

# Almost holomorphic embeddings in Grassmannians with applications to singular symplectic submanifolds

By *V. Muñoz, F. Presas* and *I. Sols* at Madrid

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**Abstract.** In this paper we use Donaldson’s approximately holomorphic techniques to build embeddings of a closed symplectic manifold with symplectic form of integer class in the Grassmannians  $\text{Gr}(r, N)$ . We assure that these embeddings are asymptotically holomorphic in a precise sense. We study first the particular case of  $\mathbb{C}\mathbb{P}^N$  obtaining control on  $N$  and we improve in a sense a classical result about symplectic embeddings [16]. The main reason of our study is the construction of singular determinantal submanifolds as the intersection of the embedding with certain “generalized Schubert cycles” defined on a product of Grassmannians. It is shown that the symplectic type of these submanifolds is quite more general than the ones obtained by Donaldson and Auroux [2], [7] as zeroes of “very ample” vector bundles.

## 1. Introduction and statement of the main results

Let  $(M, \omega)$  be a symplectic manifold of integer class, i.e.  $[\omega/2\pi] \in H^2(M; \mathbb{R})$  lifts to an integer cohomology class. Such a symplectic manifold has an associated line bundle  $L$  with first Chern class  $c_1(L) = [\omega/2\pi]$ , which is equipped with a connection  $\nabla$  of curvature  $-\i\omega$ .

In his outbreaking work [5] S. Donaldson proved the existence of symplectic submanifolds of  $M$  that realize the Poincaré dual of a large enough integer multiple of  $[\omega/2\pi]$ . These are constructed as zero sets of appropriate sections of  $L^{\otimes k}$ . This extends a classical result in Kähler geometry saying that  $L$  is ample, so  $L^{\otimes k}$  has holomorphic sections with smooth holomorphic, and so symplectic, zero sets.

Later on, D. Auroux and R. Paoletti have proved independently an extension of Donaldson’s theorem, constructing more symplectic submanifolds as the zero sets of asymptotically holomorphic sections of vector bundles. These bundles are obtained by tensoring an arbitrary complex bundle with large powers of  $L$  [2], [3], [13]. In his paper, D. Auroux also shows that, asymptotically, all the sequences of submanifolds constructed from a given vector bundle  $E$  are isotopic. (For a summary of these results see for example the review paper [6].)

The key idea to understand these works is the concept of ampleness of a complex holomorphic bundle, which allows the flexibilization of the bundles in the holomorphic category by means of increasing their curvatures. Donaldson [5] has translated the definition of ampleness to the symplectic category. For this he studies the asymptotical behaviour of sequences of sections of the bundles  $L^{\otimes k}$ . The change to the non-integrable setting is controlled by this concept. We need to fix a compatible almost complex structure  $J$  in  $(M, \omega)$ . So the pair  $(\omega, J)$  gives a metric  $g$  in the tangent bundle. We have a sequence of metrics  $g_k = kg$  indexed by integers  $k \geq 1$ . An important point in our work is the definition of the concept of asymptotic holomorphicity for sequences of embeddings.

**Definition 1.1.** Let  $X$  be a Hodge manifold with complex structure  $J_0$ . Let  $\gamma > 0$ . A sequence of embeddings  $\phi_k: M \rightarrow X$  is  $\gamma$ -asymptotically holomorphic if it satisfies the following conditions:

1.  $d\phi_k: T_x M \rightarrow T_{\phi_k(x)} X$  has a left inverse  $\theta_k$  of norm less than  $\gamma^{-1}$  at every point  $x \in M$ .
2.  $|(\phi_k)_* J - J_0|_{g_k} = O(k^{-1/2})$  on the subspace  $(\phi_k)_* T_x M$ .
3.  $|\nabla^p \phi_k|_{g_k} = O(1)$  and  $|\nabla^{p-1} \bar{\partial} \phi_k|_{g_k} = O(k^{-1/2})$ , for all  $p \geq 1$ .

A sequence of embeddings is asymptotically holomorphic if there is some  $\gamma > 0$  such that it is  $\gamma$ -asymptotically holomorphic. (The norms are taken with respect to the sequence of metrics  $g_k$ .)

The first important result we give is a generalization to the symplectic category of the classical Kodaira's embedding Theorem:

**Theorem 1.2.** *Given  $(M, \omega)$  a closed symplectic  $2n$ -dimensional manifold of integer class endowed with a compatible almost complex structure, then there exists an asymptotically holomorphic sequence of embeddings  $\phi_k: M \rightarrow \mathbb{C}\mathbb{P}^{2n+1}$  with  $\phi_k^*[\omega_{FS}] = [k\omega]$ . Moreover, given two such sequences of embeddings asymptotically holomorphic with respect to two compatible almost complex structures, then they are isotopic for  $k$  large enough.*

A sharper, in a sense, result has been obtained by Borthwick and Uribe in [4] using completely different ideas. Their result also obtains control in the symplectic part (equivalently the metric part) allowing to obtain asymptotically holomorphic embeddings which are also asymptotically symplectic. Their approach is based on ideas from Tian [15] who solved the problem in the Kähler case.

Our main interest for proving Theorem 1.2 is given by the possibility of studying “projective symplectic geometry”. We mean by this the study of sequences of asymptotically holomorphic submanifolds, namely obtained as images of asymptotically holomorphic embeddings, in the projective space. The strength of this approach is shown in the following

**Theorem 1.3.** *Let  $\phi_k$  be an asymptotically holomorphic sequence of embeddings in  $\mathbb{C}\mathbb{P}^{2n+1}$  with  $\phi_k^*[\omega_{FS}] = [k\omega]$  and let  $\varepsilon > 0$ . Let us fix a holomorphic submanifold  $N$  in  $\mathbb{C}\mathbb{P}^{2n+1}$ . Then there exists an asymptotically holomorphic sequence of embeddings  $\hat{\phi}_k$ , at distance at most  $\varepsilon$  in  $C^r$ -norm from the initial sequence and verifying that  $\hat{\phi}_k(M) \cap N$  is symplectic for  $k$  large enough.*

Actually we get that  $M \cap \hat{\phi}_k^{-1}(N)$  is a sequence of asymptotically holomorphic submanifolds, in the sense of Definition 3.9. This result will imply a projective version of the symplectic Bertini’s Theorem proved in [5]. But the constructive method could allow to find more general types of symplectic submanifolds. This is shown in a more general situation. For this we generalize Theorem 1.2 to the Grassmannian case.

**Theorem 1.4.** *Let  $(M, \omega)$  be a closed symplectic  $2n$ -dimensional manifold of integer class endowed with a compatible almost complex structure. Suppose also that we have a rank  $r$  hermitian vector bundle with connection, and that  $N > n + r - 1$  and  $r(N - r) > 2n$ . Then there exists an asymptotically holomorphic sequence of embeddings  $\phi_k: M \rightarrow \text{Gr}(r, N)$  with  $\phi_k^* \mathcal{U} = E \otimes L^{\otimes k}$ , where  $\mathcal{U} \rightarrow \text{Gr}(r, N)$  is the universal rank  $r$  bundle over the Grassmannian. Moreover, given two such sequences of embeddings asymptotically holomorphic with respect to two compatible almost complex structures, then they are isotopic for  $k$  large enough.*

In Section 5 we will take profit of this result to extend the construction of determinantal submanifolds to the symplectic category in the following way.

**Definition 1.5.** Let  $M$  be a differentiable manifold and let  $E, F$  be complex vector bundles over  $M$ . Given a morphism of vector bundles  $\varphi: E \rightarrow F$ , the  $r$ -determinantal set  $\Sigma^r(\varphi)$  is defined as

$$\Sigma^r(\varphi) = \{x \in M \mid \text{rank } \varphi_x = r\}.$$

In the algebraic category, if the vector bundle  $E^* \otimes F$  is very ample, we can find a morphism  $\varphi: E \rightarrow F$  such that  $\Sigma^r(\varphi)$  is a smooth submanifold in  $M$  of complex codimension  $(r_e - r)(r_f - r)$ , where  $r_e$  and  $r_f$  are the ranks of  $E$  and  $F$ , respectively (if this number is greater than the dimension of  $M$  then the set is empty). Our goal will be to adapt the algebraic discussion to the symplectic category to prove

**Theorem 1.6.** *Let  $(M, \omega)$  be a closed symplectic manifold of integer class. Let  $E$  and  $F$  be hermitian vector bundles of rank  $r_e$  and  $r_f$ , respectively. Then, for  $k$  large enough, there exists a morphism  $\varphi_k: E \otimes (L^*)^{\otimes k} \rightarrow F \otimes L^{\otimes k}$  verifying that:*

1.  $\Sigma^r(\varphi_k)$  is an open symplectic submanifold of  $M$ .
2.  $\text{codim } \Sigma^r(\varphi_k) = 2(r_e - r)(r_f - r)$ . The set of manifolds  $\{\Sigma^r(\varphi_k)\}_r$  constitutes a stratified submanifold, called determinantal submanifold.

*Moreover, given two stratified determinantal submanifolds constructed following the process described in the proof then there exists an ambient isotopy making the  $r$ -determinantal submanifolds associated to each stratified submanifold isotopic.*

Theorem 1.6 was the original motivation of this paper. In algebraic geometry the manifolds constructed as zeroes of sections of vector bundles have many topological restrictions, so the set of submanifolds of a given manifold constructed in this way is very special in the set of all the submanifolds. However the determinantal submanifolds are more generic. For instance, every (complex) codimension 2 submanifold of an algebraic manifold can be constructed as the determinantal degeneration loci of certain bundle homomorphism [17]. An obvious guess is that in symplectic geometry things are similar. Recall that the most

general submanifolds constructed using asymptotically holomorphic techniques were Auroux' ones [2]. These are zeroes of sections of vector bundles, so many of their topological properties are determined. In fact, Auroux cannot easily assure that these submanifolds are different from the ones constructed by Donaldson in [5]. In Section 6 we give two properties showing that the determinantal submanifolds are more general. First we show that the homology groups of the determinantal submanifolds do not satisfy the Lefschetz hyperplane theorem. In fact, they satisfy another kind of Lefschetz isomorphism. Secondly we compute some Chern numbers of determinantal submanifolds showing that they are clearly different from the Chern numbers of Auroux' and Donaldson's submanifolds. So the symplectic type, and even the topological type, of the constructed submanifolds is necessarily different. These two results show that the class of determinantal submanifolds is more general.

Remark that, in any case, all the preceding results are obtained by means of twisting vector bundles with large powers of the line bundle  $L$ . So the submanifolds constructed in this way are not generic. It would be desirable to avoid this restriction, but this generalization cannot be made with the techniques in [5].

From a symplectic point of view determinantal submanifolds are also interesting. They constitute a step in the study of singular symplectic submanifolds following the program sketched by Gromov [9]. Donaldson and Auroux have attacked this question in [7] and [3]. Donaldson studies the local symplecticity of the fibers of asymptotically holomorphic applications  $f: \mathbb{C}^n \rightarrow \mathbb{C}$  at a neighborhood of a critical point, solving it by a local perturbation argument. The conclusion of Donaldson's work is that the topological behavior of that kind of functions is similar to the holomorphic Morse functions. Auroux studies the local symplecticity of asymptotically holomorphic maps  $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$  at a neighborhood of a critical point, showing that they are topologically equivalent to one of the two generic models of a holomorphic map [1]. From this point of view Theorem 1.6 can be considered, in part, an extension of these results to generic singularities.

The organization of the paper is as follows. In Section 2 we give the basic ideas of the Donaldson-Auroux' theory needed in our work and prove Theorem 1.2. In Section 3.2 we prove Theorem 1.3. For this we explain some euclidean notions concerning the estimation of angles between subspaces. In Section 4 we generalize all the discussion to the case of the Grassmannian embeddings, proving Theorem 1.4. This allows us to prove Theorem 1.6 in Section 5 and to analyze the topological properties of the constructed submanifolds.

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## 2. Asymptotically holomorphic embeddings in projective space

As in the introduction, let  $(M, \omega)$  be a symplectic manifold of integer class with associated line bundle  $L$  and a compatible almost complex structure  $J$ . In the Kähler setting this line bundle supports a holomorphic structure and it is ample in the algebraic geometry

sense, i.e.  $L^{\otimes k}$  has a lot of holomorphic sections. This allows to embed  $M$  in the projective space  $\mathbb{C}\mathbb{P}^N$ , for some  $N$ . In this section we shall extend this classical result to the symplectic case inspired in the ideas of [5], thus proving Theorem 1.2.

**2.1. Asymptotically holomorphic sequences.** In this subsection we collect the relevant results of the asymptotically holomorphic theory, as stated by D. Auroux in [3], that we shall use extensively along this work.

**Definition 2.1.** A sequence of sections  $s_k$  of hermitian bundles  $E_k$  with connections on  $M$  is called asymptotically  $J$ -holomorphic if there exist constants  $(C_p)_{p \in \mathbb{N}}$  such that, for all  $k$ , at every point of  $M$ ,  $|s_k| \leq C_0$ ,  $|\nabla^p s_k| \leq C_p$  for all  $p \geq 0$ , and  $|\nabla^{p-1} \bar{\partial} s_k| \leq C_p k^{-1/2}$  for all  $p \geq 1$ . The norms are evaluated with respect to the metrics  $g_k$ .

In Donaldson’s first work on the subject [5],  $E_k = L^{\otimes k}$ . Donaldson imposed an additional condition of improved transversality to the sequence of sections to assure that its zero sets are symplectic submanifolds for  $k$  large enough.

**Definition 2.2.** A section  $s_k$  of the line bundle  $L^{\otimes k}$  is said to be  $\eta$ -transverse to 0 if for every point  $x \in M$  such that  $|s_k(x)| < \eta$  then  $|\nabla s_k(x)| > \eta$ .

If we get an asymptotically  $J$ -holomorphic sequence  $s_k$  of sections of  $L^{\otimes k}$  such that all of them are  $\eta$ -transverse to 0, with  $\eta > 0$  independent of  $k$  then we have  $|\hat{\partial} s_k(x)| > |\bar{\partial} s_k(x)|$  if  $x$  is a zero of  $s_k$ , for  $k$  large enough. A simple linear algebra argument shows that the zeroes of  $s_k$  are symplectic submanifolds for  $k$  large enough.

In [2] D. Auroux extended the notion of transversality to the case of higher rank bundles. Let  $E$  be a rank  $r$  hermitian bundle with connection.

**Definition 2.3.** A section  $s_k$  of the bundles  $E \otimes L^{\otimes k}$  is  $\eta$ -transverse to 0 if for every  $x \in M$  such that  $|s_k(x)| < \eta$  then  $\nabla s_k(x)$  has a right inverse  $\theta_k$  such that  $|\theta_k| < \eta^{-1}$ .

We name *universal constant* to a number which only depends on the manifold geometry and on the constants involved in the data given to start with, i.e. a number independent of  $k$  and the point  $x \in M$ . Similarly a *universal polynomial* is a polynomial only depending on the geometry of the manifold and on the constants provided in the original data. Donaldson uses highly localized asymptotically holomorphic sections, satisfying the following definition.

**Definition 2.4.** A sequence of sections  $s_k$  of hermitian bundles  $E_k$  with connections has Gaussian decay in  $C^r$ -norm away from the point  $x \in M$  if there exists a universal polynomial  $P$  and a universal constant  $\lambda > 0$  such that for all  $y \in M$ ,  $|s(y)|$ ,  $|\nabla s(y)|_{g_k}, \dots, |\nabla^r s(y)|_{g_k}$  are bounded by  $P(d_k(x, y)) \exp(-\lambda d_k^2(x, y))$ . Here  $d_k$  is the distance associated to the metric  $g_k$ .

The starting point for Donaldson’s construction is the following existence Lemma.

**Lemma 2.5** ([5], [2]). *Given any point  $x \in M$ , for  $k$  large enough, there exist asymptotically holomorphic sections  $s_{k,x}^{\text{ref}}$  of  $L^{\otimes k}$  over  $M$  satisfying the following bounds:  $|s_{k,x}^{\text{ref}}| > c_s$  at every point of a ball of  $g_k$ -radius 1 centered at  $x$ , for some universal constant  $c_s > 0$ ; the sections  $s_{k,x}^{\text{ref}}$  have Gaussian decay away from  $x$  in  $C^r$ -norm.*

Moreover, given a one-parameter family of compatible almost-complex structures  $(J_t)_{t \in [0,1]}$  and a one-parameter family of points  $(x_t)_{t \in [0,1]}$ , there exist one-parameter families of sections  $s_{t,k,x_t}^{\text{ref}}$  which depend continuously on  $t$  and satisfy the same precedent properties.  $\square$

The proof of this lemma uses in particular a refined version of Darboux' Theorem taking into account the holomorphic structure. We state this result for later use (the result is given in [3] only for  $n = 2$  but the general case is analogous).

**Lemma 2.6** ([3], section 2.1, Lemma 3). *Near any point  $x \in M$ , for any integer  $k \geq 1$ , there exist local complex Darboux coordinates  $(z_k^1, \dots, z_k^n) = \Phi_k: (M, x) \rightarrow (\mathbb{C}^n, 0)$  for the symplectic structure  $k\omega$  such that the following bounds hold universally:*

$$|\Phi_k(y)|^2 = O(d_k(x, y)^2)$$

on a ball  $B_{g_k}(x, c)$  of universal radius  $c$  around  $x$ ;  $|\nabla^r \Phi_k^{-1}|_{g_k} = O(1)$  for all  $r \geq 1$  on a ball  $B(0, c')$  of universal radius  $c'$  around 0; and, with respect to the almost-complex structure  $J$  on  $X$  and the canonical complex structure  $J_0$  on  $\mathbb{C}^n$ ,  $|\bar{\partial} \Phi_k^{-1}(z)|_{g_k} = O(k^{-1/2}|z|)$  and  $|\nabla^r \bar{\partial} \Phi_k^{-1}|_{g_k} = O(k^{-1/2})$  for all  $r \geq 1$  on  $B(0, c')$ .

Moreover, given a one-parameter continuous family of compatible almost-complex structures  $(J_t)_{t \in [0,1]}$  and a continuous family of points  $(x_t)_{t \in [0,1]}$ , there exists a continuous family of Darboux coordinates  $\Phi_{t,k}$  satisfying the same estimates and depending continuously on  $t$ .  $\square$

In [3] D. Auroux used three asymptotically holomorphic sections to set up a projection from a symplectic 4-manifold  $M$  to  $\mathbb{C}P^2$ . To control the behavior of this projection he needs to assure global transversality conditions between the sections. He develops a very useful scheme to pass from local transversality conditions to global ones by means of a globalization process inspired in the results of [5]. Now we explain his idea to formalize Donaldson's techniques.

**Definition 2.7.** A family of properties  $\mathcal{P}(\varepsilon, x)_{x \in M, \varepsilon > 0}$  of sections of bundles over  $M$  is local and  $C^r$ -open if, given a section  $s$  satisfying  $\mathcal{P}(\varepsilon, x)$ , any section  $\sigma$  such that  $|\sigma(x) - s(x)|_{C^r} < \eta$  satisfies  $\mathcal{P}(\varepsilon - C\eta, x)$ , where  $C$  is universal.

For example, the property  $|s(x)| > \varepsilon$  is local and  $C^0$ -open. The property that  $s$  be  $\varepsilon$ -transverse to 0 at a point  $x$  is local and  $C^1$ -open.

**Proposition 2.8** ([3], section 2.1, Proposition 3). *Let  $\mathcal{P}(\varepsilon, x)_{x \in M, \varepsilon > 0}$  be a local and  $C^r$ -open family of properties of sections of vector bundles  $E_k$  over  $M$ . Assume that there exist universal constants  $c, c', c''$  and  $p$  such that given any  $x \in M$ , any small  $\delta > 0$ , and asymptotically holomorphic sections  $s_k$  of  $E_k$ , there exist, for all large enough  $k$ , asymptotically holomorphic sections  $\tau_{k,x}$  of  $E_k$  with the following properties:*

1.  $|\tau_{k,x}|_{C^r, g_k} < c''\delta$ .
2. The sections  $\frac{1}{\delta} \tau_{k,x}$  have Gaussian decay away from  $x$  in  $C^r$ -norm.
3. The sections  $s_k + \tau_{k,x}$  satisfy the property  $\mathcal{P}(\eta, y)$  for all  $y \in B_{g_k}(x, c)$ , with  $\eta = c'\delta(\log(\delta^{-1}))^{-p}$ .

Then, given any  $\alpha > 0$  and asymptotically holomorphic sections  $s_k$  of  $E_k$ , there exist, for all large enough  $k$ , asymptotically holomorphic sections  $\sigma_k$  of  $E_k$  such that  $|s_k - \sigma_k|_{C^r, g_k} < \alpha$  and the sections  $\sigma_k$  satisfy  $\mathcal{P}(\varepsilon, x)$  for all  $x \in M$  with  $\varepsilon > 0$  independent of  $k$ .

Moreover, the result holds for one-parameter families of sections, provided the existence of sections  $\tau_{t,k,x}$  satisfying properties 1, 2 and 3 and depending continuously on  $t$ .  $\square$

The heart of these techniques is a series of local transversality results which allow to apply Proposition 2.8. These results are based on ideas of complexity of real polynomials coming from real algebraic geometry. The most powerful result is the following, proved in [7].

**Definition 2.9.** A function  $f: \mathbb{C}^n \rightarrow \mathbb{C}^r$  is  $\sigma$ -transverse to 0 at a point  $x \in \mathbb{C}^n$  if it satisfies at least one of the following properties:

1.  $|f(x)| > \sigma$ .
2.  $df(x)$  has a right inverse  $\theta$  such that  $|\theta| < \sigma^{-1}$ .

**Proposition 2.10** ([7], Theorem 12). *There exists a universal integer  $p$  verifying the following property: for  $0 < \delta < \frac{1}{2}$  let  $\sigma = \delta(\log(\delta^{-1}))^{-p}$ . Let  $f$  be a function with values in  $\mathbb{C}^r$  defined over the ball  $B^+ = B\left(0, \frac{11}{10}\right) \subset \mathbb{C}^n$  satisfying the following bounds over  $B^+$ :*

$$|f| \leq 1, \quad |\bar{\partial}f| \leq \sigma, \quad |\nabla \bar{\partial}f| \leq \sigma.$$

Then there exists  $w \in \mathbb{C}^r$  with  $|w| < \delta$  such that  $f - w$  is  $\sigma$ -transverse to 0 over the unit ball in  $\mathbb{C}^n$ . The same result holds for one-parameter families of functions  $f_t$  depending continuously on  $t \in [0, 1]$ , where we obtain a continuous path  $w: [0, 1] \rightarrow B(0, \delta)$ .  $\square$

We mention also that in [11] the result is refined to control the derivatives of the path  $w_t$  allowing so a generalization to the contact case of the asymptotically holomorphic techniques.

**2.2. Asymptotically holomorphic embeddings in  $\mathbb{C}\mathbb{P}^{2n+1}$ .** In this section we will study the existence of asymptotically holomorphic embeddings of a closed symplectic manifold  $(M, \omega)$  of integer class and dimension  $2n$ , endowed with a compatible almost complex structure  $J$ , in the projective space  $\mathbb{C}\mathbb{P}^{2n+1}$ . In Section 4 we will develop the techniques to study the more general Grassmannian embeddings.

**Theorem 2.11.** *Given an asymptotically  $J$ -holomorphic sequence of sections  $s_k$  of the vector bundles  $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$  and  $\alpha > 0$  then there exists another sequence  $\sigma_k$  verifying that:*

1.  $|s_k - \sigma_k|_{C^1, g_k} < \alpha$ .
2.  $\mathbb{P}(\sigma_k)$  is an asymptotically holomorphic sequence of embeddings in  $\mathbb{C}\mathbb{P}^{2n+1}$  for  $k$  large enough.
3.  $[k\omega] = [\phi_k^* \omega_{FS}]$ .

Moreover, let us have two asymptotically holomorphic sequences  $\phi_k^0$  and  $\phi_k^1$  of embeddings in  $\mathbb{C}\mathbb{P}^{2n+1}$ , with respect to two compatible almost complex structures. Then for  $k$  large enough, there exists an isotopy of asymptotically holomorphic embeddings  $\phi_k^l$  connecting  $\phi_k^0$  and  $\phi_k^1$ .

**Remark 2.12.** It is easy to check, following the proof later, that the needed  $k$ 's in Theorem 2.11 depend only on  $\alpha$ ,  $J$  and the bounds of asymptotic holomorphicity of  $s_k$ . Moreover, in the statement about isotopy, the needed integer depends on the  $\eta$ -transversality of the sequence, on  $J$  and on the bounds of the sequence. So, if we consider the moduli of  $\eta$ -asymptotically holomorphic embeddings with fixed bounds and almost complex structures varying in a compact set, then there exists a fixed  $k_0$  such that for any  $k > k_0$  all the embeddings in such moduli are isotopic.

The very same ideas do apply to all the proofs along the paper and are assumed without being detailed in each statement.

This result gives a proof of Theorem 1.2. We shall proceed by steps to obtain asymptotically holomorphic embeddings of  $M$  into  $\mathbb{C}\mathbb{P}^{2n+1}$ .

**Definition 2.13.** An asymptotically  $J$ -holomorphic sequence of sections  $s_k$  of the vector bundles  $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$  is  $\gamma$ -projectizable if for all  $x \in M$ ,  $|s_k(x)| > \gamma$ .

This is a sufficient condition to get a map to  $\mathbb{C}\mathbb{P}^{2n+1}$  defined as

$$\phi_k = \mathbb{P}(s_k): M \rightarrow \mathbb{C}\mathbb{P}^{2n+1},$$

as the  $\gamma$ -projectizability assures that the sections  $s_k = (s_k^0, \dots, s_k^{2n+1})$  are not simultaneously zero and so the  $\mathbb{P}$  operator is well defined. To get local injectivity we need to impose the following.

**Definition 2.14.** Let  $s_k$  be an asymptotically  $J$ -holomorphic  $\gamma$ -projectizable sequence of sections of the vector bundles  $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$  for some  $\gamma > 0$  and let  $0 \leq l \leq n$ . Then  $s_k$  is  $\eta$ -generic of order  $l$ , with  $\eta > 0$ , if  $|\wedge^l \partial \mathbb{P}(s_k)(x)|_{g_k} > \eta$  for all  $x \in M$ . For  $l = 0$  the condition is vacuous.

We have the following result that will be proved in the following two subsections.

**Proposition 2.15.** Let  $s_k$  be an asymptotically  $J$ -holomorphic sequence of sections of the vector bundles  $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$  and  $\alpha > 0$ . Then there exists another asymptotically holomorphic sequence  $\sigma_k$  verifying:

1.  $|s_k - \sigma_k|_{C^1, g_k} < \alpha$ .
2.  $\sigma_k$  is  $\gamma$ -projectizable and  $\gamma$ -generic of order  $n$  for some  $\gamma > 0$ .

Moreover, the result holds for one-parameter families of sections where the sections and almost complex structures depend continuously on  $t \in [0, 1]$ .

*Proof of Theorem 2.11.* We first prove the existence result. The last property is obvious since the hyperplane bundle of  $\mathbb{C}\mathbb{P}^{2n+1}$  restricts by construction to  $L^{\otimes k}$ . Let us begin with an asymptotically  $J$ -holomorphic sequence  $\sigma_k$  of sections of the bundles  $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$ . Now we perturb it using Proposition 2.15 to obtain an asymptotically holomorphic sequence  $s_k$  with  $|s_k - \sigma_k|_{C^1, g_k} < \alpha$ , which is  $\gamma$ -projectizable and  $\gamma$ -generic of order  $n$ , for some  $\gamma > 0$ . We have only to check that the sequence  $\phi_k = \mathbb{P}(s_k)$  satisfies the required properties in Definition 1.1. More specifically, we shall check that  $\phi_k$  is an immersion of  $M$  in  $\mathbb{C}\mathbb{P}^{2n+1}$ , for  $k$  large. To get rid of the possible self-intersection we take into account that  $2 \dim M < \dim \mathbb{C}\mathbb{P}^{2n+1}$  so we can make a generic  $C^r$ -perturbation of norm less than  $O(k^{-1/2})$  to get an embedding keeping the asymptotic holomorphicity and the genericity of order  $n$ .

Choose a point  $x \in M$ . By a rotation with an element of  $U(2n+2)$  acting on  $\mathbb{C}^{2n+2}$ , we can assure that  $s_k(x) = (s_k^0(x), \dots, s_k^{2n+1}(x)) = (s_k^0(x), 0, \dots, 0)$ . This produces a global isometric transformation of  $\phi_k(M)$  in  $\mathbb{C}\mathbb{P}^{2n+1}$ . Now using the  $\gamma$ -projectizable property we know that  $|s_k^0(x)| \geq \gamma$ . By the asymptotically holomorphic bounds of  $s_k^0$  there is a universal  $c$  such that  $|s_k^0| \geq \gamma/2$  on  $B_{g_k}(x, c)$  for all  $k$ . We define the application:

$$f_k: B_{g_k}(x, c) \rightarrow \mathbb{C}^{2n+1},$$

$$y \mapsto \left( \frac{s_k^1(y)}{s_k^0(y)}, \dots, \frac{s_k^{2n+1}(y)}{s_k^0(y)} \right).$$

This application can be written as  $f_k = \Phi_0 \circ \phi_k$ , where  $\Phi_0$  is the standard trivialization application in  $\mathbb{C}\mathbb{P}^{2n+1}$  defined for the chart  $U_0 = \{x = [x_0, \dots, x_{2n+1}] | x_0 \neq 0\}$ . It is well known that  $\Phi_0$  is an isometry at the point  $[1, 0, \dots, 0]$  if we use the standard metric structure of  $\mathbb{C}^{2n+1}$ . So we can compute the bounds required in Definition 1.1 using  $f_k$  instead of  $\phi_k$ . The asymptotic holomorphicity of  $s_k$  and the bound  $|s_k^0| \geq \gamma/2$  imply that  $|\nabla^p f_k(x)| = O(1)$  and  $|\nabla^p \bar{\partial} f_k(x)| = O(k^{-1/2})$ , for  $p \geq 0$ . This proves condition 3 in Definition 1.1.

Now we pass to the issue of the existence of a left inverse. We have

$$\wedge^n d\phi_k = \wedge^n \partial\phi_k + O(k^{-1/2}),$$

where the last term is obtained thanks to  $|\bar{\partial}\phi_k|_{g_k} = O(k^{-1/2})$ . By the  $\gamma$ -genericity of order  $n$  of  $\phi_k$ ,  $|\wedge^n \partial\phi_k|_{g_k} \geq \gamma$ , so  $|\wedge^n d\phi_k|_{g_k} \geq \gamma/2$  for  $k$  large. Let

$$\hat{\theta}_k = (d\phi_k)^{-1}: (\phi_k)_* T_x M \rightarrow T_x M.$$

By the asymptotic holomorphicity condition, we have  $|d\phi_k|_{g_k} \leq C_0$  for a universal constant  $C_0$ , so  $|\hat{\theta}_k| \leq C\gamma^{-1}$  for another universal constant  $C$ . Now define  $\theta_k = \hat{\theta}_k \circ \text{pr}^\perp$ , where  $\text{pr}^\perp$  is the orthogonal projection of  $T_{\phi_k(x)} \mathbb{C}\mathbb{P}^{2n+1}$  onto  $(\phi_k)_* T_x M$  to get the sought left inverse.

To prove condition 2 in Definition 1.1, we compute the norm of

$$(\phi_k)_* J - J_0: (\phi_k)_* T_x M \rightarrow T_{\phi_k(x)} \mathbb{C}\mathbb{P}^{2n+1}.$$

But it can be written as

$$(\phi_k)_*J - J_0 = d\phi_k J \hat{\theta}_k - J_0 = (d\phi_k + J_0 d\phi_k J) J \hat{\theta}_k = 2\bar{\partial}\phi_k J \hat{\theta}_k = O(k^{-1/2}).$$

For the isotopy result we follow the ideas of [2]. We need the following auxiliary result, which we prove in Subsection 2.5.

**Lemma 2.16.** *Let  $\phi_k: M \rightarrow \mathbb{C}\mathbb{P}^{2n+1}$  be a sequence of asymptotically holomorphic embeddings with  $\phi_k^*[\omega_{FS}] = [k\omega]$ . Then there exists an asymptotically holomorphic sequence of sections  $s_k$  of  $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$ , for  $k$  large enough, which is  $\gamma$ -projectizable and  $\gamma$ -generic of order  $n$ , for some  $\gamma > 0$ , such that  $\phi_k = \mathbb{P}(s_k)$ . The same holds for continuous one-parameter families of embeddings and compatible almost complex structures.*

Using Lemma 2.16, we can suppose that  $\phi_k^i = \mathbb{P}(s_k^i)$ ,  $i = 0, 1$ , where  $s_k^0$  and  $s_k^1$  are two asymptotically holomorphic sequences which are  $\gamma$ -projectizable and  $\gamma$ -generic of order  $n$ ,  $\gamma > 0$ . We construct the following family of asymptotically holomorphic sequences of sections:

$$(2.1) \quad s_k^t = \begin{cases} (1 - 3t)s_k^0, & \text{with } J_t = J_0, & t \in [0, 1/3], \\ 0, & \text{with } J_t = \text{Path}(J_0, J_1), & t \in [1/3, 2/3], \\ (3t - 2)s_k^1, & \text{with } J_t = J_1, & t \in [2/3, 1]. \end{cases}$$

Choose  $\alpha > 0$  such that any perturbation of  $s_k^i$ ,  $i = 0, 1$ , of  $C^1$ -norm less than  $\alpha$  is still  $\gamma/2$ -projectizable and  $\gamma/2$ -generic of order  $n$ . Applying Proposition 2.15 to  $s_k^t$  with this  $\alpha$ , we obtain a family  $\sigma_k^t$  which is  $\eta$ -projectizable and  $\eta$ -generic of order  $n$  for some  $\eta > 0$ . We define the family of asymptotically holomorphic sequences of sections:

$$\tau_k^t = \begin{cases} (1 - 3t)s_k^0 + 3t\sigma_k^0, & t \in [0, 1/3], \\ \sigma_k^{3t-1}, & t \in [1/3, 2/3], \\ (3t - 2)s_k^1 + (3 - 3t)\sigma_k^1, & t \in [2/3, 1]. \end{cases}$$

These are  $\varepsilon$ -projectizable and  $\varepsilon$ -generic of order  $n$  sequences of sections, with  $\varepsilon = \min\{\gamma/2, \eta\}$ , so that  $\phi_k^t = \mathbb{P}(\tau_k^t)$  are asymptotically holomorphic embeddings (maybe after a further small perturbation to get rid of self-intersections). This implies that  $\phi_k^0$  and  $\phi_k^1$  are isotopic for  $k$  large enough.  $\square$

An important corollary is the existence of symplectic embeddings of  $M$ . The following result is similar to [16], but we do not obtain an exact symplectic embedding. On the other hand we control the dimension of the projective space.

**Corollary 2.17.** *Let  $(M, \omega)$  be a closed symplectic manifold of dimension  $2n$  with symplectic form of integer class. Then there exists a symplectic embedding  $\phi: M \rightarrow \mathbb{C}\mathbb{P}^{2n+1}$  verifying that  $k\omega = \phi^*\omega_{FS}$ , for  $k$  large enough.*

*Proof.* Take a  $\gamma$ -asymptotically holomorphic sequence  $\phi_k$  of embeddings of  $M$  in  $\mathbb{C}\mathbb{P}^{2n+1}$ . The key idea is that the linear segment of forms  $\omega_t$  joining two symplectic forms compatible with a fixed  $J$  is symplectic for every  $t$ . In our case we have this condition asymptotically. Define the family of 2-forms in  $M$  given by  $\omega_t = (1 - t)k\omega + t\phi_k^*(\omega_{FS})$ ,

where  $t \in [0, 1]$ . All of them are cohomologous, so to apply Moser’s trick [12] we only need to prove that they are symplectic. Take any unitary tangent vector  $v \in T_x M$ , for some  $x \in M$ . Then the  $\gamma$ -asymptotically holomorphic embeddings yield that  $g_{FS}(\partial\phi_k v, \partial\phi_k v) \geq \gamma^2$ . Now

$$\begin{aligned} \omega_t(v, Jv) &= (1 - t)k\omega(v, Jv) + t\phi_k^* \omega_{FS}(v, Jv) \\ &= (1 - t)kg(v, v) + t\omega_{FS}(d\phi_k v, J_0 \partial\phi_k v - J_0 \bar{\partial}\phi_k v) \\ &= (1 - t)k + tg_{FS}(\partial\phi_k v, \partial\phi_k v) + O(k^{-1/2}) > 0, \end{aligned}$$

for  $k$  large enough, since  $|\bar{\partial}\phi_k| = O(k^{-1/2})$ . So  $\omega_t$  is symplectic.  $\square$

**2.3. Construction of  $\gamma$ -projectizable sections.** Our objective is to prove the following perturbation result.

**Proposition 2.18.** *Let  $s_k$  be an asymptotically  $J$ -holomorphic sequence of sections of vector bundles  $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$ . Then given  $\alpha > 0$ , there exists an asymptotically  $J$ -holomorphic sequence of sections  $\sigma_k$  verifying:*

1.  $|s_k - \sigma_k|_{C^1, g_k} < \alpha$ .
2.  $\sigma_k$  is  $\eta$ -projectizable for some  $\eta > 0$ .

Moreover, the result can be extended to continuous one-parameter families of asymptotically  $J_t$ -holomorphic sequences  $s_{t,k}$  obtaining continuous one-parameter families of sections  $\sigma_{t,k}$  satisfying the two precedent conditions.

*Proof.* The result is a generalization of Proposition 1 in [3] where the result for 4-manifolds is proved. We proceed by using the globalization argument described in Proposition 2.8. First we deal with the non-parametric case. For this we define the local and  $C^0$ -open property  $\mathcal{P}(\varepsilon, x)$  as  $|s_k(x)| > \varepsilon$ . Let  $\delta > 0$ . We only need to find for a point  $x \in M$  a section  $\tau_{k,x}$  with Gaussian decay away from  $x$ , assuring that  $s_k + \tau_{k,x}$  verifies  $\mathcal{P}(\eta, y)$  in a ball of universal  $g_k$ -radius  $c$ , with  $\eta = c'\delta(\log(\delta^{-1}))^{-p}$ ,  $c'$  and  $p$  universal constants.

For this choose a section  $s_{k,x}^{\text{ref}}$  verifying the conditions of Lemma 2.5. Then we select  $c = 1$  (obviously, universal). The lower bound of  $s_{k,x}^{\text{ref}}$  in the ball  $B_x = B_{g_k}(x, 1)$  lets us define the map

$$f_{k,x} = \frac{s_k}{s_{k,x}^{\text{ref}}}: B_x \rightarrow \mathbb{C}^{2n+2}.$$

The lower bound of  $s_{k,x}^{\text{ref}}$  and the asymptotic holomorphicity of  $s_k$  imply that

$$(2.2) \quad |f_{k,x}| < C, \quad |\bar{\partial}f_{k,x}| < Ck^{-1/2}, \quad |\nabla\bar{\partial}f_{k,x}| < Ck^{-1/2},$$

where  $C$  is a universal constant. With the aid of Lemma 2.6 we can build  $f_k = f_{k,x} \circ \Phi_k^{-1}$  defined on a fixed ball  $B(0, c') \subset \mathbb{C}^n$ . Scaling the coordinates by a universal constant  $\frac{11}{10}(c')^{-1}$  we can suppose that  $f_k$  is defined on  $B^+$ . In this ball, the bounds (2.2) yield

$$(2.3) \quad |f_k| < C_0, \quad |\bar{\partial}f_k| < C_0k^{-1/2}, \quad |\nabla\bar{\partial}f_k| < C_0k^{-1/2},$$

where  $C_0$  is a universal constant. The application  $\hat{f}_k = \frac{1}{C_0} f_k$  verifies the hypothesis of Proposition 2.10 and so there exists, for  $k$  large enough, a number  $w_k \in B(0, \delta)$  such that  $|\hat{f}_k - w_k| > \sigma = \delta(\log(\delta^{-1}))^{-p}$ . Therefore  $|f_k - C_0 w_k| > C_0 \sigma$  on  $B$ . Now define

$$\tau_{k,x} = -C_0 w_k \otimes s_{k,x}^{\text{ref}},$$

so that  $|\tau_{k,x}|_{C^r, g_k} < c'' \delta$ , for some universal constant  $c''$ . Using the lower bound of  $s_{k,x}^{\text{ref}}$  we obtain that  $|s_k + \tau_{k,x}| \geq c' \delta (\log(\delta^{-1}))^{-p}$  on a ball  $B_{g_k}(x, c)$ , for some  $c > 0$ , with  $c'$  and  $p$  universal constants. Then Proposition 2.8 applies and the proof is concluded in the non-parametric case.

The globalization to the one-parameter case is trivial because all the ingredients in the proof can be easily chosen in a continuous way.  $\square$

**2.4. Inductive construction of sections  $\gamma$ -generic of order  $l$ .** Now we study the problem of perturbing a  $\gamma$ -projectizable sequence of sections to achieve genericity of order  $n$ . We shall do this in steps. The result to be proved is the following

**Proposition 2.19.** *Let  $s_k$  be an asymptotically  $J$ -holomorphic sequence of sections of the vector bundles  $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$  which is  $\gamma$ -projectizable and  $\gamma$ -generic of order  $l$ . Then given  $\alpha > 0$ , there exists an asymptotically  $J$ -holomorphic sequence of sections  $\sigma_k$  verifying:*

1.  $|s_k - \sigma_k|_{C^1, g_k} < \alpha$ .
2.  $\sigma_k$  is  $\eta$ -generic of order  $l + 1$  for some  $\eta > 0$ .

Moreover, this can be extended to continuous one-parameter families of asymptotically  $J_t$ -holomorphic sequences  $s_{t,k}$  obtaining continuous one-parameter families of sections  $\sigma_{t,k}$  verifying conditions 1 and 2.

*Proof.* We construct local 1-forms to control the perturbations. For this at a neighborhood of a point  $x \in M$  we fix local complex Darboux coordinates  $(z_k^1, \dots, z_k^n)$  using Lemma 2.6. As in proof of Theorem 2.11, by applying a unitary transformation to  $\mathbb{C}^{2n+2}$ , we can suppose that  $s_k(x) = (s_k^0(x), 0, \dots, 0)$ . Also there exists a ball with center  $x$  and universal  $g_k$ -radius  $c$  on which  $|s_k^0| \geq \gamma/2$ . We define, following [3], a local basis of asymptotically holomorphic 1-forms:

$$\mu_k^j = \partial \left( \frac{z_k^j s_{k,x}^{\text{ref}}}{s_k^0} \right),$$

where  $s_{k,x}^{\text{ref}}$  are given by Lemma 2.5. They have Gaussian decay away from  $x$  thanks to the behavior of  $s_{k,x}^{\text{ref}}$ . At  $x$  they form an orthogonal basis of  $T_x^* M$ . We use the trivialization  $\Phi_0$  to define the application

$$(2.4) \quad f_k: B_{g_k}(x, c) \rightarrow \mathbb{C}^{2n+1},$$

$$y \mapsto \left( \frac{s_k^1(y)}{s_k^0(y)}, \dots, \frac{s_k^{2n+1}(y)}{s_k^0(y)} \right).$$

We define the following property for sections  $s_k$  which are  $\gamma/2$ -projectizable and  $\gamma/2$ -generic of order  $l$ . A section  $s_k$  has the property  $\mathcal{P}(\varepsilon, x)$  if  $|\wedge^{l+1} \partial \mathbb{P} s_k(x)| > \varepsilon$ . This property is local and open in  $C^1$ -sense. For applying Proposition 2.8 we need to build, for  $0 < \delta < \gamma/2c''C$ , a local perturbation  $\tau_{k,x}$  with  $|\tau_{k,x}| < c''\delta$  and Gaussian decay with the property  $\mathcal{P}(\eta, y)$  in a neighborhood of  $x$  of universal  $g_k$  radius  $c$ , with  $\eta = c'\delta(\log(\delta^{-1}))^{-p}$ . (Here  $C$  is the constant of the  $C^1$ -openness of  $\mathcal{P}(\varepsilon, x)$  in Definition 2.7.) We define  $f_k$  as in (2.4). Then it is easy to see that there exists a universal constant  $c$  such that

$$\frac{|\wedge^{l+1} \partial \mathbb{P}(s_k)|}{|\wedge^{l+1} \partial f_k|} > \frac{1}{2}$$

on  $B_{g_k}(x, c)$ . So we can do the computations for the applications  $f_k$ . By a unitary transformation in  $U(2n+1)$  (on  $\mathbb{C}^{2n+2}$  fixing  $(1, 0, \dots, 0)$ ) and another in  $U(n)$  (on the complex Darboux coordinate chart) we can assure that

$$(2.5) \quad \partial f_k(x) = \begin{pmatrix} u_k^{11}(x) & 0 & \dots & \dots & 0 \\ 0 & u_k^{22}(x) & 0 & \dots & 0 \\ 0 & \dots & \ddots & 0 & \dots & 0 \\ 0 & \dots & 0 & u_k^{nn}(x) & 0 & \dots & 0 \end{pmatrix},$$

where  $|u_k^{11}(x) \cdots u_k^{ll}(x)| > \gamma/C'$ ,  $C'$  a universal constant. Shrinking  $c$  if necessary we can assure that  $|(\partial f_k^1 \wedge \cdots \wedge \partial f_k^l)_{\mu_k^1 \wedge \cdots \wedge \mu_k^l}| > \gamma/2C'$  for all the points of the ball  $B_{g_k}(x, c)$ , where we denote by  $(\partial f_k^1 \wedge \cdots \wedge \partial f_k^l)_{\mu_k^1 \wedge \cdots \wedge \mu_k^l}$  the component of  $\partial f_k^1 \wedge \cdots \wedge \partial f_k^l$  in the direction of  $\mu_k^1 \wedge \cdots \wedge \mu_k^l$ . This  $l$ -form is an element of the basis composed by the  $l$ -wedge products of the 1-forms  $\mu_k^1, \dots, \mu_k^n$ . In matrix form we are denoting the order  $l$  left upper minor of  $\partial f_k$ . Now we construct the  $(l+1)$ -form

$$\theta_k(y) = (\partial f_k^1 \wedge \cdots \wedge \partial f_k^l)_{\mu_k^1 \wedge \cdots \wedge \mu_k^l} \wedge \mu_k^{l+1}.$$

Note that if  $l = 0$  then the first term in the right hand side simply does not appear. We can suppose that  $|\theta_k| > c_s \gamma$  with  $c_s > 0$  a universal constant. We also consider the following family of  $(l+1)$ -forms

$$M_k^p = (\partial f_k^1 \wedge \cdots \wedge \partial f_k^l \wedge \partial f_k^p)_{\mu_k^1 \wedge \cdots \wedge \mu_k^l \wedge \mu_k^{l+1}}, \quad l+1 \leq p \leq 2n+1.$$

These forms are components of  $\wedge^{l+1} \partial f_k$ . If we perturb so that the norm of

$$M_k = (M_k^{l+1}, \dots, M_k^{2n+1})$$

is bigger than  $\eta = c'\delta(\log(\delta^{-1}))^{-p}$  then we have finished because if  $|M_k| > \eta$  then

$$|\wedge^{l+1} \partial f_k| > C_0 \eta$$

where  $C_0$  is again a universal constant (using that the basis  $\{\mu_k^{i_1} \wedge \cdots \wedge \mu_k^{i_{l+1}}\}_{1 \leq i_1 < \cdots < i_{l+1} \leq n}$  is almost orthogonal on the ball  $B_{g_k}(x, c)$ , in fact orthogonal at  $x$ ).

We define the following sequence of asymptotically holomorphic applications,

$$h_k = (h_k^{l+1}, \dots, h_k^{2n+1}) = \left( \frac{M_k^{l+1}}{\theta_k}, \dots, \frac{M_k^{2n+1}}{\theta_k} \right).$$

So we obtain, scaling the coordinates by universal constants if necessary,  $\hat{h}_k: B^+ \rightarrow \mathbb{C}^{2n+1-l}$  which is asymptotically holomorphic thanks to the lower bound of  $\theta_k$  and to the asymptotic holomorphicity of  $M_k$  and  $\theta_k$ . We have that  $n < 2n + 1 - l$  and so we can find  $|w_k| < \delta$  such that  $|h_k - w_k| > \eta = \delta(\log(\delta^{-1}))^{-p}$ . Thus we obtain that

$$|(M_k^{l+1} - w_k^{l+1}\theta_k, \dots, M_k^{2n+1} - w_k^{2n+1}\theta_k)| > c_s\gamma\eta.$$

Recall that all the constants depend on  $\gamma$  and the asymptotic holomorphicity constants of  $s_k$ , so they are independent of  $x$  and  $k$ . The perturbation  $-(w_k^{l+1}\theta_k, \dots, w_k^{2n+1}\theta_k)$  is achieved by adding the section  $\tau_{k,x} = -(0, \dots, 0, w_k^{l+1}z_k^{l+1}s_{k,x}^{\text{ref}}, \dots, w_k^{2n+1}z_k^{2n+1}s_{k,x}^{\text{ref}})$  to  $s_k$ . This section verifies the Gaussian decay bounds required in Proposition 2.8 and  $|\tau_{k,x}|_{C^1, g_k} < c''\delta$  for some universal constant  $c''$ . This completes the proof in the non-parametric case.

Now we pass to the one-parameter case. With appropriate continuous unitary transformations, we may assume that  $s_{t,k}(x) = (s_{t,k}^0(x), 0, \dots, 0)$  and that  $\partial f_{t,k}(x)$  is written as in (2.5). The interval  $[0, 1]$  may be split in a finite number (depending on  $k$ ) of subintervals  $[t_i, t_{i+1}]$  such that, for every  $x \in M$  and for each of the subintervals, there is a fixed order  $l$  minor of  $\partial f_{t,k}(x)$  with norm bigger than  $\gamma/C'$ , for every  $t$  in the subinterval. This allows to find continuous global small perturbations of  $s_{t,k}$  in every  $[t_i, t_{i+1}]$ .

Now we work as follows. For the first subinterval  $[0, t_1]$ , consider  $s_{t,k}$ ,  $t \in [0, t_1]$ , to find a perturbation  $\sigma_{t,k}^1$ ,  $t \in [0, t_1]$ , such that  $|s_{t,k} - \sigma_{t,k}^1| < \alpha/2$  on  $[0, t_1]$  and  $\sigma_{t,k}^1$  is  $\eta_1$ -generic of order  $l + 1$ , for some  $\eta_1 > 0$ . The same can be done for all the odd subintervals  $[t_{2i}, t_{2i+1}]$ . Using that the bounds can be chosen uniformly, we obtain perturbations  $\sigma_{t,k}^{2i+1}$ ,  $t \in [t_{2i}, t_{2i+1}]$ , which are  $\eta_1$ -generic of order  $l + 1$ , with  $|s_{t,k} - \sigma_{t,k}^{2i+1}|_{C^1} < \alpha/2$ . Now we may find a global perturbation  $\sigma_{t,k}$ ,  $t \in [0, 1]$ , satisfying that:

1.  $\sigma_{t,k} = \sigma_{t,k}^{2i+1}$ , for all  $t \in [t_{2i}, t_{2i+1}]$ . So  $\sigma_{t,k}$  is  $\eta_1$ -generic of order  $l + 1$  on these subintervals.
2.  $|\sigma_{t,k} - s_{t,k}|_{C^1} < \alpha/2$ , for all  $t \in [0, 1]$ .

This perturbation can be found by connecting  $\sigma_{t_{2i+1},k}^{2i+1}$  with  $s_{t_{2i+1}+\varepsilon,k}$  through a linear segment, choosing  $\varepsilon > 0$  small enough (depending on  $k$ ). Now we choose a constant  $\gamma > 0$  with  $\gamma < \alpha/2$  and  $\eta_1 - c'\gamma > \eta_1/2$ , where  $c'$  is the constant of  $C^1$ -openness of the genericity property required. Now we look for a perturbation  $\tau_{t,k}^{2i}$  on each of the even subintervals  $[t_{2i-1}, t_{2i}]$  which achieves  $\eta_2$ -genericity of order  $l + 1$ , for some  $\eta_2 > 0$ . Again we extend these perturbations to a global continuous perturbation  $\tau_{t,k}$ ,  $t \in [0, 1]$ , which is  $\eta_2$ -generic of order  $l + 1$  on the even subintervals, and with  $|\tau_{t,k} - \sigma_{t,k}| < \gamma$ . Therefore  $|s_{t,k} - \tau_{t,k}| < \alpha$  and  $\tau_{t,k}$  is  $\min\{\eta_1/2, \eta_2\}$ -generic of order  $l + 1$ , for all  $t \in [0, 1]$ .  $\square$

**2.5. Lifting asymptotically holomorphic embeddings.** In this subsection we prove that the sequences of asymptotically holomorphic embeddings into  $\mathbb{C}\mathbb{P}^{2n+1}$  that we are

considering in Theorem 1.2 come always from asymptotically holomorphic sequences of sections  $s_k$  of  $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$  which are  $\gamma$ -projectizable and  $\gamma$ -generic of order  $n$ , for some  $\gamma > 0$  (at least for  $k$  large).

*Proof of Lemma 2.16.* Suppose that we have a sequence of  $\gamma$ -asymptotically holomorphic embeddings  $\phi_k: M \rightarrow \mathbb{C}\mathbb{P}^{2n+1}$ , for some  $\gamma > 0$ , with  $\phi_k^* \mathcal{U} = L^{\otimes k}$ . Here  $\mathcal{U}$  is the hyperplane line bundle defined over the projective space. The dual of  $\mathcal{U}$  is the universal line bundle

$$\mathcal{E} = \{(l, s) \mid s \in l\} \subset \mathbb{C}\mathbb{P}^{2n+1} \times \mathbb{C}^{2n+2} = \underline{\mathbb{C}}^{2n+2},$$

interpreted as a sub-bundle of the trivial bundle  $\underline{\mathbb{C}}^{2n+2}$ . Consider the following sequence of line bundles,  $E_k = \phi_k^* \mathcal{E} \otimes L^{\otimes k} = \underline{\mathbb{C}} \subset \mathbb{C}^{2n+2} \otimes L^{\otimes k}$ , which are topologically trivial. We look for everywhere non-zero sections  $s_k$  of  $E_k \subset \mathbb{C}^{2n+2} \otimes L^{\otimes k}$  as they satisfy  $\phi_k = \mathbb{P}(s_k)$ .

Let  $\mathcal{P}(\varepsilon, x)$  be the  $C^1$ -open property for sequences of sections  $s_k$  of  $E_k$  of being  $\varepsilon$ -transverse to 0 at the point  $x$  (see Definition 2.2). We shall use Proposition 2.10 to find sequences of sections  $s_k$  which are  $\eta$ -transverse to 0, for some  $\eta > 0$ . Fix any asymptotically holomorphic sequence  $s_k$  of  $E_k$  (e.g. the zero sections) which will act as the starting point of our perturbation process. Let  $x \in M$ . Consider the sections  $s_{k,x}^{\text{ref}}$  of  $L^{\otimes k}$  given by Lemma 2.5 and define also the local sections of the line bundle  $\phi_k^* \mathcal{E} \subset \underline{\mathbb{C}}^{2n+2}$ ,

$$\sigma_k: B_{g_k}(x, k^{1/6}) \rightarrow \mathbb{C}^{2n+2},$$

by setting  $\sigma_k(x)$  any vector of norm 1 in the direction defined by  $\phi_k(x)$  and satisfying the condition  $\nabla_r \sigma_k(y) \perp \sigma_k(y)$ , for any  $y \in B_{g_k}(x, c)$ , where  $r$  is the radial vector field from  $x$ . This determines  $\sigma_k$  uniquely. The following estimates hold:

$$(2.6) \quad |\sigma_k(y)| = 1, \quad |\nabla \sigma_k(y)| = O(1 + d_k(x, y)), \\ |\bar{\partial} \sigma_k(y)| = O(k^{-1/2}(1 + d_k(x, y))), \quad |\nabla \bar{\partial} \sigma_k(y)| = O(k^{-1/2}(1 + d_k(x, y))).$$

The first one follows from  $\nabla_r \langle \sigma_k, \sigma_k \rangle = \langle \nabla_r \sigma_k, \sigma_k \rangle + \langle \sigma_k, \nabla_r \sigma_k \rangle = 0$ . For the second one, write  $\nabla \sigma_k = \nabla \phi_k + \langle \nabla \sigma_k, \sigma_k \rangle \sigma_k$ , where we identify  $T_{\phi_k(y)} \mathbb{C}\mathbb{P}^{2n+1} = [\sigma_k(y)]^\perp \subset \mathbb{C}^{2n+2}$ , isometrically. We already know that  $|\nabla \phi_k| = O(1)$ . So

$$\begin{aligned} \nabla_r \langle \nabla \sigma_k, \sigma_k \rangle &= \langle \nabla_r \nabla \sigma_k, \sigma_k \rangle + \langle \nabla \sigma_k, \nabla_r \sigma_k \rangle = \langle \nabla \nabla_r \sigma_k, \sigma_k \rangle + \langle \nabla \sigma_k, \nabla_r \sigma_k \rangle \\ &= -\langle \nabla_r \sigma_k, \nabla \sigma_k \rangle + \langle \nabla \sigma_k, \nabla_r \sigma_k \rangle = -\langle \nabla_r \phi_k, \nabla \phi_k \rangle + \langle \nabla \phi_k, \nabla_r \phi_k \rangle = O(1). \end{aligned}$$

The first equality uses that  $\nabla$  is the standard derivative for functions with values in  $\mathbb{C}^{2n+2}$ , and hence the second derivatives commute. The second equality follows from  $\langle \nabla_r \sigma_k, \sigma_k \rangle = 0$ . So we have that  $\langle \nabla \sigma_k, \sigma_k \rangle = O(d_k(x, y))$  and hence  $|\nabla \sigma_k| = O(1 + d_k(x, y))$ . The other two cases are worked out analogously.

Now define the application

$$f_k = \frac{s_k}{s_{k,x}^{\text{ref}}}: B_{g_k}(x, c) \rightarrow \mathbb{C},$$

which is asymptotically holomorphic by construction. Using a complex Darboux chart we trivialize  $B_{g_k}(x, c)$  to obtain (scaling the coordinates by appropriate universal constants)

an application  $\hat{f}_k: B^+ \rightarrow \mathbb{C}$  to which we apply Proposition 2.10 to obtain  $w_k \in B(0, \delta)$  such that  $\hat{f}_k - w_k$  is  $\eta$ -transverse to 0 in  $B$ , where  $\eta = \delta(\log(\delta^{-1}))^{-p}$ . Rescaling and passing to the manifold, we have that  $f_k - Cw_k$  is  $C'\eta$ -transverse to 0, for some universal constants  $C$  and  $C'$ . Define the asymptotically holomorphic sequence of sections  $\tau_{k,x} = -w_k s_{k,x}^{\text{ref}} \sigma_k$  of  $E_k$ , which has Gaussian decay by (2.6), to get a perturbation satisfying the conditions in Proposition 2.8.

Thus there exists an asymptotically holomorphic sequence  $s_k$  of sections of  $E_k$  which is  $\eta$ -transverse to 0, for some  $\eta > 0$ . For  $k$  large enough, the zeroes of  $s_k$  is a symplectic submanifold representing the trivial homology class, hence the empty set. So  $s_k$  is nowhere vanishing and hence  $\phi_k = \mathbb{P}(s_k)$ .

We have that  $s_k$  is an asymptotically holomorphic sequence of sections of  $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$ . Let us check that  $s_k$  is  $\eta$ -projectizable, i.e. that  $|s_k| \geq \eta$  everywhere. Suppose that this is not the case and take the point  $x \in M$  where  $|s_k|$  attains its minimum. As  $|s_k(x)| < \eta$ ,  $\eta$ -transversality implies that  $|\nabla s_k(x)| \geq \eta$ . Also  $s_k$  is asymptotically holomorphic, so for  $k$  large  $\nabla s_k(x): T_x M \rightarrow (E_k)_x$  is surjective. Take  $v \in T_x M$  such that  $\nabla_v s_k(x) = s_k(x)$ . Evaluating the equality

$$\nabla |s_k|^2 = \langle \nabla s_k, s_k \rangle + \langle s_k, \nabla s_k \rangle$$

at the point  $x$  and along the direction of  $v$ , we obtain  $|s_k(x)|^2 = 0$ , which is impossible since we have already proved that  $s_k$  is nowhere vanishing.

Finally the extension to the one-parameter case is trivial.  $\square$

### 3. Estimated intersections of symplectic submanifolds

**3.1. Notions on estimated Euclidean geometry.** In order to set up the definitions needed in Subsection 3.2 we state the relevant notions and results on angles between subspaces of Euclidean spaces that we shall need. From now on we assume that we are in  $\mathbb{R}^n$  equipped with the standard Euclidean inner product, but all the proofs apply to a general finite dimensional Euclidean space.

The angle between two non-zero vectors  $v, w \in \mathbb{R}^n$  is defined as

$$\angle(v, w) = \arccos\left(\frac{\langle v, w \rangle}{|v||w|}\right) \in [0, \pi].$$

The angle is symmetric and satisfies the classical triangular inequality,

$$\angle(u, w) \leq \angle(u, v) + \angle(v, w),$$

for non-zero vectors  $u, v, w \in \mathbb{R}^n$ . Also the angle of a vector  $u \neq 0$  with respect to a subspace  $V \neq \{0\}$  is defined as

$$\angle(u, V) = \min_{v \in V - \{0\}} \{\angle(u, v)\} = \angle(u, v(u)) \in \left[0, \frac{\pi}{2}\right],$$

where  $v: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the orthogonal projection onto  $V$ , well understood that when  $v(u) = 0$  the angle is  $\pi/2$ .

**Definition 3.1.** The maximum angle of a subspace  $U \neq \{0\}$  with respect to a subspace  $V \neq \{0\}$  is defined as

$$\angle_M(U, V) = \max_{u \in U - \{0\}} \angle(u, V).$$

Notice that this angle is not in general symmetric. But in the case  $\dim U = \dim V$  symmetry does hold. This is easily checked by constructing an orthogonal transformation permuting the two subspaces. Indeed the maximum angle  $\angle_M(U, V)$  gives a notion of proximity between  $U$  and  $V$  whenever  $\dim U \leq \dim V$ .

**Lemma 3.2.** Given  $U, V, W$  non-zero subspaces in  $\mathbb{R}^n$  then:

$$\angle_M(U, W) \leq \angle_M(U, V) + \angle_M(V, W).$$

*Proof.* We will denote by  $v(u)$  the orthogonal projection of the vector  $u$  onto the subspace  $V$ . In the following inequalities, if  $v(u) = 0$ , we suppose that the angle in which this expression appears is  $\pi/2$ . We have

$$\begin{aligned} \angle_M(U, W) &= \max_{u \in U - \{0\}} \left\{ \min_{w \in W - \{0\}} \{ \angle(u, w) \} \right\} \\ &\leq \max_{u \in U - \{0\}} \left\{ \min_{w \in W - \{0\}} \{ \angle(u, v(u)) + \angle(v(u), w) \} \right\} \\ &= \max_{u \in U - \{0\}} \left\{ \angle(u, v(u)) + \min_{w \in W - \{0\}} \{ \angle(v(u), w) \} \right\} \\ &\leq \max_{u \in U - \{0\}} \{ \angle(u, v(u)) \} + \max_{u \in U - \{0\}} \left\{ \min_{w \in W - \{0\}} \{ \angle(v(u), w) \} \right\} \\ &\leq \angle_M(U, V) + \max_{v \in V - \{0\}} \left\{ \min_{w \in W - \{0\}} \{ \angle(v, w) \} \right\} \\ &\leq \angle_M(U, V) + \angle_M(V, W). \quad \square \end{aligned}$$

**Definition 3.3.** The minimum angle of two non-zero subspaces  $U, V$  of  $\mathbb{R}^n$  is defined as:

- If  $\dim U + \dim V < n$  then  $\angle_m(U, V) = 0$ .
- If their intersection is not transverse then  $\angle_m(U, V) = 0$ .
- If their intersection is transverse then let  $W$  be their intersection. Define  $U_c$  as the orthogonal subspace in  $U$  to  $W$ , and  $V_c$  in the same way. Then

$$\angle_m(U, V) = \min_{u \in U_c - \{0\}} \{ \angle(u, V_c) \} \in [0, \pi/2].$$

The definition is symmetric because (in the transverse case)

$$\angle_m(U, V) = \min_{u \in U_c - \{0\}} \left\{ \min_{v \in V_c - \{0\}} \{ \angle(u, v) \} \right\}$$

and the two minima commute. Also  $\angle_m(U, V) = \min_{u \in U_c - \{0\}} \{ \angle(u, V) \}$ .

**Lemma 3.4.** *For non-zero subspaces  $U$  and  $V$  of  $\mathbb{R}^n$  we have that*

$$\angle_m(U, V) = \min_{u \in U^\perp - \{0\}} \left\{ \min_{v \in V^\perp - \{0\}} \{ \angle(u, v) \} \right\}.$$

*Proof.* This is trivial in the case  $\dim U + \dim V < n$  or when  $U$  and  $V$  do not intersect transversely. In the transverse case, we can restrict ourselves to the subspace  $(U \cap V)^\perp$  to compute the angles. So without loss of generality we can suppose that  $U \oplus V = \mathbb{R}^n$ ,  $U_c = U$  and  $V_c = V$ . As  $\dim U = \dim V^\perp$ , we may construct an orthogonal transformation  $\phi$  permuting  $U$  and  $V^\perp$ , i.e.  $\phi(U) = V^\perp$  and  $\phi(V^\perp) = U$ . Therefore also  $\phi(V) = U^\perp$ . So

$$\angle_m(U, V) = \angle_m(\phi(U), \phi(V)) = \angle_m(V^\perp, U^\perp) = \min_{u \in U^\perp - \{0\}} \left\{ \min_{v \in V^\perp - \{0\}} \{ \angle(u, v) \} \right\},$$

which proves the lemma.  $\square$

**Proposition 3.5.** *For non-zero subspaces  $U, V, W$  of  $\mathbb{R}^n$  we have that*

$$\angle_m(U, V) \leq \angle_M(U, W) + \angle_m(W, V).$$

*Proof.* By Lemma 3.4 we have that

$$\begin{aligned} \angle_m(U, V) &= \min_{u \in U^\perp - \{0\}} \left\{ \min_{v \in V^\perp - \{0\}} \{ \angle(u, v) \} \right\} \\ &\leq \min_{u \in U^\perp - \{0\}} \{ \angle(u, w) \} + \min_{v \in V^\perp - \{0\}} \{ \angle(w, v) \}, \end{aligned}$$

for any  $w \in \mathbb{R}^n$ . Choose  $w_0 \in W^\perp - \{0\}$  satisfying

$$\angle_m(W, V) = \min_{w \in W^\perp - \{0\}} \left\{ \min_{v \in V^\perp - \{0\}} \{ \angle(w, v) \} \right\} = \min_{v \in V^\perp - \{0\}} \{ \angle(w_0, v) \}.$$

Then we have

$$\angle_m(U, V) \leq \min_{u \in U^\perp - \{0\}} \{ \angle(u, w_0) \} + \angle_m(W, V) \leq \angle_M(W^\perp, U^\perp) + \angle_m(W, V).$$

The result follows once we show that  $\angle_M(W^\perp, U^\perp) = \angle_M(U, W)$ . Put  $\angle_M(U, W) = \alpha$ . Let  $u \in U$  with  $\angle(u, W) = \alpha$ . Denoting by  $w$  the projection of  $u$  onto  $W^\perp$ , we have that  $\angle(u, W^\perp) = \angle(u, w) = \frac{\pi}{2} - \alpha$ . So  $\angle(w, U) \leq \frac{\pi}{2} - \alpha$  and hence  $\angle(w, U^\perp) \geq \alpha$ . This implies that  $\angle_M(W^\perp, U^\perp) \geq \alpha = \angle_M(U, W)$ . The opposite inequality follows by symmetry.  $\square$

**Corollary 3.6.** *Given non-zero subspaces  $U, U', V$  of  $\mathbb{R}^n$  with  $\angle_m(U, V) > \varepsilon$  and  $\angle_M(U, U') < \delta$  then  $\angle_m(U', V) > \varepsilon - C\delta$ , where  $C$  is a universal constant ( $C = 1$  in fact).  $\square$*

The following result will be very important for our purposes.

**Proposition 3.7.** *Given  $\varepsilon > 0$  and  $U \in \text{Gr}(m, n)$ ,  $V \in \text{Gr}(r, n)$  subspaces satisfying that  $\angle_m(U, V) > \varepsilon$ . Then there are  $\gamma_0 > 0$  and a constant  $C$ , depending only on  $\varepsilon$ , such that for any  $\gamma < \gamma_0$ , if  $U' \in \text{Gr}(m, n)$  and  $V' \in \text{Gr}(r, n)$  verify that*

$$\angle_M(U, U') < \gamma, \quad \angle_M(V, V') < \gamma,$$

*then  $U'$  and  $V'$  intersect transversely and  $\angle_M(U \cap V, U' \cap V') < C\gamma$ .*

*Proof.* By Proposition 3.5 choosing  $\gamma_0 > 0$  small enough, only depending on  $\varepsilon$ , we can assure that the following intersections are transverse:  $U \cap V = W$ ,  $U \cap V'$ ,  $U' \cap V$  and  $U' \cap V' = W'$ , and that  $\angle_m(U', V') \geq \varepsilon/2$ . By Lemma 3.2 we have

$$\angle_m(W, W') \leq \angle_M(W, U \cap V') + \angle_M(W', U \cap V').$$

We are going to bound the first term in the right hand side of the inequality, the bounding of the second term being analogous.

Put  $s = \dim W = r + m - n$ . Choose an orthonormal basis  $(e_1, \dots, e_s)$  of  $W$ , extend it to an orthonormal basis  $(e_1, \dots, e_r)$  of  $V$  and finally extend it to an orthonormal basis  $(e_1, \dots, e_n)$  of  $\mathbb{R}^n$ . Note that  $(e_{s+1}, \dots, e_r)$  is an orthonormal basis of  $V_c$ . As

$$\angle_m(U, V) = \angle_m(U_c, V) > \varepsilon \quad \text{and} \quad \angle_M(V, V') < \gamma_0$$

we have  $\angle_m(U_c, V') > \varepsilon/2$  (decreasing  $\gamma_0$  if necessary). So  $U_c \cap V' = \{0\}$ . Recalling that  $V \oplus U_c = \mathbb{R}^n$ , we see that there is a basis  $(e_1 + \varepsilon_1, \dots, e_r + \varepsilon_r)$  for  $V'$  where  $\varepsilon_j \in U_c$ . Using that  $\angle_m(U, V) > \varepsilon$  and that the decomposition  $\mathbb{R}^n = W \oplus V_c \oplus V^\perp$  is orthogonal, we have

$$\begin{aligned} \text{pr}_{W'}^\perp(\varepsilon_j) &= 0, \\ \text{pr}_{V_c}^\perp(\varepsilon_j) &\leq |\cos \varepsilon| |\varepsilon_j|, \\ \text{pr}_{V^\perp}^\perp(\varepsilon_j) &\geq \sqrt{1 - |\cos \varepsilon|^2} |\varepsilon_j| = |\sin \varepsilon| |\varepsilon_j|. \end{aligned}$$

Checking the angle of  $e_j + \varepsilon_j$  with respect to  $V$ , we get that

$$(3.1) \quad \angle_M(V, V') \geq \arctan \frac{|\sin \varepsilon| |\varepsilon_j|}{1 + |\cos \varepsilon| |\varepsilon_j|} \geq \arctan \left( \frac{\sin \varepsilon}{1 + |\varepsilon_j|} |\varepsilon_j| \right).$$

For  $\gamma_0 > 0$  small enough we get that  $|\varepsilon_j| < C \angle_M(V, V')$  for a constant  $C$  depending on  $\varepsilon$ .

Now we compute  $\angle_M(W, U \cap V')$ . The intersection  $U \cap V'$  has basis  $(e_1 + \varepsilon_1, \dots, e_s + \varepsilon_s)$ . For  $u_i = e_i + \varepsilon_i$  we have

$$\angle(u_i, W) = \arccos \frac{1}{\sqrt{1 + |\varepsilon_i|^2}} = \arctan |\varepsilon_i| \leq |\varepsilon_i|.$$

Therefore  $\angle_M(W, U \cap V') \leq \max_{1 \leq i \leq s} |\varepsilon_i| \leq C \angle_M(V, V') \leq C\gamma$ .  $\square$

Now we are going to set up the relationship between the transversality of maps in the Donaldson-Auroux approach and the angles defined above.

**Lemma 3.8.** *Let  $U, V$  be two non-zero subspaces of  $\mathbb{R}^n$  and let  $g: U \rightarrow V$  and  $h: U \rightarrow V^\perp$  be the projections from  $U$  with respect to the decomposition  $\mathbb{R}^n = V \oplus V^\perp$ . If  $h$  has a right inverse  $\theta$  satisfying  $|\theta| < \gamma^{-1}$  for some  $\gamma > 0$  then  $\angle_m(U, V) > \gamma$ .*

*Proof.* In the first place, as  $h$  is onto, the intersection between  $U$  and  $V$  is transverse. Let  $W = U \cap V$ . Define  $\hat{\theta} = \text{pr}_{U_c}^\perp \circ \theta: V^\perp \rightarrow U_c$ , which is an inverse of  $h: U_c \rightarrow V^\perp$  such that  $|\hat{\theta}| < \gamma^{-1}$ . Now consider any  $u \in U_c - \{0\}$  and put  $v = h(u)$ . Then

$$\angle(u, V) = \arcsin \frac{|h(u)|}{|u|} = \arcsin \frac{|v|}{|\hat{\theta}(v)|} > \arcsin \frac{1}{\gamma^{-1}} > \gamma,$$

and the proof is concluded.  $\square$

**3.2. Projective symplectic geometry.** In this subsection we will prove Theorem 1.3. This will provide a geometrical proof of Bertini’s theorem, the main result of [5]. Although our proof is more technical and long, it has the advantage of giving us a more general kind of symplectic submanifolds than those in [5], [2]. In fact our technique will allow us a simple generalization to solve the problem of constructing determinantal symplectic submanifolds in Section 5. First of all, in order to measure the holomorphicity of submanifolds, let us introduce the complex angle of even dimensional subspaces  $V \subset \mathbb{C}^n$  as

$$\beta: \text{Gr}_{\mathbb{R}}(2r, 2n) \rightarrow [0, \pi/2],$$

$$V \mapsto \angle_M(V, JV).$$

Clearly  $\beta(V) = 0$  if and only if  $V$  is complex and  $\beta(V) < \pi/2$  if and only if  $V$  is symplectic.

**Definition 3.9.** Let  $(M, \omega)$  be a symplectic manifold endowed with a compatible almost complex structure  $J$ . A sequence of submanifolds  $S_k \subset M$  is asymptotically holomorphic if  $\beta(TS_k) = O(k^{-1/2})$ .

Note that if the submanifolds  $S_k$  are asymptotically holomorphic then they are symplectic for  $k$  large. If  $\phi_k: M \rightarrow \mathbb{C}\mathbb{P}^N$  is a sequence of asymptotically holomorphic embeddings then  $\phi_k(M)$  is a sequence of asymptotically holomorphic submanifolds.

**Proposition 3.10.** *Let  $\phi_k^1: (M_1, J_1) \rightarrow \mathbb{C}\mathbb{P}^N$  and  $\phi_k^2: (M_2, J_2) \rightarrow \mathbb{C}\mathbb{P}^N$  be two sequences of asymptotically holomorphic embeddings. Suppose that there exists  $\varepsilon > 0$  indepen-*

dent of  $k$  such that for any  $x \in \phi_k^1(M_1) \cap \phi_k^2(M_2)$ , the minimum angle between  $(\phi_k^1)_*TM_1(x)$  and  $(\phi_k^2)_*TM_2(x)$  is greater than  $\varepsilon$ . Then  $S_k = \phi_k^1(M_1) \cap \phi_k^2(M_2)$  is a sequence of asymptotically holomorphic submanifolds (hence symplectic for  $k$  large). Also  $S_k^j = (\phi_k^j)^{-1}(S_k)$  is a sequence of asymptotically holomorphic submanifolds of  $M_j$ ,  $j = 1, 2$ . Moreover there exists a sequence of compatible almost complex structures  $J_k^j$  of  $M_j$  such that  $S_k^j$  is pseudoholomorphic for  $J_k^j$ ,  $|J_k^j - J_j| = O(k^{-1/2})$  and  $\phi_k^j$  restricted to  $(S_k^j, J_k^j)$  is a sequence of asymptotically holomorphic embeddings in  $\mathbb{C}\mathbb{P}^N$ ,  $j = 1, 2$ .

The same statement holds for the case of one-parameter families of embeddings  $(\phi_{t,k}^1)_{t \in [0,1]}$  and  $(\phi_{t,k}^2)_{t \in [0,1]}$ .

Remark that  $M_1$  and  $M_2$  are not necessarily compact manifolds.

*Proof.* Let  $J_0$  be the standard complex structure of  $\mathbb{C}\mathbb{P}^{2n+1}$ . Then

$$\angle_M((\phi_k^j)_*TM, J_0(\phi_k^j)_*TM) = O(k^{-1/2})$$

for  $j = 1, 2$ . By Proposition 3.7,  $\angle_M(TS_k, J_0TS_k) = O(k^{-1/2})$ . As  $|(\phi_k^j)_*J_j - J_0| = O(k^{-1/2})$  on  $(\phi_k^j)_*TM$ , we have  $\angle_M(TS_k, (\phi_k^j)_*J_jTS_k) = O(k^{-1/2})$  and so  $\angle_M(TS_k^j, J_jTS_k^j) = O(k^{-1/2})$ , i.e.  $S_k^j$  is a sequence of asymptotically holomorphic submanifolds of  $M_j$ .

Finally we have to build  $J_k^j$  on  $M_j$  such that  $|J_k^j - J_j| = O(k^{-1/2})$  and  $S_k^j$  is  $J_k^j$ -holomorphic. The composition of  $J_j: TS_k^j \rightarrow TM$  with the orthogonal projection  $TM \rightarrow TS_k^j$  has square close to  $-1$ , for  $k$  large enough. So we can homotop it to an almost complex structure  $J_k^j$  on  $S_k^j$ . Then we extend this  $J_k^j$  to a small tubular neighborhood of  $S_k^j$  by giving a complex structure to the normal bundle of  $S_k^j$ . Finally a homotopy between  $J_k^j$  and  $J_j$  allows us to extend  $J_k^j$  off a little bigger neighborhood of  $S_k^j$  matching with  $J_j$  on the border. This gives the required  $J_k^j$ .

The result for continuous one-parameter families is trivial from the non-parametric case.  $\square$

Let us have a smooth submanifold  $N$  of a manifold  $X$ . If we fix a metric on  $X$  we can define a geodesic flow  $\varphi_t$ . In particular, following the perpendicular directions to  $N$  we can identify a tubular neighborhood of the zero section of the normal bundle of  $N$  (defined as  $|n| < t_0$ ,  $n \in \nu(N)$ , for some small  $t_0 > 0$ ) with a tubular neighborhood  $U_N \subset X$  of  $N$ . So we can define a distribution  $D_N$  in  $U_N$  as

$$D_N(\varphi_n(x)) = (\varphi_n)_*T_xN, \quad \forall x \in N, n \in \nu(N), |n| < t_0.$$

where  $(\varphi_n)_*$  denotes parallel transport along the geodesic tangent to  $n$ .

**Definition 3.11.** Suppose  $\phi_k: M \rightarrow X$  is a sequence of asymptotically holomorphic embeddings into a Hodge manifold  $X$ . Let us fix a complex submanifold  $N \subset X$ . We say that  $\phi_k$  is  $\sigma$ -transverse to  $N$ , with  $\sigma < t_0$ , if for all  $x \in M$  and all  $k$ ,

$$d(\phi_k(x), N) < \sigma \Rightarrow \angle_m((\phi_k)_*(T_xM), D_N(\phi_k(x))) > \sigma.$$

This property is  $C^1$ -open, i.e. given  $\phi_k$  an embedding  $\eta$ -transverse to  $N$ , then a perturbation of  $\hat{\phi}_k$  with  $d_{C^1}(\phi_k, \hat{\phi}_k) < \delta$  is  $(\eta - C\delta)$ -transverse to  $N$ , where  $C$  is a universal constant.

Obviously a  $\sigma$ -transverse sequence of embeddings  $\phi_k$  verifies the conditions of Proposition 3.10 with  $\phi_k^1 = \phi_k: M \rightarrow X$  and  $\phi_k^2 = i: N \hookrightarrow X$ . The following result then completes the proof of Theorem 1.3.

**Theorem 3.12.** *Let  $\phi_k = \mathbb{P}(s_k)$ , where  $s_k$  is an asymptotically holomorphic sequence of sections of  $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$  which is  $\gamma$ -projectizable and  $\gamma$ -generic of order  $n$ , for some  $\gamma > 0$ . Let us fix a holomorphic submanifold  $N$  in  $\mathbb{C}\mathbb{P}^{2n+1}$ . Then for any  $\delta > 0$  there exists an asymptotically holomorphic sequence of sections  $\sigma_k$  of  $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$  such that:*

1.  $|\sigma_k - s_k|_{g_k, C^1} < \delta$ .
2.  $\hat{\phi}_k = \mathbb{P}(\sigma_k)$  is an  $\eta$ -asymptotically holomorphic embedding in  $\mathbb{C}\mathbb{P}^{2n+1}$  which is  $\varepsilon$ -transverse to  $N$ , for some  $\eta > 0$  and  $\varepsilon > 0$ . In the case  $\dim_{\mathbb{C}} M + \dim_{\mathbb{C}} N < 2n + 1$  we actually have that  $d_{FS}(\hat{\phi}_k(M), N) > \varepsilon$ , for  $k$  large enough.

Moreover the result can be extended to one-parameter continuous families of complex submanifolds  $(N_t)_{t \in [0,1]}$ , taking in this case as starting point a continuous family  $\phi_{t,k} = \mathbb{P}(s_{t,k})$  where  $s_{t,k}$  are asymptotically  $J_t$ -holomorphic sections of  $\mathbb{C}^{2n+2} \otimes L^{\otimes k}$  which are  $\gamma$ -projectizable and  $\gamma$ -generic of order  $n$ , for some  $\gamma > 0$ .

The proof of this result will be the content of Subsection 3.3. As a corollary we get a geometric proof of the main Theorem of [5].

**Corollary 3.13.** *Given a compact symplectic manifold  $(M, \omega)$ , suppose that  $[\omega/2\pi] \in H^2(M, \mathbb{R})$  is the reduction of an integral class  $h$ . Then for  $k$  large enough there exists symplectic submanifolds realizing the Poincaré dual of  $kh$ . Moreover, perhaps by increasing  $k$ , all the symplectic submanifolds realizing this Poincaré dual, constructed as transverse intersections with a fixed complex hyperplane of asymptotically holomorphic sequences of embeddings with respect to two compatible almost complex structures, are isotopic. The isotopy can be made by symplectomorphisms.*

*Proof.* By Theorem 2.11 we build an asymptotically holomorphic sequence of embeddings to  $\mathbb{C}\mathbb{P}^{2n+1}$ . In  $\mathbb{C}\mathbb{P}^{2n+1}$  we choose a complex hyperplane  $H$ . By Theorem 3.12 we perturb the sequence of embeddings to find a new asymptotically holomorphic sequence of embeddings  $\phi_k$  such that  $\phi_k(M)$  intersects  $H$  with minimum angle greater than  $\varepsilon > 0$ . Finally Proposition 3.10 implies that  $\phi_k(M) \cap H = H_M$  is an asymptotically holomorphic sequence of submanifolds, which are symplectic for  $k$  large enough. Also  $\phi_k^{-1}(H_M)$  is a symplectic submanifold of  $M$  for  $k$  large enough. A direct topological argument shows us that it is Poincaré dual of  $kh$ .

For the isotopy statement, we take two sequences of symplectic submanifolds,  $W_k^0$  and  $W_k^1$ , obtained as intersections between two  $\eta$ -asymptotically  $J_j$ -holomorphic sequences  $\mathbb{P}(s_{k,j})$ ,  $j = 0, 1$ , and two fixed complex hyperplanes  $H_0$  and  $H_1$  in  $\mathbb{C}\mathbb{P}^{2n+1}$  with angles greater than a fixed  $\varepsilon > 0$ . Then we prove that they are isotopic. We construct the straight segment  $H_t$ , in the dual space, of hyperplanes connecting  $H_0$  and  $H_1$ . We construct

the family of asymptotically holomorphic sequences  $s_{t,k}$  as in (2.1). Using Theorem 3.12, we obtain a family  $\phi_{t,k} = \mathbb{P}(\sigma_{t,k})$  of asymptotically  $J_t$ -holomorphic embeddings which are  $\eta/2$ -transverse to  $N$ , choosing the perturbation  $\delta > 0$  small enough. This gives a family of symplectic isotopic submanifolds  $(W_k^t)'$  in  $M$  for each fixed large  $k$ . The problem is that  $W_k^0$  does not coincide with  $(W_k^0)'$  (and respectively for  $t = 1$ ). For  $\delta > 0$  small enough, they are isotopic, in fact the linear segment  $((1-t)\sigma_{0,k} + t\sigma_{0,k})_{t \in [0,1]}$  provides a family of asymptotically holomorphic embeddings transverse to  $H_0$ , for  $k$  large enough giving the desired isotopy.  $\square$

The constructive technique of Theorem 3.12 is more general as we can take any algebraic subvariety of  $\mathbb{C}\mathbb{P}^{2n+1}$  to make the intersection. However, the difficulty in finding topological information about the constructed submanifolds makes it hard to prove that they are more general than those produced in [2]. To overcome this problem we shall construct in Section 5 a special kind of submanifolds where we can compute symplectic invariants using formulas from algebraic geometry.

**3.3. Estimated intersections in  $\mathbb{C}\mathbb{P}^{2n+1}$ .** Now we aim to prove Theorem 3.12. Our objective is to find sequences  $\phi_k$  of asymptotically holomorphic embeddings which are  $\sigma$ -transverse to  $N$ .

*Proof of Theorem 3.12.* As usual we begin with the simplest case, when the complex codimension of  $N$  is 1. Also we consider the non-parametric case, being the parametric one a simple generalization. We say that a sequence of sections  $s_k$  which is  $\gamma/2$ -projectizable and  $\gamma/2$ -generic of order  $n$  verifies  $\mathcal{P}(\varepsilon, x)$  if  $\mathbb{P}(s_k)$  is  $\varepsilon$ -transverse to  $N$  at the point  $x$ . This property is local and open in  $C^1$ -sense, for  $\varepsilon < t_0$ . To make use of Proposition 2.8 we need to find local sections with Gaussian decay obtaining local transversality. To achieve this local transversality we are going to use Proposition 2.10.

As  $N$  is a fixed holomorphic submanifold, we may fix a finite covering of  $\mathbb{C}\mathbb{P}^{2n+1}$  by balls  $U_j$  such that  $N$  is defined as the zero set of a holomorphic function  $f_j: U_j \rightarrow \mathbb{C}$  in each  $U_j$  and such that for any  $z_1, z_2 \in U_j \cap U_N$ ,  $\angle_M(D_N(z_1), D_N(z_2)) \leq \varepsilon$ , and for any  $z_1, z_2 \in U_j$ ,  $\angle_M(\ker df_j(z_1), \ker df_j(z_2)) \leq \varepsilon$ , with  $\varepsilon > 0$  an arbitrarily small number fixed along the proof.

We choose a constant  $C$  independent of  $k$  such that  $|\nabla\phi_k|_{g_k} \leq C$ . Therefore  $\phi_k(B_{g_k}(x, c)) \subset B_{g_{FS}}(\phi_k(x), Cc)$ , for any  $c$ . Now we choose  $c > 0$  small enough satisfying the following premises:

1. Let  $x \in M$ . With a transformation of  $U(2n+2)$  in  $\mathbb{C}^{2n+2}$ , we may suppose that  $s_k(x) = (s_k^0(x), 0, \dots, 0)$ . As  $s_k$  is  $\gamma$ -projectizable and asymptotically holomorphic, we can choose a universal  $g_k$ -radius  $c$  with  $|s_k^0| \geq \gamma/2$  on  $B_{g_k}(x, 20c)$ . Also the sections  $s_{k,x}^{\text{ref}}$  of Lemma 2.5 satisfy  $|s_{k,x}^{\text{ref}}| \geq c_s$  on  $B_{g_k}(x, 20c)$ . Note that  $\phi_k(B_{g_k}(x, 20c)) \subset B_{g_{FS}}(\phi_k(x), 20Cc)$ .

2. We use the standard chart  $\Phi_0$  for  $\mathbb{C}\mathbb{P}^{2n+1}$  around  $p = \phi_k(x) = [1, 0, \dots, 0]$  to trivialize the ball  $B_{g_{FS}}(p, 20Cc)$ . We may choose  $c$  small enough so that  $\Phi_0$  is near an isometry, in the sense that

$$\frac{2}{3}|\Phi_0(q)| \leq d_{FS}(p, q) \leq 2|\Phi_0(q)|$$

for  $q \in B_{g_{FS}}(p, 20Cc)$ . Also we require  $|\nabla\Phi_0| \leq 2$  in such ball. With respect to this trivialization the map  $\phi_k$  is given locally as

$$f_k = \Phi_0 \circ \phi_k: B_{g_k}(x, 20c) \rightarrow B(0, 40Cc),$$

$$y \mapsto \left( \frac{s_k^1(y)}{s_k^0(y)}, \dots, \frac{s_k^{2n+1}(y)}{s_k^0(y)} \right).$$

Clearly  $|\nabla f_k| \leq 2C$  uniformly in  $k$ .

3. We can reduce  $c$  so that, for any  $p$ ,  $B_{g_{FS}}(p, 20Cc) \subset U_j$  for some  $U_j$ . Therefore  $N$  is defined in  $B(0, 15Cc)$  by a function  $f: B(0, 15Cc) \rightarrow \mathbb{C}$ . Call  $Z = Z(f)$  in such ball. The angle condition means that  $\ker df(z_1), \ker df(z_2)$  are close enough (say less than  $\pi/6$ ) for  $z_1, z_2 \in Z$ .

Let  $x \in M$ . In the case  $d(\phi_k(x), N) \geq 2Cc$ , as we perform a small perturbation, say of norm  $\delta > 0$  such that  $d_{FS}(\phi_k(x), \hat{\phi}_k(x)) < \frac{1}{2}Cc$ , for all  $x \in M$ , there is still  $\frac{1}{2}Cc$ -transversality at a  $c$ -neighborhood of  $x$ . So we are finished.

Suppose  $d(\phi_k(x), N) < 2Cc$ . Then take a point  $z_0 \in B(0, 4Cc) \cap Z$  giving the minimum distance from 0 to  $Z$ . If  $0 \notin Z$ , take  $v = (v_1, \dots, v_{2n+1}) \in \mathbb{C}^{2n+1}$  a unitary vector in the direction of the line from 0 to  $z_0$ . This vector is perpendicular to  $T_{z_0}Z$ . If  $0 \in Z$  then let  $v$  be a unitary vector orthogonal to  $T_0Z$ . Therefore

$$(3.2) \quad \langle df(z), v \rangle \geq \frac{1}{2}|df(z)|$$

for any  $z \in Z \cap B(0, 15Cc)$ , by the condition on the angle (taking  $\varepsilon > 0$  small enough).

Let  $r_0 \in \mathbb{C}$  with  $r_0v = z_0 \in Z$ . We look for a function  $r_k = r_k(y): B_{g_k}(x, c) \rightarrow \mathbb{C}$  such that  $r_k(x) = r_0$  and

$$(3.3) \quad f(f_k^1(y) + r_kv_1, \dots, f_k^{2n+1}(y) + r_kv_{2n+1}) = 0.$$

This corresponds to tracing a straight line from the image of the point  $y \in B_{g_k}(x, c)$  to  $Z$  with direction  $v$ . Such  $r_k$  can be found with the use of the implicit function theorem applied to the function  $F: B_{g_k}(x, c) \times B(r_0, 4Cc) \rightarrow \mathbb{C}$  given as the left hand side of (3.3). This  $F$  is well-defined since  $f$  is defined on  $B(0, 10Cc) \subset \Phi_0(U_j)$ . To guarantee the existence of  $r_k = r_k(y)$  for all  $y \in B_{g_k}(x, c)$  we have to check that

$$\left| \frac{\nabla_y F}{\partial F / \partial r_k} \right| = \left| \frac{\langle df, \nabla f_k \rangle}{\langle df, v \rangle} \right| \leq 4C,$$

which holds thanks to (3.2). This gives the existence of  $r_k$  in the whole of the ball  $B_{g_k}(x, c)$  as well as the bound  $|\nabla r_k| \leq 4C$ , and hence  $|r_k| \leq 8Cc$ .

Now our task is to prove that  $r_k$  is asymptotically holomorphic, so we trade a geometrical transversality problem into a local one. For this let us compute  $\bar{\partial}r_k$ . Recall

that  $f_k$  is asymptotically holomorphic and  $f$  is holomorphic. Differentiate the equality  $f(f_k(y) + r_k(y)v) = 0$ , so

$$(3.4) \quad \begin{aligned} 0 &= \bar{\partial}(f(f_k(y) + r_k(y)v)) = \partial f(z) \cdot (\bar{\partial}f_k(y) + \bar{\partial}r_k(y)v) \\ &= O(k^{-1/2}) + \langle df(z), v \rangle \bar{\partial}r_k(y), \end{aligned}$$

with  $z = f_k(y) + r_k(y)v$ . Using (3.2) we get that  $\bar{\partial}r_k = O(k^{-1/2})$ . We already know that  $|\nabla r_k| = O(1)$ . Differentiating (3.4) one easily obtains also that  $|\nabla \bar{\partial}r_k| = O(k^{-1/2})$ . So  $r_k$  is asymptotically holomorphic. Now the function

$$h_k = r_k \frac{s_k^0}{s_{k,x}^{\text{ref}}} : B_{g_k}(x, c) \rightarrow \mathbb{C},$$

is also asymptotically holomorphic.

Dividing  $h_k$  by an appropriate constant, using the chart  $\Phi_k$  defined in Lemma 2.6 and scaling the coordinates by a universal constant, we obtain a function  $\tilde{h}_k$  defined on  $B^+$  satisfying the hypothesis of Proposition 2.10, for  $k$  large enough. So going back to  $h_k$  through universal constants, we find  $|w_k| < \delta$  such that  $h_k - w_k$  is  $\eta$ -transverse to 0 in a ball of universal  $g_k$ -radius around  $x$  with  $\eta = c'\delta(\log(\delta^{-1}))^{-p}$ .

Now we have a direction  $v$  and a modulus  $w_k$  for a perturbation. The perturbation we give is

$$\tau_{k,x} = (0, w_k v_1 s_{k,x}^{\text{ref}}, \dots, w_k v_{2n+1} s_{k,x}^{\text{ref}}).$$

Let us look at the perturbed map  $\hat{\phi}_k = \mathbb{P}(s_k - \tau_{k,x})$ . It is asymptotically holomorphic and  $\gamma'$ -projectizable and  $\gamma'$ -generic of order  $n$ , for some  $\gamma' > 0$ , with  $|\tau_{k,x}| < c''\delta$  (for  $\delta > 0$  small enough). Let us check that  $\hat{\phi}_k$  is  $\eta$ -transverse to  $N$  with  $\eta = c'\delta(\log(\delta^{-1}))^{-p}$  and  $c'$  a constant depending only on  $c$  and the asymptotically holomorphic bounds of  $s_k$ . With this, applying Proposition 2.8, the proof in this case is concluded. (If  $\hat{\phi}_k$  is an immersion, we take a perturbation of order  $O(k^{-1/2})$  to make it an embedding.)

The  $h_k$  associated to  $\hat{\phi}_k$  is  $\hat{h}_k = h_k - w_k$ . The final point is to set up the relationship between the transversality of  $\hat{h}_k$  to 0 and the transversality of  $\hat{\phi}_k$  to  $N$ . Note that we have  $\hat{r}_k = \hat{h}_k \frac{s_{k,x}^{\text{ref}}}{s_k^0}$ ,  $\hat{f}_k = \Phi_0 \circ \hat{\phi}_k$  and  $\hat{\pi}_k = \hat{f}_k + \hat{r}_k v = \pi_k$ .

Using that  $|s_{k,x}^{\text{ref}}/s_k^0|$  is bounded above and below uniformly and that

$$|\nabla(s_{k,x}^{\text{ref}}/s_k^0)| = O(1),$$

it is easy to prove that if  $\hat{h}_k$  is  $\eta$ -transverse to 0 then  $\hat{r}_k$  is  $c_0\eta$ -transverse to 0, for some universal constant  $c_0$ .

Let  $y \in B_{g_k}(x, c)$ . If  $|\hat{r}_k(y)| \geq c_0\eta$  then  $d(\hat{\phi}_k(y), N) \geq c_1\eta$ , for some universal constant  $c_1$ . Otherwise  $|\nabla \hat{r}_k(y)| > c_0\eta$ . We shall use Lemma 3.8 for the subspaces  $U = (d\hat{f}_k)_* T_y M$

and  $V = T_{\pi_k(y)}Z$  of  $\mathbb{C}^{2n+1}$ . Let  $V' = [v]$ . The projections from  $U$  to the summands of the decomposition  $\mathbb{C}^{2n+1} = V \oplus V'$  are given respectively by  $g = d\pi_k \circ (df_k)^{-1}$  and

$$h = -v d\hat{r}_k \circ (d\hat{f}_k)^{-1}.$$

This follows from  $d\pi_k = d\hat{f}_k + d\hat{r}_k v$  which gives  $\text{Id} = d\pi_k \circ (d\hat{f}_k)^{-1} - v d\hat{r}_k \circ (d\hat{f}_k)^{-1}$ . The map  $h$  has a right inverse of norm bounded by  $C'\eta^{-1}$ , for some universal constant  $C'$  (here we use that  $\phi_k$  is generic of order  $n$  and that the perturbations are small). It is easy to check that Lemma 3.8 is still valid when  $V$  and  $V'$  are almost orthogonal (and not just orthogonal), so we have

$$\angle_m((d\hat{f}_k)_* T_y M, T_{\pi_k(y)} Z) \geq c_2 \eta.$$

Push forward the distribution  $D_N$  through the chart  $\Phi_0$  to a distribution  $D_Z$  in  $B(0, 15Cc)$ . Then there exists a constant  $C''$  independent of  $k$  such that

$$\angle_M(T_z Z, D_Z(z + \lambda v)) < C'' d(z + \lambda v, Z),$$

for  $z \in Z$ ,  $\lambda \in \mathbb{C}$  with  $|z| < 14Cc$ ,  $|\lambda| < Cc$ . Now use Proposition 3.5 to get

$$\angle_m((d\hat{f}_k)_* T_y M, D_Z(\hat{f}_k(y))) > c_2 \eta - C'' d(\hat{f}_k(y), Z).$$

For  $d(\hat{f}_k(y), Z) < c_2 \eta / 2C''$  we get  $\angle_m((d\hat{f}_k)_* T_y M, D_Z(\hat{f}_k(y))) > c_2 \eta / 2$ . Passing to the manifold we get  $\angle_m((d\hat{\phi}_k)_* T_y M, D_N(\hat{\phi}_k(y))) > c'_2 \eta$ , whenever  $d(\hat{\phi}_k(y), N) < c'_1 \eta$ , for some universal constants  $c'_1$  and  $c'_2$ .

To achieve the solution when the codimension of  $N$  is  $r > 1$ , we follow the same ideas as in the precedent case. In this case  $f: B(0, 15Cc) \rightarrow \mathbb{C}^r$  and one chooses the point  $z_0$  giving the minimum distance from 0 to  $Z$  which yields a vector  $v_1$  orthogonal to  $Z$  at  $z_0$ . Then one completes to a unitary basis  $(v_1, \dots, v_r)$  for the orthogonal to  $T_{z_0} Z$ . The function  $r_k: B_{g_k}(x, c) \rightarrow \mathbb{C}^r$  is defined by the condition  $f(f_k + r_k^1 v_1 + \dots + r_k^r v_r) = 0$ . The perturbation will be of the form

$$\tau_{k,x} = (0, w_k^1 v_1^1 s_{k,x}^{\text{ref}} + \dots + w_k^r v_r^1 s_{k,x}^{\text{ref}}, \dots, w_k^1 v_1^{2n+1} s_{k,x}^{\text{ref}} + \dots + w_k^r v_r^{2n+1} s_{k,x}^{\text{ref}}),$$

where  $v_i = (v_i^1, \dots, v_i^{2n+1})$ ,  $i = 1, \dots, r$  and  $w_k = (w_k^1, \dots, w_k^r)$ . The proof above works again.

The parametric case is easy and we leave the reader to fill in the details.  $\square$

#### 4. Asymptotically holomorphic embeddings to Grassmannians

Let  $(M, \omega)$  be a symplectic manifold of integer class and let  $L$  stand for the hermitian line bundle with a connection  $\nabla$  with curvature  $-i\omega$ . Let  $E$  be a rank  $r$  hermitian bundle over  $M$  endowed with a hermitian connection. Fix a compatible almost complex structure  $J$  on  $M$ . In this section we shall deal with the issue of constructing sequences of embeddings of  $M$  into the Grassmannian  $\text{Gr}(r, N)$  which are asymptotically  $J$ -holomorphic in the sense of Definition 1.1. More specifically, we aim to prove the following result from which Theorem 1.4 follows.

**Theorem 4.1.** *Suppose  $N > n + r - 1$  and  $r(N - r) > 2n$ . Given an asymptotically  $J$ -holomorphic sequence of sections  $s_k$  of the vector bundles  $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$  and  $\alpha > 0$  then there exists another sequence  $\sigma_k$  verifying that:*

1.  $|s_k - \sigma_k|_{C^1, g_k} < \alpha$ .
2.  $\phi_k = \text{Gr}(\sigma_k)$  is an asymptotically holomorphic sequence of embeddings in  $\text{Gr}(r, N)$  for  $k$  large enough.
3.  $\phi_k^* \mathcal{U} = E \otimes L^{\otimes k}$ , where  $\mathcal{U} \rightarrow \text{Gr}(r, N)$  is the universal rank  $r$  bundle.

Moreover given two asymptotically holomorphic sequences  $\phi_k^0$  and  $\phi_k^1$  of embeddings in  $\text{Gr}(r, N)$  with respect to two compatible almost complex structures, then for  $k$  large enough there exists an isotopy of asymptotically holomorphic embeddings  $\phi_k^t$  connecting  $\phi_k^0$  and  $\phi_k^1$ .

**4.1. Proof of main result.** First let us fix some notation. A point  $s \in \text{Gr}(r, N)$  corresponds to an  $r$ -dimensional subspace  $V_s \subset \mathbb{C}^N$ . Choosing a basis  $s_1, \dots, s_r$  for  $V_s$ , we denote

$$s = \begin{bmatrix} s_1 \\ \vdots \\ s_r \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1N} \\ \vdots & & \ddots & \vdots \\ s_{r1} & s_{r2} & \cdots & s_{rN} \end{bmatrix}.$$

This identifies  $s$  as the equivalence class of  $r \times N$  matrices of rank  $r$  under the action of  $\text{GL}(r, \mathbb{C})$  on the left. The standard metric  $g_{\text{Gr}}$  for  $\text{Gr}(r, N)$  is the metric induced by the Fubini-Study metric  $g_{\text{FS}}$  under the Plücker embedding [8], Chapter 1, Section 5. We proceed by steps to obtain asymptotically holomorphic embeddings.

**Definition 4.2.** Let  $\gamma > 0$  and  $0 \leq l \leq r$ . A sequence of asymptotically  $J$ -holomorphic sections  $s_k = (s_k^1, \dots, s_k^N)$  of the vector bundles  $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$  is said to be  $\gamma$ -grassmannizable of order  $l$  if for all  $x \in M$ ,  $|\bigwedge^l s_k(x)| > \gamma$ . It is  $\gamma$ -grassmannizable when it is  $\gamma$ -grassmannizable of order  $r$ . (Here  $s_k = (s_k^1, \dots, s_k^N)$  is interpreted as a morphism of bundles  $\mathbb{C}^N \rightarrow E \otimes L^{\otimes k}$  and  $\bigwedge^l s_k$  is the corresponding  $l$ -fold wedge product.)

If we have the condition of  $\gamma$ -grassmannizability for a section  $s_k$  then we obtain a morphism  $\phi_k = \text{Gr}(s_k): M \rightarrow \text{Gr}(r, N)$ , called the grassmannization of  $s_k$ , as follows. At a point  $x$  take a basis  $(e_1, \dots, e_r)$  for the fiber of  $E$  at  $x$ . Then

$$\phi_k(x) = [s_k^1(x), \dots, s_k^N(x)] = \begin{bmatrix} s_k^{11} & s_k^{12} & \cdots & s_k^{1N} \\ \vdots & & \ddots & \vdots \\ s_k^{r1} & s_k^{r2} & \cdots & s_k^{rN} \end{bmatrix},$$

where  $s_k^i(x) = s_k^{i1}e_1 + \dots + s_k^{ir}e_r$ . This is well-defined and independent of the chosen basis.

**Definition 4.3.** Let  $\eta > 0$  and  $0 \leq l \leq n$ . A sequence of asymptotically  $J$ -holomorphic  $\gamma$ -grassmannizable sections  $s_k$  of vector bundles  $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$  is  $\eta$ -generic of order  $l$  if  $\text{Gr}(s_k)$  satisfies  $|\bigwedge^l \partial \text{Gr}(s_k)(x)|_{g_k} > \eta$ , for all  $x \in M$ .

In order to prove Theorem 4.1 we shall use the following auxiliary proposition that will be proved in the following subsections.

**Proposition 4.4.** *Suppose  $N > n + r - 1$  and  $r(N - r) > 2n$ . Let  $s_k$  be an asymptotically  $J$ -holomorphic sequence of sections of the vector bundles  $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$  and  $\alpha > 0$ . Then there exists another sequence  $\sigma_k$  verifying:*

1.  $|s_k - \sigma_k|_{C^1, g_k} < \alpha$ .
2.  $\sigma_k$  is  $\gamma$ -grassmannizable and  $\gamma$ -generic of order  $n$  for some  $\gamma > 0$ .

*Moreover, the result holds for one-parameter families of sections where the sections and the compatible almost complex structures depend continuously on  $t \in [0, 1]$ .*

There is an analogue of Lemma 2.16 in the case of Grassmannian embeddings. We state it without proof, since we shall not make use of it in this work. Details of the proof can be found in [14]. If we do not use it then we need to impose in the isotopy part of the statement of Theorem 4.1 that  $\phi_k^i$ ,  $i = 0, 1$ , come from sequences of sections of  $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$ .

**Lemma 4.5.** *Let  $\phi_k: M \rightarrow \text{Gr}(r, N)$  be a sequence of asymptotically holomorphic embeddings with  $\phi_k^* \mathcal{U} = E \otimes L^{\otimes k}$ . Then there exists a sequence of asymptotically holomorphic sections  $s_k$  of  $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$ , for  $k$  large enough, which is  $\gamma$ -grassmannizable and  $\gamma$ -generic of order  $n$ , for some  $\gamma > 0$ , such that  $\phi_k = \text{Gr}(s_k)$ . The same holds for continuous one-parameter families of embeddings and compatible almost complex structures.*

*Proof of Theorem 4.1.* Note that the last property is obvious by the construction. Let us begin with an asymptotically  $J$ -holomorphic sequence  $\sigma_k$  of sections of the bundles  $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$  and perturb it using Proposition 4.4 to obtain an asymptotically holomorphic  $\gamma$ -grassmannizable and  $\gamma$ -generic of order  $n$  sequence of sections  $s_k$ . The first property implies that  $\phi_k = \text{Gr}(s_k)$  is well-defined, the second that it is an immersion. To get an embedding we use that  $2 \dim M < \dim \text{Gr}(r, N) = 2r(N - r)$  to find a generic  $C^p$ -perturbation of norm less than  $O(k^{-1/2})$  to get rid of the self-intersections and keeping the asymptotic holomorphicity, the grassmannizability and the genericity of order  $n$ . Now we only have to check that the sequence  $\phi_k = \text{Gr}(s_k)$  satisfies the required conditions in Definition 1.1.

Choose a point  $x \in M$  and trivialize  $E$  in a neighborhood of  $x$  by fixing an orthonormal basis  $e_1, \dots, e_r$ . Now by a rotation with an element of  $U(N)$  acting on  $\mathbb{C}^N$  and an element of  $U(r)$  acting on  $E$ , we can assure that

$$(4.1) \quad s_k(x) = \begin{pmatrix} s_k^{11}(x) & 0 & \dots & \dots & 0 \\ 0 & s_k^{22}(x) & 0 & \dots & 0 \\ 0 & \dots & \ddots & 0 & \dots & 0 \\ 0 & \dots & & s_k^{rr}(x) & 0 & \dots & 0 \end{pmatrix}$$

where  $s_k^{ij}$  are sections of  $L^{\otimes k}$ . This corresponds to an isometric transformation of  $\text{Gr}(r, N)$ . The  $\gamma$ -grassmannizable property implies that  $|s_k^{11} \dots s_k^{rr}| \geq \gamma$ . By the asymptotic holomorphicity bounds it is  $|s_k| = O(1)$ , so that  $|s_k^{ii}| \geq \gamma/C$ , for some universal constant  $C$ . Therefore on a ball  $B_{g_k}(x, c)$  of fixed universal radius  $c$ , the first  $r \times r$  minor of  $s_k(y)$  has an inverse of norm bounded by  $C'\gamma^{-1}$ , for some universal constant  $C'$ .

Let  $v_1, \dots, v_N$  be the canonical basis of  $\mathbb{C}^N$ . We consider the standard local chart for  $\text{Gr}(r, N)$  around  $\Pi_0 = \phi_k(x) = [v_1, \dots, v_r]$  for the open set  $U_0 = \{\Pi \mid \Pi \cap [v_{r+1}, \dots, v_N] = \{0\}\}$  as

$$\Phi_0: U_0 \rightarrow \mathbb{C}^{r \times (N-r)},$$

$$\begin{bmatrix} s_{11} & \cdots & s_{1N} \\ \vdots & \ddots & \vdots \\ s_{r1} & \cdots & s_{rN} \end{bmatrix} \mapsto \begin{pmatrix} s_{11} & \cdots & s_{1r} \\ \vdots & \ddots & \vdots \\ s_{r1} & \cdots & s_{rr} \end{pmatrix}^{-1} \begin{pmatrix} s_{1,r+1} & s_{1,r+2} & \cdots & s_{1N} \\ \vdots & & \ddots & \vdots \\ s_{r,r+1} & s_{r,r+2} & \cdots & s_{rN} \end{pmatrix}.$$

It is easy to check that  $\Phi_0$  is an isometry at the point  $\Pi_0$ . In this chart, we have the map  $f_k = \Phi_0 \circ \phi_k$ . We can compute the bounds required in Definition 1.1 using  $f_k$  instead of  $\phi_k$ . The proof of Theorem 2.11 now carries, over verbatim. For the isotopy result we use Lemma 4.5.  $\square$

**4.2. Construction of  $\gamma$ -grassmannizable sections.** Our goal is to prove:

**Proposition 4.6.** *Suppose  $N > n + r - 1$ . Let  $s_k$  be an asymptotically  $J$ -holomorphic sequence of sections of the vector bundles  $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$  which is  $\gamma$ -grassmannizable of order  $l$ , for some  $\gamma > 0$ . Then given  $\alpha > 0$ , there exists an asymptotically  $J$ -holomorphic sequence of sections  $\sigma_k$  verifying:*

1.  $|s_k - \sigma_k|_{C^1, g_k} < \alpha$ .
2.  $\sigma_k$  is  $\eta$ -grassmannizable of order  $l + 1$  for some  $\eta > 0$ .

Moreover, the result can be extended to continuous one-parameter families depending continuously of  $t \in [0, 1]$ .

*Proof.* Again we use the globalization argument in Proposition 2.8. Let us do the non-parametric case, the other one being a trivial extension by now. Define the local and  $C^0$ -open property  $\mathcal{P}(\varepsilon, x)$  as  $|\wedge^{l+1} s_k(x)| > \varepsilon$ . We need to find for a point  $x \in M$  a section  $\tau_{k,x}$  with Gaussian decay away from  $x$ , with  $s_k + \tau_{k,x}$  satisfying  $\mathcal{P}(\eta, y)$  in a ball of universal  $g_k$ -radius  $c$ .

Choose a point  $x \in M$ . Fix an orthonormal basis  $e_1, \dots, e_r$  trivializing  $E$  in a neighborhood of  $x$ , so  $s_k$  may be interpreted as a morphism  $\mathbb{C}^N \rightarrow \mathbb{C}^r \otimes L^{\otimes k}$ . Again we can suppose that  $s_k(x)$  is as in (4.1), with  $|s_k^{I1}(x) \cdots s_k^{II}(x)| \geq \gamma$ . Define

$$M_k^p = (\wedge^{l+1} s_k)_{(e_1 \wedge \cdots \wedge e_{l+1}) \otimes (e_1 \wedge \cdots \wedge e_l \wedge e_p)}$$

as the component of  $\wedge^{l+1} s_k$  along  $(e_1 \wedge \cdots \wedge e_{l+1}) \otimes (e_1 \wedge \cdots \wedge e_l \wedge e_p)$ . Also

$$|(\wedge^l s_k)_{(e_1 \wedge \cdots \wedge e_l) \otimes (e_1 \wedge \cdots \wedge e_l)}| > \gamma/2$$

on a ball  $B_{g_k}(x, c)$  of fixed radius  $c$ . Let  $s_{k,x}^{\text{ref}}$  be the sections given by Lemma 2.5 and define  $\theta_k = (\wedge^l s_k)_{(e_1 \wedge \cdots \wedge e_l) \otimes (e_1 \wedge \cdots \wedge e_l)} s_{k,x}^{\text{ref}}$ . Clearly  $|\theta_k| > c_s \gamma/2$  on  $B_{g_k}(x, c)$ .

If we perturb  $s_k$  so that the norm of  $M_k = (M_k^{l+1}, \dots, M_k^N)$  is bigger than  $\eta = c'\delta(\log(\delta^{-1}))^{-p}$  then we have finished. For this we define

$$h_k = (h_k^{l+1}, \dots, h_k^N) = \left( \frac{M_k^{l+1}}{\theta_k}, \dots, \frac{M_k^N}{\theta_k} \right).$$

We obtain, scaling the coordinates by universal constants if necessary,  $h_k: B^+ \rightarrow \mathbb{C}^{N-l}$  which is asymptotically holomorphic. As  $n < N - l$ , we can find  $|w_k| < \delta$  so that

$$|h_k - w_k| > \delta(\log(\delta^{-1}))^{-p}.$$

Then we obtain that  $|(M_k^{l+1} - w_k^{l+1}\theta_k, \dots, M_k^N - w_k^N\theta_k)| > \eta = c'\delta(\log(\delta^{-1}))^{-p}$ , for some universal  $c'$ . This perturbation term is achieved with the section

$$\tau_{k,x} = -(0, \dots, 0, w_k^{l+1}e_{l+1}s_{k,x}^{\text{ref}}, \dots, w_k^N e_{l+1}s_{k,x}^{\text{ref}})$$

of the bundles  $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$ . This finishes the proof.  $\square$

**Remark 4.7.** We cannot improve the condition  $N > n + r - 1$  in Proposition 4.6. As we shall see in Section 5, we expect the locus of points of  $M$  where the rank of  $s_k: \mathbb{C}^N \rightarrow E \otimes L^{\otimes k}$  is not maximum to have codimension  $N - r + 1$ .

**4.3. Inductive construction of sections  $\gamma$ -generic of order  $l$ .** Now we study the problem of perturbing the sequence  $s_k$  to achieve genericity of order  $n$ .

**Proposition 4.8.** *Suppose  $r(N - r) > 2n$ . Let  $s_k$  be an asymptotically  $J$ -holomorphic sequence of sections of the vector bundles  $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$ , which is  $\gamma$ -grassmannizable and  $\gamma$ -generic of order  $l$ . Then given  $\alpha > 0$ , there exists an asymptotically  $J$ -holomorphic sequence of sections  $\sigma_k$  verifying:*

1.  $|s_k - \sigma_k|_{C^1, g_k} < \alpha$ .
2.  $\sigma_k$  is  $\eta$ -generic of order  $l + 1$  for some  $\eta > 0$ .

Moreover, this result can be extended to continuous one-parameter families of sections and almost complex structures.

*Proof.* Define the property  $\mathcal{P}(\varepsilon, x)$  for a section  $s_k$  which is  $\gamma/2$ -grassmannizable and  $\gamma/2$ -generic of order  $l$  as  $|\wedge^{l+1} \partial \text{Gr}(s_k)(x)| > \varepsilon$ . A perturbation of our initial section verifies the hypothesis if we perturb by adding sections of  $C^1$ -norm smaller than  $\gamma/2C$ ,  $C$  some universal constant. For applying Proposition 2.8 we need to build, for  $0 < \delta < \gamma/2Cc''$ , a local perturbation  $\tau_{k,x}$  with  $|\tau_{k,x}| < c''\delta$  and Gaussian decay with the property  $\mathcal{P}(\eta, y)$  on  $B_{g_k}(x, c)$  with  $\eta = c'\delta(\log(\delta^{-1}))^{-p}$ .

Choose a point  $x \in M$ . We can suppose that  $s_k(x)$  is as in (4.1) with  $|s_k^{11}(x) \cdots s_k^{rr}(x)| \geq \gamma$ . The asymptotically holomorphic bounds imply that  $|s_k| = O(1)$ , so that  $|s_k^{ii}(x)| \geq \gamma/C$  for some universal constant  $C$ . There is a fixed universal radius  $c$  such

that the first  $r \times r$  minor of  $s_k(y)$  has an inverse of norm bounded by  $C'\gamma^{-1}$ , for some universal constant  $C'$ , on  $B_{g_k}(x, c)$ . Then we can use the trivialization  $\Phi_0$  to define the maps

$$f_k: B_{g_k}(x, c) \rightarrow \mathbb{C}^{r \times (N-r)},$$

$$y \mapsto \begin{pmatrix} s_k^{11}(y) & \cdots & s_k^{1r}(y) \\ \vdots & & \vdots \\ s_k^{r1}(y) & \cdots & s_k^{rr}(y) \end{pmatrix}^{-1} \begin{pmatrix} s_k^{1,r+1}(y) & \cdots & s_k^{1N}(y) \\ \vdots & & \vdots \\ s_k^{r,r+1}(y) & \cdots & s_k^{rN}(y) \end{pmatrix}.$$

Let  $A_k$  be the first  $r \times r$  minor of  $s_k$ . Then we can define  $\mathbb{C}^r$ -valued 1-forms, for  $1 \leq i \leq r, 1 \leq j \leq n$ , as

$$\mu_k^{ij} = \partial((A_k)^{-1} B z_k^j s_{k,x}^{\text{ref}} e_i),$$

where  $B$  is a constant matrix of  $\text{GL}(r, \mathbb{C})$  chosen to assure that  $\mu_k^{ij}$  is an orthonormal basis of  $T^*M \otimes \mathbb{C}^r \otimes L^{\otimes k}$  at  $x$  (and hence approximately orthonormal at a universal neighborhood). By the lower and upper bounds of  $s_k(x)$  it is easy to check that all the coefficients of  $B$  are universally bounded.

Denoting by  $f_k^t, 1 \leq t \leq N - r$ , the  $t$ -th column of  $f_k$ , we have  $\partial f_k^t = \sum_{i,j} u_k^{ijt} \mu_k^{ij}$ . So

$$\partial f_k = (u_k^{111} \mu_k^{11} + u_k^{121} \mu_k^{12} + \cdots + u_k^{1n1} \mu_k^{1n}, u_k^{211} \mu_k^{21} + \cdots, \dots, \dots + u_k^{rn(N-r)} \mu_k^{rn}).$$

Now we spread out the coefficients  $(i, t)$  in one single row to get an  $n \times r(N - r)$  matrix  $u_k$ . After a suitable unitary transformation of  $U(n)$  on the complex Darboux coordinate chart and a horizontal relabeling of the coordinates, we can suppose that

$$(4.2) \quad \partial \tilde{f}_k(x) = \begin{pmatrix} u_k^{11}(x) & * & \cdots & \cdots & * \\ 0 & u_k^{22}(x) & * & \cdots & * \\ 0 & \cdots & \ddots & * & \cdots & * \\ 0 & \cdots & 0 & u_k^{mm}(x) & * & \cdots & * \end{pmatrix},$$

where  $|u_k^{11}(x) \cdots u_k^{ll}(x)| > \gamma/C_0$ ,  $C_0$  a universal constant. The relabeling is given by a function  $\alpha \in \{1, \dots, r(N - r)\} \mapsto (i(\alpha), t(\alpha)) \in \{1, \dots, r\} \times \{1, \dots, N - r\}$ . Consider the function  $\theta_k(y)$  given by the determinant of the first  $l \times l$  minor of (4.2). Shrinking  $c$  if necessary we can assure that  $\theta_k(y) > \gamma/2C_0$  on the ball  $B_{g_k}(x, c)$ . Also we define the functions  $M_k^p(y)$  given as the determinant of the  $(l + 1) \times (l + 1)$  minor given by the first  $l + 1$  rows of (4.2) and the first  $l$  columns together with the  $p$ -th column,  $l + 1 \leq p \leq r(N - r)$ . If we perturb so that the norm of  $M_k = (M_k^{l+1}, \dots, M_k^{r(N-r)})$  gets bigger than  $\eta = c'\delta(\log(\delta^{-1}))^{-p}$  then we are done. We define the asymptotically holomorphic maps  $h_k = \left(\frac{M_k^{l+1}}{\theta_k}, \dots, \frac{M_k^{r(N-r)}}{\theta_k}\right)$ .

So we obtain, scaling the coordinates by universal constants if necessary,  $h_k: B^+ \rightarrow \mathbb{C}^{r(N-r)-l}$  which is asymptotically holomorphic. As  $n < r(N - r) - l$  we can find  $|w_k| < \delta$  such that  $|h_k - w_k| > \delta(\log(\delta^{-1}))^{-p}$ . Thus

$$|(M_k^{l+1} - w_k^{l+1} \theta_k, \dots, M_k^{r(N-r)} - w_k^{r(N-r)} \theta_k)| > \eta = c'\delta(\log(\delta^{-1}))^{-p}.$$

The perturbation term  $-(w_k^{l+1}\theta_k, \dots, w_k^{r(N-r)}\theta_k)$  is achieved by adding the section

$$\tau_{k,x} = -B \cdot \left( 0, \overset{(r)}{\dots}, 0, \sum_{i(\alpha)=r+1, \alpha>l} w_k^{\alpha} z_{l+1} e_{i(\alpha)} s_{k,x}^{\text{ref}}, \dots, \sum_{i(\alpha)=N, \alpha>l} w_k^{\alpha} z_{l+1} e_{i(\alpha)} s_{k,x}^{\text{ref}} \right).$$

This finishes the proof in the non-parametric case. The other case is left to the reader.  $\square$

**4.4. Zero sets of vector bundles.** Following the ideas of Subsection 3.3 and using Proposition 3.10 we can prove the following two results.

**Theorem 4.9.** *Given  $\phi_k = \text{Gr}(s_k)$ , where  $s_k$  is a sequence of asymptotically holomorphic sections of  $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$ , which are  $\gamma$ -grassmannizable and  $\gamma$ -generic of order  $n$ , for some  $\gamma > 0$ . Fix a holomorphic submanifold  $V$  in  $\text{Gr}(r, N)$ . Then for any  $\alpha > 0$  there exists a sequence of asymptotically holomorphic sections  $\sigma_k$  of  $\mathbb{C}^N \otimes E \otimes L^{\otimes k}$  such that:*

1.  $|\sigma_k - s_k|_{g_k, C^1} < \alpha$ .
2.  $\text{Gr}(\sigma_k)$  is an  $\eta$ -asymptotically holomorphic embedding in  $\text{Gr}(r, N)$  which is  $\varepsilon$ -transverse to  $V$ , with  $\eta > 0$  and  $\varepsilon > 0$  independent of  $k$ . In the case  $\dim M + \dim V < 2r(N - r)$  we have that  $d_{\text{Gr}}(\phi_k(M), V) > \varepsilon$ , for  $k$  large enough.

Moreover the result can be extended to one-parameter continuous families of complex submanifolds  $(V_t)_{t \in [0,1]}$ , taking in this case as starting point a continuous family  $\phi_{t,k} = \text{Gr}(s_{t,k})$ , where  $s_{t,k}$  is a continuous family of asymptotically  $J_t$ -holomorphic sequences of sections which are  $\gamma$ -grassmannizable and  $\gamma$ -generic of order  $n$ .  $\square$

We call universal planes to the zero sets of algebraic sections transverse to zero of the universal bundle  $\mathcal{U}$  over the grassmannian  $\text{Gr}(r, N)$ . Now we can deduce the main result of [2].

**Corollary 4.10.** *Let  $(M, \omega)$  be a compact symplectic manifold of integer class. Let  $E$  be a hermitian vector bundle over  $M$ . Then for  $k$  large enough there exist symplectic submanifolds obtained as zero sets of the bundles  $E \otimes L^{\otimes k}$ . Moreover, perhaps by increasing  $k$ , we have that all the symplectic submanifolds constructed as transverse intersections of asymptotically holomorphic sequences with a fixed universal plane are isotopic. The isotopy can be made through symplectomorphisms.*

### 5. Determinantal submanifolds of closed symplectic manifolds

Let  $(M, \omega)$  be a symplectic manifold of integer class, endowed with a compatible almost complex structure. Let  $E$  and  $F$  be two vector bundles of ranks  $r_e$  and  $r_f$ , respectively. Recall that for any morphism  $\varphi: E \rightarrow F$  we have defined in Definition 1.5 the  $r$ -determinantal set as

$$\Sigma^r(\varphi) = \{x \in M \mid \text{rank } \varphi_x = r\}.$$

We want to prove Theorem 1.6, which allows to construct  $\Sigma^r(\varphi)$  as a symplectic submanifold, after twisting  $E$  and  $F$  with large powers of  $L$ . We prove this by embedding  $M$  in a product of two Grassmannians and cutting its image with suitable “generalized Schubert cycles”. The next section is devoted to this task.

**Remark 5.1.** A naive approach to proving Theorem 1.6 consists on taking the  $r$ -fold wedge product of  $\varphi_k$ ,  $\bigwedge^r \varphi_k$ , whose zero set is the stratified submanifold  $\Sigma^0(\varphi_k) \cup \dots \cup \Sigma^r(\varphi_k)$ . If  $\varphi_k$  is an asymptotically  $J$ -holomorphic sequence of sections of the bundle  $E^* \otimes F \otimes L^{\otimes 2k}$ , one could try to use Donaldson’s techniques to get a new sequence of sections transverse in an adequate sense to assure symplecticity. The following example shows the main obstacle to this approach: Take a symplectic 8-manifold in the hypothesis of Theorem 1.6 with two hermitian vector bundles  $E$  and  $F$  of rank 2. Using Auroux’ techniques we can assure that the zero sets of  $\varphi_k$  are  $\eta$ -transverse to 0, for some  $\eta > 0$ . For  $\bigwedge^2 \varphi_k$ , we need the condition

$$|\bar{\partial} \bigwedge^2 \varphi_k| < |\partial \bigwedge^2 \varphi_k|.$$

But at any  $x \in Z(\varphi_k)$  both terms vanish, so we cannot impose a global transversality property for the section  $\bigwedge^2 \varphi_k$ . This case is very similar to that in [5] and can be treated with an “ad hoc” argument, but more general cases do not admit a treatment based on the use of normal forms of the singularities, because for higher dimensions the problem becomes intractable [1].

**5.1. Bigrassmannian embeddings.** Choose two sequences of sections  $s_k^e$  and  $s_k^f$  of the bundles  $\mathbb{C}^N \otimes E^* \otimes L^{\otimes k}$  and  $\mathbb{C}^N \otimes F \otimes L^{\otimes k}$  respectively, which are  $\gamma$ -grassmannizable and  $\gamma$ -generic of order  $n$ , for some  $\gamma > 0$ , providing by Theorem 4.1, asymptotically holomorphic sequences of embeddings  $\text{Gr}(s_k^e)$  and  $\text{Gr}(s_k^f)$  of  $M$  in  $\text{Gr}(r_e, N)$  and  $\text{Gr}(r_f, N)$ , respectively, for  $N$  a large integer number. Thus we obtain an asymptotically holomorphic sequence of embeddings of  $M$  into the bigrassmannian  $\text{Bi}(r_e, r_f, N) = \text{Gr}(r_e, N) \times \text{Gr}(r_f, N)$ ,

$$\phi_k = \text{Gr}(s_k^e) \times \text{Gr}(s_k^f): M \rightarrow \text{Gr}(r_e, N) \times \text{Gr}(r_f, N) = \text{Bi}(r_e, r_f, N).$$

Let  $\mathcal{U}_e$  and  $\mathcal{U}_f$  be the universal bundles over  $\text{Gr}(r_e, N)$  and  $\text{Gr}(r_f, N)$  respectively. Write  $\pi_e: \text{Bi}(r_e, r_f, N) \rightarrow \text{Gr}(r_e, N)$  for the projection onto the first factor (and analogously  $\pi_f$ ). Recall that  $\text{Gr}(s_k^e)^*(\mathcal{U}_e) = E^* \otimes L^{\otimes k}$  and  $\text{Gr}(s_k^f)^*(\mathcal{U}_f) = F \otimes L^{\otimes k}$ . Then  $\mathcal{U}_{ef} = \pi_e^*(\mathcal{U}_e) \otimes \pi_f^*(\mathcal{U}_f)$  satisfies  $\phi_k^* \mathcal{U}_{ef} = E^* \otimes F \otimes L^{\otimes 2k}$  and it has a holomorphic section  $s$  verifying that:

1.  $D_r = \Sigma^r(s)$  is an open complex submanifold in  $\text{Bi}(r_e, r_f, N)$ .
2.  $\text{codim}_{\mathbb{C}} D_r = (r_e - r)(r_f - r)$ .

Actually this section comes naturally: pull back the inclusion and projection morphisms  $\mathcal{U}_e^* \hookrightarrow \underline{\mathbb{C}}^N$  and  $\underline{\mathbb{C}}^N \twoheadrightarrow \mathcal{U}_f$  to the bigrassmannian, and compose them after having taken a suitable identification of the trivial bundles  $\underline{\mathbb{C}}^N$ .

Now if we assure that, for each  $r$ ,  $\phi_k$  is transverse to  $D_r$  with an angle  $\varepsilon > 0$  independent of  $k$ , we have finished the proof of Theorem 1.6 by Proposition 3.10.

**Lemma 5.2.** *Let  $\phi_k: M \rightarrow \text{Bi}(r_e, r_f, N)$  be a  $\gamma$ -asymptotically holomorphic sequence of embeddings. Suppose that  $\phi_k$  is  $\sigma$ -transverse to  $D_r$ . Then there exists  $\varepsilon > 0$ , depending only on  $\gamma$ ,  $\sigma$  and the universal bounds of the derivatives of the sequence, such that  $\phi_k$  is  $\sigma/2$ -transverse to  $D_{r'}$ ,  $r' > r$ , when we restrict to an  $\varepsilon$ -neighborhood of  $D_r$ .*

In other words we do not have to care about the behavior of the angle near the border of the strata.

*Proof.* Choose a point  $x \in D_r \cap \phi_k(M)$ . Recall that by  $\sigma$ -transversality, the minimum angle between  $T_x D_r$  and  $T_x \phi_k(M)$  is greater than  $\sigma$ . We trivialize  $\text{Bi}(r_e, r_f, N)$  by a chart  $\Phi_0$  defined as the cartesian product of two standard charts in the Grassmannians, which is an isometry at the origin and satisfies that  $\Phi_0(x) = 0$ . Since  $D_r$  is contained in the closure of  $D_{r'}$  and the  $\{D_r\}_r$  form a Whitney stratification, we have

$$(5.1) \quad |y| < \delta \Rightarrow \angle_M(T_0\Phi_0(D_r), T_y\Phi_0(D_{r'})) < c_D\delta, \quad \forall y \in B(0, c_u) \cap \Phi_0(D_{r'}).$$

The angles are measured with respect to the standard Euclidean metric which is close to that induced by the bigrassmannian if we choose  $c_u$  small enough. Here  $c_D$  is universal. Also by the asymptotic holomorphicity bounds of  $\phi_k$  we know that

$$(5.2) \quad |y| < \delta \Rightarrow \angle_M(T_0\Phi_0(\phi_k(M)), T_y\Phi_0(\phi_k(M))) < c_\phi\delta, \\ \forall y \in B(0, c_u) \cap \Phi_0(\phi_k(M)),$$

where  $c_\phi$  is universal. Now Proposition 3.5 says that

$$\begin{aligned} \angle_M(T_0\Phi_0(D_r), T_0\Phi_0(\phi_k(M))) &\leq \angle_M(T_0\Phi_0(D_r), T_y\Phi_0(D_{r'})) \\ &\quad + \angle_M(T_y\Phi_0(D_{r'}), T_y\Phi_0(\phi_k(M))) \\ &\quad + \angle_M(T_y\Phi_0(\phi_k(M)), T_0\Phi_0(\phi_k(M))). \end{aligned}$$

Using inequalities (5.1) and (5.2) and recalling that the angles are measured with respect to the bigrassmannian metric (which is related to the standard metric in the ball  $B(0, c_u)$  by non zero universal constants), we get the required result.  $\square$

With Lemma 5.2 the proof of Theorem 1.6 reduces to the following result, whose proof is similar to that of Theorem 4.9.

**Proposition 5.3.** *Let  $s_k^e$  and  $s_k^f$  be two asymptotically holomorphic sequences of the vector bundles  $\mathbb{C}^N \otimes E^* \otimes L^{\otimes k}$  and  $\mathbb{C}^N \otimes F \otimes L^{\otimes k}$  which are  $\gamma$ -grassmannizable and  $\gamma$ -generic of order  $n$ , defining so an asymptotically holomorphic embedding in  $\text{Bi}(r_e, r_f, N)$ . Fix an algebraic open submanifold  $V$  in  $\text{Bi}(r_e, r_f, N)$  with compactification  $\bar{V} = V \cup W$ . Then for any  $\varepsilon, \alpha > 0$ , there exist  $\eta > 0$  and two asymptotically holomorphic sequences  $\sigma_k^e$  and  $\sigma_k^f$  of sections of the vector bundles  $\mathbb{C}^N \otimes E^* \otimes L^{\otimes k}$  and  $\mathbb{C}^N \otimes F \otimes L^{\otimes k}$  respectively, verifying:*

1.  $|\sigma_k^e - s_k^e|_{g_k, C^1} < \alpha$  and  $|\sigma_k^f - s_k^f|_{g_k, C^1} < \alpha$ .
2.  $\phi_k = \text{Gr}(\sigma_k^e) \times \text{Gr}(\sigma_k^f)$  is a sequence of  $\eta$ -asymptotically holomorphic embeddings in  $\text{Bi}(r_e, r_f, N)$ .
3. Denoting by  $V_{\varepsilon^-}$  the compact submanifold of  $V$  obtained by removing an  $\varepsilon$ -neighborhood of  $W$ , we obtain that  $\phi_k$  is  $\eta$ -transverse to  $V_{\varepsilon^-}$ .

Moreover the result can be extended to continuous one-parameter families of sections  $(s_{k,t}^e)_{t \in [0,1]}$  and  $(s_{k,t}^f)_{t \in [0,1]}$  providing embeddings to the bigrassmannian and to continuous one-parameter families of open submanifolds  $V_t$ . Thus we obtain continuous families of sequences  $\sigma_{k,t}^e$  and  $\sigma_{k,t}^f$  verifying the required conditions.  $\square$

**5.2. Dependence loci of sections of a vector bundle.** Suppose that  $E$  is a hermitian vector bundle of rank  $n$  and consider  $s_1, \dots, s_m$  sections of  $E$ . Then we can interpret  $s = (s_1, \dots, s_m)$  as a morphism of bundles  $s: \mathbb{C}^m \rightarrow E$ . The  $r$ -determinantal set of  $s$  is

$$\Sigma^r(s) = \{x \in M \mid \dim[s_1(x), \dots, s_m(x)] = r\},$$

and it is called the  $r$ -dependence locus of the sections  $s_1, \dots, s_m$ . The following result is proved with arguments similar to those of Subsection 5.1.

**Theorem 5.4.** *Let  $(M, \omega)$  be a closed symplectic manifold of integer class and let  $E$  be a rank  $n$  hermitian vector bundle. Then, for  $k$  large enough, there exist  $s_k = (s_k^1, \dots, s_k^m)$  sections of  $\mathbb{C}^m \otimes E$  such that:*

1.  $\Sigma^r(s_k)$  is an open symplectic submanifold of  $M$ .
2.  $\text{codim} \Sigma^r(s_k) = 2(m - r)(n - r)$ . The set of manifolds  $\{\Sigma^r(s_k)\}_r$  constitutes a stratified submanifold.

Moreover, any two stratified submanifolds constructed as intersections of the image of asymptotically holomorphic embeddings of  $M$  into the Grassmannian with the relevant Schubert cycles are isotopic.  $\square$

## 6. Topological considerations

**6.1. Homology and homotopy groups of determinantal submanifolds.** In this subsection we prove a weaker version for the topology of smooth determinantal submanifolds of the symplectic Lefschetz hyperplane theorem proven in Proposition 39 in [5] and Proposition 2 in [2]. The main result is

**Proposition 6.1.** *Let  $E, F$  be vector bundles of ranks  $r_e, r_f$ , respectively, over a closed symplectic manifold  $(M, \omega)$  of integer class and let  $D_r^k$  be a sequence of determinantal submanifolds constructed, by using the vector bundles  $E \otimes (L^*)^{\otimes k}$  and  $F \otimes L^{\otimes k}$ , as a transverse intersection of an asymptotically holomorphic sequence of embeddings in  $\text{Bi}(r_e, r_f, N)$  with the determinantal varieties of a fixed generic section  $s$  of the universal bundle  $\mathcal{U}_{ef}$  over  $\text{Bi}(r_e, r_f, N)$ . Assume that there is only one stratum  $D_r^k$  in the stratified determinantal submanifolds. Then the inclusion  $i: D_r^k \rightarrow M$  induces, for  $k$  large enough, an isomorphism on homotopy groups  $\pi_j$  for*

$$j \leq \min \left\{ 1, \frac{1}{2} \dim D_r^k - 1 \right\}$$

and a surjection on  $\pi_j$  with  $j = \min \left\{ 2, \frac{1}{2} \dim D_r^k \right\}$ . The same property also holds for homology groups.

If there is only one stratum in the determinantal submanifolds then it must be  $r = \min\{r_e, r_f\} - 1$  and  $D_{r-1} = \emptyset$ . We shall assume that  $r_e \leq r_f$ , as otherwise we may change the roles of  $E$  and  $F$ . Therefore  $r + 1 = r_e \leq r_f$ . Nonetheless, most of the arguments in this subsection work for any  $r = 0, 1, \dots, \min\{r_e, r_f\} - 1$ .

The idea for proving Proposition 6.1 is the following. Consider an asymptotically holomorphic sequence of embeddings of  $\phi_k: M \rightarrow \text{Bi}(r_e, r_f, N)$  which are  $\varepsilon$ -transverse to  $D_r = \Sigma^r(s)$ . Then  $\varphi_k = \phi_k^*s: E \otimes (L^*)^{\otimes k} \rightarrow F \otimes L^{\otimes k}$  is an asymptotically holomorphic sequence of vector bundle maps such that  $D_r^k = \Sigma^r(\varphi_k)$ . We may consider  $\varphi_k: E \rightarrow F \otimes L^{\otimes 2k}$ . Let

$$(6.1) \quad \pi: \text{Gr}(r_e - r, E) \rightarrow M$$

be the fibration whose fiber at any point  $x \in M$  is the Grassmannian  $\text{Gr}(r_e - r, E_x)$ . Using the hermitian connection  $\nabla^E$ , we have a natural almost complex structure  $J$  for  $\text{Gr}(r_e - r, E)$  such that  $\pi$  is pseudo-holomorphic and the fibers are complex manifolds. The metric  $\hat{g}_k$  on  $\text{Gr}(r_e - r, E)$  is obtained by lifting the metric  $g_k$  on  $M$  with the connection and using the standard metric for the Grassmannian in every fiber. Note also that  $\dim \text{Gr}(r_e - r, E) = \dim M + 2r(r_e - r)$ .

We associate to  $\varphi_k$  a morphism of bundles  $\sigma_k: \mathcal{U}^* \rightarrow \pi^*F \otimes L^{\otimes 2k}$  on  $\text{Gr}(r_e - r, E)$  given as the composition of the inclusion  $\mathcal{U}^* \subset \pi^*E$  with  $\pi^*\varphi_k$ , where  $\mathcal{U}$  is the universal bundle. So  $\sigma_k$  is a sequence of sections of the bundles  $\mathcal{U} \otimes \pi^*F \otimes L^{\otimes 2k}$  over  $\text{Gr}(r_e - r, E)$ . It is easy to see that  $\sigma_k$  is asymptotically holomorphic, using that  $\varphi_k$  is asymptotically holomorphic and that the connection  $\nabla^E$  is within  $O(k^{-1/2})$  of being flat in any ball  $B_{g_k}(x, c)$  of fixed radius  $c$ . Moreover  $\sigma_k$  is holomorphic along the fibers.

As  $D_k^{r-1}$  is empty, there is a one-to-one correspondence between the zeroes of  $\sigma_k$  and the points in  $D_k^r$ . Also  $\text{codim } Z(\sigma_k) = 2r(r_e - r) + 2(r_f - r)(r_e - r) = 2r_f(r_e - r)$  which is the rank of  $\mathcal{U} \otimes \pi^*F \otimes L^{\otimes 2k}$ . Our idea is to prove that  $\sigma_k$  is a transverse sequence of sections and then to use the proof of the Lefschetz hyperplane theorem ([2], Proposition 2) in our case.

**Lemma 6.2.**  $\sigma_k$  is a sequence of sections  $\gamma$ -transverse to 0, for some  $\gamma > 0$ .

*Proof.* Consider a universal construction over the bigrassmannian. Let

$$\pi: \text{Gr}(r_e - r, \mathcal{U}_e^*) \rightarrow \text{Bi}(r_e, r_f, N).$$

The section  $s$  of  $\mathcal{U}_{ef} = \mathcal{U}_e \otimes \mathcal{U}_f$  gives a morphism  $\sigma: \mathcal{U}^* \subset \pi^*\mathcal{U}_e^* \rightarrow \pi^*\mathcal{U}_f$  as above. The embeddings  $\phi_k: M \rightarrow \text{Bi}(r_e, r_f, N)$  produce embeddings

$$\hat{\phi}_k: \text{Gr}(r_e - r, E \otimes (L^*)^{\otimes k}) \rightarrow \text{Gr}(r_e - r, \mathcal{U}_e^*)$$

such that  $\sigma_k = \hat{\phi}_k^*\sigma: \mathcal{U}^* \subset \pi^*E \otimes (L^*)^{\otimes k} \rightarrow \pi^*F \otimes L^{\otimes k}$ . Let us see that the embeddings  $\hat{\phi}_k$  are transverse to  $Z(\sigma)$ .

Fix a point  $p$  in  $\text{Gr}(r_e - r, E \otimes (L^*)^{\otimes k})$  lying over a point  $x = \pi(p) \in M$ . Then  $\hat{\phi}_k(p) \in \text{Gr}(r_e - r, \mathcal{U}_e^*)$  corresponds to a rank  $r_e - r$  subspace  $\Pi \subset (\mathcal{U}_e^*)_{\hat{\phi}_k(x)}$ . Choose orthonormal trivializations of  $\mathcal{U}_e$  and  $\mathcal{U}_f$  over a ball  $B(\hat{\phi}_k(x), c)$ . We may suppose that  $\Pi = [v_1, \dots, v_{r_e-r}]$ . With these trivializations, the map  $s: \mathcal{U}_e^* \rightarrow \mathcal{U}_f$  is a map  $\mathbb{C}^{r_e} \rightarrow \mathbb{C}^{r_f}$ .

Decomposing  $\mathbb{C}^{r_e} = \mathbb{C}^{r_e-r} \oplus \mathbb{C}^r$ , we may write  $s = [s_1, s_2]$ . Choose a trivialization around  $\hat{\phi}_k(p)$  given by

$$B(\phi_k(x), c) \times \mathbb{C}^{(r_e-r) \times r} \rightarrow \text{Gr}(r_e - r, \mathcal{U}_e^*),$$

$$(y, A) \mapsto (y, [I_{r_e-r}, A]).$$

In this trivialization, the map  $\sigma: \mathcal{U} \rightarrow \pi^* \mathcal{U}_f$  is written as  $\sigma = s_1 + s_2 A^T: \mathbb{C}^{r_e-r} \rightarrow \mathbb{C}^r$ . If  $\hat{\phi}_k(p) \in Z(\sigma)$  then the tangent space  $T_{\hat{\phi}_k(p)} Z(\sigma)$  is defined as

$$(6.2) \quad \{u = (v, W) \in T_{\phi_k(x)} \text{Bi}(r_e, r_f, N) \times \mathbb{C}^{(r_e-r) \times r} \mid \nabla_v s_1 + s_2 W = 0\}.$$

If  $\hat{\phi}_k(p)$  does not belong to  $Z(\sigma)$  but it is close enough to this subvariety, we may define the distribution  $D_{Z(\sigma)}$  by the same formula (6.2).

The embedding  $\hat{\phi}_k$  gives us trivializations for the corresponding fiber bundles over a ball  $B_{g_k}(x, c')$  in  $M$ . With respect to these trivializations, we may write  $\varphi_k = [\varphi_k^1, \varphi_k^2]$ . As  $D_k^{r-1}$  is empty,  $|\wedge^r \varphi_k| \geq \eta$ , for some  $\eta > 0$ . We may choose a universal constant  $C$  such that for  $p$  close enough to  $Z(\sigma)$ , say satisfying  $|\varphi_k^1| < C\eta$ , we have  $|\wedge^r \varphi_k^2| \geq \eta/2$ . Therefore  $\varphi_k^2$  has a left inverse  $\theta_k: \mathbb{C}^r \rightarrow \mathbb{C}^r$  of norm less than  $C'\eta^{-1}$ , for some constant  $C'$ . For any  $u = (v, W) \in D_{Z(\sigma)}(\hat{\phi}_k(p))$ , one has

$$(6.3) \quad W = -\theta_k \nabla_v s_1$$

and then  $|W| \leq C''\eta^{-1}|v|$ , for some universal constant  $C''$ .

This implies that the angle of  $D_{Z(\sigma)}(\hat{\phi}_k(p))$  and the fiber  $\text{Gr}(r_e - r, (E \otimes (L^*)^{\otimes k})_{\phi_k(x)})$  is bigger than a universal  $\varepsilon > 0$ . Using that  $M$  is transverse to  $D_r$  in the bigrassmannian, we get that the angle between  $(\hat{\phi}_k)_* T_p \text{Gr}(r_e - r, E \otimes (L^*)^{\otimes k})$  and  $D_{Z(\sigma)}(\hat{\phi}_k(p))$  is bigger than a universal  $\varepsilon' > 0$ . Thus the embeddings  $\hat{\phi}_k$  are  $\varepsilon'$ -transverse to  $Z(\sigma)$ .

Now it follows that  $\sigma_k = (\hat{\phi}_k)^* s$  is a sequence of sections of  $\mathcal{U} \otimes \pi^* F \otimes L^{\otimes k}$  over  $\text{Gr}(r_e - r, E \otimes (L^*)^{\otimes k})$ , which are  $\alpha$ -transverse to 0, for some  $\alpha > 0$ . It is easy to check that this implies that the corresponding section  $\sigma_k$  of  $\mathcal{U} \otimes \pi^* F \otimes L^{\otimes 2k}$  over  $\text{Gr}(r_e - r, E)$  is  $\alpha'$ -transverse to 0, by working in balls  $B_{g_k}(x, c)$ , where we may trivialize  $L$  by a section whose norm is universally bounded below and above.  $\square$

*Proof of Proposition 6.1.* First note that any Grassmannian  $\text{Gr}(r, N)$  is connected, simply connected and  $\pi_2(\text{Gr}(r, N)) = \mathbb{Z}$ . Therefore (6.1) induces an isomorphism in  $\pi_0$  and  $\pi_1$  and an epimorphism in  $\pi_2$ . Next we shall prove that the inclusion  $Z(\sigma_k) \subset \text{Gr}(r_e - r, E)$  satisfies the Lefschetz hyperplane theorem for  $k$  large in our case. As  $\pi$  induces a diffeomorphism  $Z(\sigma_k) \cong D_k^r$ , this concludes the proof of the proposition.

We consider the (possibly degenerate) Morse function  $f_k = \log|s_k|^2$ . Let  $p$  be a critical point of  $f_k$ . Consider the subspace

$$\mathcal{V} = \{u = (v, w) \in T_p \text{Gr}(r_e - r, E) = T_x M \times T_p \text{Gr}(E_x) \mid \nabla_u \sigma_k(x) = 0\}$$

of dimension at least  $\dim D_k^r + 1$ . By the discussion in Section 5.1 of [2], we have for any unitary vector  $u = (v, w) \in \mathcal{V}$ ,

$$H_f(u) + H_f(Ju) = -2i\bar{\partial}\partial f_k(u, Ju) = -2i \frac{1}{|\sigma|^2} \langle R(u, Ju)\sigma, \sigma \rangle + O(k^{-1/2}),$$

where  $R$  is the  $(1, 1)$ -part of the curvature of  $\mathcal{U} \otimes \pi^*F \otimes L^{\otimes 2k}$ . The curvature form is, with respect to the metric  $\hat{g} = \hat{g}_1$ ,

$$R(u, Ju) = R_{\mathcal{U}}(w, Jw) - i2k\omega(v, Jv) + O(|v||w|) + O(|v|^2).$$

Using the inequality  $2|v||w| \leq k^{1/2}|v|^2 + k^{-1/2}|w|^2$ , we have that

$$R(u, Ju) = R_{\mathcal{U}}(w, Jw) - 2ik\omega(v, Jv) + O(k^{1/2}|v|^2) + O(k^{-1/2}|w|^2).$$

Now we restrict to the case  $r = r_e - 1$ . Then  $\text{Gr}(r_e - r, E_x)$  is a projective space and  $\mathcal{U}$  is its  $\mathcal{O}(1)$ -bundle. Therefore  $R_{\mathcal{U}}(w, Jw) = -i\omega_{FS}(w, Jw)$ , where  $\omega_{FS}$  is the standard symplectic form for a projective space. Thus we get that  $H_f(u) + H_f(Ju)$  is negative for  $k$  large enough and  $u \in \mathcal{V}$ . As in Section 5.1 of [2], this implies that the index of  $f_k$  at  $p$  is at least  $\frac{1}{2} \dim D_k^r + 1$ . A standard Morse theory argument gives us the claimed result.  $\square$

**Remark 6.3.** In the case  $r + 1 = r_e \leq r_f$ , the proof above gives a Lefschetz hyperplane theorem for the inclusion  $D_k^r \cong Z(\sigma_k) \subset \mathbb{P}(E)$ . Therefore for  $\dim D_k^r \geq 6$ , we have that the inclusion  $i: D_k^r \rightarrow M$  does not induce isomorphisms in the homotopy and homology groups in general.

**6.2. Chern classes of the constructed submanifolds.** For computing the Chern classes of determinantal submanifolds, we use the results of Harris and Tu in [10]. All their results are stated for holomorphic determinantal submanifolds in a holomorphic manifold, but they are valid without the condition of integrability of the complex structure. We state the formulas that we shall use. Following Subsection 5.1 we denote  $r_e = \text{rank } E$ ,  $r_f = \text{rank } F$ ,  $2n = \dim M$  and  $D_r$  is the  $r$ -determinantal locus of a bundle map  $\varphi: E \rightarrow F$  constructed in Theorem 1.6. First of all, set

$$(6.4) \quad \Delta_{i_1, \dots, i_{r_e-r}} = \begin{vmatrix} c_{r_f-r+i_1} & c_{r_f-r+i_1+1} & \cdots & & \\ c_{r_f-r+i_2-1} & c_{r_f-r+i_2} & \cdots & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & c_{r_f-r+i_{r_e-r}} \end{vmatrix},$$

where  $c_j = c_j(F - E)$ . For instance,  $\Delta_{0, \dots, 0} = \Delta = \text{PD}([D_r])$ , which is the classical Porteous formula for the homology class of a determinantal locus. We can suppose that the indices  $i_j$  are decreasing, and so we do not write any index  $i_j = 0$ , e.g.  $\Delta_{2,1,0} = \Delta_{2,1}$ .

In [10] a complete description of the Chern numbers of the tangent bundle of a determinantal submanifold is performed, supposing that  $D_{r-1} = \emptyset$  and so  $D_r$  is smooth. We shall give examples in the cases  $\dim_{\mathbb{C}} D_r = 1$  and  $\dim_{\mathbb{C}} D_r = 2$ .

**6.2.1. Example 1.** Choose  $\dim_{\mathbb{C}} M = (r_e - r)(r_f - r) + 1$ . We apply the formula for  $\dim_{\mathbb{C}} D_r = 1$  in page 474 of [10]. Also suppose that  $r = 1$  and  $r_e = 2$ , so  $\dim_{\mathbb{C}} M = r_f = n > 1$ . By Proposition 6.1 the submanifolds  $D_1$  are connected. Now  $\text{PD}[D_1] = \Delta = c_{n-1}$  and  $\Delta_1 = c_n$ . Computing we get

$$\begin{aligned} \text{vol}_{\omega_k}(D_1) &= \Delta\omega_k = (n2^{n-1} + O(k^{-1})) \text{vol}_{\omega_k}(M), \\ n_1(D_1) &= -(n+2)\omega_k\Delta + (2-n)\Delta_1 + O(k^{-1}) \text{vol}_{\omega_k}(M) \\ &= ((2-n-n^2)2^n + O(k^{-1})) \text{vol}_{\omega_k}(M), \\ \frac{n_1(D_1)}{\text{vol}_{\omega_k}(D_1)} &= -2 - 2n + \frac{4}{n} + O(k^{-1}). \end{aligned}$$

To compare with Auroux' case we compute the precedent symplectic invariants for this situation. Denote by  $Z$  the zero set of a transverse section of the bundle  $E \otimes L^{\otimes k}$ . We choose  $\text{rank } E = n - 1$  to set up the comparison. Suppose that  $Z$  is symplectic. Using Proposition 5 in [2] we obtain

$$\begin{aligned} \text{vol}_{\omega_k}(Z) &= (1 + O(k^{-1})) \text{vol}_{\omega_k}(M), \\ n_1(Z) &= (1 - n + O(k^{-1})) \text{vol}_{\omega_k}(M), \\ \frac{n_1(Z)}{\text{vol}_{\omega_k}(Z)} &= 1 - n + O(k^{-1}). \end{aligned}$$

Therefore there does not exist any  $n \geq 2$  such that the quotients  $\frac{n_1(D_1)}{\text{vol}_{\omega_k}(D_1)}$  coincide with  $\frac{n_1(Z)}{\text{vol}_{\omega_k}(Z)}$ , obviously for  $k$  large enough. So Auroux' sequences of submanifolds are not symplectomorphic to our sequences of determinantal submanifolds.

To check that, for  $k$  large, our determinantal submanifolds do not coincide with Auroux' examples we work as follows. Suppose that for integers  $k_1, k_2$  the submanifold  $D_1 = D_1^{k_1}$  is isotopic to  $Z = Z_{k_2}$ . Then they define the same cohomology class and hence  $n2^{n-1}k_1^{n-1} = k_2^{n-1} + O(k_1^{n-2} + k_2^{n-2})$ . Also  $n_1(D_1) = n_1(Z)$  implies

$$\left(-2 - 2n + \frac{4}{n}\right)k_1 = (1 - n)k_2 + O(1).$$

So, for large enough  $k$ 's,  $(1 - n)^{n-1}n2^{n-1} = \left(-2 - 2n + \frac{4}{n}\right)^{n-1}$  which gives  $n^n = (n + 2)^{n-1}$  and hence  $n = 2$ . Therefore for  $n > 2$  we get new examples of symplectic submanifolds.

Note that for  $n = r_e = r_f = 2$ , the determinantal set  $D_1$  for a morphism  $\varphi: E \otimes (L^*)^{\otimes k} \rightarrow F \otimes L^{\otimes k}$  is the zero set of the section  $\wedge^2 \varphi$  of  $\wedge^2 E^* \otimes \wedge^2 F \otimes L^{\otimes 4k}$ . Since this zero set is smooth of the expected codimension, our example is just one of Auroux' examples.

**6.2.2. Example 2.** Now choose  $\dim_{\mathbb{C}} M = (r_e - r)(r_f - r) + 2$ . We apply the formula for  $\dim_{\mathbb{C}} D_r = 2$  in page 474 of [10]. Again we suppose that  $r = 1$  and  $r_e = 2$ , so

$\dim_{\mathbb{C}} M = r_f + 1 = n > 2$ . By Proposition 6.1 these submanifolds are connected. In this case we have

$$\begin{aligned}\text{vol}_{\omega_k}(D_1) &= ((n-1)2^{n-2} + O(k^{-1})) \text{vol}_{\omega_k}(M), \\ n_{11}(D_1) &= (4(n-1)(n^2-5)2^{n-2} + O(k^{-1})) \text{vol}_{\omega_k}(M), \\ n_2(D_1) &= (2(n^2+n-4)(n-1)2^{n-2} + O(k^{-1})) \text{vol}_{\omega_k}(M) \\ \frac{n_2(D_1)}{n_{11}(D_1)} &= \frac{n^2+n-4}{2(n^2-5)} + O(k^{-1}).\end{aligned}$$

For the Auroux' case with rank  $E = n - 2$  we obtain

$$\begin{aligned}\text{vol}_{\omega_k}(Z) &= (1 + O(k^{-1})) \text{vol}_{\omega_k}(M), \\ n_{11}(Z) &= ((n-2)^2 + O(k^{-1})) \text{vol}_{\omega_k}(M), \\ n_2(Z) &= \left( \frac{(n-1)(n-2)}{2} + O(k^{-1}) \right) \text{vol}_{\omega_k}(M), \\ \frac{n_2(Z)}{n_{11}(Z)} &= \frac{n-1}{2(n-2)} + O(k^{-1}).\end{aligned}$$

For 4-manifolds the numbers  $n_2 = \chi$  and  $n_{11} = (2\chi + 3\sigma)/4$  are topological invariants. Therefore  $\frac{n_2}{n_{11}}$  is a topological invariant. For  $n > 3$ , comparing the Auroux' case and the determinantal example we see that these symplectic submanifolds are not even *homeomorphic*, for  $k$  large enough (even choosing different  $k$ 's in either case). In the case  $n = 3$ , the manifolds  $D_1^{k_1}$  and  $Z_{k_2}$  coincide if we take  $k_2 = 4k_1$ . This is due to the same reason as in the first example.

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Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049, Madrid, Spain  
e-mail: vicente.munoz@uam.es

Departamento de Álgebra, Universidad Complutense de Madrid, 28040 Madrid, Spain  
e-mail: fpm@eucmos.sim.ucm.es  
e-mail: sols@eucmax.sim.ucm.es

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