Misuse of Kuo-Eliassen Equation in Studies of the Climatological Mean Meridional Circulation

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ABSTRACT: The Kuo-Eliassen equation provides the mean meridional circulation that must be present for the axisymmetric component of a flow forced by heat and momentum sources to remain balanced as it evolves. It does not tell us whether or not the flow is steady. Using this equation to explain how the mean meridional circulation is perturbed due to a change in thermal or momentum forcing, including the forcing due to large-scale eddies, requires a division of the forcing into prescribed and reactive parts and, most importantly, a physical theory for the latter. It should not be used without explicit discussion of the assumptions being made about the reactive component of the forcing and justification for the choice being made.

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1. Introduction

One goal of general circulation theory is the clarification of the roles of heat and momentum sources, including eddy flux convergences, in controlling the climatological mean meridional circulation (MMC). An historical introduction to this issue focusing on the earliest work from the late 19th and early 20th centuries is provided by Lorenz (1967). In the 1950's, Eliassen (1951) and then Kuo (1956) described a diagnostic equation for the MMC that must be satisfied to maintain a balanced axisymmetric vortex in the presence of heat and momentum sources. There followed a series of papers in which this Kuo-Eliassen (K-E) equation was applied to the climatological MMC and its sensitivity to momentum and heat fluxes, despite the fact that this equation has no direct bearing on the question of whether the balanced vortex is or is not steady. Examples include Sasamori (1978), Pfeffer (1981), Crawford and Sasamori (1981), and Salustri and Stone (1983). More recently this methodology has been resurrected in studies of climate change and various other questions regarding how the MMC responds to perturbations [e.g., Chemke and Polvani (2019), Chemke et al. (2019), Zaplotnik et al. (2022), Thakur et al. (2024)].

In contrast, simulations with atmospheric models have provided a framework for directly assessing the role of different aspects of heat and momentum sources for the MMC. In particular, one can introduce prescribed extratropical eddy fluxes into a model for the axisymmetric component of the flow and then perturb these fluxes systematically, explicitly integrating to a new steady state and the corresponding change in the MMC. Examples of this approach include Schneider (1984), Satoh et al. (1995), Becker et al. (1997), Kim and Lee (2001), Walker and Schneider (2006), Singh and Kuang (2016), Singh et al. (2017). These studies can be thought of as following on from studies focusing on the limit of vanishing free tropospheric eddy momentum fluxes initiated by Schneider (1977) and Held and Hou (1980).

One might naturally wonder why this direct computational approach is necessary if the simpler diagnostic procedure based on the K-E equation is available. In fact, limitations to the use of the K-E equation to the study of the climatological MMC have been emphasized in the literature on several occasions, including by Kim and Lee (2001) and Chang (1996). But the use of the K-E equation for studies of the MMC persists despite these critiques. We do not claim any novelty in

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this short note, but by reiterating the fundamental misinterpretation underlying the use of the K-E equation in general circulation theory, we hope to discourage this usage more definitively.

2. A QG steady state

To retain the utmost simplicity, we restrict our discussion to the low Rossby number QG limit of a zonally symmetric Bousinesq fluid with constant Coriolis parameter and constant buoyancy frequency. We also use the traditional rather than Transformed Eulerian Mean framework. Our discussion is equally relevant to either formulation of the zonal mean equations.

Denoting the zonal mean buoyancy by B, with the notation otherwise standard, we have

$$\frac{\partial U}{\partial t} = fV + F \tag{1}$$

$$\frac{\partial B}{\partial t} = -N^2 W + Q \tag{2}$$

$$0 = \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} \tag{3}$$

$$f\frac{\partial U}{\partial z} = -\frac{\partial B}{\partial y} \tag{4}$$

Eliminating the time tendencies by combining the z-derivative of (1) and the y-derivative of (2) with the thermal wind equation (4), we have

$$f^2\frac{\partial^2 \Psi}{\partial z^2} + N^2\frac{\partial^2 \Psi}{\partial y^2} = f\frac{\partial F}{\partial z} + \frac{\partial Q}{\partial y} \tag{5}$$

where the streamfunction Ψ of the MMC determines the velocity field through $[V,W] = [-\partial_z \Psi, \partial_y \Psi]$. We refer to this equation as our simplified K-E equation in this context; it is simply the standard QG ω-equation specialized to the axisymmetric component of the flow, for which the solution can be written in terms of the streamfunction.

One of the attractive features of the K-E equation is that it generalizes elegantly to the case of an arbitrary basic state $(U(y,z), B(y,z))$, arbitrary Rossby number (with f^2 on the left hand side replaced by $f(f - \partial U/\partial y)$, a meaure of inertial instability), and nonlinear balance rather than linear geostrophic balance, as demonstrated by Eliassen (1951). The spatially varying inertial stability is a central feature when applying this equation to axisymmetric models of tropical cyclones (Schubert and Hack 1982) and is also important when applied to the Hadley Cell. But none of

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these generalizations affect the essence of the discussion here, so we limit ourselves to this simplest special case.

Eliminating the ageostrophic flow in the usual way by differentiating (1) by y and (2) by z, we obtain the potential vorticity equation:

$$\frac{\partial Z}{\partial t} = -\frac{\partial F}{\partial y} + \frac{f}{N^2}\frac{\partial Q}{\partial z} \tag{6}$$

where $Z = -\partial_y U + (f/N^2)\partial_z B$. The condition for a steady state is that the potential vorticity tendency equal zero, so the momentum and thermal forcing need to satisfy

$$\frac{\partial F}{\partial y} = \frac{f}{N^2}\frac{\partial Q}{\partial z} \tag{7}$$

In a steady state, one can obtain the MMC either from the steady momentum equation or the steady buoyancy equation. In particular, one can immediately obtain $V = -F/f$ from the momentum equation and $W = Q/N^2$ from the buoyancy equation, with (7) being the condition that ensures that the resulting V and W are consistent with the continuity equation. Gilman (1964) provides an early and clear discussion of the importance of this consistency, which tends to get lost in discussions of the climatological MMC that are focused on the K-E equation in isolation.

We write the solution for the streamfunction given F and Q as $\Psi(F,Q)$. Since K-E is a linear equation one can always write $\Psi(F,Q) = \Psi(0,Q) + \Psi(F,0)$. But if the pair $[F,Q]$ satisfy the steady state condition (7), $[F,0]$ and $[0,Q]$ clearly do not. Therefore, this is not a decomposition of the steady solution into the sum of two steady solutions forced by the heating and the momentum source in isolation.

More modestly, one can start with a steady solution $\Psi(F,Q)$ and, for example, perturb Q to be $Q+\delta Q$. While one can solve for $\Psi(Q+\delta Q,F)$, this does not correspond to a steady solution and is not able to predict the steady response to the perturbation in heating. What the K-E equation tells us, in this instance, is how the MMC must change to ensure that the circulation remains balanced as it begins to evolve to its new steady state. Its relevance to questions about this steady state requires some method for inferring the final steady state from the initial tendency, or equivalently some understanding of how F and Q adjust to satisfy the steady state condition.

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Alternatively, one can consider two steady states produced, say, by numerical simulations of a control climate, with $\Psi_{control} = \Psi(F,Q)$ and a perturbed climate with $\Psi_{control} + \delta\Psi = \Psi(F + \delta F, Q + \delta Q)$. By construction, both $[F,Q]$ and $[F + \delta F, Q + \delta Q]$ satisfy the K-E equation and the equilibrium condition (7). But $[F + \delta F, Q]$ and $[F, Q + \delta Q]$ do not satisfy the equilibrium condition. Exactly the same issue arises if we split F or Q into parts corresponding to distinct physical processes. The relevance of the K-E equation to questions about the decomposition of the difference between two steady state MMCs into contributions from different causes requires, once again, some method for inferring how F and Q mutually adjust to satisfy the steady state condition.

3. Splitting F and Q into prescribed and reactive parts

A modification of the approach described above that is sometimes considered, implicitly if not explicitly, as a basis for using the K-E equation for studies of steady states is to break Q and F into prescribed parts (F_{pre}, Q_{pre}) and reactive parts (F_{react}, Q_{react}), the simplest assumption for the later being $(F_{react}, Q_{react}) = (-\alpha U, -\alpha B)$, with α a constant. The equations (1, 2) are then simply modified by replacing the time derivative with multiplication by α. Using thermal wind and continuity as before, the value of α drops out and the equation for the streamfunction is identical to (5) except that it is forced only by the prescribed parts of F and Q.

This approach seems especially attractive when one is trying to understand the difference in the streamfunction between two steady states, $\delta\Psi$, resulting from perturbations to the prescribed forcing, $(\delta F_{pre}, \delta Q_{pre})$. For example, these steady states could be those in a GCM's control simulation and a simulation with perturbed CO_2. The hope is that this simplest linear relaxation might be more justifiable for perturbations to the system than the full climatology. With $(\delta F_{react}, \delta Q_{react}) = (-\alpha\delta U, -\alpha\delta B)$ we have

$$0 = f\delta V + \delta F_{pre} - \alpha\delta U \tag{8}$$

$$0 = -N^2\delta W + \delta Q_{pre} - \alpha\delta B \tag{9}$$

and the familiar manipulation gives

$$f^2\frac{\partial^2\delta\Psi}{\partial z^2} + N^2\frac{\partial^2\delta\Psi}{\partial y^2} = f\frac{\partial\delta F_{pre}}{\partial z} + \frac{\partial\delta Q_{pre}}{\partial y} \tag{10}$$

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While this approach seems intuitive, the retrieval of the original K-E equation (5) is only applicable for this particular form of the reactive parts of the forcing. For example, if one retains the spatially uniform relaxation but with different strengths for momentum, α_M, and for buoyancy (i.e, temperature), α_T, the result is instead,

$$f^2 \frac{\alpha_T}{\alpha_M} \frac{\partial^2 \delta\Psi}{\partial z^2} + N^2 \frac{\partial^2 \delta\Psi}{\partial y^2} = f \frac{\alpha_T}{\alpha_M} \frac{\partial \delta F_{pre}}{\partial z} + \frac{\partial \delta Q_{pre}}{\partial y} \tag{11}$$

Sasamori (1978) discusses the potential importance of this modification of the K-E equation, with reference to an earlier paper by Dickinson (1971).

While Sasamori (1978) attempts to define the most appropriate choice for the ratio α_M/α_T the choice of spatially uniform damping is itself very suspect, especially for the momentum damping. Strong mechanical damping is a plausible qualitative model of the reactive part of the stresses in the planetary boundary layer, but in the free troposphere the magnitude and form of reactive stresses are unclear. A useful limit to consider is that in which the reactive momentum stresses are confined to the boundary layer, with $\alpha_M = 0$ in the free troposphere. It is straightforward to generalize (11) to handle spatially varying damping rates, but for this special case it is easier to just refer back to the momentum equation. In the free troposphere we have simply

$$\delta V \equiv -\frac{\partial \delta\Psi}{\partial z} = -\delta F_{pre}/f \tag{12}$$

which can be integrated down from an upper level z_{up} at which we can set $\delta\Psi(z = z_{up}) = 0$. The boundary layer then provides the compensating meridional flow required to satisfy the lower boundary condition $\delta\Psi(z = 0) = 0$. This is the essence of the "downward control" principle that has played a particularly important role in stratospheric dynamics (Haynes et al. 1991). The downward control limit illustrates how different assumptions about the reactive parts of the thermal and momentum forcing profoundly change the balance of thermal vs. momentum control of the MMC response.

Simple damping prescriptions may be of qualitative value for representing radiative responses and boundary layer stresses. And the responses of eddy buoyancy fluxes with a diffusive character may also be captured qualitatively with linear damping, although the damping rate would depend

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on the horizontal scale of the forcing. But responses in eddy momentum fluxes are more resistant to being captured in as simple a manner.

An illustrative example is provided by the response of equilibrated climate models to the addition of a prescribed zonal torque δF_{pre}. Experiments of this sort performed by Chen and Zurita-Gotor (2008) in an idealized dry global model show zonal wind responses that project strongly on the model's annular mode (Ring and Plumb 2007), with wind responses of opposite sign at different latitudes. Treating the reactive part of the momentum forcing, the eddy momentum flux convergence, as a local damping cannot mimic this response.

One way of rationalizing this non-local response is through a quasi-linear perspective involving the space-time spectrum of eddies responsible for the momentum fluxes – the response being determined by changes in the amplitude and spectrum of waves excited by baroclinic production and changes in the index of refraction controlling the propagation and then dissipation of these waves through wave breaking. The changes in propagation and dissipation are in turn a consequence of changes in critical latitudes and turning latitudes, as described by Lorenz (2015). Since the index of refraction differs depending on the zonal wavelength and the phase speed of the eddies, it is a challenge to somehow condense this quasi-linear dynamics into a form that can be incorporated into a Kuo-Eliassen-like equation. The upshot is that one needs a physically based model to justify a particular form for the reactive part of the momentum forcing in the free troposphere. The assumption of spatially uniform damping throughout the troposphere is not a useful starting point.

4. Conclusion

Analyses of the zonally averaged mean meridional circulation (MMC) in observations and models often make use of the Kuo-Eliassen (K-E) equation. The K-E equation is a linear elliptic equation for the streamfunction of the MMC required to maintain balanced zonal winds and temperatures as the flow evolves. In the QG limit, which we assume in this note for simplicity, K-E is simply the classic ω-equation specialized to the axisymmetric case. Since it is a linear equation, it provides a decomposition of the MMC into parts associated with different parts of the forcing, but this decomposition is often misinterpreted.

In itself the K-E equation does not ensure that the zonal flow, temperatures, and the MMC are steady, but if the sources are taken from observations or a model of a steady circulation then the

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K-E equation will be satisfied by this specific forcing. If the forcing is perturbed (for example, if one part is removed) the resulting MMC no longer corresponds to a steady state in general. In a steady state, the flow must satisfy the steady state zonal wind and temperature equations separately. In the simple QG case, the zonal mean QG potential vorticity tendency must vanish.

When the forcing associated with a steady state is perturbed, the K-E equation provides information on how the system will evolve in search of a new steady state. Dividing the forcing into prescribed and reactive parts makes explicit the assumptions required to demonstrate the relevance of this decomposition for the new steady state. Inferences about the MMC in the new steady state based on the K-E equation are justified if this damping is spatially uniform and identical in the heat and momentum equations. But this assumption about the reactive component of the forcing is unphysical.

In papers on tropical cyclone dynamics, the K-E equation is used in its most general form with spatially varying gravitational and inertial stability and is commonly referred to as Eliassen's balanced vortex model or the Sawyer-Eliassen equation. It is typically applied when trying to understand the evolution of the cyclone, especially its intensification (e.g., Schubert and Hack (1982)). Its use in this context is not subject to the critique discussed here, although there may be other issues, such as those related to how the heating is influenced by the circulation, that may limit its utility.

In contrast, in analyses of the MMC in the general circulation the interest is typically on the maintenance of steady state circulations, for example on the response of the steady circulation to a perturbation in the forcing. Justification of the use of the K-E equation in this context requires a division of the heat and momentum sources into prescribed and reactive parts with the form of the latter being critical.

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References

Becker, E., G. Schmitz, and R. Geprägs, 1997: The feedback of midlatitude waves onto the Hadley cell in a simple general circulation model. *Tellus A*, **49 (2)**, 182–199.

Chang, E. K., 1996: Mean meridional circulation driven by eddy forcings of different timescales. *Journal of the Atmospheric Sciences*, **53 (1)**, 113–125.

Chemke, R., L. Polvani, and C. Deser, 2019: The effect of Arctic sea ice loss on the Hadley circulation. *Geophysical Research Letters*, **46 (2)**, 963–972.

Chemke, R., and L. M. Polvani, 2019: Opposite tropical circulation trends in climate models and in reanalyses. *Nature Geoscience*, **12 (7)**, 528–532.

Chen, G., and P. Zurita-Gotor, 2008: The tropospheric jet response to prescribed zonal forcing in an idealized atmospheric model. *Journal of the Atmospheric Sciences*, **65 (7)**, 2254–2271.

Crawford, S. L., and T. Sasamori, 1981: A study of the sensitivity of the winter mean meridional circulation to sources of heat and momentum. *Tellus*, **33 (4)**, 340–350.

Dickinson, R. E., 1971: Analytic model for zonal winds in the tropics: I. Details of the model and simulation of gross features of the zonal mean troposphere. *Monthly Weather Review*, **99 (6)**, 501–510.

Eliassen, A., 1951: Slow thermally or frictionally controlled meridional circulation in a circular vortex. *Astrophisica Norvegica*, **5**, 19–60.

Gilman, P. A., 1964: On the mean meridional circulation in the presence of a steady state, symmetric, circumpolar vortex. *Tellus*, **16 (2)**, 160–167.

Haynes, P., M. McIntyre, T. Shepherd, C. Marks, and K. P. Shine, 1991: On the “downward control” of extratropical diabatic circulations by eddy-induced mean zonal forces. *Journal of the Atmospheric Sciences*, **48 (4)**, 651–678.

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Held, I. M., and A. Y. Hou, 1980: Nonlinear axially symmetric circulations in a nearly inviscid atmosphere. *Journal of the Atmospheric Sciences*, **37 (3)**, 515–533.

Kim, H.-k., and S. Lee, 2001: Hadley cell dynamics in a primitive equation model. Part I: Axisymmetric flow. *Journal of the Atmospheric Sciences*, **58 (19)**, 2845–2858.

Kuo, H., 1956: Forced and free meridional circulations in the atmosphere. *Journal of the Atmospheric Sciences*, **13 (6)**, 561–568.

Lorenz, D. J., 2015: Understanding midlatitude jet variability and change using Rossby wave chromatography: Methodology. *Journal of the Atmospheric Sciences*, **72 (1)**, 369–388.

Lorenz, E., 1967: The nature and theory of the general circulation of the atmosphere. *World Meteorological Organization*, **161**.

Pfeffer, R. L., 1981: Wave-mean flow interactions in the atmosphere. *Journal of Atmospheric Sciences*, **38 (7)**, 1340–1359.

Ring, M. J., and R. A. Plumb, 2007: Forced annular mode patterns in a simple atmospheric general circulation model. *Journal of the Atmospheric Sciences*, **64 (10)**, 3611–3626.

Salustri, G., and P. H. Stone, 1983: A diagnostic study of the forcing of the Ferrel cell by eddies, with latent heat effects included. *Journal of the Atmospheric Sciences*, **40 (5)**, 1101–1109.

Sasamori, T., 1978: A parameterization of large-scale heat transport in mid-latitudes. part ii. stationary waves and the ferrel cell. *Tellus*, **30 (4)**, 300–312.

Satoh, M., M. Shiobara, and M. Takahashi, 1995: Hadley circulations and their rôles in the global angular momentum budget in two-and three-dimensional models. *Tellus A*, **47 (5)**, 548–560.

Schneider, E. K., 1977: Axially symmetric steady-state models of the basic state for instability and climate studies. Part II. Nonlinear calculations. *Journal of the Atmospheric Sciences*, **34 (2)**, 280–296.

Schneider, E. K., 1984: Response of the annual and zonal mean winds and temperatures to variations in the heat and momentum sources. *Journal of Atmospheric Sciences*, **41 (7)**, 1093–1115.

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Schubert, W. H., and J. J. Hack, 1982: Inertial stability and tropical cyclone development. *Journal of the Atmospheric Sciences*, **39 (8)**, 1687–1697.

Singh, M. S., and Z. Kuang, 2016: Exploring the role of eddy momentum fluxes in determining the characteristics of the equinoctial Hadley circulation: Fixed-SST simulations. *Journal of the Atmospheric Sciences*, **73 (6)**, 2427–2444.

Singh, M. S., Z. Kuang, and Y. Tian, 2017: Eddy influences on the strength of the Hadley circulation: Dynamic and thermodynamic perspectives. *Journal of the Atmospheric Sciences*, **74 (2)**, 467–486.

Thakur, A., J. Sukhatme, and N. Harnik, 2024: Investigating the role of tropical and extratropical waves in the Hadley circulation via present-day earth-like to globally uniform sea-surface temperature forcing. *Quarterly Journal of the Royal Meteorological Society*, https://doi.org/10.1002/qj.4784.

Walker, C. C., and T. Schneider, 2006: Eddy influences on Hadley circulations: Simulations with an idealized GCM. *Journal of the Atmospheric Sciences*, **63 (12)**, 3333–3350.

Zaplotnik, Ž., M. Pikovnik, and L. Boljka, 2022: Recent Hadley circulation strengthening: a trend or multidecadal variability? *Journal of Climate*, **35 (13)**, 4157–4176.

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