frac{1}{R}}$. To conclude notice that using Step 1,

$$M_R(\varphi) := \frac{N}{R^{N+1}}\varphi_{\frac{1}{R}} + \frac{1}{R^{N+2}}(\partial_R\varphi)_{\frac{1}{R}} = \frac{1}{R^{N+1}}\big(N\varphi + x\nabla\varphi\big)_{\frac{1}{R}} = \frac{1}{R^{N+1}}(\nabla(x\varphi))_{\frac{1}{R}}.$$

Also, using Step 1 above we can alternatively write

$$M_R(\varphi) = \frac{1}{R^{N+1}}\big(N\varphi_{\frac{1}{R}} + x\nabla(\varphi_{\frac{1}{R}})\big) = \frac{1}{R^{N+1}}\nabla(x\varphi_{\frac{1}{R}}).$$

Using the first expression above, we get

$$\langle \partial_R \phi_R, \varphi\rangle = \frac{-1}{R^{N+1}}\langle \phi, (\nabla(x\varphi))_{\frac{1}{R}}\rangle = \frac{-1}{R}\langle \phi_R, \nabla(x\varphi)\rangle = \frac{1}{R}\langle x\nabla(\phi_R), \varphi\rangle.$$

Using the second expression we get $\langle \partial_R \phi_R, \varphi\rangle = \frac{1}{R^{N+1}}\langle x\nabla\phi, \varphi_{\frac{1}{R}}\rangle = \frac{1}{R}\langle (x\nabla\varphi)_R, \varphi\rangle$ and the result is proved. □

Below, we turn our attention to homogeneous distributions, which in Sect. 5 play an important role in producing self-similar solutions to the evolution problem (1.1).

Recall that a function $h(x)$ in $\mathbb{R}^N$ is homogeneous of degree $\sigma \in \mathbb{R}$, if $h(Rx) = R^\sigma h(x)$, for $R > 0, x \in \mathbb{R}^N$. For example, $h(x) = |x|^\sigma$ or $h(x) = x^\alpha$ for some multi-index such that $|\alpha| = \sigma$. Analogously, a distribution is homogeneous of degree $\sigma \in \mathbb{R}$ if $\phi_R = R^\sigma \phi$. For example, the Dirac delta is homogeneous of degree $-N$, since for a compactly supported function $\langle \delta_R, \varphi\rangle = \langle \delta, \frac{1}{R^N}\varphi_{\frac{1}{R}}\rangle = \frac{1}{R^N}\varphi(0) = \frac{1}{R^N}\langle \delta, \varphi\rangle$, i.e. $\delta_R = R^{-N}\delta$.

Observe, for $2m < N$ or odd N with $N \le 2m$, for the polyharmonic operator $(-\Delta)^m$ the fundamental solution $\frac{1}{|x|^{N-2m}}$ is homogeneous of degree $-(N-2m)$ see [17, p. 7].

Lemma A.2. *Assume X is a linear space of functions or distributions in $\mathbb{R}^N$ invariant by rescaling and $\|\cdot\|_X$ is a homogenous norm in X of degree $\nu \in \mathbb{R}$. Then, if $\phi \in X\setminus\{0\}$ is homogeneous of degree σ, then $\sigma = \nu$.*

Proof. Just note $\|\phi_R\| = R^\nu\|\phi\| = R^\sigma\|\phi\|$ for all $R > 0$.

The next result characterises homogeneous distributions. According to the result below, homogeneous distributions are the eigendistributions of the operator $G_0 = -x\nabla$.

Proposition A.3. (Euler's theorem) *A distribution is homogeneous of degree $\sigma \in \mathbb{R}$ if and only if $x\nabla\phi = \sigma\phi$ in $\mathscr{D}'(\mathbb{R}^N)$.*

Proof. The result follows by Proposition A.1 differentiating $\phi_R = R^\sigma\phi$ at $R = 1$. For the converse, given $\varphi \in \mathscr{D}(\mathbb{R}^N)$, setting $f(R) = \langle \phi_R, \varphi\rangle$ one gets, again by Proposition A.1,

$$f'(R) = \langle x(\nabla\phi)_R, \varphi\rangle = \frac{1}{R}\langle (x\nabla\phi)_R, \varphi\rangle = \frac{\sigma}{R} f(R)$$

which gives $f(R) = f(1)R^\sigma$ and then $\phi_R = R^\sigma\phi$.

The following results determine homogeneous distributions in some of the spaces appearing before. Observe that the function $\phi(x) = \frac{1}{|x|^\beta}$, which is homogeneous of degree $\sigma = -\beta$ and locally integrable for $\beta < N$, captures all the possibilities in the Lemma.

Lemma A.4. *For $1 \le p < \infty$, the space $L^p(\mathbb{R}^N)$ contains no nonzero homogeneous functions. In $L^\infty(\mathbb{R}^N)$ all homogeneous functions have degree 0 and are of the form $g(\frac{x}{|x|})$, $x \neq 0$, for some bounded function in the unit sphere S^N.*

The space $M^{p,\ell}(\mathbb{R}^N)$, for $\ell \in [0, N)$, contains homogeneous functions only of degree $-\frac{\ell}{p}$.

For $1 \le p < \infty$, the space $L^p_U(\mathbb{R}^N)$ contains homogeneous functions only of degree $-\frac{N}{p} < \sigma \le 0$ and they actually belong to $M^{p,\ell}(\mathbb{R}^N)$ with $\ell = -p\sigma \in [0, N)$.

Proof. First, observe that $\phi \in L^p_{\text{loc}}(\mathbb{R}^N)$, $1 \le p < \infty$, can only be σ-homogeneous for $\sigma > -\frac{N}{p}$. To see this note that for such a ϕ and for $R > 0$, setting $y = Rx$,

$$\int_{B(0,1)} |\phi(y)|^p \, \mathrm{d}y = R^{\sigma p+N} \int_{B(0,\frac{1}{R})} |\phi(x)|^p \, \mathrm{d}x.$$

So, if $\phi = 0$ in $B(0, 1)$, homogeneity $\phi(Rx) = R^\sigma \phi(x)$ yields $\phi = 0$. Otherwise, as $R \to \infty$

$$\int_{B(0,\frac{1}{R})} |\phi(x)|^p \, d\mathrm{d} = \frac{1}{R^{\sigma p+N}} \int_{B(0,1)} |\phi(y)|^p \, d\mathrm{d} \to 0$$

which implies $\sigma > -\frac{N}{p}$.

From Lemma A.2, if $\phi \in L^p(\mathbb{R}^N)$, $1 \le p < \infty$ is homogeneous, then $\sigma = -\frac{N}{p}$ which contradicts the argument above. For $p = \infty$, the argument above implies $\sigma = 0$. Thus, $\phi(Rx) = \phi(x)$ for $R > 0$ and ϕ is constant along rays. Therefore, $\phi(x) = g(\frac{x}{|x|})$, $x \neq 0$, for some bounded function.

In Morrey spaces, if $\phi \in M^{p,\ell}(\mathbb{R}^N)$ is σ-homogeneous, then Lemma A.2 implies $\sigma = -\frac{\ell}{p}$. To conclude we check that if ϕ is homogeneous of degree $\sigma = -\frac{\ell}{p}$ and $\phi \in L^p_U(\mathbb{R}^N)$, as, e.g. $\phi(x) = \frac{1}{|x|^\beta}$ with $\beta = \frac{\ell}{p}$, then ϕ belongs to $M^{p,\ell}(\mathbb{R}^N)$. In fact for $x_0 \in \mathbb{R}^N$, we get

$$R^{\ell-N} \int_{B(x_0,R)} |\phi(y)|^p \, \mathrm{d}y = R^{\ell+\sigma p} \int_{B(\frac{x_0}{R},1)} |\phi(x)|^p \, \mathrm{d}x = \int_{B(\frac{x_0}{R},1)} |\phi(x)|^p \, \mathrm{d}x \le C$$

independent of x_0, R.

Finally, assume $\phi \in L^p_U(\mathbb{R}^N)$, $1 \le p < \infty$, is σ-homogeneous with $\sigma > -\frac{N}{p}$. Then, for $x_0 \in \mathbb{R}^N$ and $R > 0$, setting $y = Rx$,

$$\int_{B(Rx_0,1)} |\phi(y)|^p \, \mathrm{d}y = R^{\sigma p+N} \int_{B(x_0,\frac{1}{R})} |\phi(x)|^p$$

where the left-hand side is uniformly bounded independently of x_0 and R and the Lebesgue differentiation theorem implies that, as $R \to \infty$, $R^N \int_{B(x_0,\frac{1}{R})} |\phi(x)|^p \to \phi(x_0)$ for a.e. $x_0 \in \mathbb{R}^N$. Hence, we get $\sigma \le 0$. By the argument above, $\phi \in M^{p,\ell}(\mathbb{R}^N)$ for $\ell = -p\sigma \in [0, N)$.

Clearly the function $\phi(x) = \frac{1}{|x|^{\beta}}$ is homogeneous of degree $\sigma = -\beta$ and belongs to $L^p_U(\mathbb{R}^N)$ as soon as $-\frac{N}{p} < \sigma \le 0$ (and thus to $M^{p,\ell}(\mathbb{R}^N)$ for $\ell = p\beta \in [0, N)$). □

We complete this analysis now with the case of measures.

Lemma A.5. *The space $\mathcal{M}^{\ell}(\mathbb{R}^N)$, for $\ell \in [0, N)$, contains homogeneous measures only of degree $-\ell$.*

The only homogeneous Radon measures of degree $-N$ are multiples of Dirac's delta δ. In particular, in $\mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^N)$ the only homogeneous measures are multiples of δ.

The space $\mathcal{M}_U(\mathbb{R}^N)$ contains homogeneous measures only of degree $-N \le \sigma \le 0$ and they actually belong to $\mathcal{M}^{\ell}(\mathbb{R}^N)$ with $\ell = -\sigma \in [0, N)$ or are multiples of δ if $\sigma = -N$.

Proof. Observe that a Radon measure $\mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^N)$ is homogeneous of degree $\sigma \in \mathbb{R}$ if and only if, for every open set $\mathcal{O} \subset \mathbb{R}^N$

$$\mu(R\mathcal{O}) = R^{\sigma+N}\mu(\mathcal{O}). \tag{6.5}$$

In such a case, $|\mu|$ is also homogeneous and satisfies (6.5). Hence, if we take the unit ball $\mathcal{O} = B(0, 1)$, we get

$$|\mu|(B(0, R)) = R^{\sigma+N}|\mu|(B(0, 1))$$

and since the left-hand side must be non-decreasing in R, we get $\sigma \ge -N$ or $|\mu|$ is identically zero.

Now we prove that any $-N$-homogeneous Radon measure is a multiple of δ. If $\sigma = -N$, then $|\mu|(B(0, R)) = |\mu|(B(0, 1))$ and taking $R \to \infty$ we get $\mu \in \mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^N)$. Additionally, taking $0 < R < 1$ and $R > 1$ we get, respectively,

$$|\mu|(B(0, 1) \setminus B(0, R)) = 0, \quad |\mu|(B(0, R) \setminus B(0, 1)) = 0$$

which implies that μ is supported at $\{0\}$. So either $\mu = 0$ or $\mu = c\delta$.

If $\mu \in \mathcal{M}_{\mathrm{BTV}}(\mathbb{R}^N)$ is homogeneous of degree σ, then Lemma A.2 gives $\sigma = -N$ and μ is a multiple of δ.

Therefore, hereafter we can assume $\sigma > -N$. Now observe that if $\mu \in \mathcal{M}_U(\mathbb{R}^N)$ is homogeneous of degree σ, then $\sigma \le 0$ and $\mu \in \mathcal{M}^{\ell}(\mathbb{R}^N)$ with $\ell = -\sigma$. In fact, taking $\mathcal{O} = B(\frac{x_0}{R}, 1)$ in (6.5) we get $R^{-\sigma-N}|\mu|(B(x_0, R)) = |\mu|(B(\frac{x_0}{R}, 1)) \le c$ for some constant independent of $x_0 \in \mathbb{R}^N$ and $R > 0$.

If $\sigma > 0$, then for every compact set and $\varepsilon > 0$ there exists a finite covering of K with balls of radius not exceeding ε. Thus, $|\mu|(K) \le c\sum_i r_i^{N+\sigma} \le c\varepsilon^{\sigma}\sum_i r_i^N$. Minimising over all finite coverings of K, we get $\varepsilon^{-\sigma}|\mu|(K) \le cH_{N,\varepsilon}(K)$. As $\varepsilon \to 0$, the right-hand side converges to the Lebesgue measure of K and therefore $|\mu|(K) = 0$ for any compact set. As μ is regular, we get $\mu = 0$. Hence, $\sigma \le 0$ and then $\mu \in \mathcal{M}^{\ell}(\mathbb{R}^N)$ for $\ell = -\sigma$.

Finally, if a Morrey measure is homogeneous of degree σ, Lemma A.2 gives $\sigma = -\ell$. As in Lemma A.4, $\mu = \frac{dx}{|x|^\beta}$ is homogeneous of degree $\sigma = -\beta$ and in $\mathcal{M}_U(\mathbb{R}^N)$, $0 \leq \beta < N$. □

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