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GAUGE INVARIANCE ON PRINCIPAL $SU(2)$-BUNDLES

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Abstract. Given a principal $SU(2)$-bundle $\pi\colon P \to M$, the structure of differential forms on $J^1(P)$ which are invariant under the natural representation of the gauge algebra of P, is analyzed.

1. Introduction

Given a principal $SU(2)$-bundle $\pi\colon P \to M$, we study the structure of the algebra of differential forms on $J^1(P)$ (the 1-jet bundle of local sections of π) which are invariant under the natural representation of the Lie algebra of all infinitesimal automorphisms of P and also under the subalgebra of π-vertical infinitesimal automorphisms, the so-called gauge algebra of P. In [HM1], [HM2] the structure of the invariant differential forms with respect to the above two algebras has been determined in the case of a principal $U(1)$-bundle. In the non-abelian case however this structure seems to be much more complex. Because of this we proceed to analyze the invariant differential forms on the 1-jet bundle of sections of P as a first step to obtain the invariant differential forms on the bundle of connections. This is a well-known technique in gauge theories (*e.g.*, see [A2], [B], [G], [GS], [K]) which is based on the fact that the bundle of connections can be identified to the quotient bundle of $J^1(P)$ modulo de action of the structure group, so that we recover the action of the gauge algebra and also the algebra of all infinitesimal automorphisms of P on connections as a quotient action of these algebras on $J^1(P)$ by infinitesimal contact transformations.

Key words and phrases: Automorphism of a principal bundle, bundle of connections, contact forms, gauge algebra, gauge group, infinitesimal contact transformation, invariant differential forms,jet bundle.

2. Preliminaries and Notations

2.1. Automorphisms

An *automorphism* of a principal G-bundle $\pi\colon P \to M$ is an equivariant diffeomorphism $\Phi\colon P \to P$; *i.e.*,

$$\Phi(u \cdot g) = \Phi(u) \cdot g, \quad \forall u \in P, \forall g \in G.$$

The set of all automorphisms of P is a group denoted by $\mathrm{Aut}P$. An automorphism $\Phi \in \mathrm{Aut}P$ induces a diffeomorphism on the ground manifold $\phi\colon M \to M$, such that $\pi \circ \Phi = \phi \circ \pi$. If ϕ is the identity map then Φ is said to be a *gauge transformation*. The set of all gauge transformations is a normal subgroup $\mathrm{Gau}P \subset \mathrm{Aut}P$, and we have an exact sequence of groups

$$1 \to \mathrm{Gau}P \to \mathrm{Aut}P \to \mathrm{Diff}M.$$

In the case of the trivial bundle $\mathrm{pr}_1\colon P = M \times G \to M$, every $\Phi \in \mathrm{Aut}P$ can be written as

$$\Phi(x, g) = (\phi(x), \psi(x) \cdot g),$$

with $x \in M$, $g \in G$, $\psi \in C^\infty(M, G)$. Hence

$$\mathrm{Gau}(M \times G) \simeq C^\infty(M, G).$$

2.2. G-invariant vector fields

A vector field $X \in \mathfrak{X}(P)$ is G-invariant if and only if

$$R_g \cdot X = X, \quad \forall g \in G.$$

Let Φ_t be the flow of X. Then,

$$X \text{ is } G\text{-invariant } \Leftrightarrow \Phi_t \in \mathrm{Aut}P, \quad \forall t \in \mathbb{R}.$$

The G-invariant vector fields are a Lie algebra denoted by $\mathrm{aut}P$. G-invariant vector fields are π-projectable. The *gauge algebra* of P is the ideal $\mathrm{gau}P$ of π-vertical vector fields in the Lie algebra $\mathrm{aut}P$; *i.e.*,

$$\mathrm{gau}P = \{X \in \mathrm{aut}P \mid \pi_* X = 0\}.$$

The group G acts naturally on $T(P)$ by setting:

$$X \cdot g = (R_g)_*(X), \quad \forall X \in T(P), \forall g \in G.$$

The quotient manifold $T(P)/G$ exists and we have

$$\mathrm{aut}P \simeq \Gamma(M, T(P)/G).$$

If $\mathrm{ad}P = (P \times \mathfrak{g})/G$ is the bundle associated to P by the adjoint representation of G on $\mathfrak{g}$ (*cf.* [A1]), then

$$\mathrm{gau}P \simeq \Gamma(M, \mathrm{ad}P).$$

By passing to the quotient modulo G in the exact sequence of vector bundles over P,

$$0 \to V(P) \to T(P) \xrightarrow{\pi_*} \pi^* TM \to 0,$$

where $V(P)$ stands for the *vertical subbundle*, we obtain an exact sequence of vector bundles over M (the so-called *Atiyah sequence*, [A1], [Bl], [G], [GS], [K]),

$$0 \to \mathrm{ad}P \to T(P)/G \xrightarrow{\pi_*} TM \to 0.$$

2.3. $SU(2)$ NOTATIONS

We consider the *standard basis* of the Lie algebra $\mathfrak{su}(2)$ normalized by the factor $\frac{1}{2}$; *i.e.*,

$$B_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; B_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; B_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

with $i = \sqrt{-1}$. Note that $2\,i\,B_a$, $1 \leq a \leq 3$ are the *Pauli matrices*. We have

$$[B_1, B_2] = B_3, \quad [B_2, B_3] = B_1, \quad [B_3, B_1] = B_2.$$

As is well-known (*e.g.*, see [NS]), $SU(2)$ can be identified to the 3-sphere; *i.e.*,

$$SU(2) \simeq S^3 \subset \mathbb{C}^2.$$

Let $(y^0 + iy^1, y^2 + iy^3)$ be the standard coordinates in $\mathbb{C}^2$. Then, a matrix $g \in SU(2)$ is uniquely written as

$$\begin{cases} g = \begin{pmatrix} y^0(g) + iy^1(g) & y^2(g) + iy^3(g) \\ -y^2(g) + iy^3(g) & y^0(g) - iy^1(g) \end{pmatrix} \\ y^0(g)^2 + y^1(g)^2 + y^2(g)^2 + y^3(g)^2 = 1 \end{cases}$$

3. THE BUNDLE OF CONNECTIONS

3.1. CONNECTIONS AND SPLITTINGS

A connection Γ on a principal G-bundle $\pi \colon P \to M$ gives an splitting

$$\sigma_\Gamma \colon TM \to T(P)/G, \quad \pi_* \circ \sigma_\Gamma = 1_{TM}$$

of the Atiyah sequence (*cf.* [A1], [K], [MV]), by setting $\sigma_\Gamma(X) = X^*$, where X^* stands for the Γ-horizontal lift of X to P, as the horizontal lift of a vector field $X \in \mathfrak{X}(M)$ is G-invariant (see [KN]). The *bundle of connections*,

$$p \colon \mathcal{C}(P) \to M$$

is the bundle of all $\mathbb{R}$-linear maps

$$\lambda \colon T_xM \to (T(P)/G)_x \text{ such that } \pi_* \circ \lambda = 1_{T_xM}$$

Connections on P are the global sections of $p \colon \mathcal{C}(P) \to M$. Moreover, $\mathcal{C}(P)$ is an affine bundle modelled over the vector bundle

$$\mathrm{Hom}(TM, \mathrm{ad}P) \simeq T^*M \otimes \mathrm{ad}P.$$

3.2. Coordinates on $\mathcal{C}(P)$

Let $(U; x^1, \ldots, x^n)$ be a coordinate open domain in M such that a given principal $SU(2)$-bundle $\pi: P \to M$ is trivial over U. For every $B \in \mathfrak{su}(2)$, we define

$$\varphi_t^B: \pi^{-1}(U) \simeq U \times SU(2) \to \pi^{-1}(U)$$

by the formula:

$$\varphi_t^B(x, g) = (x, \exp(tB) \cdot g)$$

and we denote by $\tilde{B}$ the infinitesimal generator of φ_t^B. It follows that $\tilde{B}$ is a π-vertical $SU(2)$-invariant vector field; *i.e.*, $\tilde{B} \in \mathrm{gau}P$. Furthermore, $\left(\tilde{B}_1, \tilde{B}_2, \tilde{B}_3\right)$ is a basis of $\Gamma(U, \mathrm{ad}\pi^{-1}(U))$. Hence for each σ_Γ there exist unique functions $A_j^a(\Gamma) \in C^\infty(U)$ such that,

$$\sigma_\Gamma\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial x^j} - A_j^a(\Gamma)\,\tilde{B}_a, \quad 1 \leq j \leq n.$$

The functions

$$(x^j; A_j^a), \quad 1 \leq j \leq n,\ 1 \leq a \leq 3$$

are a coordinate system on $p^{-1}(U) = \mathcal{C}(\pi^{-1}U)$, which will be called the natural coordinate system induced by the coordinate system $(U; x^1, \ldots, x^n)$ on the bundle of connections.

4. Aut$P$ acting on $\mathcal{C}(P)$

Each $\Phi \in \mathrm{Aut}P$ acts on the connections of P as follows: given Γ, $\Gamma' = \Phi(\Gamma)$ is the connection corresponding to the connection form

$$\omega_{\Gamma'} = (\Phi^{-1})^* \omega_\Gamma.$$

If $\Psi \in \mathrm{Aut}P$, then

$$(\Psi \circ \Phi)(\Gamma) = \Psi(\Phi(\Gamma)).$$

For each $\Phi \in \mathrm{Aut}P$ there exists a unique diffeomorphism

$$\Phi_{\mathcal{C}}: \mathcal{C}(P) \to \mathcal{C}(P)$$

such that $p \circ \Phi_{\mathcal{C}} = \varphi \circ p$, where $\varphi \in \mathrm{Diff}\, M$ is the diffeomorphism induced by Φ. Furthermore,

$$\Phi_{\mathcal{C}} \circ \sigma_\Gamma = \sigma_{\Phi(\Gamma)},$$

for every connection Γ. We obtain a group homomorphism

$$\mathrm{Aut}P \to \mathrm{Diff}\, \mathcal{C}(P), \quad \Phi \mapsto \Phi_{\mathcal{C}}.$$

If Φ_t is the flow of a G-invariant vector field $X \in \mathrm{aut}P$, then $(\Phi_t)_{\mathcal{C}}$ is a one-parameter group in $\mathcal{C}(P)$. We set

$$X_{\mathcal{C}} = \text{infinitesimal generator of } (\Phi_t)_{\mathcal{C}}.$$

In this way we obtain a Lie algebra representation of the Lie algebra of all infinitesimal automorphisms of P on the vector fields of its bundle of connections,

$$\mathrm{aut}P \to \mathfrak{X}\left(\mathcal{C}\left(P\right)\right), \quad X \mapsto X_{\mathcal{C}}.$$

Note that X and $X_{\mathcal{C}}$ both are projectable onto the same vector field of M. By using a coordinate domain $(U; x^1, \ldots, x^n)$ in M and the basis $\left(\tilde{B}_1, \tilde{B}_2, \tilde{B}_3\right)$, each $X \in \mathrm{aut}\pi^{-1}(U)$ can be written as

$$X = f_j \frac{\partial}{\partial x^j} + g^a \tilde{B}_a, \quad f_j, g^a \in C^\infty(U).$$

The local expression of the above representation is as follows:

$$X_{\mathcal{C}} = f_j \frac{\partial}{\partial x^j} - \mathfrak{S}_{123}\left(\frac{\partial g^1}{\partial x^j} + \frac{\partial f_i}{\partial x^j} A_i^1 + g^3 A_j^2 - g^2 A_j^3\right)\frac{\partial}{\partial A_j^1},$$

where $\mathfrak{S}_{123}$ stands for cyclic sum over the indices $1, 2, 3$. In particular, if X is an infinitesimal gauge transformation then $f_j = 0$. Hence

$$X_{\mathcal{C}} = -\mathfrak{S}_{123}\left(\frac{\partial g^1}{\partial x^j} + g^3 A_j^2 - g^2 A_j^3\right)\frac{\partial}{\partial A_j^1}.$$

5. The identification $\left(J^1P\right)/G \simeq \mathcal{C}\left(P\right)$

Let $\pi\colon P \to M$ be an arbitrary principal G-bundle and let $\pi_1\colon J^1P \to M$ be the 1-jet bundle of local sections of π. The group G acts (on the right) on J^1P by

$$j_x^1 s \cdot g = j_x^1\left(R_g \circ s\right),$$

where s is a local section of π, $g \in G$ and R_g stands for the right translation. The quotient $\left(J^1P\right)/G$ exists as a fibred differentiable manifold over M which can be identified to $\mathcal{C}\left(P\right)$. Let us describe this identification. Let

$$q\colon J^1P \to \mathcal{C}\left(P\right)$$

be the mapping defined as follows. Each local section s defines a retract

$$\Gamma_{s(x)}\colon T_{s(x)}P \to V_{s(x)}P = \ker\left(\pi_*\right)_{s(x)}$$

of the inclusion

$$V_{s(x)}P \subset T_{s(x)}P,$$

by setting

$$\Gamma_{s(x)}\left(X\right) = X - s_*\pi_*\left(X\right).$$

For every $u \in \pi^{-1}(x)$ there exists a unique $g \in G$ such that $u = s(x) \cdot g$ and we define $\Gamma_u\colon T_uP \to V_uP$ as

$$\Gamma_u = \left(R_g\right)_* \circ \Gamma_{s(x)} \circ \left(R_{g^{-1}}\right)_*.$$

In this way we obtain a "connection Γ at x"; that is, an element of $\mathcal{C}\left(P\right)$ which only depends on $j_x^1 s$. Hence we set

$$q(j_x^1 s) = \Gamma.$$

5.1. Aut$P$ acting on J^1P

Let X be a π-projectable vector field on P, let X' be its projection onto M and let Φ_t, ϕ_t be the flows of X, X', respectively. A flow $\Phi_t^{(1)}$ can be defined on J^1P by the formula

$$\Phi_t^{(1)}\left(j_x^1 s\right) = j^1_{\phi_t(x)}\left(\Phi_t \circ s \circ \phi_{-t}\right).$$

If X is π-vertical ($i.e.$, $X' = 0$ or even $\phi_t = \mathrm{id}_M$) then

$$\Phi_t^{(1)} = J^1(\Phi_t).$$

We denote by $X^{(1)}$ the infinitesimal generator of the flow $\Phi_t^{(1)}$ which is called the *infinitesimal contact transformation* associated to X (or also the *natural lift* of X to the 1-jet bundle). The mapping $X \mapsto X^{(1)}$ is a Lie algebra monomorphism and $X^{(1)}$ is π_{10}-projectable onto X, where π_{10} is the canonical projection,

$$\pi_{10} \colon J^1P \to P, \quad \pi_{10}\left(j_x^1 s\right) = s(x).$$

For every $\Phi \in \mathrm{Aut}P$ we have

$$q \circ J^1(\Phi) = \Phi_{\mathcal{C}} \circ q.$$

Hence for every $X \in \mathrm{aut}P$ the vector field $X^{(1)}$ is q-projectable and its projection is $X_{\mathcal{C}}$; $i.e.$,

$$q_* \circ X^{(1)} = X_{\mathcal{C}} \circ q.$$

Therefore

Proposition. *The representation*

$$\mathrm{aut}P \to \mathfrak{X}\left(\mathcal{C}\left(P\right)\right), \quad X \mapsto X_{\mathcal{C}}.$$

can be obtained "by projecting" the natural representation of the algebra aut$P$ *on* J^1P *by infinitesimal contact transformations by means of the identification* $\left(J^1P\right)/G \simeq \mathcal{C}\left(P\right)$.

6. Gauge invariance in J^1P

6.1. Contact forms on J^1P

Let $\pi \colon P \to M$ be a principal $SU(2)$-bundle. We define a $\mathfrak{su}(2)$-valued 1-form θ on J^1P as follows. For every $Y \in T_{j_x^1 s}(J^1P)$ we have

$$q(j_x^1 s)\left((\pi_{10})_* Y\right) \in V_{s(x)}P.$$

If $B^* \in \mathfrak{X}(P)$ is the *fundamental vector field* (*cf.* [KN]) associated to $B \in \mathfrak{su}(2)$ we have a vector bundle isomorphism

$$P \times \mathfrak{su}(2) \to V(P),$$

given by

$$(u, B) \mapsto B_u^*.$$

Hence there exists a unique $B \in \mathfrak{su}(2)$ such that

$$q(j_x^1 s)\left((\pi_{10})_* Y\right) = B^*_{s(x)},$$

and we set

$$\theta(Y) = B.$$

In the standard basis we have

$$\theta = \theta^a \otimes B_a,$$

where θ^1, θ^2, θ^3 are global ordinary 1-forms on J^1P called the *standard contact forms*. The following properties of the form θ are easily checked:

1. For every $\Phi \in \mathrm{Gau}P$, we have
$$J^1(\Phi)^* \theta = \theta.$$
2. Hence for every $X \in \mathrm{gau}P$,
$$L_{X^{(1)}} \theta^a = 0, \;\; 1 \leq a \leq 3,$$
3. For every $B \in \mathfrak{su}(2)$, let $B^\bullet$ be the fundamental vector field associated to B under the action of $SU(2)$ on J^1P. Then,
$$L_{B^\bullet}\theta = [\theta, B]\,.$$

6.2. Invariant differential forms

A differential form ω_r on J^1P of degree $r \in \mathbb{N}$ is said to be $\mathrm{gau}P$*-invariant* if for every $X \in \mathrm{gau}P$ we have

$$L_{X^{(1)}} \omega_r = 0.$$

We denote by $\mathcal{I}_{\mathrm{gau}P}$ the set of $\mathrm{gau}P$-invariant differential forms. (Usually, $\mathrm{gau}P$-invariant differential forms are called gauge invariant forms.)

Example. The standard contact forms are gauge invariant.

Note that $\mathcal{I}_{\mathrm{gau}P}$ is a $\mathbb{Z}$-graded algebra over $\Omega^\bullet(M)$.

A differential form ω_r on $\mathcal{C}(P)$ of degree $r \in \mathbb{N}$ is said to be $\mathrm{aut}P$ *-invariant* if for every $X \in \mathrm{aut}P$ we have

$$L_{X_{\mathcal{C}}} \omega_r = 0.$$

We denote by $\mathcal{I}_{\mathrm{aut}P}$ the set of $\mathrm{aut}P$-invariant differential forms. Note that $\mathcal{I}_{\mathrm{gau}P}$ is a $\mathbb{Z}$-graded algebra over $\Omega^\bullet(M)$ and that $\mathcal{I}_{\mathrm{aut}P} \subset \mathcal{I}_{\mathrm{gau}P}$ is a subalgebra.

6.3. Statement of the main result

Theorem. *The algebra of gauge invariant forms on* J^1P *is generated over* $\pi_1^*\Omega^\bullet(M)$ *by the forms*

$$(\theta^a, \mathrm{d}\theta^a),\ 1 \leq a \leq 3,$$

that is,

$$\mathcal{I}_{\mathrm{gau}P} = \pi_1^*\Omega^\bullet(M)\,[\theta^a, \mathrm{d}\theta^a]_{1\leq a\leq 3}\,.$$

The only aut$P$*-invariant differential forms on* J^1P *are:*

$$\mathcal{I}_{\mathrm{aut}P} = \mathbb{R}\,[\theta^a, \mathrm{d}\theta^a]_{1\leq a\leq 3}\,.$$

6.4. Conclusions

If $\pi\colon P \to M$ is a principal $SU(2)$-bundle, we have seen that the algebra of gauge invariant (resp. aut$P$-invariant) differential forms on $J^1(P)$ is differentiably generated over the graded algebra of differential forms on M (resp. over the real numbers) by the standard structure forms. In a previous paper ([EM, Theorem 3]) it was proved that if $\pi\colon P \to M$ is an arbitrary principal G-bundle with G and M connected, then on $J^1(P)$ the unique gauge invariant Lagrangian densities are the differential n-forms on M. This result was originally motivated by Utiyama's theorem ([U]; also see [B], [Bl], [G] for the geometric interpretation of Utiyama's theorem) which stated the structure of gauge invariant Lagrangian densities on $J^1(\mathcal{C}(P))$. In the case $G = SU(2)$ Theorem 3 in [EM] can be obtained as a consequence of the theorem above remarking that contact forms on the 1-jet bundle are not horizontal over the base manifold.

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