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Maximally superintegrable systems in flat three-dimensional space are linearizable

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ABSTRACT

All maximally superintegrable Hamiltonian systems in three-dimensional flat space derived in the work of Evans [Phys. Rev. A **41**, 5666–5676 (1990)] are shown to possess hidden symmetries leading to their linearization, likewise the maximally superintegrable Hamiltonian systems in two-dimensional flat space as shown in the work of Gubbiotti and Nucci [J. Math. Phys. **58**, 012902 (2017)]. We conjecture that even minimally superintegrable systems in three-dimensional flat space have hidden symmetries that make them linearizable.

I. INTRODUCTION

The superintegrability of (classical) Hamiltonian systems has generally been analyzed from the perspective of separating coordinates of the associated Hamilton–Jacobi equation, an approach that motivated the systematic classification of (orthogonal) coordinate systems in $\mathbb{R}^3$ for which a separation of variables can be obtained. For the Hamilton–Jacobi equation in the potential-free case, 11 coordinate systems with the separability property were found, and for each of these classes, Eisenhart determined the most general form of the potential that can be added such that the separability is preserved.[1] Although some of these systems, such as the Kepler problem, were recognized to have relevant symmetry properties related to Lie's approach to differential equations,[2,3] the Lie group analysis has not been exploited systematically in the context of superintegrability, in particular, its relation to the linearization problem.

Several important classes of superintegrable Hamiltonian systems in flat two-dimensional spaces have been shown in Ref. 4 to be linearizable by means of some hidden symmetries of the system using a powerful method originally developed in Ref. 2 in the context of the Kepler problem. This reduction technique, valid for any first-order autonomous system, is based on the observation that one of the unknown functions can always be taken to be the new independent variable. Rewriting the system in these new coordinates and applying the Lie group analysis allow us to determine symmetries that cannot be detected in the original coordinates and therefore correspond to hidden symmetries of the system (see, e.g., Refs. 2 and 4 for details). This procedure further allowed us to show that the hidden linearity is completely independent of the degree of the first integrals of the system, as well as the separability properties of the associated Hamilton–Jacobi equation. In Ref. 5, the linearizability of a Hamiltonian system devoid of first integrals quadratic in the momenta but possessing constants of the motion of third- and fourth-order was shown, providing a first example exhibiting hidden linearity but with no second-order integrals. The symmetry analysis in the case of the non-Euclidean plane, considered in Ref. 6, points out that the symmetry approach remains valid, regardless of the space curvature. The success of the Lie method in linearizing these systems suggests us to inspect higher-order systems along the same lines in an attempt to vindicate the usefulness of the Lie group analysis in the context of superintegrability.

The main purpose of this work is to show that the maximally superintegrable Hamiltonian systems in $\mathbb{R}^3$ classified in Ref. 7 and possessing linear or quadratic first integrals in the momenta can also be linearized by means of their hidden symmetries. The main tool used in this context is the Lie criterion that establishes the equivalence between the existence of a point transformation that reduces an nth-order differential equation (or system) to the free equation $w^{(n)} = 0$ and a Lie algebra of point symmetries of the maximal possible dimension (see Refs. 8 and 9 and references therein). As can be expected, the linearizing transformations are generally nonlinear in their arguments, thus corresponding to a more general class than the canonical transformations that preserve the Hamiltonian.

As will be seen, some of the superintegrable Hamiltonian systems in Ref. 7 can be directly linearized, without requiring a change of the independent variable. This approach is sometimes the simplest and computationally shortest among the various possible ways to determine the hidden symmetries. For comparison purposes with the linearization using the reduction, those systems that are solved without changing the independent variable are reconsidered in the Appendix, illustrating that the amount of calculations increases considerably when new independent variables are introduced. On the other hand, for certain systems, it is more convenient to express the Hamiltonian systems in coordinates different from the Cartesian ones, with the purpose of simplifying the symmetry analysis. Indeed, the procedure to linearize a system is far from being unique, and for each of the superintegrable systems, the most efficient way in terms of calculations has been chosen.

II. MAXIMALLY SUPERINTEGRABLE SYSTEMS IN $\mathbb{R}^3$

In Ref. 7, it was shown that there are five equivalence classes of maximally superintegrable systems in $\mathbb{R}^3$ admitting integrals that are at most quadratic in the canonical momenta. If $H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + V(w_1, w_2, w_3)$ denotes the Hamiltonian of the system in Cartesian coordinates, then these potentials were given by

1. $V_I(w_1, w_2, w_3) = k\left(w_1^2 + w_2^2 + w_3^2\right) + \frac{k_1}{w_1^2} + \frac{k_2}{w_2^2} + \frac{k_3}{w_3^2}$,
2. $V_{II}(w_1, w_2, w_3) = -\frac{k}{\sqrt{w_1^2+w_2^2+w_3^2}} + \frac{k_1}{w_1^2} + \frac{k_2}{w_2^2}$,
3. $V_{III}(w_1, w_2, w_3) = \frac{k_1 w_1}{w_2^2\sqrt{w_1^2+w_2^2}} + \frac{k_2}{w_2^2} + \frac{k_3}{w_3^2}$,
4. $V_{IV}(w_1, w_2, w_3) = \frac{k_1 w_1}{w_2^2\sqrt{w_1^2+w_2^2}} + \frac{k_2}{w_2^2} + k_3 w_3$,
5. $V_V(w_1, w_2, w_3) = k\left(w_1^2 + w_2^2\right) + 4kw_3^2 + \frac{k_1}{w_1^2} + \frac{k_2}{w_2^2}$,

where $k, k_1, k_2,$ and k_3 are arbitrary constants.

These Hamiltonian systems with the above potentials are all autonomous so that the general reduction method of Ref. 2 to detect hidden symmetries is potentially applicable to them. We analyze each of these potentials separately and prove that the corresponding Hamiltonian system is linearizable.

A. The potential $V_I(w_1, w_2, w_3)$

The Hamiltonian

$$H_1 = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + k\left(w_1^2 + w_2^2 + w_3^2\right) + \frac{k_1}{w_1^2} + \frac{k_2}{w_2^2} + \frac{k_3}{w_3^2} \tag{1}$$

yields the Hamiltonian equations

$$\begin{cases} \dot{w}_1 = p_1, \\ \dot{w}_2 = p_2, \\ \dot{w}_3 = p_3, \\ \dot{p}_1 = 2\dfrac{k_1 - kw_1^4}{w_1^3}, \\ \dot{p}_2 = 2\dfrac{k_2 - kw_2^4}{w_2^3}, \\ \dot{p}_3 = 2\dfrac{k_3 - kw_3^4}{w_3^3}. \end{cases} \tag{2}$$

We apply the reduction method developed in Ref. 2 to this system. If we choose w_1 as a new independent variable y, then system (2) reduces to the following five equations:

$$\begin{cases} \frac{\mathrm{d}w_2}{\mathrm{d}y} = \frac{p_2}{p_1}, \\ \frac{\mathrm{d}w_3}{\mathrm{d}y} = \frac{p_3}{p_1}, \\ \frac{\mathrm{d}p_1}{\mathrm{d}y} = 2\frac{k_1 - ky^4}{y^3 p_1}, \\ \frac{\mathrm{d}p_2}{\mathrm{d}y} = 2\frac{k_2 - kw_2^4}{w_2^3 p_1}, \\ \frac{\mathrm{d}p_3}{\mathrm{d}y} = 2\frac{k_3 - kw_3^4}{w_3^3 p_1}. \end{cases} \tag{3}$$

The third equation of system (3) can be integrated to yield

$$\frac{2ky^4 + 2k_1 + p_1^2 y^2}{4y^2} = A = \text{const.} \implies p_1 = \pm\frac{\sqrt{2}}{y}\sqrt{2Ay^2 - ky^4 - k_1}. \tag{4}$$

This coincides with the first integral I_1 determined in Ref. 7. Deriving p_2 from the first equation of system (3) and replacing it into the fourth equation yield the following second-order equation in w_2:

$$\frac{\mathrm{d}^2 w_2}{\mathrm{d}y^2} = \frac{(ky^4 - k_1)w_2^3 \frac{\mathrm{d}w_2}{\mathrm{d}y} + (k_2 - kw_2^4)y^3}{yw_2^3(2Ay^2 - ky^4 - k_1)}. \tag{5}$$

Analogously, deriving p_3 from the second equation of system (3) and replacing it into the fifth equation give the following second-order equation in w_3:

$$\frac{\mathrm{d}^2 w_3}{\mathrm{d}y^2} = \frac{(ky^4 - k_1)w_3^3 \frac{\mathrm{d}w_3}{\mathrm{d}y} + (k_3 - kw_3^4)y^3}{yw_3^3(2Ay^2 - ky^4 - k_1)}, \tag{6}$$

which is equal to Eq. (5) if k_3 is substituted with k_2. Equation (5) admits a three-dimensional Lie symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. This suggests us to apply the general method described in Ref. 10, valid for any second-order ordinary differential equation exhibiting $\mathfrak{sl}(2,\mathbb{R})$ symmetry. If we solve Eq. (5) with respect to the constant k_2 and derive it once with respect to y, then we obtain the following third-order equation,

$$\frac{\mathrm{d}^3 w_2}{\mathrm{d}y^3} = -\frac{3}{w_2}\frac{\mathrm{d}w_2}{\mathrm{d}y}\frac{\mathrm{d}^2 w_2}{\mathrm{d}y^2} + \frac{3(ky^4 - k_1)}{y^2 w_2(2Ay^2 - ky^4 - k_1)}\left(yw_2\frac{\mathrm{d}^2 w_2}{\mathrm{d}y^2} + y\left(\frac{\mathrm{d}w_2}{\mathrm{d}y}\right)^2 - w_2\frac{\mathrm{d}w_2}{\mathrm{d}y}\right), \tag{7}$$

which admits a seven-dimensional Lie symmetry algebra and is therefore linearizable (see, e.g., Ref. 9, p. 244). A two-dimensional Abelian intransitive subalgebra is that generated by the operators

$$\frac{1}{w_2}\partial_{w_2}, \qquad \frac{A - ky^2}{w_2}\partial_{w_2}. \tag{8}$$

Following Lie's classification (Ref. 8, p. 405), if we transform them into their canonical form, i.e., $\partial_u, v\partial_u$, then we obtain that the new dependent and independent variables are given by $u = w_2^2/2,\ v = A - ky^2$, and Eq. (7) becomes linear, i.e.,

$$\frac{\mathrm{d}^3 u}{\mathrm{d}v^3} = \frac{3v}{A^2 - kk_1 - v^2}\frac{\mathrm{d}^2 u}{\mathrm{d}v^2}. \tag{9}$$

Integration of the latter yields the general solution

$$u = b_1\sqrt{A^2 - kk_1 - v^2} + b_2 v + b_3 \tag{10}$$

with $b_j (j = 1, 2, 3)$ arbitrary constants. Thus, the general solution of Eq. (7) is

$$w_2 = \pm\sqrt{2(\sqrt{2Aky^2 - k^2y^4 - kk_1}b_1 + Ab_2 - b_2ky^2 - b_3)}, \tag{11}$$

which substituted into (5) obviously yields that one of the three arbitrary constants b_j is not at all arbitrary and, in particular, depends on k_2, i.e.,

$$b_3 = \frac{1}{2\sqrt{k}}\sqrt{4A^2b_1^2k + 4\,A^2b_2^2k - 4b_1^2k^2k_1 - 4b_2^2k^2k_1 + k_2}. \tag{12}$$

Consequently, the general solution of Eq. (5) is

$$w_2 = \pm\sqrt{2\left(\sqrt{2Aky^2 - k^2y^4 - kk_1}b_1 + Ab_2 - b_2ky^2 - \frac{1}{2\sqrt{k}}\sqrt{4A^2b_1^2k + 4\,A^2b_2^2k - 4b_1^2k^2k_1 - 4b_2^2k^2k_1 + k_2}\right)}, \tag{13}$$

and replacing k_2 with k_3, and b_j with a_j, yields the general solution of Eq. (6), i.e.,

$$w_3 = \pm\sqrt{2\left(\sqrt{2Aky^2 - k^2y^4 - kk_1}a_1 + Aa_2 - a_2ky^2 - \frac{1}{2\sqrt{k}}\sqrt{4A^2a_1^2k + 4\,A^2a_2^2k - 4a_1^2k^2k_1 - 4a_2^2k^2k_1 + k_3}\right)}, \tag{14}$$

with a_1, a_2 being arbitrary constants. Thus, we have derived w_2 and w_3 as functions of $y = w_1$, namely, the general solution of system (3), since p_2 and p_3 can be obtained by substituting w_2 and w_3 into the first and second equations in (3), respectively. Of course, integrating the first Hamiltonian equation in (2), i.e.,

$$\dot{w}_1 = \frac{\sqrt{2}}{w_1}\sqrt{2Aw_1^2 - kw_1^4 - k_1}, \tag{15}$$

will yield the general solution of the Hamiltonian equations (2).

Therefore, we have shown that the maximally superintegrable Hamiltonian system (2) hides twice a third-order linear equation (9) and, consequently, that its general solution can be derived by substitutions and a final integration by quadratures. All of these have been accomplished without making use of any of the other four first integrals derived in Ref. 7 of which two are given in cartesian and two in spherical polar coordinates.

However, we are more interested in showing that all the maximally superintegrable Hamiltonian systems hide linear equations, than in determining their general solutions.

We observe that as the Hamiltonian H_1 contains three copies of the one-dimensional caged oscillator, all variables w_i lead to the same reduction. The procedure does not depend whether the frequencies are commensurable or not, indicating that the same reduction is valid for the case where the oscillator is not isotropic. This shows, in particular, that the superintegrable caged anisotropic oscillator with Hamiltonian,

$$H_1^{(\ell,m,n)} = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + k\left(\ell^2w_1^2 + m^2w_2^2 + n^2w_3^2\right) + \frac{k_1}{w_1^2} + \frac{k_2}{w_2^2} + \frac{k_3}{w_3^2}, \tag{16}$$

introduced in Ref. 11 also hides a third-order linear equation of the type (9), leading to the linearization of the system. In addition, as the Hamiltonian (16) always admits two independent integrals of orders $2(\ell + m - 1)$ and $2(\ell + n - 1)$ in the momenta, respectively, it follows that the reduction method is not dependent on the degree of the first integrals, or whether these arise from the separation of variables of the Hamilton–Jacobi equation, as it was shown in other such instances in Refs. 5 and 4.

B. The potential $V_{II}(w_1, w_2, w_3)$

In Cartesian coordinates, the Hamiltonian is given by

$$H_2^{(cc)} = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) - \frac{k}{\sqrt{w_1^2 + w_2^2 + w_3^2}} + \frac{k_1}{w_1^2} + \frac{k_2}{w_2^2}. \tag{17}$$

As the variables w_i in the potential are nontrivially coupled, it is convenient to reformulate the Hamiltonian in spherical coordinates in order to largely simplify the computations in the reduction. In spherical polar coordinates, we obtain

$$H_2 = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{r^2\sin^2(\theta)} + \frac{p_\theta^2}{r^2}\right) - \frac{k}{r} + \frac{k_1}{r^2\sin^2(\theta)\cos^2(\phi)} + \frac{k_2}{r^2\sin^2(\theta)\sin^2(\phi)}, \tag{18}$$

yielding the following Hamiltonian equations:

$$\begin{cases} \dot{r} &= p_r, \\ \dot{\phi} &= \dfrac{p_\phi}{r^2\sin^2(\theta)}, \\ \dot{\theta} &= \dfrac{p_\theta}{r^2}, \\ \dot{p}_r &= \dfrac{p_\phi^2}{r^3\sin^2(\theta)} + \dfrac{p_\theta^2}{r^3} - \dfrac{k}{r^2} + \dfrac{2k_1}{r^3\sin^2(\theta)\cos^2(\phi)} + \dfrac{2k_2}{r^3\sin^2(\theta)\sin^2(\phi)}, \\ \dot{p}_\phi &= \dfrac{2k_2\cos(\phi)}{r^2\sin^2(\theta)\sin^3(\phi)} - \dfrac{2k_1\sin(\phi)}{r^2\sin^2(\theta)\cos^3(\phi)}, \\ \dot{p}_\theta &= \cos(\theta)\left(\dfrac{p_\phi^2}{r^2\sin^3(\theta)} + \dfrac{k_1}{r^2\sin^3(\theta)\cos^2(\phi)} + \dfrac{k_2}{r^2\sin^3(\theta)\sin^2(\phi)}\right). \end{cases} \tag{19}$$

We apply again the reduction method.[2] If we choose ϕ as a new independent variable y, then system (19) reduces to the following five equations:

$$\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}y} &= \dfrac{r^2\sin^2(\theta)p_r}{p_\phi}, \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}y} &= \dfrac{\sin^2(\theta)p_\theta}{p_\phi}, \\ \dfrac{\mathrm{d}p_r}{\mathrm{d}y} &= \dfrac{p_\phi}{r} + \dfrac{\sin^2(\theta)p_\theta^2}{rp_\phi} - \dfrac{k\sin^2(\theta)}{p_\phi} + \dfrac{2k_1}{r\cos^2(y)p_\phi} + \dfrac{2k_2}{r\sin^2(y)p_\phi}, \\ \dfrac{\mathrm{d}p_\phi}{\mathrm{d}y} &= \dfrac{2k_2\cos(y)}{\sin^3(y)p_\phi} - \dfrac{2k_1\sin(y)}{\cos^3(y)p_\phi}, \\ \dfrac{\mathrm{d}p_\theta}{\mathrm{d}y} &= \dfrac{\cot(\theta)}{p_\phi}\left(p_\phi^2 + \dfrac{k_1}{\cos^2(y)} + \dfrac{k_2}{\sin^2(y)}\right). \end{cases} \tag{20}$$

The fourth equation of system (20) can be integrated to yield

$$\frac{p_\phi^2}{2} + \frac{k_1}{\cos^2(y)} + \frac{k_2}{\sin^2(y)} = A = \text{const.} \implies p_\phi = \pm\sqrt{2}\sqrt{A - \frac{k_1}{\cos^2(y)} - \frac{k_2}{\sin^2(y)}}. \tag{21}$$

We observe that this is identical to the first integral I_2 given in Ref. 7. Deriving p_θ from the second equation of system (20), i.e.,

$$p_\theta = \frac{\sqrt{2}}{\sin^2(\theta)}\frac{\mathrm{d}\theta}{\mathrm{d}y}\sqrt{A - \frac{k_1}{\cos^2(y)} - \frac{k_2}{\sin^2(y)}}, \tag{22}$$

and replacing it into the fifth equation lead to the following second-order equation in θ:

$$\begin{aligned} \frac{\mathrm{d}^2\theta}{\mathrm{d}y^2} &= 2\cot(\theta)\left(\frac{\mathrm{d}\theta}{\mathrm{d}y}\right)^2 + \frac{A\cos^2(y)\sin^2(y)}{A\cos^2(y)\sin^2(y) - k_1\sin^2(y) - k_2\cos^2(y)}\sin(\theta)\cos(\theta) \\ &\quad + \frac{k_1\sin^4(y) - k_2\cos^4(y)}{\cos(y)\sin(y)\left(A\cos^2(y)\sin^2(y) - k_1\sin^2(y) - k_2\cos^2(y)\right)}\frac{\mathrm{d}\theta}{\mathrm{d}y}. \end{aligned} \tag{23}$$

This equation admits an eight-dimensional Lie symmetry algebra and is therefore linearizable. One simple symmetry is $-\cos(\theta)\sin(\theta)\partial_\theta$, which can be transformed into the normal form $u\partial_u$ by the change of the dependent variable $u = -\cot(\theta)$. Then, Eq. (23) becomes the following linear equation:

$$u'' = \frac{(k_1\sin(y)^4 - k_2\cos(y)^4)u' - A\sin(y)^3\cos(y)^3 u}{\sin(y)\cos(y)(A\sin(y)^2\cos(y)^2 - k_1\sin(y)^2 - k_2\cos(y)^2)}. \tag{24}$$

Now, deriving p_r from the first equation of system (20) and replacing it into the third equation yield a second-order equation in r, the coefficients of which involve the general solution of Eq. (23). It is explicitly given by

$$\frac{d^2 r}{dy^2} - \frac{2}{r}\left(\frac{dr}{dy}\right)^2 + \mathcal{F}(y)\frac{dr}{dt} + \mathcal{G}(y)r + \mathcal{H}(y)r^2 = 0, \tag{25}$$

where

$$\begin{aligned} \mathcal{F}(y) &= \frac{2(k_1 - k_2)\cos^4(y) - k_1\cos(2y)}{\sin(2y)\left(A\sin^2(y)\cos^2(y) - k_1\sin^2(y) - k_2\cos^{(2)}y\right)} - 2\cot(\theta)\frac{d\theta}{dy}, \\ \mathcal{G}(y) &= \frac{\sin^2(\theta)\sin^2(y)\cos^2(y)}{A\cos^4(y) + (k_2 - k_1 - A)\cos^2(y) + k_1} - \left(\frac{d\theta}{dy}\right)^2, \\ \mathcal{H}(y) &= \frac{k\sin^2(\theta)\sin^2(y)}{2A\cos^4(y) + 2(k_2 - k_1 - A)\cos^2(y) + 2k_1}. \end{aligned}$$

Equation (25) admits an eight-dimensional Lie symmetry algebra $\mathfrak{sl}(3,\mathbb{R})$, and therefore, it is linearizable. Indeed, it can be transformed into the following linear equation by means of the change of the dependent variable $z = -r^{-1}$:

$$\frac{d^2 z}{dy^2} + \mathcal{F}(y)\frac{dz}{dy} - \mathcal{G}(y)z + \mathcal{H}(y) = 0. \tag{26}$$

The reduction to the canonical form $w''(s) = 0$ can be obtained using a generalized Kummer–Liouville transformation

$$z = P(y)w(s) + Q(y), \quad s = s(y), \tag{27}$$

where

$$\frac{ds}{dy} = \frac{2(A\cos^4(y) + k_1 + (k_2 - k_1 - A)\cos^2(y))}{\sin^2(\theta)\sin(2y)P^2(y)} \tag{28}$$

and $P(y)$ and $Q(y)$ are a solution of the homogeneous part of Eq. (26) and a particular solution, respectively.[12] We conclude that the maximally superintegrable Hamiltonian system (19) is linearizable since it hides two second-order linear equations (24) and (26).

C. The potential $V_{III}(w_1, w_2, w_3)$

The Hamiltonian

$$H_3 = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{k_1 w_1}{w_2^2\sqrt{w_1^2 + w_2^2}} + \frac{k_2}{w_2^2} + \frac{k_3}{w_3^2} \tag{29}$$

yields the Hamiltonian equations

$$\begin{cases} \dot{w}_1 = p_1, \\ \dot{w}_2 = p_2, \\ \dot{w}_3 = p_3, \\ \dot{p}_1 = -\dfrac{k_1}{(w_1^2 + w_2^2)^{3/2}}, \\ \dot{p}_2 = \dfrac{k_1 w_1(2w_1^2 + 3w_2^2)}{w_2^3(w_1^2 + w_2^2)^{3/2}} + \dfrac{2k_2}{w_2^3}, \\ \dot{p}_3 = \dfrac{2k_3}{w_3^3}. \end{cases} \tag{30}$$

If we derive p_3 from the third equation of system (30) and replace it into the sixth equation, we obtain the following second-order equation in w_3:

$$\ddot{w}_3 = \frac{2k_3}{w_3^3}. \tag{31}$$

It admits a three-dimensional Lie symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$ generated by the following operators:

$$t^2\partial_t + tw_3\partial_{w_3}, \quad 2t\partial_t + w_3\partial_{w_3}, \quad \partial_t. \tag{32}$$

Solving Eq. (31) with respect to k_3 and deriving it once with respect to t lead to the third-order equation,

$$\dddot{w}_3 = -\frac{3\dot{w}_3\ddot{w}_3}{w_3}, \tag{33}$$

which is easily seen to admit a seven-dimensional Lie symmetry algebra generated by the following operators:

$$\mathbf{X}_1 = t^2\partial_t + tw_3\partial_{w_3},\ \mathbf{X}_2 = t\partial_t,\ \mathbf{X}_3 = \partial_t,\ \mathbf{X}_4 = w_3\partial_{w_3},$$
$$\mathbf{X}_5 = \frac{t^2}{w_3}\partial_{w_3},\ \mathbf{X}_6 = \frac{t}{w_3}\partial_{w_3},\ \mathbf{X}_7 = \frac{1}{w_3}\partial_{w_3}$$

and is therefore linearizable. We find that a two-dimensional non-Abelian intransitive subalgebra is that generated by $\mathbf{X}_4$ and $\mathbf{X}_7$, and following Lie's classification,[8] if we transform these operators into their canonical form, i.e., $\partial_u, u\partial_u$, then we obtain that the new dependent variable $u = w_3^2/2$ transforms equation (33) into the linear equation

$$\dddot{u} = 0, \tag{34}$$

which, solved and replaced into Eq. (31), yields

$$w_3 = \pm\sqrt{A_1t^2 + A_2t + \frac{A_2^2 + 8k_3}{4A_1}} \tag{35}$$

with $A_n(n = 1, 2)$ arbitrary constants.

About the other four equations of system (30), we make a simplifying substitution, i.e.,

$$w_2 = \sqrt{r_2^2 - w_1^2}. \tag{36}$$

If we derive p_1 from the first equation of system (30) and replace it into the fourth equation, we obtain the following second-order equation in w_1:

$$\ddot{w}_1 = -\frac{k_1}{r_2^3}. \tag{37}$$

Similarly, deriving p_2 from the second equation of system (30) and replacing it into the fifth equation yield the second-order equation in r_2,

$$\ddot{r}_2 = \frac{w_1^2\dot{r}_2^2}{r_2(r_2^2 - w_1^2)} - \frac{2w_1\dot{w}_1\dot{r}_2}{r_2^2 - w_1^2} + \frac{\dot{w}_1^2r_2^3 + 2k_1w_1 + 2k_2r_2}{r_2^2(r_2^2 - w_1^2)}. \tag{38}$$

The system of Eqs. (37) and (38) admits a three-dimensional Lie symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$ generated by the following operators:

$$t^2\partial_t + tw_1\partial_{w_1} + tr_2\partial_{r_2}, \quad 2t\partial_t + w_1\partial_{w_1} + r_2\partial_{r_2}, \quad \partial_t. \tag{39}$$

If we solve system (37) and (38) with respect to k_1 and k_2 and derive once with respect to t, the following system of two separated third-order equations is obtained:

$$\dddot{w}_1 = -\frac{3\dot{w}_1\ddot{w}_1}{w_1}, \tag{40}$$
$$\dddot{r}_2 = -\frac{3\dot{r}_2\ddot{r}_2}{r_2}, \tag{41}$$

namely, both w_1 and r_2 satisfy the same Eq. (33) as w_3. As a consequence, the transformations $u_1 = w_1^2/2$ and $u_2 = r_2^2/2$ take Eqs. (40) and (41) into the linear equations,

$$\dddot{u}_1 = 0, \tag{42}$$
$$\dddot{u}_2 = 0. \tag{43}$$

We conclude that the maximally superintegrable Hamiltonian system (30) hides (three times) the third-order linear equation (34).

We remark that, as shown in Ref. 7, the Hamilton–Jacobi equation corresponding to the Hamiltonian (29) is not separable in Cartesian coordinates.

In contrast to the previous Hamiltonians, in this case, the shortest and simplest ansatz to linearize the system is a direct approach, i.e., without replacing the independent variable t by a new one and reducing the number of equations, although the reduction method can also be applied (see the Appendix).

D. The potential $V_{IV}(w_1, w_2, w_3)$

The Hamiltonian

$$H_4 = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{k_1 w_1}{w_2^2\sqrt{w_1^2 + w_2^2}} + \frac{k_2}{w_2^2} + k_3 w_3 \tag{44}$$

can also be treated directly as the case H_3, which turns out to be the shortest way to linearize the system. Its linearization using the reduction method is given in the Appendix.

The equations associated with the Hamiltonian are given by

$$\begin{cases} \dot{w}_1 = p_1, \\ \dot{w}_2 = p_2, \\ \dot{w}_3 = p_3, \\ \dot{p}_1 = -\dfrac{k_1}{(w_1^2 + w_2^2)^{3/2}}, \\ \dot{p}_2 = \dfrac{k_1 w_1 (2w_1^2 + 3w_2^2)}{w_2^3 (w_1^2 + w_2^2)^{3/2}} + \dfrac{2k_2}{w_2^3}, \\ \dot{p}_3 = -k_3. \end{cases} \tag{45}$$

The last linear equation can be immediately integrated to give $p_3 = -k_3 t + A_1$, which replaced into the third equation of the Hamiltonian system (45) yields a linear equation in w_3 whose solution is $w_3 = -\frac{k_3}{2}t^2 + A_1 t + A_2$.

The remaining four equations of system (45) are treated in exactly the same way as in the preceding case, starting with the simplifying substitution (36). Deriving p_1 from the first equation of system (45) and replacing it into the fourth equation yield the second-order equation (37), as well as deriving p_2 from the second equation of system (45) and replacing it into the fifth equation lead to the second-order equation (38).

It follows that the maximally superintegrable Hamiltonian system (45) hides (twice) the third-order linear equation (34).

E. The potential $V_V(w_1, w_2, w_3)$

Although the Hamiltonian

$$H_5 = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + k\left(w_1^2 + w_2^2\right) + 4kw_3^2 + \frac{k_1}{w_1^2} + \frac{k_2}{w_2^2} \tag{46}$$

can be seen, as H_1, as a three-dimensional extension of a plane system containing two copies of the one-dimensional caged oscillator, the simplest way to linearize the system is, again, the direct approach. For H_5, we obtain the Hamiltonian equations,

$$\begin{cases} \dot{w}_1 = p_1, \\ \dot{w}_2 = p_2, \\ \dot{w}_3 = p_3, \\ \dot{p}_1 = 2\dfrac{k_1 - kw_1^4}{w_1^3}, \\ \dot{p}_2 = 2\dfrac{k_2 - kw_2^4}{w_2^3}, \\ \dot{p}_3 = -8kw_3. \end{cases} \tag{47}$$

Deriving p_3 from the third equation of system (47) and replacing it into the sixth equation yield the following linear second-order equation in w_3:

$$\dddot{w}_3 = -8kw_3, \tag{48}$$

whose general solution is given by

$$w_3 = A_1\cos(2\sqrt{2k}t) + A_2\sin(2\sqrt{2k}t), \tag{49}$$

if we assume $k > 0$.

Deriving p_1 from the first equation of system (47) and replacing it into the fourth equation yield the following second-order equation in w_1:

$$\ddot{w}_1 = 2\frac{k_1 - kw_1^4}{w_1^3}. \tag{50}$$

Deriving p_2 from the second equation of system (47) and replacing it into the fifth equation yield the following second-order equation in w_2:

$$\ddot{w}_2 = 2\frac{k_2 - kw_2^4}{w_2^3}, \tag{51}$$

which is equal to Eq. (50) if k_2 is substituted with k_1. Equation (50) admits a three-dimensional Lie symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$ generated by the following operators:

$$\partial_t,\ \cos(2\sqrt{2k}t)\partial_t - \sqrt{2k}\sin(2\sqrt{2k}t)w_1\partial_{w_1},\ \sin(2\sqrt{2k}t)\partial_t + \sqrt{2k}\cos(2\sqrt{2k}t)w_1\partial_{w_1}. \tag{52}$$

If we solve Eq. (50) with respect to k_1 and derive once with respect to t, then we obtain the following third-order equation:

$$\dddot{w}_1 = -\dot{w}_1\frac{3\ddot{w}_1 + 8kw_1}{w_1}, \tag{53}$$

which admits a seven-dimensional Lie symmetry algebra generated by the following operators:

$$\begin{aligned}
&\mathbf{Y}_1 = \partial_t, \quad \mathbf{Y}_2 = \cos(2\sqrt{2k}t)\partial_t - \sqrt{2k}\sin(2\sqrt{2k}t)w_1\partial_{w_1},\\
&\mathbf{Y}_3 = \sin(2\sqrt{2k}t)\partial_t + \sqrt{2k}\cos(2\sqrt{2k}t)w_1\partial_{w_1}, \quad \mathbf{Y}_4 = w_1\partial_{w_1},\\
&\mathbf{Y}_5 = \frac{\cos(2\sqrt{2k}t)}{w_1}\partial_{w_1}, \quad \mathbf{Y}_6 = \frac{\sin(2\sqrt{2k}t)}{w_1}\partial_{w_1}, \quad \mathbf{Y}_7 = \frac{1}{w_1}\partial_{w_1}
\end{aligned}$$

and is therefore linearizable. A two-dimensional non-Abelian intransitive subalgebra is generated by $\mathbf{Y}_4$ and $\mathbf{Y}_7$. Bringing them into the canonical form $u\partial_u$, ∂_u we have that the new dependent variable $u = w_1^2/2$ transforms equation (53) into the linear equation,

$$\dddot{u} = -8k\dot{u}. \tag{54}$$

The case $k < 0$ is completely analogous and leads to the same result. We conclude that the maximally superintegrable Hamiltonian system (47) hides (twice) the third-order linear equation (54).

We observe that system (47) is quite similar to system (2), as H_5 contains two copies of a one-dimensional caged oscillator with the addition of an ordinary oscillator. Therefore, as both systems contain a common two-dimensional subsystem and merely differ in the extension to three dimensions, it would have been conceivable to linearize system (47) by using the same reduction as for system (2), although the computations are more involved. Indeed, the second-order equation that can be obtained by considering $w_1 = y$ a new independent variable is identical to Eq. (5), while the resulting second-order equation in w_3 would be

$$\frac{\mathrm{d}^2w_3}{\mathrm{d}y^2} = -\frac{(k_1 - ky^4)}{k_1 + ky^5 - 2Ay^3}\frac{\mathrm{d}w_3}{\mathrm{d}y} - \frac{4ky^2}{k_1 + ky^4 - 2Ay^2}. \tag{55}$$

This linear equation admits the maximal symmetry $\mathfrak{sl}(3,\mathbb{R})$ and thus can be reduced to the free form $u'' = 0$ by a point transformation.[8,9] Hence, we conclude that system (47) hides two linear equations of second- and third-order, respectively.

III. CONCLUDING REMARKS

Using a combination of various techniques of Lie point symmetries, notably the classical Lie criterion addressing the linearization of a system that admits a Lie point symmetry algebra of maximal dimension,[8,9] the reduction method for autonomous systems introduced in Ref. 2, as well as a method to linearize second-order ordinary differential equations with $\mathfrak{sl}(2,\mathbb{R})$ symmetry,[10] we have shown that all maximally superintegrable systems on flat space classified in Refs. 7 and 11 admit hidden symmetries leading to linearization. For some of the potentials, two possible ways for linearization have been presented, a direct approach that does not involve a change of the independent variable, as well as the application of the reduction method of Ref. 2, which may result in more complicated equations and computations. It has been illustrated that the linearization process applied to each of the potential is independent on the separating coordinates of the system and does not rely on the degree of the first integrals. We recall that although in Ref. 7, the admitting integrals are at most quadratic in the canonical momenta, on the other end in Ref. 11, there are two independent integrals of higher orders. Consequently, it follows that the reduction method is not dependent on the degree of the first integrals or whether these arise from the separation of variables of the Hamilton–Jacobi equation, as it was shown in other such instances in Refs. 5 and 4. This may be a hint that other maximally superintegrable systems for which the first integrals are of higher-order may also hide some linear equations.

Regardless of their particular degree, we have not used explicitly the fact that the number of independent first integrals is maximal, which leads us to ask if the procedure is also valid for Hamiltonian systems in three-dimensional space having less than five independent constants of the motion. The following example illustrates that a minimally superintegrable system can also be linearizable. To this extent, consider the Hamiltonian[7]

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + 4kw_1^2 + kw_2^2 + \frac{k_2}{w_2^2} + F(w_3), \tag{56}$$

where k, k_2 are constants and $F(w_3)$ is an arbitrary function. The Hamiltonian system

$$\begin{cases} \dot{w}_1 = p_1, \\ \dot{w}_2 = p_2, \\ \dot{w}_3 = p_3, \\ \dot{p}_1 = -8kw_1, \\ \dot{p}_2 = -2kw_2 + \dfrac{2k_2}{w_2^3}, \\ \dot{p}_3 = -\dfrac{\mathrm{d}F(w_3)}{\mathrm{d}w_3} \end{cases} \tag{57}$$

is minimally superintegrable with four globally defined first integrals, separable in both Cartesian and parabolic cylindrical coordinates.[7] Applying the reduction method of Ref. 2 with w_3 as new independent variable y, the system is reduced to the following five equations:

$$\begin{cases} \dfrac{\mathrm{d}w_1}{\mathrm{d}y} = \dfrac{p_1}{p_3}, \\ \dfrac{\mathrm{d}w_2}{\mathrm{d}y} = \dfrac{p_2}{p_3}, \\ \dfrac{\mathrm{d}p_1}{\mathrm{d}y} = -\dfrac{8kw_1}{p_3}, \\ \dfrac{\mathrm{d}p_2}{\mathrm{d}y} = -\dfrac{2kw_2}{p_3} + \dfrac{2k_2}{w_2^3 p_3}, \\ \dfrac{\mathrm{d}p_3}{\mathrm{d}y} = -\dfrac{F'(y)}{p_3}. \end{cases} \tag{58}$$

The last equation is easily integrable and provides

$$p_3 = \pm\sqrt{A - 2F(y)}. \tag{59}$$

We take into consideration the positive root, without loss of generality. If we now derive p_1 and p_2 from the first and second equations of system (58) and insert them into the third and fourth equations, respectively, then we are led to the following two second-order equations:

$$w_1''(y) = \frac{F'(y)w_1'(y) - 8kw_1(y)}{A - 2F(y)} \tag{60}$$

and

$$w_2''(y) = \frac{F'(y)w_2'(y)w_2^3(y) - 2kw_2^4(y) + 2k_2}{w_2^3(y)(A - 2F(y))}. \tag{61}$$

Equation (60) is linear and thus, as shown in Ref. 8, admits an eight-dimensional Lie symmetry algebra, i.e., $\mathfrak{sl}(3,\mathbb{R})$. Equation (61) can be shown to admit a three-dimensional Lie symmetry algebra [unless $k_2 = 0$, for Eq. (61) becomes linear], i.e., $\mathfrak{sl}(2,\mathbb{R})$, generated by the following operators:

$$2S(y)\partial_y + \left(S'(y) + \frac{F'(y)S(y)}{A - 2F(y)}\right)w_2\partial_{w_2}, \tag{62}$$

where $S(y)$ is the solution of the following third-order linear equation:

$$\begin{aligned}(A - 2F)S''' &+ \left(2(A - 2f)^2F'' + 3(A - 2F)F'^2 + 8k(A - 2F)^2\right)S' \\ &+ \left((A - 2F)^2F''' + 5(A - 2F)F'F'' + 6F'^3 + 8k(A - 2F)\right)S = 0.\end{aligned} \tag{63}$$

If we solve Eq. (61) with respect to the constant k_2, i.e.,

$$k_2 = \frac{w_2^3}{2}\left((A - 2F)w_2'' - F'w_2' + 2kw_2\right), \tag{64}$$

and derive it with respect to y, the resulting third-order equation, i.e.,

$$w_2''' = \frac{1}{(A - 2F)w_2}\left(3\left(F'w_2 - (A - 2F)w_2'\right)w_2'' + \left(3F'w_2' + (F'' - 8k)w_2\right)w_2'\right), \tag{65}$$

admits a seven-dimensional Lie symmetry algebra, showing that it is linearizable. Indeed, if we make the transformation $u = w_2^2$, then Eq. (65) becomes the following linear equation:

$$u''' = \frac{1}{(A - 2F)}\left(3F'u'' + (F'' - 8k)u'\right). \tag{66}$$

We conclude that the Hamiltonian system (57) hides a second-order linear equation and a third-order linear equation, regardless of the arbitrary function $F(w_3)$.

In this context, it constitutes a natural question whether the remaining equivalence classes of minimally superintegrable Hamiltonian systems classified in Ref. 7 are also linearizable. Work in this direction is currently in progress. An answer in the positive would indicate that the Lie symmetry method, a technique that has somehow been neglected in the context of superintegrable systems, is an approach potentially relevant to their analysis as the separability problem of the Hamilton–Jacobi equation.

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APPENDIX: LINEARIZATION OF THE HAMILTONIANS H_3 AND H_4 BY THE REDUCTION METHOD

We show that the superintegrable systems with Hamiltonians H_3 and H_4 that have been treated directly, without explicitly reducing the system with respect to a new variable, can also be linearized using the reduction method. However, this procedure leads to more complicated computations than the direct approach.

First of all, we observe that the Hamiltonians H_3 and H_4 can be seen as an extension of the two-dimensional Hamiltonian

$$H_e = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{k_1 w_1}{w_2^2\sqrt{w_1^2 + w_2^2}} + \frac{k_2}{w_2^2}. \tag{A1}$$

More precisely, we have

$$H_3 = H_e + \frac{1}{2}p_3^2 + \frac{k_3}{w_3^2}, \quad H_4 = H_e + \frac{1}{2}p_3^2 + k_3 w_3. \tag{A2}$$

Then, we should begin by looking at the common two-dimensional Hamiltonian H_e and then analyzing the three-dimensional cases. In this case, the use of polar coordinates is best suited, and indeed, if we introduce them, i.e.,

$$w_1 = r\cos(\phi), \quad w_2 = r\sin(\phi),$$

then the Hamiltonian H_e is transformed as

$$H_{ep} = \frac{1}{2}\left(p_r^2 + \frac{p_\phi}{r^2}\right) + \frac{k_1\cot(\phi)}{\sin(\phi)r^2} + \frac{k_2}{r^2\sin(\phi)^2}. \tag{A3}$$

This Hamiltonian is a particular case of a more general two-dimensional Hamiltonians

$$H_{ep} = \frac{1}{2}\left(p_r^2 + \frac{p_\phi}{r^2}\right) + \frac{\alpha}{r} + \frac{1}{r^2}F(\phi) \tag{A4}$$

which was shown in Ref. 4 to hide a second-order linear equation for any function $F(\phi)$. The reduction is carried out considering ϕ as a new independent variable y from which it follows that [see Eq. (67) in Ref. 4 and the following discussion fora detailed discussion of this potential]

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{p_\phi}{r^2}, \quad p_\phi = \pm\sqrt{A - 2F(\phi)}. \tag{A5}$$

This same reduction can be applied to the extended three-dimensional case if we express the Hamiltonians H_3 and H_4 in cylindrical coordinates,

$$w_1 = r\cos(\phi), \quad w_2 = r\sin(\phi), \quad w_3 = z,$$

so that the Hamiltonians are given by

$$H_3 = H_{ep} + \frac{1}{2}p_z^2 + \frac{k_3}{z^2}, \quad H_4 = H_{ep} + \frac{1}{2}p_z^2 + k_3 z, \tag{A6}$$

respectively. Then, the equations for $\dot{z}$ and $\dot{p}_z$ with respect to the new independent variable $\phi = y$ become the following equations for the Hamiltonian H_3:

$$\frac{\mathrm{d}z}{\mathrm{d}y} = \frac{r^2 p_z}{p_\phi}, \quad \frac{\mathrm{d}p_z}{\mathrm{d}y} = \frac{2k_3 r^2}{p_\phi z^3}, \tag{A7}$$

while those for H_4 become

$$\frac{\mathrm{d}z}{\mathrm{d}y} = \frac{r^2 p_z}{p_\phi}, \quad \frac{\mathrm{d}p_z}{\mathrm{d}y} = -\frac{k_3 r^2}{p_\phi}. \tag{A8}$$

Deriving now p_z from the first equation in (A7), inserting it into the second equation taking into account the expression for p_ϕ, and simplifying the resulting expression yield the following second-order equation for z in the case of the Hamiltonian H_3:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}y^2} = \frac{\mathrm{d}}{\mathrm{d}y}\ln\left(\frac{r^2\sin(y)^{\frac{3}{2}}}{\sqrt{A\sin(y)^3 - k_1\cos(y) - k_2\sin(y)}}\right)\frac{\mathrm{d}z}{\mathrm{d}y} + \frac{2k_3 r^4\sin(y)^3}{(A\sin(y)^3 - k_1\cos(y) - k_2\sin(y))z^3}, \tag{A9}$$

while for the Hamiltonian H_4, it is

$$\frac{\mathrm{d}^2 z}{\mathrm{d}y^2} = \frac{\mathrm{d}}{\mathrm{d}y}\ln\left(\frac{r^2\sin(y)^{\frac{3}{2}}}{\sqrt{A\sin(y)^3 - k_1\cos(y) - k_2\sin(y)}}\right)\frac{\mathrm{d}z}{\mathrm{d}y} - \frac{k_3 r^4\sin(y)^3}{(A\sin(y)^3 - k_1\cos(y) - k_2\sin(y))}. \tag{A10}$$

Equation (A9) can be simplified by means of the additional change of the dependent variable,

$$z = \frac{u\, r\sin(y)^{\frac{3}{4}}}{(A\sin(y)^3 - k_1\cos(y) - k_2\sin(y))^{\frac{1}{4}}}, \tag{A11}$$

leading to the differential equation in u,

$$\frac{\mathrm{d}^2u}{\mathrm{d}y^2} = \mathcal{T}(y)u + \frac{2k_3}{u^3}, \tag{A12}$$

with $\mathcal{T}(y)$ expressed in terms of the known functions r and p_ϕ as

$$\mathcal{T}(y) = \frac{2}{r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}y}\right)^2 - \frac{1}{r}\frac{\mathrm{d}^2r}{\mathrm{d}y^2} + \frac{1}{2p_\phi}\frac{\mathrm{d}^2p_\phi}{\mathrm{d}y^2} - \frac{1}{4p_\phi^2}\left(\frac{\mathrm{d}p_\phi}{\mathrm{d}y}\right)^2 - \frac{1}{rp_\phi}\frac{\mathrm{d}r}{\mathrm{d}y}\frac{\mathrm{d}p_\phi}{\mathrm{d}y}. \tag{A13}$$

Equation (A12) admits a symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, since it is a Pinney equation (see, e.g., Ref. 10). Solving this equation with respect to k_3 and deriving with respect to y lead to the third-order differential equation,

$$\frac{\mathrm{d}^3u}{\mathrm{d}y^3} = -\frac{\mathrm{d}\mathcal{T}}{\mathrm{d}y}u - 4\mathcal{T}(y)\frac{\mathrm{d}u}{\mathrm{d}y} - \frac{3}{u}\frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}^2u}{\mathrm{d}y^2}, \tag{A14}$$

which can be further reduced to a linear equation, regardless of the function $\mathcal{T}(y)$, by means of a change of dependent variable $U = \frac{1}{2}u^2$,

$$\frac{\mathrm{d}^3U}{\mathrm{d}y^3} = -2\frac{\mathrm{d}\mathcal{T}}{\mathrm{d}y}U - 4\mathcal{T}(y)\frac{\mathrm{d}U}{\mathrm{d}y}. \tag{A15}$$

Consequently, the maximally superintegrable Hamiltonian system (30) hides a second-order linear equation [see Eq. (80) in Ref. 4] and a linear third-order equation (A15), while the Hamiltonian system (45) hides two linear second-order equations, i.e., Eq. (80) in Ref. 4 and Eq. (A10).

Thus, we have shown that systems (30) and (45) hide linear equations by two different approaches.

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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