ON ŁOJASIEWICZ'S INEQUALITY AND THE NULLSTELLENSATZ FOR RINGS OF SEMIALGEBRAIC FUNCTIONS

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Abstract. In this article we present versions of Łojasiewicz's inequality and the Nullstellensatz for the ring of bounded semialgebraic functions on an arbitrary semialgebraic set M. We also prove that the classical Łojasiewicz inequality and Nullstellensatz for the ring of semialgebraic functions on a semialgebraic set M work if and only if M is locally compact.

1. Introduction

A subset $M \subset \mathbb{R}^n$ is said to be *basic semialgebraic* if it can be written as

$$M = \{x \in \mathbb{R}^n : \ f(x) = 0, \ g_1(x) > 0, \dots, g_m(x) > 0\}$$

for some polynomials $f, g_1, \dots, g_m \in \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_n]$. The finite unions of basic semialgebraic sets are called *semialgebraic sets*. A continuous function $f : M \to \mathbb{R}$ is said to be *semialgebraic* if its graph is a semialgebraic subset of $\mathbb{R}^{n+1}$. Usually, semialgebraic function just means a function, non necessarily continuous, whose graph is semialgebraic. However, since most of the semialgebraic functions occurring in this article are continuous we will omit for simplicity the continuity condition when we refer to them and we will write *functions whose graph is semialgebraic* for those which are non necessarily continuous. For further readings about semialgebraic sets and functions we refer the reader to [BCR, §2].

The sum and product of functions, defined pointwise, endow the set $\mathcal{S}(M)$ of semialgebraic functions on M with a natural structure of commutative ring whose unity is the function with constant value 1. In fact $\mathcal{S}(M)$ is an $\mathbb{R}$-algebra, if we identify each real number r with the constant function which just attains this value. The simplest examples of semialgebraic functions on M are the restrictions to M of polynomials in n variables. Other relevant ones are the Euclidean distance function $\operatorname{dist}(\,\cdot\,, N)$ to a given semialgebraic set $N \subset M$, the absolute value of a semialgebraic function, the maximum and the minimum of a finite family of semialgebraic functions, the inverse and the k-root of a semialgebraic function whenever these operations are well-defined.

It is obvious that the subset $\mathcal{S}^*(M)$ of bounded semialgebraic functions on M is a real subalgebra of $\mathcal{S}(M)$. For the time being we denote by $\mathcal{S}^\diamond(M)$, indistinctly, either $\mathcal{S}(M)$ or $\mathcal{S}^*(M)$, in case the involved statements or arguments are valid for both rings. For each

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$f \in \mathcal{S}^\diamond(M)$ and each semialgebraic subset $N \subset M$, we denote $Z_N(f) = \{x \in N : f(x) = 0\}$ and $D_N(f) = M \setminus Z_N(f)$. In case $N = M$, we say that $Z_M(f)$ is the *zeroset* of f.

Łojasiewicz's inequality is one of the main results in Real Algebraic Geometry. Its first versions are due, independently, to L. Hörmander [H] and S. Łojasiewicz [L], who invented them as the main ingredient in their solutions to the so called "division problem", stated by L. Schwartz [S], concerning the division of a distribution by a polynomial or, more generally, by an analytic function.

Precisely, Hörmander's version states that given a polynomial $f \in \mathbb{R}[\mathtt{x}_1, \ldots, \mathtt{x}_n]$ there exist positive real numbers c, μ such that $c\,\mathrm{dist}(x, Z_M(f))^\mu \leq |f(x)|$ for every $x \in \mathbb{R}^n$ with $\|x\| \leq 1$. On the other hand, Łojasiewicz stated (without proof) that given a compact set $K \subset \mathbb{R}^n$, an open neighbourhood $\Omega \subset \mathbb{R}^n$ of K, and an analytic function $f : \Omega \to \mathbb{R}$, there exist positive real numbers c, μ so that $c\,\mathrm{dist}(x, Z_M(f))^\mu \leq |f(x)|$ for all $x \in K$.

A useful version of this classical result, when dealing with semialgebraic functions, and which produces as a byproduct a Nullstellensatz for semialgebraic functions (see 3.4), appears in [BCR, 2.6.6-7]. Namely,

Theorem 1.1 (Łojasiewicz's inequality)**.** *Let $M \subset \mathbb{R}^n$ be a locally compact semialgebraic set and let $f, g \in \mathcal{S}(M)$ be such that $Z_M(f) \subset Z_M(g)$. Then,*

(i) *There exist a positive integer ℓ and $h \in \mathcal{S}(M)$ such that $g^\ell = fh$.*
(ii) *Moreover, if $c = \sup\{|h(x)| : x \in M\}$ exists, then $|g(x)|^\ell \leq c|f(x)|$ for each $x \in M$.*

Remarks 1.2. (i) The previous result, and in fact the corresponding Nullstellensatz, is no longer true if M is not locally compact, see 3.5. A very representative example of this situation is the following one, proposed in [BCR, 2.6.5]. Consider the semialgebraic set $M = \{y > 0\} \cup \{(0,0)\} \subset \mathbb{R}^2$ and the semialgebraic functions $g(x,y) = x^2 + y^2$ and $f(x,y) = y$. Their zerosets are $Z_M(f) = Z_M(g) = \{(0,0)\}$. However, for each $\ell \in \mathbb{N}$ the limit at the origin of the semialgebraic function $h_\ell = \frac{g^\ell}{f} = \frac{(x^2+y^2)^\ell}{y}$ does not exist.

(ii) Observe that statement 1.1(ii) says nothing if $c = +\infty$. However, if $c < +\infty$ it is equivalent to 1.1(i), even if $M \subset \mathbb{R}^n$ is an arbitrary semialgebraic set. More precisely, *let $M \subset \mathbb{R}^n$ be a semialgebraic set and let $f, g \in \mathcal{S}^\diamond(M)$ be such that $Z_M(f) \subset Z_M(g)$ and there exist a constant $c > 0$ and a positive integer $\ell \geq 1$ such that $|g(x)|^\ell \leq c|f(x)|$ for each $x \in M$. Then, there exists $h \in \mathcal{S}^\diamond(M)$ such that $g^{2\ell+1} = fh$.*

Indeed, for each $x \in M$ we have $g^{2\ell}(x) \leq c^2 f^2(x)$. Thus, the function $h_0 : M \to \mathbb{R}$ given by

$$h_0(x) = \begin{cases} \frac{g^{2\ell+1}(x)}{f^2(x)} & \text{if } x \in D_M(f), \\ 0 & \text{if } x \in Z_M(f). \end{cases}$$

is continuous because $Z_M(f) \subset Z_M(g)$ and the quotient $g^{2\ell}/f^2$ is bounded on $D_M(f)$. Moreover, $h_0 \in \mathcal{S}(M)$, and in fact it is bounded if g is so. Now, since $h_0 f^2 = g^{2\ell+1}$, we deduce that $h = fh_0 \in \mathcal{S}^\diamond(M)$ satisfies the required condition. □

In view of 1.2(ii), in what follows we will say that Łojasiewicz's inequality does not hold for a semialgebraic set M if there exist semialgebraic functions $f, g \in \mathcal{S}(M)$ such that $Z_M(f) \subset Z_M(g)$ but $g \notin \sqrt{f\mathcal{S}(M)}$.

Of course, 1.1(i) can be understood as a Nullstellensatz for principal ideals. To approach the announced Nullstellensatz for arbitrary ideals (see 3.4), and since the common zeroset

Z of the semialgebraic functions of a prime ideal $\mathfrak{p}$ of $\mathcal{S}(M)$ provides almost no information about such $\mathfrak{p}$ because Z is either empty or a singleton (see 2.3), we are led to consider the *z-filter* consisting of the collection of the zerosets of all functions in $\mathfrak{p}$ (see 3.1). As it is well-known, this is a classical idea used to study rings of continuous functions, which was compiled full in detail in [GJ]. On the other hand, the use of these kinds of filters is a usual technique in Real Algebra (see for instance [ABR, II.1.6] and [BCR, 7.1, 7.5]).

In any case, the main goal of this work is to develop a similar theory (Łojasiewicz's inequality and Nullstellensatz) to approach the case of bounded semialgebraic functions. The existence of non-units in $\mathcal{S}^*(M)$ with empty zeroset will require to generalize these z-filters to obtain a similar Łojasiewicz's inequality, which has been revealed as a crucial tool in Real Geometry. Even more, the bounded case can be done without the local compactness assumption. Namely,

Theorem 1.3. *Let f, g be two bounded semialgebraic functions on the semialgebraic set M such that each maximal ideal of $\mathcal{S}^*(M)$ containing f contains g too. Then, $g^\ell = fh$ for a suitable positive integer ℓ and a function $h \in \mathcal{S}^*(M)$. In particular, $|g|^\ell \leq \sup_M(|h|)|f|$ on M.*

Clearly, this result (translated to the language of maximal spectra of semialgebraic rings in 3.12) can be understood as the counterpart for rings of bounded semialgebraic functions of the classical Łojasiewicz inequality 1.1(ii). Its importance lies, among other things, behind the fact that it provides as a byproduct a Nullstellensatz for the ring $\mathcal{S}^*(M)$, where M is an arbitrary semialgebraic set (see 3.11). In contrast, as we have already commented, the Nullstellensatz for $\mathcal{S}(M)$ is only true if M is locally compact (see 3.5). To prove this fact it will be indispensable to analyze the set $M_{\rm lc} \subset M$ of those points in M having a compact neighbourhood in M. In fact, such set is moreover semialgebraic (see 2.10).

The article is organized as follows. In Section 2 we introduce most of the terminology used in the sequel, and we also prove that every non locally compact semialgebraic set M contains a semialgebraic subset C, closed in M, which is semialgebraically homeomorphic to the triangle $T = \{(x, y) \in \mathbb{R}^2 : 0 < y \leq x \leq 1\} \cup \{(0, 0)\}$. This last result is the key to prove that Łojasiewicz inequality and the corresponding Nullstellensatz are no longer true for non locally compact semialgebraic sets. Finally, in Section 3 we develop the main results of this work concerning Łojasiewicz inequality and the Nullstellensatz for rings of semialgebraic and bounded semialgebraic functions on semialgebraic sets. In fact, we prove that Theorem 1.1 can be also obtained as a byproduct of Theorem 1.3.

To finish this Introduction, we would like to point out that Łojasiewicz's inequalities and Nullstellensätze are crucial tools for the study of chains of prime ideals in rings of semialgebraic and bounded semialgebraic functions (see [Fe1]), and to determine the Krull dimension of the rings of semialgebraic and bounded semialgebraic functions on a semialgebraic set (see [FG1] for further details).

2. Preliminaries on semialgebraic sets and functions

In this section we present some preliminary terminology and results that will be useful in the rest of the sequel. We point out first that some times it will be advantageous to assume that the semialgebraic set M we are working with is bounded. Such assumption can be done without loss of generality because of the next remark. Along this work, we denote by $\mathbb{B}_n(x,\varepsilon)$ and $\overline{\mathbb{B}}_n(x,\varepsilon)$ the open and closed balls of $\mathbb{R}^n$ of center $x \in \mathbb{R}^n$ and radius ε. Their common boundary is denoted by $\mathbb{S}^{n-1}(x,\varepsilon)$.

Remark 2.1. Let $M \subset \mathbb{R}^n$ be a semialgebraic set. The semialgebraic homeomorphism

$$\varphi : \mathbb{B}_n(0,1) \to \mathbb{R}^n,\ x \mapsto \frac{x}{\sqrt{1-\|x\|^2}},$$

induces a ring isomorphism $\mathcal{S}(M) \to \mathcal{S}(N)$, $f \mapsto f \circ \varphi$, where $N = \varphi^{-1}(M)$, that maps $\mathcal{S}^*(M)$ onto $\mathcal{S}^*(N)$. Hence, if necessary, we may always assume that M is bounded.

The following result, which concerns the representation of closed semialgebraic subsets of a semialgebraic set as zerosets of semialgebraic functions, is well-known and it will be used freely along this work.

Lemma 2.2. *Let Z be a closed semialgebraic subset of the semialgebraic set $M \subset \mathbb{R}^n$. Then, there exists $h \in \mathcal{S}^*(M)$ such that $Z = Z_M(h)$.*

Proof. Take for instance $h = \min\{1, \operatorname{dist}(\cdot, Z)\}$. □

In contrast to what happens in dealing with ideals of polynomial rings, the zeroset of a prime ideal $\mathfrak{p}$ of $\mathcal{S}^\diamond(M)$ provides no substantial information about $\mathfrak{p}$, because it is either a point or the emptyset.

Proposition 2.3. *Let $M \subset \mathbb{R}^n$ be a semialgebraic set and let $\mathfrak{p}$ be a prime ideal of $\mathcal{S}^\diamond(M)$. Then, the set $Z = \{x \in M : f(x) = 0\ \forall f \in \mathfrak{p}\}$ is either empty or a singleton.*

Proof. Suppose, by way of contradiction, that Z contains two distinct points p, q. Let $r > 0$ be the Euclidean distance between p and q and let B_1 and B_2 be the open balls centered at p of respective radii $r_1 = r/3$ and $r_2 = 2r/3$. Consider the closed semialgebraic sets in $\mathbb{R}^n$ defined as $C_1 = \mathbb{R}^n \setminus B_1$ and $C_2 = \operatorname{Cl}_{\mathbb{R}^n}(B_2)$. By 2.2, there exist $f_1, f_2 \in \mathcal{S}^*(\mathbb{R}^n)$ such that $Z_{\mathbb{R}^n}(f_i) = C_i$. Clearly, the product $f_1 f_2$ vanishes identically on $\mathbb{R}^n$; hence, on M. Thus, if we write $g_i = f_i|_M$ for $i = 1, 2$ we have $g_1 g_2 \in \mathfrak{p}$, and therefore either g_1 or g_2 belongs to $\mathfrak{p}$. But g_1 does not vanish at p and g_2 does not vanish at q, a contradiction. □

This result suggests to introduce some classical definitions.

Definitions and Notations 2.4. An ideal $\mathfrak{a}$ of $\mathcal{S}^\diamond(M)$ is said to be *fixed* if all functions in $\mathfrak{a}$ vanish simultaneously at some point of M. Otherwise the ideal $\mathfrak{a}$ is *free*.

Given a point $p \in M$ we denote by $\mathfrak{m}_p$ (resp. $\mathfrak{m}^*_p$) the fixed ideal of $\mathcal{S}(M)$ (resp. $\mathcal{S}^*(M)$) consisting of those functions vanishing at p. Distinct points produce distinct maximal ideals, and $\{\mathfrak{m}_p\}_{p\in M}$ (resp. $\{\mathfrak{m}^*_p\}_{p\in M}$) constitutes the collection of all fixed maximal ideals of $\mathcal{S}(M)$ (resp. $\mathcal{S}^*(M)$).

We will denote by $\beta^*_{\rm s}M$ the collection of all maximal ideals of $\mathcal{S}^*(M)$. Given a function $f \in \mathcal{S}^*(M)$ we write

$$\mathcal{Z}_{\beta^*_{\rm s}M}(f) = \{\mathfrak{m} \in \beta^*_{\rm s}M : f \in \mathfrak{m}\} \quad \text{and} \quad \mathcal{D}_{\beta^*_{\rm s}M}(f) = \beta^*_{\rm s}M \setminus \mathcal{Z}_{\beta^*_{\rm s}M}(f).$$

Notice that the map $\phi : M \to \beta^*_{\rm s}M$, $p \mapsto \mathfrak{m}^*_p$ is injective; thus, for the time being, we identify M with $\phi(M)$. This provides the equalities $D_M(f) = \mathcal{D}_{\beta^*_{\rm s}M}(f) \cap M$ and $Z_M(f) = \mathcal{Z}_{\beta^*_{\rm s}M}(f) \cap M$.

Concerning free maximal ideals of $\mathcal{S}^*(M)$, which are deeply studied in [Fe2] and [FG2], we are mainly interested in simplest class of them: those associated to semialgebraic paths. Namely,

(2.5) Maximal ideals associated to semialgebraic paths. Let $M \subset \mathbb{R}^n$ be a semialgebraic set. Consider a semialgebraic path $\alpha : (0,1] \to M$, that is, a continuous map whose components are semialgebraic functions. We claim that: *The set* $\mathfrak{m}_\alpha^* = \{f \in \mathcal{S}^*(M) : \lim_{t\to 0}(f\circ\alpha)(t) = 0\}$ *is a maximal ideal of* $\mathcal{S}^*(M)$. Of course, the ideal $\mathfrak{m}_\alpha^*$ is free if and only if α cannot be extended to a (continuous) semialgebraic path $[0,1] \to M$.

Before proving 2.5, we need the following preliminary result. Recall that given an open semialgebraic set $U \subset \mathbb{R}^n$, a function $f \in \mathcal{S}(U)$ is said to be a *Nash* function on U if it is, moreover, analytic (see [BCR, 8.1.6-8]).

Lemma 2.6. *Let $I = (a,b) \subset \mathbb{R}$ be an open interval, with $-\infty \le a < b \le +\infty$, and let $f \in \mathcal{S}(I)$ be a semialgebraic function. Then,*

(i) *There exists a finite subset $F \subset I$ such that the restriction $h = f|_{I\setminus F}$ is a Nash function.*
(ii) *There exists $c \in I$ such that the restriction $f|_{[c,b)}$ is a monotone function.*
(iii) *If f is moreover bounded, there exist* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to b} f(x)$.

Proof. (i) The graph of f being a 1-dimensional semialgebraic subset of $\mathbb{R}^2$, it is a finite union of singletons $\{p_1,\dots,p_n\}$ and 1-dimensional Nash manifolds (see [BCR, 2.9.10]). Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto x$ be the projection onto the first coordinate. Then, the set $F = \{\pi_1(p_1),\dots,\pi_1(p_n)\}$ satisfies the statement.

(ii) If f is constant on a subinterval (c,b) of I the result is evident. If not, since the zeroset $Z_{I\setminus F}(f')$ of the derivative f' of f is semialgebraic, it is a union of singletons and intervals, none of them of the form (c,b). In other words, $Z_{I\setminus F}(f') \subset (a,c_0)$ for some $c_0 < b$, and it is enough to choose $c = c_0$. Note that in this case $f|_{[c,b)}$ is either increasing or decreasing, according to the sign of f' in $[c,b)$.

(iii) It suffices to prove that there exists the limit of f at b. This is obvious if f is constant on a subinterval $J = [c,b) \subset I$. Hence, we can suppose, without loss of generality, that f is decreasing on J. Since f is a bounded function, $f(J)$ is a bounded interval and, f being decreasing on J, there exists $\lambda \in \mathbb{R}$ such that $f(J) = (\lambda, f(c)]$. Note that $\mathrm{Cl}_{\mathbb{R}}(f(J)) \setminus f(J) = \{\lambda\}$, and so $\lim_{x\to b} f(x) = \lambda$. □

Now, the claim in 2.5 follows straightforwardly from 2.6:

Proof of 2.5. It follows from 2.6 that there exists $\lim_{t\to 0}(f\circ\alpha)(t) \in \mathbb{R}$ for each function $f \in \mathcal{S}^*(M)$. Once this is done, note that $\mathfrak{m}_\alpha^*$ is the kernel of the ring epimorphism $\mathcal{S}^*(M) \to \mathbb{R},\ f \mapsto \lim_{t\to 0}(f\circ\alpha)(t)$. □

Remark 2.7. With the notation above, suppose there exists $\lim_{t\to 0}\alpha(t) = p \in M$. This includes the case in which the path α is locally constant around 0. Then, $\mathfrak{m}_\alpha^* = \mathfrak{m}_p^*$.

Finally in this section, we study some properties about local compactness of semialgebraic sets.

(2.8) Local compactness. Locally compact Hausdorff spaces are characterized as those spaces which admit a Hausdorff compactification by a single point ([Mu, 3.29.1]). On the other hand, locally closed semialgebraic subsets of $\mathbb{R}^n$ are those which can be embedded as closed semialgebraic subsets of some $\mathbb{R}^m$. It must be pointed out that local closedness has revealed to be, in the semialgebraic setting, an important property for the validity of results which are in the core of semialgebraic geometry, as Łojasiewicz inequality. But, as it is well-known, locally closed subsets of $\mathbb{R}^n$ coincide with the locally compact ones (see [Bo, §9.7. Prop.12-13]). In fact, if $M \subset \mathbb{R}^n$ is locally compact, then $M = U \cap \mathrm{Cl}_{\mathbb{R}^n}(M)$ where $U = \mathbb{R}^n \setminus (\mathrm{Cl}_{\mathbb{R}^n}(M) \setminus M)$ is an open subset of $\mathbb{R}^n$. Of course, if $M \subset \mathbb{R}^n$ is a semialgebraic set, both $\mathrm{Cl}_{\mathbb{R}^n}(M)$ and U are semialgebraic; hence, each locally compact semialgebraic set $M \subset \mathbb{R}^n$ is the intersection of a closed and an open semialgebraic subsets of $\mathbb{R}^n$.

As we have already announced in the Introduction, we will see in Section 3 that only the locally compact semialgebraic sets satisfy a Łojasiewicz inequality or a Nullstellensatz for its ring of semialgebraic functions. The clue result to prove this is the following:

Lemma 2.9. *Let $M \subset \mathbb{R}^n$ be a semialgebraic set which is not locally compact. Then, M contains a semialgebraic set C, closed in M, semialgebraically homeomorphic to the triangle $T = \{(x, y) \in \mathbb{R}^2 : 0 < y \leq x \leq 1\} \cup \{(0, 0)\}$.*

The proof of this fact requires a certain analysis of the set of points of M having a compact neighbourhood in M.

Lemma 2.10. *Let $M \subset \mathbb{R}^n$ be a semialgebraic set. Define*

$$\rho_0(M) = \mathrm{Cl}_{\mathbb{R}^n}(M) \setminus M \quad \textit{and} \quad \rho_1(M) = \rho_0(\rho_0(M)) = \mathrm{Cl}_{\mathbb{R}^n}(\rho_0(M)) \cap M.$$

Then, $M_{\mathrm{lc}} = M \setminus \rho_1(M)$ is a locally compact semialgebraic set which coincides with the set of points of M having a compact neighbourhood in M.

Assume we have already proved 2.10 and let as show 2.9.

Proof of Lemma 2.9. We may assume that $0 \in \rho_1(M)$. By 2.10, the origin is not an isolated point of M. By [BCR, 9.3.6], there exist a positive real number $\varepsilon > 0$ and a semialgebraic homeomorphism $\varphi : \overline{\mathbb{B}}_n(0, \varepsilon) \to \overline{\mathbb{B}}_n(0, \varepsilon)$ such that:

(i) $\|\varphi(x)\| = \|x\|$ for every $x \in \overline{\mathbb{B}}_n(0, \varepsilon)$,
(ii) $\varphi|_{\mathbb{S}^{n-1}(0,\varepsilon)}$ is the identity map,
(iii) $\varphi^{-1}(M \cap \overline{\mathbb{B}}_n(0, \varepsilon))$ is the cone with vertex 0 and basis $M \cap \mathbb{S}^{n-1}(0, \varepsilon)$.

Consider the semialgebraic homeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\psi(x) = \begin{cases} x & \text{if } x \in \mathbb{R}^n \setminus \overline{\mathbb{B}}_n(0, \varepsilon), \\ \varphi(x) & \text{if } x \in \overline{\mathbb{B}}_n(0, \varepsilon). \end{cases}$$

In what follows we identify M with $\psi^{-1}(M)$. Since $0 \in \rho_1(M)$, this point has no compact neighbourhood in M, see 2.10. In particular, $M \cap \overline{\mathbb{B}}_n(0, \varepsilon)$, which is the cone with vertex 0 and basis $N = \mathbb{S}^{n-1}(0, \varepsilon) \cap M$, is not compact. This implies that also the basis N is not compact, hence it is not closed in $\mathbb{R}^n$, and we choose a point $q \in \mathrm{Cl}_{\mathbb{R}^n}(N) \setminus N$. By the Curve Selection Lemma [BCR, 2.5.5], there exists a semialgebraic path $\alpha : [0, 1] \to \mathbb{R}^n$ such that $\alpha(0) = q$ and $\alpha((0, 1]) \subset N$. After shrinking the domain of α if necessary, we may assume that $\alpha|_{(0,1]}$ is a homeomorphism onto its image $K = \alpha((0,1]) \subset N$. Thus, K is a closed subset of N, and it is homeomorphic to the interval $(0,1]$.

Let C be the cone with vertex 0 and basis K. A straightforward computation shows that C, which is a closed semialgebraic subset of M, is homeomorphic to T via the semialgebraic homeomorphism

$$T \to C,\ (s,t) \mapsto \begin{cases} s\alpha(t/s) & \text{if } s \neq 0, \\ 0 & \text{if } s = 0, \end{cases}$$

whose inverse map is defined by

$$C \to T,\ x \mapsto \begin{cases} (\|x\|/\varepsilon)(1, \alpha^{-1}(\varepsilon x/\|x\|)) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We are done. □

Next, we proceed to prove the remaining result 2.10.

Proof of Lemma 2.10. We check first that $M \setminus \rho_1(M) = \operatorname{Cl}_{\mathbb{R}^n}(M) \setminus \operatorname{Cl}_{\mathbb{R}^n}(\rho_0(M))$. Observe that $\operatorname{Cl}_{\mathbb{R}^n}(M) = M \sqcup \rho_0(M)$ and $\operatorname{Cl}_{\mathbb{R}^n}(\rho_0(M)) = \rho_0(M) \sqcup \rho_1(M)$. Thus,

$$\operatorname{Cl}_{\mathbb{R}^n}(M) \setminus \operatorname{Cl}_{\mathbb{R}^n}(\rho_0(M)) = (M \sqcup \rho_0(M)) \setminus (\rho_0(M) \sqcup \rho_1(M)) = M \setminus \rho_1(M).$$

Consequently, $M_{\mathrm{lc}} = M \setminus \rho_1(M) = \operatorname{Cl}_{\mathbb{R}^n}(M) \setminus \operatorname{Cl}_{\mathbb{R}^n}(\rho_0(M))$ is a locally closed set, and so it is locally compact, by 2.8. Next, note that

$$\rho_1(M) = \operatorname{Cl}_{\mathbb{R}^n}(\operatorname{Cl}_{\mathbb{R}^n}(M) \setminus M) \cap M$$

is a closed subset of M. Thus, if N denotes the set of points of M having a compact neighbourhood in M, we deduce, since M_{lc} is locally compact and open in M, that $M_{\mathrm{lc}} = M \setminus \rho_1(M)$ is contained in N.

Conversely, let $x \in N$ and let K be a compact neighbourhood of x in M. Let W be an open subset of $\mathbb{R}^n$ such that $x \in W$ and $M \cap W \subset K$. Thus,

$$x \in \operatorname{Cl}_{\mathbb{R}^n}(M) \cap W = \operatorname{Cl}_{\mathbb{R}^n}(M \cap W) \cap W \subset K \subset M,$$

or equivalently, W is a neighbourhood of x in $\mathbb{R}^n$ such that $W \cap (\operatorname{Cl}_{\mathbb{R}^n}(M) \setminus M) = \varnothing$. Hence, $x \notin \operatorname{Cl}_{\mathbb{R}^n}(\operatorname{Cl}_{\mathbb{R}^n}(M) \setminus M) \cap M = \rho_1(M)$, that is, $x \in M_{\mathrm{lc}} = M \setminus \rho_1(M)$, as wanted. □

3. Łojasiewicz's inequalities and Nullstellensätze

We begin this section by introducing several preliminary notions and remarks which allow us to state properly the Nullstellensatz for the ring of semialgebraic functions on a semialgebraic set. Along this section whenever we consider an ideal of $\mathcal{S}^{\diamond}(M)$ we mean a proper ideal of $\mathcal{S}^{\diamond}(M)$.

(3.1) Filters in rings of semialgebraic functions. Let $\mathcal{Z}_M$ be the collection of all closed semialgebraic subsets of M, which coincides, by 2.2, with the family of zerosets of semialgebraic functions on M. Let $\mathcal{P}(\mathcal{Z}_M)$ be the set of all subsets of $\mathcal{Z}_M$. Recall that a subset $\mathcal{F}$ of $\mathcal{P}(\mathcal{Z}_M)$ is a z-filter on M if it satisfies the following properties:

(i) $\varnothing \notin \mathcal{F}$.
(ii) Given $Z_1, Z_2 \in \mathcal{F}$ then $Z_1 \cap Z_2 \in \mathcal{F}$.
(iii) Given $Z \in \mathcal{F}$ and $Z' \in \mathcal{Z}_M$ such that $Z \subset Z'$ then $Z' \in \mathcal{F}$.

Let $\mathfrak{a}$ be an ideal of $\mathcal{S}(M)$. One can check straightforwardly that:

(i) *The family* $\mathcal{Z}[\mathfrak{a}] = \{Z_M(f) : f \in \mathfrak{a}\}$ *is a z-filter on* M.
(ii) *If* $\mathcal{F}$ *is a z-filter, then* $\mathcal{J}(\mathcal{F}) = \{f \in \mathcal{S}(M) : Z_M(f) \in \mathcal{F}\}$ *is an ideal of* $\mathcal{S}(M)$ *satisfying* $\mathcal{Z}[\mathcal{J}(\mathcal{F})] = \mathcal{F}$.

Definition 3.2. An ideal $\mathfrak{a}$ of $\mathcal{S}(M)$ is a *z-ideal* if $\mathcal{J}(\mathcal{Z}[\mathfrak{a}]) = \mathfrak{a}$, that is, whenever there exist $f \in \mathfrak{a}$ and $g \in \mathcal{S}(M)$ satisfying $Z_M(f) \subset Z_M(g)$, we have $g \in \mathfrak{a}$.

Remark 3.3. Notice that the equality $\mathcal{Z}[\mathcal{J}(\mathcal{F})] = \mathcal{F}$ implies that $\mathcal{J}(\mathcal{F})$ is a z-ideal whenever $\mathcal{F}$ is a z-filter. Note also that each z-ideal is a radical ideal because $Z_M(f) = Z_M(f^k)$ for each $f \in \mathcal{S}(M)$ and each $k \geq 1$.

We are now ready to present the Nullstellensatz for the ring of semialgebraic functions on a semialgebraic set.

Corollary 3.4 (Nullstellensatz). *Let* $M \subset \mathbb{R}^n$ *be a locally compact semialgebraic set. Let* $\mathfrak{a}$ *be an ideal of* $\mathcal{S}(M)$. *Then* $\mathcal{J}(\mathcal{Z}[\mathfrak{a}]) = \sqrt{\mathfrak{a}}$, *and* $\mathfrak{a}$ *is a z-ideal if and only if* $\mathfrak{a}$ *is a radical ideal. In particular, each prime ideal of* $\mathcal{S}(M)$ *is a z-ideal.*

Proof. Let $g \in \mathcal{S}(M)$ be such that $Z_M(g) \in \mathcal{Z}[\mathfrak{a}]$. Then, there exists $f \in \mathfrak{a}$ such that $Z_M(f) = Z_M(g)$ and, by 1.1, there exist $\ell \geq 1$ and $h \in \mathcal{S}(M)$ such that $g^\ell = fh \in \mathfrak{a}$, that is, $g \in \sqrt{\mathfrak{a}}$. The rest of the statement follows from 3.3 and the fact that all prime ideals are radical ideals. $\square$

Next, let us see that if M is not locally compact, Łojasiewicz's inequality 1.1 does not hold for M and, in addition, there exist prime ideals in $\mathcal{S}(M)$ which are not z-ideals. More precisely,

Proposition 3.5. Let $M \subset \mathbb{R}^n$ be a semialgebraic set which is not locally compact. Then,

(i) Łojasiewicz's inequality does not hold for M.
(ii) The ring $\mathcal{S}(M)$ has fixed prime ideals which are not z-ideals.

Before proving this, we need some preliminary results. Namely,

Lemma 3.6. *Let* $N \subset M \subset \mathbb{R}^m$ *be semialgebraic sets. Write* $Y = M \setminus N$ *and take* $b \in \mathcal{S}^*(N)$. *Let* $h \in \mathcal{S}^\diamond(M)$ *be such that* $Y \subset Z_M(h)$. *Then, the product* $(h|_N)b$ *can be continuously extended by* 0 *to a semialgebraic function belonging to* $\mathcal{S}^\diamond(M)$.

Proof. Since b is bounded on N and h vanishes identically on Y, we deduce that

$$\lim_{x\to p}(h|_N b)(x) = 0$$

for all $p \in Y \cap \operatorname{Cl}_M(N)$. Thus, $(h|_N)b$ can be continuously extended by 0 to the whole M. The graph of such extension, being the union $\operatorname{graph}(h|_N b) \cup (Y \times \{0\})$, is a semialgebraic set. This means that such extension is an element of $\mathcal{S}^\diamond(M)$. $\square$

Lemma 3.7. *Let* $N \subset M \subset \mathbb{R}^n$ *be semialgebraic sets such that* N *is closed in* M, *and let* $\mathfrak{a}$ *be a radical ideal of* $\mathcal{S}(N)$ *which is not a z-ideal. Let* $j : N \hookrightarrow M$ *be the inclusion map and let* $\phi : \mathcal{S}(M) \to \mathcal{S}(N)$, $f \mapsto f|_N = f \circ j$ *be the induced homomorphism. Then,* $\mathfrak{b} = \phi^{-1}(\mathfrak{a})$ *is a radical ideal but not a z-ideal.*

Proof. It is immediate to check that $\mathfrak{b}$ is radical, so let us prove that it is not a z-ideal. Since N is closed in M, the homomorphism ϕ is surjective, by the semialgebraic version of the Tietze–Urysohn Lemma [DK]. Suppose now, by way of contradiction, that $\mathfrak{b}$ is a z-ideal. Since $\mathfrak{a}$ is not a z-ideal, there exist $f \in \mathfrak{a}$ and $g \in \mathcal{S}(N)\setminus\mathfrak{a}$ such that $Z_N(f) \subset Z_N(g)$. Let $F, G \in \mathcal{S}(M)$ such that $\phi(F) = f$ and $\phi(G) = g$. By 2.2 there exists $H \in \mathcal{S}(M)$ such that $Z_M(H) = N$. Consider the semialgebraic functions $F_1 = F^2 + H^2$ and $G_1 = G^2 + H^2$. Then,

$$F_1|_N = f^2, \quad G_1|_N = g^2 \quad \text{and} \quad Z_M(F_1) = Z_N(f) \subset Z_N(g) = Z_M(G_1).$$

Moreover, $F_1 \in \mathfrak{b}$ because $\phi(F_1) = f^2 \in \mathfrak{a}$. Thus, $G_1 \in \mathfrak{b}$ and therefore $g^2 = \phi(G_1) \in \mathfrak{a}$. Since $\mathfrak{a}$ is radical, we conclude that $g \in \mathfrak{a}$, a contradiction. □

Now, we are ready to prove 3.5.

Proof of Proposition 3.5. By 2.9, there exists a semialgebraic subset $C \subset M$ which is closed in M, and a semialgebraic homeomorphism

$$\psi : C \to T = \{(x, y) \in \mathbb{R}^2 : 0 < y \le x \le 1\} \cup \{p = (0, 0)\}.$$

By 2.2, there exists $c \in \mathcal{S}^*(M)$ such that $Z_M(c) = C$.

(i) Consider the semialgebraic functions $g(x, y) = y$ and $h(x, y) = x^2 + y^2$ on T. Let $g_1 = g \circ \psi$, $h_1 = h \circ \psi \in \mathcal{S}(C)$. Let $G_1, H_1 \in \mathcal{S}(M)$ be semialgebraic functions which extend[1] g_1, h_1 respectively. The semialgebraic functions $G = G_1^2 + c^2$ and $H = H_1^2 + c^2$ satisfy $Z_M(G) = Z_M(H) = \{\psi^{-1}(p)\}$. Suppose, by way of contradiction, that there exist $\ell \ge 2$ and $F \in \mathcal{S}(M)$ such that $H^\ell = GF$, and so $(H|_C)^\ell = (G|_C)(F|_C)$. After composition with ψ^{-1} we deduce the existence of $f \in \mathcal{S}(T)$ such that $h^{2\ell} = g^2 f$, that is, the quotient

$$f = \frac{h^{2\ell}}{g^2} = \frac{(x^2 + y^2)^{2\ell}}{y^2}$$

is continuous on T, a contradiction. Therefore, Łojasiewicz's inequality does not hold for M.

(ii) Since C is closed, it is enough, by 3.7, to find a fixed prime ideal in $\mathcal{S}(C)$ which is not a z-ideal. Even more, the semialgebraic homeomorphism $\psi : C \to T$ induces a ring isomorphism $\psi^* : \mathcal{S}(T) \to \mathcal{S}(C)$, $f \mapsto f \circ \psi$, and $Z_T(f) = \psi(Z_C(\psi^*(f)))$ for every $f \in \mathcal{S}(T)$. Thus, we have just to prove the existence of a fixed prime ideal in $\mathcal{S}(T)$ which is not a z-ideal.

(3.5.1) We claim that:

$$\mathfrak{p} = \{f \in \mathcal{S}(T) : \exists\, \varepsilon > 0 \,|\, f \textit{ extends continuously by } 0 \textit{ to } T \cup ((0, \varepsilon] \times \{0\})\}$$

is a fixed prime ideal of $\mathcal{S}(T)$ which is not a z-ideal.

Indeed, it is clear that $\mathfrak{p}$ is closed under addition. Next, let $f \in \mathfrak{p}$ and $g \in \mathcal{S}(T)$. Since the origin $p \in T$, there exists a neighbourhood W of p in T on which g is bounded. Thus, by 3.6, there exists $\varepsilon > 0$ such that fg extends continuously by 0 to $T \cup ([0, \varepsilon] \times \{0\})$, that is, $fg \in \mathfrak{p}$, and so $\mathfrak{p} \subset \mathfrak{m}_p$ is a fixed ideal of $\mathcal{S}(T)$. Moreover, $\mathfrak{p}$ is not a z-ideal, because the semialgebraic functions $g_1 = x^2 + y^2$ and $g_2 = y$ satisfy $Z_T(g_1) = Z_T(g_2) = \{p\}$ and $g_2 \in \mathfrak{p}$ while $g_1 \notin \mathfrak{p}$.

[1]Recall the already mentioned semialgebraic version of the Tietze–Urysohn Lemma [DK].

We check now that $\mathfrak{p}$ is prime. Let $h_1, h_2 \in \mathcal{S}(T)$ such that $h_1h_2 \in \mathfrak{p}$. Since $1/(1+|h_1|)$ and $1/(1+|h_2|)$ are units in $\mathcal{S}(T)$, it is enough to check that either $f_1 = h_1/(1+|h_1|)$ or $f_2 = h_2/(1+|h_2|)$ lies in $\mathfrak{p}$. Note that both f_1 and f_2 are bounded functions.

Let $X_1 = \mathrm{Cl}_{\mathbb{R}^3}(\mathrm{graph}(f_1))$ and $X_2 = \mathrm{Cl}_{\mathbb{R}^3}(\mathrm{graph}(f_2))$, which are compact bidimensional semialgebraic sets. By [BCR, 2.8.13], each $C_i = X_i \setminus \mathrm{graph}(f_i)$ is a semialgebraic curve whose projection onto the plane $\{z = 0\}$ is the segment $(0, 1] \times \{0\}$.

By [BCR, 2.9.10], each curve $C_i \subset \mathbb{R} \times \{0\} \times \mathbb{R}$ is the disjoint union of finitely many points $p_{i\ell}$ and a finite number of Nash curves M_{ik}, each of them Nash diffeomorphic to an open interval $(0, 1)$. Note that each curve M_{ik} is either contained in a vertical line $\{(a, 0)\} \times \mathbb{R}$ or it has just finitely many points with vertical tangent. Thus, there exist just finitely many values $a \in (0, 1]$ such that the line $\{(a, 0)\} \times \mathbb{R}$ either passes through one of the points $p_{i\ell}$, or it contains some curve M_{ik}, or it is the tangent line to some M_{ik} at one of its points. Denote by J the set of such values and let $b \in (0, 1] \setminus J$. Let us see that we can extend continuously both functions f_1, f_2 to the point $(b, 0)$. Fix $i = 1, 2$ and observe that the line $\{(b, 0)\} \times \mathbb{R}$ intersects the curve C_i into finitely many points. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection onto the first two coordinates.

Let $\delta > 0$ be such that the closure $\overline{B}$ of the open ball B of center $(b, 0)$ and radius δ has the following properties:

(1) $\overline{B}_1 = \overline{B} \cap \{y \geq 0\} \subset \mathrm{Cl}_{\mathbb{R}^2}(T) \setminus \{p\}$.
(2) There exists an index k such that the closed interval $[b-\delta, b+\delta]$ is Nash diffeomorphic, via the projection onto the first coordinate, to a closed subset of the Nash curve M_{ik}.

This way, one can check that the restriction

$$\varphi = \pi|_Z : Z = \mathrm{Cl}_{\mathbb{R}^3}(\pi^{-1}(B \cap T)) \to \pi(Z) = \overline{B}_1$$

is a semialgebraic bijection and, Z being compact, φ is a semialgebraic homeomorphism. Let $q = (b, 0, s) = \varphi^{-1}(b, 0)$. It is clear that f_i can be continuously extended to the point $(b, 0)$ by setting $f_i(b, 0) = s$.

Therefore, there exists a finite set $J \subset (0, 1]$ such that both f_1 and f_2 can be continuously extended to $T \cup ((0, 1] \setminus J) \times \{0\}$. Thus, they can be continuously extended to $T \cup I_1$ for some interval $I_1 = (0, \varepsilon_1] \times \{0\}$ with $\varepsilon_1 > 0$. Since $f_1 f_2 \in \mathfrak{p}$, we may assume that $f_1 f_2$ can be continuously extended by 0 to $T \cup I_1$. By the semialgebraicity of f_1 and f_2, we may assume the existence of $\varepsilon_2 \in (0, \varepsilon_1)$ such that the continuous extension of, say f_1, to $T \cup ((0, \varepsilon_2] \times \{0\})$ vanishes identically on $(0, \varepsilon_2] \times \{0\}$, that is, $f_1 \in \mathfrak{p}$. Consequently, $\mathfrak{p}$ is a fixed prime ideal of $\mathcal{S}(T)$ which is not a z-ideal, as wanted. □

Our next aim is to develop a similar theory to approach the case of bounded semialgebraic functions. The existence of non-units in $\mathcal{S}^*(M)$ with empty zeroset will require to generalize the z-filters used above to obtain a similar Łojasiewicz's inequality. It is worthwhile mentioning that, in contrast with what happens for the ring $\mathcal{S}(M)$, this can be done without the local compactness assumption on M.

(3.8) Filters in rings of bounded semialgebraic functions. Recall that a function $f \in \mathcal{S}(M)$ is a unit if and only if $Z_M(f) = \varnothing$. However, this is not longer true in the bounded case, because given a bounded semialgebraic function with empty zeroset its inverse in $\mathcal{S}(M)$ could be unbounded. Recall that in a general commutative ring with unity an element is a unit if and only if it occurs in no maximal ideal. This leads us to handle all maximal ideals in $\mathcal{S}^*(M)$ and not only the ones corresponding to points in M. Observe that, with the notations in 2.4, a function $f \in \mathcal{S}^*(M)$ is a unit if and only if $\mathcal{Z}_{\beta_s^* M}(f) = \varnothing$. The family of all sets $\mathcal{Z}_{\beta_s^* M}(f)$ for $f \in \mathcal{S}^*(M)$ is denoted by $\mathcal{Z}_{\beta_s^* M}$. Recall that a subset $\mathcal{F}$ of $\mathcal{P}(\mathcal{Z}_{\beta_s^* M})$ is a z^*-filter on M if it satisfies the following properties:

(i) $\varnothing \notin \mathcal{F}$.
(ii) Given $Z_1, Z_2 \in \mathcal{F}$ then $Z_1 \cap Z_2 \in \mathcal{F}$.
(iii) Given $Z \in \mathcal{F}$ and $Z' \in \mathcal{Z}_{\beta_s^* M}$ such that $Z \subset Z'$ then $Z' \in \mathcal{F}$.

Let $\mathfrak{a}$ be an ideal of $\mathcal{S}^*(M)$. One can check almost straightforwardly that:

(i) *The family* $\mathcal{Z}_{\beta_s^* M}[\mathfrak{a}] = \{\mathcal{Z}_{\beta_s^* M}(f) : f \in \mathfrak{a}\}$ *is a* z^**-filter on* M.
(ii) *If* $\mathcal{F}$ *is a* z^**-filter, then* $\mathcal{J}(\mathcal{F}) = \{f \in \mathcal{S}^*(M) : \mathcal{Z}_{\beta_s^* M}(f) \in \mathcal{F}\}$ *is an ideal of* $\mathcal{S}^*(M)$ *such that* $\mathcal{Z}_{\beta_s^* M}[\mathcal{J}(\mathcal{F})] = \mathcal{F}$.

Definition 3.9. An ideal $\mathfrak{a}$ of the ring $\mathcal{S}^*(M)$ is a z^**-ideal* if $\mathcal{J}(\mathcal{Z}_{\beta_s^* M}[\mathfrak{a}]) = \mathfrak{a}$, that is, whenever there exist $f \in \mathfrak{a}$ and $g \in \mathcal{S}^*(M)$ satisfying $\mathcal{Z}_{\beta_s^* M}(f) \subset \mathcal{Z}_{\beta_s^* M}(g)$, we have $g \in \mathfrak{a}$.

Remark 3.10. Notice that the equality $\mathcal{Z}_{\beta_s^* M}[\mathcal{J}(\mathcal{F})] = \mathcal{F}$ implies that $\mathcal{J}(\mathcal{F})$ is a z^*-ideal whenever $\mathcal{F}$ is a z^*-filter. Note also that each z^*-ideal is a radical ideal because $\mathcal{Z}_{\beta_s^* M}(f) = \mathcal{Z}_{\beta_s^* M}(f^k)$ for all $f \in \mathcal{S}^*(M)$ and all $k \geq 1$.

The analogous result to 3.4 concerning bounded semialgebraic functions is the following Nullstellensatz, whose proof requires to state some preliminary results.

Corollary 3.11 (Nullstellensatz)**.** *Let* $M \subset \mathbb{R}^n$ *be a semialgebraic set and let* $\mathfrak{a}$ *be an ideal of* $\mathcal{S}^*(M)$. *Then* $\mathcal{J}(\mathcal{Z}_{\beta_s^* M}[\mathfrak{a}]) = \sqrt{\mathfrak{a}}$, *and* $\mathfrak{a}$ *is a* z^**-ideal if and only if* $\mathfrak{a}$ *is a radical ideal. In particular, each prime ideal of* $\mathcal{S}^*(M)$ *is a* z^**-ideal.*

Again, the crucial tool to prove the Nullstellensatz is a Łojasiewicz inequality that in this context takes the following formulation (equivalent to the one already stated in 1.3).

Theorem 3.12 (Łojasiewicz's inequality)**.** *Let* $M \subset \mathbb{R}^n$ *be a semialgebraic set and let* $f, g \in \mathcal{S}^*(M)$ *be such that* $\mathcal{Z}_{\beta_s^* M}(f) \subset \mathcal{Z}_{\beta_s^* M}(g)$. *Then, there exist* $h \in \mathcal{S}^*(M)$ *and a positive integer* ℓ *such that* $g^\ell = fh$. *In particular,* $|g|^\ell \leq \sup_M(|h|)|f|$ *on* M.

Remarks 3.13. (i) As we have already observed in 1.2(ii), the existence of an integer $\ell \geq 1$ and a constant $c > 0$ such that $|g|^\ell \leq cf$ on M guarantees, in our context, the existence of $h \in \mathcal{S}^*(M)$ such that $g^{2\ell+1} = hf$.

(ii) The previous result plays an important role in the study of nonrefinable chains of prime ideals in rings of bounded semialgebraic functions (see [Fe1] for further details). In fact 3.12 is crucial to prove a useful criterion of primality of ideals of $\mathcal{S}(M)$ (see [Fe1, 5.4]), strongly inspired in the corresponding result in [GJ, 2.9] concerning rings of continuous functions.

On the other hand, it follows from [BCR, 7.1.23] that given a free maximal ideal $\mathfrak{m}$ of $\mathcal{S}(M)$, the family of prime ideals of $\mathcal{S}^*(M)$ containing the prime ideal $\mathfrak{m} \cap \mathcal{S}^*(M)$ constitutes a chain, and Łojasiewicz's inequality 3.12 is an essential tool to describe the *immediate successor* of $\mathfrak{m} \cap \mathcal{S}^*(M)$, that is, the smallest prime ideal of $\mathcal{S}^*(M)$ containing properly $\mathfrak{m} \cap \mathcal{S}^*(M)$. This is done in [Fe1, §6] and it is strongly inspired in the corresponding result for rings of continuous functions developed in [M, 6] and [GJ, 14.25-27].

Assume for a while that 3.12 is proved, and let us use it to prove the Nullstellensatz 3.11 as its straightforward consequence.

Proof of Corollary 3.11. Let $g \in \mathcal{S}^*(M)$ be such that $\mathcal{Z}_{\beta_s^*M}(g) \in \mathcal{Z}_{\beta_s^*M}[\mathfrak{a}]$. Then, there exists $f \in \mathfrak{a}$ such that $\mathcal{Z}_{\beta_s^*M}(f) = \mathcal{Z}_{\beta_s^*M}(g)$. By 3.12, there exist a positive integer ℓ and $h \in \mathcal{S}^*(M)$ such that $g^\ell = fh \in \mathfrak{a}$, that is, $g \in \sqrt{\mathfrak{a}}$. The rest of the statement follows from 3.10 nd the fact that all prime ideals are radical ideals. □

Therefore, we are led to prove 3.12. The proof we present here is inspired in [BCR, 2.6.4].

Proof of Theorem 3.12. As observed in 2.1 we may assume that $M \subset \mathbb{B}_n(0,1)$. For each $u \in \mathbb{R}$, we define the semialgebraic subset $M_u = \{y \in M : u|g(y)| = 1\}$. Let us see that:

(3.12.1) If $M_u \neq \varnothing$, then $\sup\{1/|f(y)| : y \in M_u\} < +\infty$.

Otherwise, there exists a sequence $\{y_m\}_{m\geq 1} \subset M_u$ such that $\lim_{m\to+\infty} f(y_m) = 0$. Consider the graph Γ of the restriction $h = f|_{M_u} : M_u \to \mathbb{R}$. Since $M_u \subset \mathbb{B}_n(0,1)$ is a bounded subset of $\mathbb{R}^n$, its closure $\operatorname{Cl}_{\mathbb{R}^n}(M_u)$ is compact. Thus, we may assume, after substituting $\{y_m\}_{m\geq 1}$ by one of its subsequences, if necessary, that there exists $\lim_{m\to+\infty} y_m = y \in \operatorname{Cl}_{\mathbb{R}^n}(M_u)$. Note that the point $(y,0) \in \operatorname{Cl}_{\mathbb{R}^n}(\Gamma)$. Hence, by the Curve Selection Lemma [BCR, 2.5.5], there exists a semialgebraic path $\gamma : [0,1] \to \mathbb{R}^{n+1}$ such that $\gamma(0) = (y,0)$ and $\gamma((0,1]) \subset \Gamma$. For each $t \in [0,1]$ we write $\gamma(t) = (\alpha(t), \nu(t)) \in \mathbb{R}^n \times \mathbb{R}$. Then, $\alpha : [0,1] \to \mathbb{R}^n$ is a semialgebraic path such that $\alpha(0) = y$, $\alpha((0,1]) \subset M_u$ and $\nu(t) = (f\circ\alpha)(t)$ for all $t \in (0,1]$. Hence, $\lim_{t\to 0}(f\circ\alpha)(t) = 0$, that is, $f \in \mathfrak{m}_\alpha^*$. This implies, since $\mathcal{Z}_{\beta_s^*M}(f) \subset \mathcal{Z}_{\beta_s^*M}(g)$, that also $g \in \mathfrak{m}_\alpha^*$ or, equivalently, that $\lim_{t\to 0}(g\circ\alpha)(t) = 0$. But this is impossible because $|g|_{M_u}| \equiv 1/u \in \mathbb{R}$. This proves 3.12.1.

Next, consider the non necessarily continuous function,

$$v : \mathbb{R} \to [0,+\infty),\ u \mapsto v(u) = \begin{cases} 0 & \text{if } M_u = \varnothing, \\ \sup\{1/|f(y)| :\ y \in M_u\} & \text{otherwise,} \end{cases}$$

whose graph is semialgebraic. Note that the function v is identically 0 on $(-\infty,0]$. We claim that:

(3.12.2) *The restriction $v_r = v|_{[0,r]}$ is bounded for every $r > 0$.*

Indeed, assume, by way of contradiction, the existence of $r > 0$ and a sequence $\{u_m\}_{m\geq 1} \subset [0,r]$ such that $v(u_m) > m$ for all $m \geq 1$. Thus, by the definition of the function v, there exists a sequence $\{y_m\}_{m\geq 1}$ such that $1/|f(y_m)| > m$ and $y_m \in M_{u_m}$ for all $m \geq 1$. Since $\operatorname{Cl}_{\mathbb{R}^n}(M)$ is compact, we may assume that the sequence $\{y_m\}_{m\geq 1}$ converges to a point $z \in \operatorname{Cl}_{\mathbb{R}^n}(M)$, and so the sequence $\{(y_m, f(y_m))\}_{m\geq 1}$ converges to the point $(z,0)$. On the other hand, since $[0,r]$ is compact, we may assume that $\{u_m\}_{m\geq 1}$ converges to a point $a \in [0,r]$, and $|g(y_m)|u_m = 1$ because $y_m \in M_{u_m}$. Therefore,

$$\lim_{m\to+\infty} |g(y_m)| = \lim_{m\to+\infty} \frac{1}{u_m} = \frac{1}{a}.$$

Since g is bounded, a cannot be 0 and this way the previous limit is well-defined. Next, consider the semialgebraic set

$$T = \{(u, y, f(y)) \in [0, r] \times M \times \mathbb{R} : \, u|g(y)| = 1\}.$$

The points $(u_m, y_m, f(y_m)) \in T$, and so $(a, z, 0) \in \mathrm{Cl}_{\mathbb{R}^n}(T)$. By the Curve Selection Lemma [BCR, 2.5.5] there exists a semialgebraic path $\varphi = (\rho, \eta, \mu) : [0, 1] \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ such that $\varphi(0) = (\rho(0), \eta(0), \mu(0)) = (a, z, 0)$ and

$$\varphi|_{(0,1]} = \Big(\frac{1}{|(g \circ \eta)|_{(0,1]}|}, \eta|_{(0,1]}, (f \circ \eta)|_{(0,1]}\Big).$$

Therefore, $\lim_{t\to 0}(f \circ \eta)(t) = 0$, that is, $f \in \mathfrak{m}^*_\eta$, and so $g \in \mathfrak{m}^*_\eta$, because $\mathcal{Z}_{\beta^*_s M}(f) \subset \mathcal{Z}_{\beta^*_s M}(g)$. This means $\lim_{t\to 0}(g \circ \eta)(t) = 0$, which is impossible, because

$$\lim_{t\to 0} \frac{1}{|(g \circ \eta)(t)|_{(0,1]}|} = a \in \mathbb{R}.$$

This proves 3.12.2.

(3.12.3) On the other hand, by [BCR, 2.6.1], there exist $c, s \in \mathbb{R}$ and a positive integer $p \geq 1$ such that $v(u) \leq cu^p$ for every u such that $|u| \geq s$ and, as we have just seen, there exists $L > 0$ such that $0 \leq v|_{[-s,s]} \leq L$. Now, let us prove that the function

$$h_1 : M \to \mathbb{R}, \; y \mapsto h_1(y) = \begin{cases} g^p(y)/f(y) & \text{if } y \in D_M(f), \\ 0 & \text{if } y \in Z_M(f), \end{cases}$$

is bounded. Of course, it is enough to check that h_1 is bounded on $D_M(f)$. Let $y_0 \in D_M(f)$. If $g(y_0) = 0$, then $h_1(y_0) = 0$. Thus, we may assume that $g(y_0) \neq 0$, and denote $u_0 = 1/|g(y_0)|$. Suppose first that $|g(y_0)| \leq 1/s$ or, equivalently, $u_0 \geq s$. Then,

$$\Big|\frac{g^p(y_0)}{f(y_0)}\Big| \leq \frac{1}{u_0^p} \sup\{1/|f(y)| : \, y \in M_{u_0}\} = \frac{v(u_0)}{u_0^p} \leq c.$$

Suppose now that $|g(y_0)| > 1/s$, that is, $u_0 < s$. Then,

$$\begin{aligned}\Big|\frac{g^p(y_0)}{f(y_0)}\Big| &\leq |g(y_0)|^p \sup\{1/|f(y)| : \, y \in M_{u_0}\} \\ &= |g(y_0)|^p v(u_0) \leq \sup\{|g(y)|^p : \, y \in M\} \cdot L.\end{aligned}$$

Since g is bounded, we conclude that also the function h_1 is bounded. Therefore, by 3.6, $h = gh_1 \in \mathcal{S}^*(M)$, because $Z_M(f) \subset Z_M(g)$ since $\mathcal{Z}_{\beta^*_s M}(f) \subset \mathcal{Z}_{\beta^*_s M}(g)$. Finally, if $\ell = p + 1$ we get $g^\ell = fgh_1 = fh$, as wanted. □

Remark 3.14. The proof above shows that to get an equality of the form $g^\ell = fh$, it suffices to require that $g \in \mathfrak{m}^*_\alpha$ for each semialgebraic path $\alpha : (0, 1] \to M$ such that $f \in \mathfrak{m}^*_\alpha$.

We finish this work with an alternative proof of 1.1 obtained as an almost straightforward consequence of 3.12 and 3.14. Namely,

Alternative proof of Theorem 1.1. First, since M is locally compact, it is locally closed (see 2.8) and so, by [BCR, 2.2.9], M can be embedded in some $\mathbb{R}^m$ as a closed semialgebraic subset. Thus, in what follows, we assume that $M \subset \mathbb{R}^n$ is a closed semialgebraic subset of $\mathbb{R}^n$. Next, let $f, g \in \mathcal{S}(M)$ such that $Z_M(f) \subset Z_M(g)$ and consider the bounded semialgebraic functions on M

$$f_1 = \frac{f}{(1+\|x\|)(1+|f|)} \in \mathcal{S}^*(M) \quad \text{and} \quad g_1 = \frac{g}{(1+\|x\|)(1+|g|)} \in \mathcal{S}^*(M).$$

Taking 3.14 into account, to apply 3.12 to f_1 and g_1 it is enough to check that $g_1 \in \mathfrak{m}^*_\alpha$ for each semialgebraic path $\alpha : (0,1] \to M$ such that $f_1 \in \mathfrak{m}^*_\alpha$. Indeed, let $\alpha : (0,1] \to M$ be a semialgebraic path such that $f_1 \in \mathfrak{m}^*_\alpha$. If $\mathfrak{m}^*_\alpha$ is a fixed maximal ideal of $\mathcal{S}^*(M)$, there exists a point $p \in M$ such that $\mathfrak{m}^*_\alpha = \mathfrak{m}^*_p$, and

$$0 = f_1(p) = \frac{f(p)}{(1+\|p\|)(1+|f(p)|)}$$

because $f_1 \in \mathfrak{m}^*_p$. Hence, $f(p) = 0$ and so $g(p) = 0$, because $Z_M(f) \subset Z_M(g)$. Thus,

$$g_1(p) = \frac{g(p)}{(1+\|p\|)(1+|g(p)|)} = 0,$$

that is, $g_1 \in \mathfrak{m}^*_p = \mathfrak{m}^*_\alpha$.

Next, if $\mathfrak{m}^*_\alpha$ is a free ideal, the semialgebraic path $\alpha : (0,1] \mapsto M$ cannot be extended to a continuous semialgebraic path $[0,1] \mapsto M$ (see 2.5). Since M is closed in $\mathbb{R}^n$ this implies that α cannot be extended to a semialgebraic path $[0,1] \mapsto \mathbb{R}^n$. Thus, by 2.6, the semialgebraic function $\|\alpha\| : (0,1] \to \mathbb{R}$ is unbounded. On the other hand, the semialgebraic function $\frac{1}{1+\|\alpha\|} : (0,1] \to \mathbb{R}$ being bounded, there exists, by 2.6, the limit $\lim_{t\to 0} \frac{1}{1+\|\alpha\|} = c \in \mathbb{R}$. In fact $c = 0$, because $\|\alpha\| : (0,1] \to \mathbb{R}$ is unbounded. Thus, using 2.6 once more,

$$\lim_{t\to 0} (g_1 \circ \alpha)(t) = \lim_{t\to 0} \Big(\frac{1}{1+\|\alpha(t)\|}\Big)\Big(\frac{g(\alpha(t))}{1+|g(\alpha(t))|}\Big) = 0.$$

Therefore, also $g_1 \in \mathfrak{m}^*_\alpha$ and, by 3.12 and 3.14, there exist $h_1 \in \mathcal{S}^*(M)$ and a positive integer $\ell \geq 1$ such that $g_1^\ell = f_1 h_1$. Hence, $g^\ell = fh$, where

$$h = h_1 \frac{(1+\|x\|)^{\ell-1}(1+|g|)^\ell}{1+|f|} \in \mathcal{S}(M),$$

and we are done. □

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