Parabolicity and Cheeger's constant on graphs

Álvaro Martínez-Pérez[1*] and José M. Rodríguez[2†]

[1*]Departamento de Análisis Económico y Finanzas, Universidad de Castilla-La Mancha, Avda. Real Fábrica de Seda, s/n., Talavera de la Reina, 45600, Toledo, Spain. ORCID 0000-0002-1344-6189.
[2]Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, Leganés, 28911, Madrid, Spain. ORCID 0000-0003-2851-7442.

*Corresponding author(s). E-mail(s): alvaro.martinezperez@uclm.es;
Contributing authors: jomaro@math.uc3m.es;
†These authors contributed equally to this work.

Abstract

Herein we study p-parabolicity on graphs. We prove that if a uniform graph satisfies the (Cheeger) isoperimetric inequality, then it is non-p-parabolic for every $1 < p < \infty$. Moreover, we give sufficient conditions for a uniform graph to be non-parabolic and to be p-parabolic for every $1 < p < \infty$.

Keywords: Parabolicity, p-parabolicity, quasi-isometries, isoperimetric inequality.

MSC Classification: Primary 53C21 , 53C23; Secondary 58C40. 30F45 , 53C20 , 30F20 , 31C12.

1 Introduction

Quasi-isometries preserve many of the large scale properties of metric spaces, starting with Gromov hyperbolicity (see, e.g., [18], [15]). M. Kanai proved that a useful strategy to study large scale properties in manifolds is to consider a quasi-isometric discrete approximation. In [22], [23], [24], he studied several geometric properties (such as volume growth rate, isoperimetric inequalities, Liouville type theorems, Poincaré-Sobolev inequalities and parabolicity) for a large class of Riemannian manifolds with certain conditions on their local geometry. Kanai proved that these properties are preserved under quasi-isometries between these manifolds and between a manifold and its approximating graph.

Kanai's ideas have inspired many works relating the large scale behavior of a manifold and some associated graph or allowing to characterize some large scale properties using some graph approximation (see, e.g., [1], [2], [6], [14], [17], [27], [28], [29], [32], [33], [34], [36], [37], [40]).

Quasi-isometries also preserve the parabolic Harnack inequality (see [11]) and several estimates on transition probabilities of random walks, such as heat kernel estimates. Also, the existence of non-trivial solutions of a wide class of partial differential equations is invariant under quasi-isometry (see, e.g., [20], [21], [39]).

A manifold M is said to be p-parabolic if all positive p-superharmonic functions on M are constant. This is equivalent to not having p-Green's function (i.e. a positive fundamental solution of the p-Laplace-Beltrami operator). The classical definition of parabolicity is just the case where $p = 2$. It is a classic problem in complex analysis to give criteria for a Riemannian surface to be parabolic. For a deeper exposition see the book by Sario and Nakai [38].

In [30] we studied the stability of p-parabolicity (with $1 < p < \infty$) by quasi-isometries between Riemannian manifolds weakening Kanai's assumptions. Also, we obtained some results on the p-parabolicity of graphs and trees; in particular, we characterized p-parabolicity for a large class of trees. Then, in [31], we continued the study of p-parabolicity on (uniform) graphs considering the behavior of p-parabolicity through certain decompositions or vertex identifications. We also gave necessary and sufficient conditions for a uniform hyperbolic graph to be p-parabolic in terms of the boundary at infinity of the graph. Finally, we proved that Cheeger isoperimetric inequality implies non-p-parabolicity in uniform hyperbolic graphs.

The main result of this paper extends this last result proving that Cheeger isoperimetric inequality implies non-p-parabolicity for every uniform graph, see Section 4. To obtain this, first in Section 3 we need to extend some known results relating the Cheeger isoperimetric constant and the bottom of the spectrum of the p-Laplacian and prove that if M is a complete Riemannian manifold with isoperimetric inequality, then M is non-p-parabolic for any $1 < p < \infty$; this result, which is interesting by itself, was already known for the case $p = 2$ (see [10], [13]).

Finally, in Section 5 we give sufficient conditions for a (uniform) graph to be non-parabolic and p-parabolic.

2 Definitions and background

A function between two metric spaces $f : X \to Y$ is said to be an (a, b)-*quasi-isometric embedding* with constants $a \geq 1$, $b \geq 0$, if

$$\frac{1}{a}\, d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a\, d_X(x_1, x_2) + b\,, \qquad \text{for every } x_1, x_2 \in X.$$

Such a quasi-isometric embedding f is a *quasi-isometry* if there exists a constant $c \geq 0$ such that f is *c-full*, i.e., if for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \leq c$.

Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry between them. It is well-known that to be quasi-isometric is an equivalence relation (see, e.g., [22]).

Given a complete Riemannian manifold M and a relatively compact domain with smooth boundary $\Omega \subset M$, define

$$\operatorname{cap}_p \Omega = \operatorname{cap}_p(\Omega, M) = \inf \Big\{ \int_M |\nabla u|^p \,:\, u \in C_c^\infty(M),\ u|_\Omega = 1 \Big\}.$$

A useful characterization of the existence of p-Green's function is:

Theorem 1 *Given $1 < p < \infty$, a complete Riemannian manifold is p-parabolic if and only if $\operatorname{cap}_p \Omega = 0$ for some (and then for every) relatively compact domain with smooth boundary $\Omega \subset X$.*

The proof of Theorem 1 appears in [23] for $p = 2$ and in [19] for $1 < p < \infty$.

Given a function u on the set of vertices $V(G)$ of a graph G, define the *p-modulus of its discrete gradient* $|\nabla_G u|_p$ and its *discrete p-Dirichlet integral* $D_{p,G}(u)$, respectively, by

$$|\nabla_G u|_p(x) := \Big(\sum_{y \in N(x)} |u(y) - u(x)|^p \Big)^{1/p},$$

$$D_{p,G}(u) := \sum_{x \in V(G)} |\nabla_G u|_p^p(x) = 2 \sum_{vw \in E(G)} |u(v) - u(w)|^p,$$

where the edges are considered non-oriented.

Note that the definition of discrete p-Dirichlet integral in [20] is slightly different, but both are equivalent.

For a finite subset S of $V(G)$, the *p-capacity* of S is defined by

$$\operatorname{cap}_p S = \operatorname{cap}_p(S, G) = \inf \Big\{ D_{p,G}(u) : u \text{ function on } V(G) \text{ with finite support},\ u|_S = 1 \Big\}.$$

A graph G is said to be μ-*uniform* if each vertex p of $V(G)$ has at most μ neighbors, i.e.,

$$\sup\left\{|N(p)|\,:\,p\in V(G)\right\}\le\mu.$$

If a graph G is μ-uniform for some constant μ we say that G is *uniform*.

Theorem 2 *Given $1<p<\infty$, a uniform graph G is p-parabolic if and only if $\operatorname{cap}_p S=0$ for some (and then for every) non-empty finite subset of $S\subset V(G)$.*

For a proof of Theorem 2, see [24, Proposition 6] and [20, Final remark 5.16].

Corollary 1 Every finite graph is p-parabolic for every $1<p<\infty$.

The following result is an interesting remark.

Proposition 3 *Let $1<p<\infty$ and let G be a uniform graph. If G is p-parabolic, then it is p'-parabolic for every $p'\ge p$.*

Proof Fix a finite subset S of $V(G)$. We claim that

$$\operatorname{cap}_p S=\inf\left\{D_{p,G}(u) : u \text{ function on } V(G) \text{ with finite support, } u|_S=1,\ 0\le u\le 1\right\}.$$

Let us consider a function u on $V(G)$ with finite support and $u|_S=1$. Define the function u_0 on $V(G)$ by

$$u_0(x):=\begin{cases} u(v) & \text{if } 0\le u(v)\le 1,\\ 0 & \text{if } u(v)<0,\\ 1 & \text{if } u(v)>1.\end{cases}$$

Thus, $0\le u_0\le 1$ and $|u_0(v)-u_0(w)|\le|u(v)-u(w)|$ for every $v,w\in V(G)$. Hence, $D_{p,G}(u_0)\le D_{p,G}(u)$ and the claim holds.

If G is p-parabolic, then Theorem 2 implies $\operatorname{cap}_p S=0$. Let us consider a function u on $V(G)$ with finite support, $0\le u\le 1$ and $u|_S=1$. Since $|u(v)-u(w)|\le 1$ for every $v,w\in V(G)$, we have $|u(v)-u(w)|^{p'}\le|u(v)-u(w)|^p$ for every $p'\ge p$. Therefore, $D_{p',G}(u)\le D_{p,G}(u)$ and $0\le\operatorname{cap}_{p'} S\le\operatorname{cap}_p S=0$. Hence, $\operatorname{cap}_{p'} S=0$ and Theorem 2 implies that G is p'-parabolic for every $p'\ge p$. □

Also, Corollary 7 in [23] can be trivially extended to the general case to obtain the following:

Proposition 4 *If two uniform graphs P and Q are quasi-isometric, then P is p-parabolic if so is Q.*

3 Eigenvalues and non-parabolicity

Let $1 < p < \infty$, and assume that M is a complete Riemannian manifold. The bottom of the spectrum of the p-Laplacian is

$$\lambda_p(M) := \inf_{v \in W_0^{1,p} \setminus \{0\}} \frac{\int_M |\nabla v|^p}{\int_M |v|^p} .$$

Given a Riemannian manifold M, the *linear isoperimetric constant* or *Cheeger's constant* $h(M)$ is defined by

$$h(M) = \inf \left\{ \frac{|\partial\Omega|}{|\Omega|} \ : \ \Omega \subset M \text{ is a nonempty relatively compact domain in } M \right\}.$$

Throughout, $|\Omega|$ (respectively, $|\partial\Omega|$) refers to Riemannian volume of Ω (respectively, $(n-1)$-Riemannian volume of $\partial\Omega$).

A manifold M satisfies the (Cheeger) *isoperimetric inequality* if $h(M) > 0$, since this means that

$$|\Omega| \le h(M)^{-1} |\partial\Omega|$$

for every nonempty relatively compact domain Ω in M.

Definition 1 The *combinatorial Cheeger isoperimetric constant* of a graph G is defined to be

$$h(G) = \inf_U \frac{|\partial U|}{|U|} ,$$

where U ranges over all non-empty finite subsets of vertices in G,

$$\partial U = \{ v \in G \,|\, d_G(v, U) = 1 \}$$

and $|U|$ denotes the cardinality of the set U.

A graph G satisfies the (Cheeger) *isoperimetric inequality* if $h(G) > 0$, since this means that

$$|U| \le h(G)^{-1} |\partial U|$$

for every finite set of vertices U.

Inspired by the proof of [26, Appendix] for the compact case (see also [25, Theorem 3]) we can prove the following result.

Theorem 5 *If M is a complete Riemannian manifold and $1 < p < \infty$, then*

$$\lambda_p(M) \ge \left(\frac{h(M)}{p} \right)^p .$$

Proof First of all consider a non-negative function $u \in C_c^\infty(M)$. Define

$$W_t := \{ x \in M : u(x) > t \}.$$

The co-area formula and Fubini's theorem imply

$$\int_M |\nabla u| = \int_{-\infty}^{\infty} |\partial W_t|\, dt \geq \int_{-\infty}^{\infty} h(M)\, |W_t|\, dt = h(M) \int_M |u|\,. \tag{1}$$

Since $W_0^{1,1}(M)$ is the closure of $C_c^\infty(M)$, this previous inequality (1) holds for every $u \in W_0^{1,1}(M)$.

For each $1 < p < \infty$ and $v \in W_0^{1,p}(M)$, consider $w = |v|^{p-1}v$. Thus, $\|w\|_1 = \|v\|_p^p$ and Hölder inequality gives

$$\begin{aligned}\int_M |\nabla w| &= p \int_M |v|^{p-1} |\nabla v| \\ &\leq p \Big(\int_M \big(|v|^{p-1}\big)^{p/(p-1)} \Big)^{(p-1)/p} \Big(\int_M |\nabla v|^p \Big)^{1/p} \\ &= p\, \|v\|_p^{p-1} \|\nabla v\|_p\end{aligned}$$

and so, $w \in W_0^{1,1}(M)$. Hence, we can apply (1)

$$\int_M |\nabla w| \geq h(M) \int_M |w| = h(M) \int_M |v|^p\,.$$

These two last inequalities imply

$$h(M) \leq \frac{\int_M |\nabla w|}{\int_M |v|^p} \leq \frac{p\, \|v\|_p^{p-1} \|\nabla v\|_p}{\int_M |v|^p} = \frac{p\, \|\nabla v\|_p}{\|v\|_p}$$

and this finishes the proof. □

Theorem 6 *If M is a complete Riemannian manifold with isoperimetric inequality, then M is non-p-parabolic for any $1 < p < \infty$. In fact,*

$$\operatorname{cap}_p \Omega \geq |\Omega|\, \lambda_p(M) \geq |\Omega| \left(\frac{h(M)}{p} \right)^p$$

for any relatively compact domain with smooth boundary $\Omega \subset M$.

Proof Since M satisfies the isoperimetric inequality, $h(M) > 0$ and Theorem 5 gives $\lambda_p(M) > 0$.

Fix $1 < p < \infty$, a relatively compact domain with smooth boundary $\Omega \subset M$ and any function $u \in C_c^\infty(M)$ with $u|_\Omega = 1$. We have

$$0 < \lambda_p(M) \leq \frac{\int_M |\nabla u|^p}{\int_M |u|^p} \leq \frac{\int_M |\nabla u|^p}{\int_\Omega |u|^p} \leq \frac{\int_M |\nabla u|^p}{|\Omega|}$$

and so, $\int_M |\nabla u|^p \geq |\Omega| \lambda_p(M)$. Since u is any fixed function in $C_c^\infty(M)$ with $u|_\Omega = 1$, we conclude that

$$\operatorname{cap}_p \Omega \geq |\Omega| \lambda_p(M) \geq |\Omega| \left(\frac{h(M)}{p} \right)^p > 0.$$

Hence, Theorem 1 gives that M is non-p-parabolic for any $1 < p < \infty$. □

4 Cheeger isoperimetric inequality and *p*-parabolicity on graphs

Let us recall the following result in [4, Theorem 19].

Theorem 7 *Every uniform graph G is quasi-isometric to a 3-regular graph.*

The idea is to transform locally the graph as it is shown in figure 1.

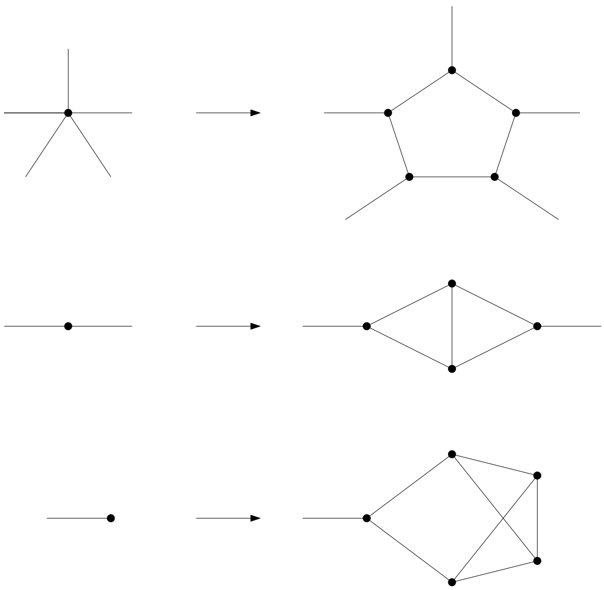

Fig. 1 Given any uniform graph G the following local transformations generate a 3-regular graph G'.

A Riemannian n-manifold M has *bounded local geometry* if there exist positive constants r, c, such that for every $x \in M$ there is a diffeomorphism $F : B(x, r) \to \mathbb{R}^n$ with

$$\frac{1}{c}\, d(x_1, x_2) \leq \|F(x_1) - F(x_2)\| \leq c\, d(x_1, x_2)$$

for every $x_1, x_2 \in B(x, r)$.

The *injectivity radius* $\operatorname{inj}(x)$ *of* $x \in M$ is the largest radius for which the exponential map at x is a diffeomorphism. If M has non-positive sectional curvatures, then the injectivity radius can be defined, also, as the supremum of those $r > 0$ such that the ball $B(p, r)$ is simply connected or, equivalently, as half the infimum of the lengths of the (homotopically non-trivial) loops based at x. The *injectivity radius* $\operatorname{inj}(M)$ *of* M is the infimum over $x \in M$ of $\operatorname{inj}(x)$.

Remark 1 If M has positive injectivity radius and a lower bound on its Ricci curvature, then M has bounded local geometry [3]. The arguments in the proofs of the results in [22], [23] and [24] for manifolds with positive injectivity radius and a lower bound on its Ricci curvature allows to obtain the same conclusions for manifolds with bounded local geometry.

A subset A in a metric space (X, d) is called ε-*separated*, $\varepsilon > 0$, if $d(a, a') \geq \varepsilon$ for any distinct $a, a' \in A$. Note that if A is maximal with this property, then the union $\cup_{a \in A} B_\varepsilon(a)$ covers X. A maximal ε-separated set A in a metric space X is called an ε-*approximation* of X.

Let X be a complete Riemannian manifold and denote by d the induced metric. Given any ε-approximation A of X, the graph $\Gamma_A = (A, E)$ with $E := \{xy \,|\, x, y \in A \text{ with } 0 < d(x, y) \leq 2\varepsilon\}$ is called an ε-*net*.

The following results are trivial extensions of [23, Theorems 2 and Corollary 7] for the general case of p-parabolicity.

Theorem 8 *Given $1 < p < \infty$ and* $\mathbf{e}{>}0$, *let X be a complete Riemannian manifold with bounded local geometry, and let P be an ε-net in X. Then X is p-parabolic if and only if P is p-parabolic.*

Theorem 9 *Suppose that P and Q are uniform graphs quasi-isometric to each other (with respect to their combinatorial metrics). Then P is p-parabolic if so is Q for any fixed $1 < p < \infty$.*

Kanai proved in [22] that quasi-isometries also preserve isoperimetric inequalities between Riemannian manifolds with bounded local geometry and uniform graphs. This result also holds with weaker hypotheses in the context of Riemann surfaces, see [7], [16], [29]. As a particular case, we can state the following:

Theorem 10 *Let X be a manifold with bounded local geometry, G be a uniform graph and suppose X and G are quasi-isometric. Then, $h(X) > 0$ if and only if $h(G) > 0$.*

A *Y-piece* (or a pair of pants) is a compact bordered Riemann surface with curvature $K = -1$ which is topologically a sphere without three disks and whose border is the union of three simple closed geodesics. Given three positive numbers a, b, c, there is a unique (up to conformal mapping) Y-piece such that its boundary curves have lengths a, b, c (see, e.g., [35, p.410]). Y-pieces are a standard tool for constructing Riemann surfaces (see [8, Chapter X.3] and [5, Chapter 1]).

Theorem 11 *If G is a uniform graph with isoperimetric inequality, then G is non-p-parabolic for every $1 < p < \infty$.*

Proof By Theorem 7, we can consider a 3-regular graph G' quasi-isometric to G. Let Y_0 be a Y-piece with boundary curves of length 1 (and constant curvature $K = -1$). Now, let us build a surface X by considering for each vertex in G' a Y-piece isometric to Y_0, and pasting these Y-pieces with the same structure of G' (two boundaries are identified if and only if the corresponding vertices are adjacent). One can check that X is quasi-isometric to G'. Since Y_0 is a compact surface, $\text{inj}(Y_0) > 0$ and so, $\text{inj}(X) > 0$. Also, X has constant curvature $K = -1$ and so, X has bounded local geometry. Therefore, by Theorem 10, $h(X) > 0$ and by Theorem 6, X is non-p-parabolic for any $1 < p < \infty$. Thus, by theorems 8 and 9, it follows that G is non-p-parabolic for any $1 < p < \infty$. □

In [13] the authors proved that for a hyperbolic Riemann surface S (with constant curvature $K = -1$), the bottom of the spectrum of the Laplacian is greater than 0 if and only if $h(S) > 0$. Thus, it is a natural question if for hyperbolic graphs, or at least for trees, the converse of Theorem 11 could be true. The answer is negative as we can see from the following result for Cantor trees.

Given two sequences of positive integers $L = \{\ell_n\}_{n=1}^{\infty}$ and $R = \{r_n\}_{n=1}^{\infty}$, with $2 \leq r_n \leq N$ for every $n \geq 1$ and some constant N, the *Cantor tree* $(T_{L,R}, v_0)$ is a rooted tree such that the root, v_0, has degree r_1, the vertices at distance $\ell_1 + \cdots + \ell_{n-1}$ have degree $r_n + 1$, and any other vertex has degree two. Note that $(T_{L,R}, v_0)$ is uniform since $R = \{r_n\}_{n=1}^{\infty}$ is a bounded sequence.

The *Cantor tree* (T_C, v_0) is a rooted tree such that the root, v_0, has degree two and any other vertex has degree three, i.e., $(T_C, v_0) = (T_{L,R}, v_0)$ with $\ell_n = 1$ and $r_n = 2$ for every $n \geq 1$.

Theorem 12 *[30, Theorem 4.21] Given $1 < p < \infty$ and sequences $L = \{\ell_n\}_{n=1}^{\infty}$ and $R = \{r_n\}_{n=1}^{\infty}$, the Cantor tree $(T_{L,R}, v_0)$ is p-parabolic if and only if*

$$\sum_{k=1}^{\infty} \frac{\ell_k}{(r_1 \cdots r_k)^{1/(p-1)}} = \infty.$$

Theorem 12 allows to show the following non-p-parabolic graph (for every $1 < p < \infty$) without isoperimetric inequality.

Example 1 Consider the Cantor tree $(T_{L,R}, v_0)$ with $\ell_k = k$ and $r_k = 2$ for every $k \in \mathbb{N}$. Thus, by Theorem 12, to see that this tree is non-p-parabolic for every $1 < p < \infty$ it suffices to check that

$$\sum_{k=1}^{\infty} \frac{k}{(2^k)^{1/(p-1)}} = \sum_{k=1}^{\infty} \frac{k}{(2^{1/(p-1)})^k} < \infty.$$

This follows immediately from D'Alembert criterion since

$$\lim_{k\to\infty} \frac{k+1}{k \cdot 2^{1/(p-1)}} = \frac{1}{2^{1/(p-1)}}$$

and

$$p > 1 \quad \Rightarrow \quad \frac{1}{p-1} > 0 \quad \Rightarrow \quad 2^{1/(p-1)} > 1.$$

However, by construction, $(T_{L,R}, v_0)$ contains arbitrarily long path graphs where every vertex has degree two. Thus, we can find subsets $C_k \subset (T_{L,R}, v_0)$ with $|C_k| = k$ for k arbitrarily large and so that $|\partial C_k| = 2$. Therefore, $h(T_{L,R}, v_0) = 0$.

5 Sufficient conditions for non-parabolicity and p-parabolicity on graphs

Let us recall the following result from [12]:

Given a Riemannian manifold M, define

$$\varphi_M(t) = \inf\{|\partial\Omega| : \ \Omega \text{ is a smooth relatively compact domain in } M \text{ with } |\Omega| \geq t\}.$$

Theorem 13 *[12] If M is a complete Riemannian manifold satisfying $\int^{|M|} \frac{dt}{\varphi_M(t)^2} < \infty$, then M has a Green's function (i.e., it is non-parabolic).*

Recall that every finite graph is p-parabolic for every $1 < p < \infty$, by Corollary 1. Given an infinite uniform graph G, let us define:

$$\varphi_G(t) = \inf\{|\partial A| : \ A \subset V(G) \text{ with } |A| \geq t\},$$

where $|A|$ denotes the cardinality of A.

Then, let us prove the following discrete version of Theorem 13:

Theorem 14 *If G is an infinite uniform graph such that*

$$\sum_{k=1}^{\infty} \frac{1}{\varphi_G(k)^2} < \infty,$$

then G is non-parabolic.

Proof Let $\mu \in \mathbb{Z}^+$ such that G is μ-uniform. By Theorem 7, there is a 3-regular graph G' which is quasi-isometric to G. Moreover, from the construction in [4] (see Figure 1) it follows that every vertex in G' corresponds to a vertex in G and a vertex v in G gives rise to $k \leq \mu$ vertices in G' if $\deg(v) = k > 2$, 4 vertices in G' if $\deg(v) = 2$ and 5 vertices in G' if $\deg(v) = 1$. Thus, given $v \in V(G)$, let us define the equivalence class $[v]$ in $V(G')$ as the vertices corresponding to v in the construction of G'. Then, the cardinal of every class satisfies that $|[v]| \leq n_0 = \max\{\mu, 5\}$.

Let A' be any finite subset of $V(G')$ and let

$$A := \{v \in V(G) \,|\, v' \in [v] \text{ for some } v' \in A'\}.$$

Since $|[v]| \leq n_0$, we have $\frac{1}{n_0}|A'| \leq |A| \leq |A'|$.

Also, notice that if a vertex w in ∂A is adjacent to some vertex u in A one of the following occurs:

- there exist two adjacent vertices in $V(G')$, $w' \in [w]$ and $u' \in [u] \cap A'$, and thus $w' \in [w] \cap \partial A'$.
- there exist two adjacent vertices in $V(G')$, $u_1' \in [u] \cap A'$, $u_2' \in [u] \setminus A'$, and thus $u_2' \in [u] \cap \partial A'$.

In the first case, different vertices w_1, w_2 in ∂A induce two different points w_1', w_2' in $\partial A'$. In the second case, two different vertices w_1, w_2 in ∂A may induce the same vertex $u_2' \in \partial A'$ only if w_1, w_2 are both adjacent to the same vertex u. Since G is μ-uniform, it follows that $|\partial A| \le \mu\, |\partial A'|$. Thus,

$$|A'| \ge k \quad \Rightarrow \quad |A| \ge \frac{k}{n_0} \quad \Rightarrow \quad \mu\, |\partial A'| \ge |\partial A| \ge \varphi_G\Big(\frac{k}{n_0}\Big).$$

Then,

$$\varphi_{G'}(k) \ge \frac{1}{\mu}\, \varphi_G\Big(\frac{k}{n_0}\Big),$$

and since for each $m \in \mathbb{N}$ and each $(m-1)\cdot n_0 < j \le m\cdot n_0$ we have $\varphi_G\big(\frac{j}{n_0}\big) = \varphi_G(m)$, we conclude that

$$\sum_{k=1}^{\infty} \frac{1}{\varphi_{G'}(k)^2} \le \sum_{k=1}^{\infty} \frac{\mu^2}{\varphi_G(\frac{k}{n_0})^2} = n_0 \sum_{k=1}^{\infty} \frac{\mu^2}{\varphi_G(k)^2} < \infty.$$

Let Y_0 be a Y-piece with boundary curves of length 1 (and constant curvature $K = -1$). Now, let us build a surface X by considering for each vertex in G' a Y-piece isometric to Y_0, and pasting these Y-pieces with the same structure of G' (two boundaries are identified if and only if the corresponding vertices are adjacent). One can check that X is quasi-isometric to G'. Since Y_0 is a compact surface, $\operatorname{inj}(Y_0) > 0$ and so, $\operatorname{inj}(X) > 0$. Also, X has constant curvature $K = -1$ and so, X has bounded local geometry. Since G is an infinite graph and every Y-piece has area 2π, we have $|X| = \infty$.

Let us define

$$\phi_X(t) = \inf\{|\partial\hat{\Omega}| : \ |\hat{\Omega}| \ge t\},$$

where $\hat{\Omega} = \cup_{i\in I} Y_i$ is connected, for some finite family $\{Y_i\}_{i\in I}$ of Y-pieces from the construction of X. If $|\hat{\Omega}| \ge t$ with $2\pi k \le t < 2\pi(k+1)$, then $\hat{\Omega}$ is the union of k Y-pieces, and since every boundary curve of these Y-pieces has length 1, then $|\partial\hat{\Omega}| \ge \varphi_{G'}(k)$ and so, $\phi_X(t) \ge \varphi_{G'}(k)$ and

$$\int_{2\pi}^{\infty} \frac{dt}{\phi_X(t)^2} \le \sum_{k=1}^{\infty} \frac{1}{\varphi_{G'}(k)^2} < \infty.$$

From this, we are going to prove

$$\int_{2\pi}^{\infty} \frac{dt}{\varphi_X(t)^2} < \infty.$$

Let Ω be a nonempty relatively compact domain in X with $|\Omega| \ge t$. We are going to find a lower bound for $|\partial\Omega|$.

We say that a domain is doubly connected if its fundamental group is isomorphic to $\mathbb{Z}$.

Assume that Ω is either a simply or doubly connected domain. Thus, [13, Lemma 1.1] gives $|\partial\Omega| \ge |\Omega|$ and so, $|\partial\Omega| \ge t$.

Assume that $\partial\Omega$ is the union of the simple closed curves $g_1, \dots, g_m$. If every g_j is homotopically trivial, then we can remove $m-1$ boundary curves obtaining a simply connected domain with more area and less perimeter. If some g_j is homotopically non-trivial, then we can remove the trivial curves in $\partial\Omega$ obtaining a new domain with more area and less perimeter; if this new domain has either one or two boundary curves, we have proved that $|\partial\Omega| \geq t$. Hence, we can assume that each g_j is homotopically non-trivial and that $m \geq 3$.

Denote by I_1 the set of indices i such that $Y_i \cap \Omega \neq \emptyset$. For each $1 \leq j \leq m$, let A_j be the set of indices $i \in I_1$ such that $Y_i \cap g_j \neq \emptyset$, and let T_j be the set of indices $i \in A_j$ such that $g_j \cap Y_i$ does not contain a curve joining two different simple closed geodesics in ∂Y_i.

Claim 1. If $i \in A_j \setminus T_j$ for some $1 \leq j \leq m$, then $L(g_j \cap Y_i) \geq c_1$ for some universal constant c_1.

In order to prove Claim 1, denote by $\gamma_1, \gamma_2, \gamma_3$ the simple closed geodesics in ∂Y_i and by $\sigma_{j_1 j_2}$ the shortest geodesic in Y_i joining γ_{j_1} and γ_{j_2} for $1 \leq j_1 < j_2 \leq 3$. If we cut Y_i along $\sigma_{12}, \sigma_{13}, \sigma_{23}$, then we obtain two isometric right-angled geodesic hexagons H_1, H_2; by symmetry, the three sides of H_1 contained in $\gamma_1, \gamma_2, \gamma_3$ have length $\frac{1}{2}$. The usual trigonometric formulas in the hyperbolic plane (see e.g. [5, p.40]) give

$$\frac{\sinh L(\sigma_{12})}{\sinh 1/2} = \frac{\sinh L(\sigma_{13})}{\sinh 1/2} = \frac{\sinh L(\sigma_{23})}{\sinh 1/2} .$$

Thus, $L(\sigma_{12}) = L(\sigma_{13}) = L(\sigma_{23})$ and so, $L(g_j \cap Y_i) \geq L(\sigma_{12})$. We also have (see e.g. [5, p.40])

$$\cosh 1/2 = \sinh^2 1/2 \, \cosh L(\sigma_{12}) - \cosh^2 1/2 \, ,$$

$$L(\sigma_{12}) = \operatorname{arg\,cosh} \frac{\cosh 1/2 + \cosh^2 1/2}{\sinh^2 1/2} =: c_1 .$$

Claim 2. We have

$$d(\gamma_1, \sigma_{23}) = d(\gamma_2, \sigma_{13}) = d(\gamma_3, \sigma_{12}) = \operatorname{arg\,cosh} \big(\sinh c_1 \, \sinh 1/2 \big) =: c_2 .$$

Denote by σ the shortest geodesic in H_1 joining γ_1 and σ_{23}. If we cut H_1 along σ, then we obtain two right-angled geodesic pentagons P_1, P_2. The usual trigonometric formulas (see e.g. [5, p.39]) and Claim 1 give

$$\cosh d(\gamma_1, \sigma_{23}) = \cosh L(\sigma) = \sinh L(\sigma_{13}) \, \sinh 1/2 \, ,$$
$$d(\gamma_1, \sigma_{23}) = \operatorname{arg\,cosh} \big(\sinh c_1 \, \sinh 1/2 \big) .$$

The same argument gives the other equalities.

Since Ω is a connected set, $\hat{\Omega} := \cup_{i \in I_1} Y_i$ is also connected and we have $|\hat{\Omega}| \geq |\Omega|$. We are going to prove $|\partial\hat{\Omega}| \leq c_0 |\partial\Omega|$, with $c_0 := 3 \max\{2, \frac{4}{c_1}, \frac{1}{c_2}\}$.

The boundaries of Ω and $\hat{\Omega}$ are contained in $\cup_{1 \leq j \leq m} \cup_{i \in A_j} Y_i$. Hence, $|\partial\hat{\Omega}| \leq 3 \sum_{1 \leq j \leq m} |A_j|$ because $|\partial Y_i| = 3$ for every i.

If $i \in A_j \setminus T_j$ for some $1 \leq j \leq m$, then Claim 1 implies $L(g_j \cap Y_i) \geq c_1$; in this case, there are at most three Y-pieces in T_j for each Y-piece in $A_j \setminus T_j$ and so, $|T_j| \leq 3|A_j \setminus T_j|$, $|A_j| \leq 4|A_j \setminus T_j|$ and $L(g_j) \geq c_1 |A_j \setminus T_j| \geq \frac{1}{4} c_1 |A_j|$.

If $A_j = T_j$ for some $1 \leq j \leq m$, then $|T_j| = |A_j| \leq 2$. Recall that, by hypothesis, g_j is homotopically non-trivial. If g_j is freely homotopic to some closed geodesic γ in ∂Y_i with $i \in T_j$, then $L(g_j) \geq L(\gamma) = 1$. Otherwise, $|T_j| = |A_j| = 2$ and there exist $i \in T_j$, γ in ∂Y_i and $g_j^* \subset g_j \cap T_i$ such that g_j^* is a path starting and ending in γ and the connected components of $Y_i \setminus g_j^*$ are not simply connected. Hence, Claim 2 implies that $L(g_j) \geq L(g_j^*) \geq 2c_2$. We have in both cases $L(g_j) \geq \min\{1, 2c_2\} \geq \frac{1}{2} \min\{1, 2c_2\} |A_j|$.

Therefore,

$$|\partial\hat{\Omega}| \leq 3 \sum_{1\leq j\leq m} |A_j| \leq \frac{3}{\min\{\frac{c_1}{4}, \frac{1}{2}, c_2\}} \sum_{1\leq j\leq m} L(g_j) = 3\max\Big\{2, \frac{4}{c_1}, \frac{1}{c_2}\Big\}|\partial\Omega|.$$

Since $|\hat{\Omega}| \geq |\Omega| \geq t$, we conclude

$$\frac{1}{c_0}\,\phi_X(t) \leq \frac{1}{c_0}|\partial\hat{\Omega}| \leq |\partial\Omega|,$$

if Ω is neither simply nor doubly connected. Since $|\partial\Omega| \geq t$ if Ω is either simply or doubly connected, we have

$$\min\Big\{t, \frac{1}{c_0}\,\phi_X(t)\Big\} \leq \varphi_X(t).$$

Hence, we conclude

$$\int_{2\pi}^{\infty} \frac{dt}{\varphi_X(t)^2} \leq \int_{2\pi}^{\infty} \max\Big\{\frac{1}{t^2}, \frac{c_0^2}{\phi_X(t)^2}\Big\}\,dt \leq \int_{2\pi}^{\infty} \frac{dt}{t^2} + c_0^2\int_{2\pi}^{\infty} \frac{dt}{\phi_X(t)^2} < \infty$$

and X is non-parabolic, by Theorem 13. Since X is a complete Riemannian surface with constant curvature -1 and it has positive injectivity radius, Theorem 8 implies that any ε-net P in X is non-parabolic. Since both uniform graphs P and G are quasi-isometric to X, they are also quasi-isometric and Theorem 9 gives that G is non-parabolic. □

Note that requiring the graph to be infinite in the hypothesis in Theorem 14 is not a restriction, since Corollary 1 gives that every finite graph is parabolic.

We have the following sufficient condition for p-parabolicity on manifolds.

Theorem 15 *[9, Corollary 3.2] If M is a complete Riemannian manifold satisfying*

$$\int_{1}^{\infty} \Big(\frac{t}{|B_M(u,t)|}\Big)^{1/(p-1)} dt = \infty$$

for some $1 < p < \infty$ and $u \in M$, then M is p-parabolic.

Let us prove the following discrete version of Theorem 15:

Theorem 16 *If G is a uniform graph such that*

$$\sum_{k=1}^{\infty} \Big(\frac{k}{|B_G(u,k)|}\Big)^{1/(p-1)} = \infty \tag{2}$$

for some $1 < p < \infty$ and $u \in V(G)$, then G is p-parabolic.

Proof First of all note that if G is a finite graph, then $|B_G(u,k)| = |V(G)|$ for every $k > \operatorname{diam} G$ and so, (2) holds and G is p-parabolic. Assume now that G is an infinite graph.

Given any fixed $v \in V(G)$, since $B_G(v,k) \subset B_G(u,k+d(u,v))$, we have

$$\begin{aligned}\sum_{k=1}^{\infty}\Big(\frac{k}{|B_G(v,k)|}\Big)^{1/(p-1)} &\geq \sum_{k=1}^{\infty}\Big(\frac{k}{|B_G(u,k+d(u,v))|}\Big)^{1/(p-1)} \\ &= \sum_{n=1+d(u,v)}^{\infty}\Big(\frac{n-d(u,v)}{|B_G(u,n)|}\Big)^{1/(p-1)} \\ &\geq \sum_{n=1+2d(u,v)}^{\infty}\Big(\frac{n-d(u,v)}{|B_G(u,n)|}\Big)^{1/(p-1)} \\ &\geq \sum_{n=1+2d(u,v)}^{\infty}\Big(\frac{n}{2|B_G(u,n)|}\Big)^{1/(p-1)} = \infty.\end{aligned}$$

Hence, condition (2) does not depend on the vertex u.

Since G is a uniform graph, Theorem 7 implies that there exist a 3-regular graph G' and an (a,b)-quasi-isometry $g: G' \to G$ for some constants $a \geq 1$ and $b \geq 0$. The argument in the proof of [22, Lemma 3.4] gives that there exists a positive constant μ such that $|B_{G'}(v,k)| \leq \mu |B_G(g(v),ak+b)|$ for every $v \in V(G')$ and $k \geq 1$.

If $\lceil a \rceil$ denotes the upper integer part of a (i.e., the least integer greater than or equal to a), then there are at most $\lceil a \rceil$ integers n satisfying $ak+b \leq n < a(k+1)+b$ for each $k \geq 1$; in this case, $B_G(g(v),ak+b) \subseteq B_G(g(v),n)$ and

$$\begin{aligned}\sum_{ak+b\leq n<a(k+1)+b}\Big(\frac{n}{|B_G(g(v),n)|}\Big)^{1/(p-1)} &\leq \sum_{ak+b\leq n<a(k+1)+b}\Big(\frac{a(k+1)+b}{|B_G(g(v),ak+b)|}\Big)^{1/(p-1)} \\ &\leq \lceil a \rceil \Big(\frac{(2a+b)k}{|B_G(g(v),ak+b)|}\Big)^{1/(p-1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty}\Big(\frac{k}{|B_{G'}(v,k)|}\Big)^{1/(p-1)} &\geq \sum_{k=1}^{\infty}\Big(\frac{k}{\mu|B_G(g(v),ak+b)|}\Big)^{1/(p-1)} \\ &\geq \frac{1}{\lceil a \rceil}\Big(\frac{1}{\mu(2a+b)}\Big)^{1/(p-1)}\sum_{n\geq a+b}\Big(\frac{n}{|B_G(g(v),n)|}\Big)^{1/(p-1)} = \infty\end{aligned}$$

for every $v \in V(G')$ and so, G' satisfies the same hypothesis than G.

Let Y_0 be a Y-piece with boundary curves of length 1 (and constant curvature $K=-1$). Recall that the constant c_1 in the proof of Theorem 14 is the distance between two closed geodesics in ∂Y_0. As in the proof of Theorem 14, let us build a surface X by considering for each vertex in G' a Y-piece isometric to Y_0, and pasting these Y-pieces with the same structure of G'. One can check that X is quasi-isometric to G'. Since Y_0 is a compact surface, $\operatorname{inj}(Y_0) > 0$ and so, $\operatorname{inj}(X) > 0$. Also, X has constant curvature $K=-1$ and so, X has bounded local geometry.

We are going to prove that

$$\int_1^{\infty}\Big(\frac{t}{|B_X(x,t)|}\Big)^{1/(p-1)}dt = \infty$$

for any $x \in X$. Fix $v \in V(G')$ and $x \in X$ a point in the Y-piece Y_v corresponding to the vertex v. Define $c := d(x,\partial Y_v)$. The ball $B_X(x,c+kc_1)$ is contained in the union of the Y-pieces corresponding to the vertices in $\overline{B_{G'}(v,k)} = B_{G'}(v,k+1)$. Therefore,

$|B_X(x, c + kc_1)| \le 2\pi |B_{G'}(v, k + 1)|$ because $|Y_0| = 2\pi$. Since $k - 1 \ge (k + 1)/4$ for every $k \ge 2$, we have

$$\int_{c+c_1}^{\infty} \Big(\frac{t}{|B_X(x,t)|}\Big)^{1/(p-1)} dt = \sum_{k=2}^{\infty} \int_{c+(k-1)c_1}^{c+kc_1} \Big(\frac{t}{|B_X(x,t)|}\Big)^{1/(p-1)} dt$$
$$\ge \sum_{k=2}^{\infty} \int_{c+(k-1)c_1}^{c+kc_1} \Big(\frac{(k-1)c_1}{|B_X(x,c+kc_1)|}\Big)^{1/(p-1)} dt$$
$$\ge \sum_{k=2}^{\infty} c_1 \Big(\frac{(k+1)c_1}{4|B_X(x,c+kc_1)|}\Big)^{1/(p-1)}$$
$$\ge \sum_{k=2}^{\infty} c_1 \Big(\frac{(k+1)c_1}{8\pi|B_{G'}(v,k+1)|}\Big)^{1/(p-1)} = \infty.$$

Hence, we have

$$\int_{1}^{\infty} \Big(\frac{t}{|B_X(x,t)|}\Big)^{1/(p-1)} dt = \infty$$

and Theorem 15 gives that X is p-parabolic. Since X has bounded local geometry, Theorem 8 implies that any ε-net P in X is p-parabolic. Since both uniform graphs P and G are quasi-isometric to X, they are also quasi-isometric and Theorem 9 gives that G is p-parabolic. □

Acknowledgments. First author supported in part by a grant from Ministerio de Economía y Competitividad (PID2021-126124NB-I00), Spain. Second author supported in part by a grant from Agencia Estatal de Investigación (PID2019-106433GB-I00 / AEI / 10.13039/501100011033), Spain, and by the Madrid Government (Comunidad de Madrid-Spain) under the Multiannual Agreement with UC3M in the line of Excellence of University Professors (EPUC3M23), and in the context of the V PRICIT (Regional Programme of Research and Technological Innovation).

Declarations

- Funding: First author supported in part by a grant from Ministerio de Economía y Competitividad (PID2021-126124NB-I00), Spain. Second author supported in part by a grant from Agencia Estatal de Investigación (PID2019-106433GB-I00 / AEI / 10.13039/501100011033), Spain, and by the Madrid Government (Comunidad de Madrid-Spain) under the Multiannual Agreement with UC3M in the line of Excellence of University Professors (EPUC3M23), and in the context of the V PRICIT (Regional Programme of Research and Technological Innovation).
- Conflict of interest/Competing interests: none.
- Ethics approval: Not applicable.
- Consent to participate: Not applicable.
- Consent for publication: Not applicable.
- Availability of data and materials: Not applicable.
- Code availability: Not applicable.
- Authors' contributions: Not applicable.

References

[1] Alvarez, V., Pestana, D., Rodríguez, J. M., Isoperimetric inequalities in Riemann surfaces of infinite type. *Rev. Mat. Iberoamericana* **15** (1999), 353–427.

[2] Alvarez, V., Rodríguez, J. M., Yakubovich, V. A., Subadditivity of p-harmonic "measure" on graphs, *Michigan Math. J.* **49** (2001), 47–64.

[3] Anderson, M., Cheeger, J., C^{α}-compactness for manifolds with Ricci curvature and injectivity radius bounded below, *J. Diff. Geom.* **35** (1992) 265–281.

[4] Bermudo, S., RodrÃguez, J. M., Sigarreta, J. M., Vilaire, J.-M., Gromov hyperbolic graphs, *Discr. Math.* **313** (2013) 1575–1585.

[5] Buser, P., Geometry and Spectra of Compact Riemann Surfaces. Birkhäuser, Boston, 1992.

[6] Cantón, A., Fernández, J. L., Pestana, D., Rodríguez, J. M., On harmonic functions on trees, *Potential Analysis* **15** (2001), 199–244.

[7] Cantón, A., Granados, A., Portilla, A., Rodríguez, J. M., Quasi-isometries and isoperimetric inequalities in planar domains, *J. Math. Soc. Japan* **67** (2015), 127–157.

[8] Chavel, I., Eigenvalues in Riemannian Geometry. Academic Press, New York, 1984.

[9] Coulhon, T., Holopainen, I., Saloff-Coste, L., Harnack inequality and hyperbolicity for subelliptic p-Laplacians with applications to Picard type theorems, *GAFA Geom. Funct. Anal.* **11** (2001) 1139–1191.

[10] Dodziuk, J., Every covering of a compact Riemannian surface of genus greater that one nontrivial L^2 harmonic differential, *Acta Math.* **152** (1984) 49–56.

[11] Delmotte, T., Parabolic Harnack inequality and estimates of Markov chains on graphs, *Rev. Mat. Iberoamericana* **15** (1999), 181–232.

[12] Fernández, J. L., On the existence of Green's function in Riemannian manifolds, *Proc. Amer. Math. Soc.* **96 (2)** (1986), 284–286.

[13] Fernández, J. L., Rodríguez, J. M., The exponent of convergence of Riemann surfaces: Bass Riemann surfaces, *Ann. Acad. Sci. Fenn. A. I.* **15** (1990), 165–183.

[14] Fernández, J. L., Rodríguez, J. M., Area growth and Green's function of Riemann surfaces, *Arkiv för matematik* **30** (1992), 83–92.

[15] Ghys, E., de la Harpe, P., *Sur les Groupes Hyperboliques d'après Mikhael Gromov.* Progress in Mathematics, Volume 83, Birkhäuser, 1990.

[16] Granados, A., Pestana, D., Portilla, A., Rodríguez, J. M., Tourís, E., Stability of the injectivity radius under quasi-isometries and applications to isoperimetric inequalities, *RACSAM* **112** (2018) 1225–1247.

[17] Granados, A., Pestana, D., Portilla, A., Rodríguez, J. M., Tourís, E., Stability of the volume growth rate under quasi-isometries, *Rev. Matem. Complut.* **33(1)** (2020), 231–270.

[18] Gromov, M., Hyperbolic groups, in "Essays in group theory". Edited by S. M. Gersten, M. S. R. I. Publ. **8**. Springer, 1987, 75-263.

[19] Holopainen, I., Nonlinear potential theory and quasiregular mappings on Riemannian manifolds, *Ann. Acad. Sci. Fenn.* **74** (1990), 1–45.

[20] Holopainen, I., Rough isometries and p-harmonic functions with finite Dirichlet integral, *Rev. Mat. Iberoam.* **10** (1994), 143–176.

[21] Holopainen, I., Soardi, P. M., p-harmonic functions on graphs and manifolds, *Manuscr. Math.* **94** (1997), 95–110.

[22] Kanai, M., Rough isometries and combinatorial approximations of geometries of non-compact Riemannian manifolds, *J. Math. Soc. Japan* **37** (1985), 391–413.

[23] Kanai, M., Rough isometries and the parabolicity of Riemannian manifolds, *J. Math. Soc. Japan* **38** (1986), 227–238.

[24] Kanai, M., Analytic inequalities and rough isometries between non-compact Riemannian manifolds. Curvature and Topology of Riemannian manifolds (Katata, 1985). Lecture Notes in Math. **1201**. Springer (1986), 122–137.

[25] Kawohl, B., Fridman, V., Isoperimetric estimates for the first eigenvalue of the p-Laplace operator and the Cheeger constant, *Comment. Math. Univ. Carolin.* **44(4)** (2003) 659–667.

[26] Lefton, L., Wei, D., Numerical approximation of the first eigenpair of the p-Laplacian using finite elements and the penalty method, *Numer. Funct. Anal. Optim.* **18** (1997), 389–399.

[27] Martínez-Pérez, A., Rodríguez, J. M., Cheeger isoperimetric constant of Gromov hyperbolic manifolds and graphs, *Commun. Contemp. Math.*

20:5 (2018) 1750050 (33 pages).

[28] Martínez-Pérez, A., Rodríguez, J. M., A note on isoperimetric inequalities of Gromov hyperbolic manifolds and graphs, *RACSAM* **115**, 154 (2021). https://doi.org/10.1007/s13398-021-01096-2.

[29] Martínez-Pérez, A., Rodríguez, J. M., Isoperimetric inequalities in Riemann surfaces and graphs, *J. Geom. Anal.* **31** (2021), 3583–3607. https://doi.org/10.1007/s12220-020-00407-0.

[30] Martínez-Pérez, A., Rodríguez, J. M., On p-parabolicity of Riemannian manifolds and graphs, *Rev. Mat. Complut.* **35**, 179–198 (2022). https://doi.org/10.1007/s13163-021-00387-x.

[31] Martínez-Pérez, A., Rodríguez, J. M., Parabolicity on graphs, Submitted.

[32] Melián, M. V., Rodríguez, J. M., Tourís, E., Escaping geodesics in Riemannian surfaces with pinched variable negative curvature. *Advances in Math.* **345**, 17 (2019), 928–971. https://doi.org/10.1016/j.aim.2019.01.032.

[33] Portilla, A., Rodríguez, J. M., Tourís, E., Gromov hyperbolicity through decomposition of metric spaces II, *J. Geom. Anal.* **14** (2004), 123–149.

[34] Portilla, A., Tourís, E., A characterization of Gromov hyperbolicity of surfaces with variable negative curvature, *Publ. Mat.* **53** (2009), 83–110.

[35] Ratcliffe, J. G., Foundations of Hyperbolic Manifolds. Springer-Verlag, New York, 1994.

[36] Rodríguez, J. M., Isoperimetric inequalities and Dirichlet functions of Riemann surfaces, *Publ. Mat.* **38** (1994), 243–253.

[37] Rodríguez, J. M., Two remarks on Riemann surfaces, *Publ. Mat.* **38** (1994), 463–477.

[38] Sario L., Nakai, M., Classification Theory of Riemann Surfaces, *Grundlehren der mathematischen Wissenschaften* **164**, (1970).

[39] Soardi, P. M., Potential Theory in infinite networks. Lecture Notes in Math., Volume 1590, Springer-Verlag, 1994.

[40] Tourís, E., Graphs and Gromov hyperbolicity of non-constant negatively curved surfaces, *J. Math. Anal. Appl.* **380** (2011), 865–881.