

On the Conformal Geometry of Transverse Riemann-Lorentz Manifolds

E. Aguirre, V. Fernández, J. Lafuente.

Dept. Geometría y Topología, Fac. CC. Matemáticas, UCM.

Plaza de las Ciencias 3, Madrid, Spain.

Mathematics Subject Classification: 53C50, 53B30, 53C15.

November 19, 2018

Abstract

Physical reasons suggested in [2] for the *Quantum Gravity Problem* lead us to study *type-changing metrics* on a manifold. The most interesting cases are *Transverse Riemann-Lorentz Manifolds*. Here we study the conformal geometry of such manifolds.

1 Preliminaries

Let M be a connected manifold, $\dim M = m \geq 2$, and let g be a symmetric covariant tensor field of order 2 on M . Assume that the set Σ of points where g degenerates is not empty. Consider $p \in \Sigma$ and (U, x) a coordinate system around p . We say that g is a *transverse type-changing metric* on p if $d_p(\det(g_{ab})) \neq 0$ (this condition does not depend on the choice of the coordinates). We call (M, g) *transverse type-changing pseudoriemannian manifold* if g is transverse type-changing on every point of Σ . In this case, Σ is a hypersurface of M . Moreover, at every point of Σ there exists a one-dimensional *radical*, that is the subspace $Rad_p(M)$ of $T_p M$ which is g -ortogonal to the whole $T_p M$ (and it can be transverse or tangent to the hypersurface Σ). The *index* of g is constant on every connected component of $\mathbb{M} = M - \Sigma$, thus \mathbb{M} is a union of connected pseudoriemannian manifolds. Locally, Σ separates two pseudoriemannian manifolds whose indices differ in one unit (so we call

Σ *transverse type-changing hypersurface*, in particular Σ is orientable). The most interesting cases are those in which Σ separates a riemannian part from a lorentzian one. We call these cases *transverse Riemann-Lorentz manifolds*.

Let $\tau \in C^\infty(M)$ be such that $\tau|_\Sigma = 0$ and $d\tau|_\Sigma \neq 0$. We say that (locally, around Σ) $\tau = 0$ is an equation for Σ . Given $f \in C^\infty(M)$, it holds: $\tau|_\Sigma = 0 \Leftrightarrow f = k\tau$, for some $k \in C^\infty(M)$. In what follows we shall use this fact extensively.

On \mathbb{M} we have naturally defined all the objects associated to pseudoriemannian geometry, derived from the Levi-Civita connection. In [4], [5], [6], [7] and [1], the extendibility of geodesics, parallel transport and curvatures have been studied. Our aim in the present paper is to study the conformal geometry of transverse Riemann-Lorentz manifolds, including criteria for the extendibility of the *Weyl conformal curvature*.

Let (M, g) be a transverse Riemann-Lorentz manifold. First of all, note that we do not have any Levi-Civita connection ∇ defined on the whole M . However we have ([4]) a unique torsion-free metric *dual connection*

$$\square : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}^*(M)$$

on M defined by a *Koszul-like formula*. On \mathbb{M} it holds $\square_X Y(Z) = g(\nabla_X Y, Z)$, and thus the concepts derived from Levi-Civita connection ∇ (on \mathbb{M}) coincide with those derived from the dual connection \square .

We say that a vectorfield $R \in \mathfrak{X}(M)$ is *radical* if $R_p \in \text{Rad}_p(M) - \{0\}$ for all $p \in \Sigma$. Given a radical vectorfield $R \in \mathfrak{X}(M)$, $\square_X Y(R)|_\Sigma$ only depends on $X|_\Sigma$ and $Y|_\Sigma$, thus we obtain the following well-defined map

$$II^R : \mathfrak{X}_\Sigma \times \mathfrak{X}_\Sigma \rightarrow C^\infty(\Sigma), (X, Y) \mapsto \square_X Y(R)$$

Note that the II^R -orthogonal complement to $\text{Rad}_p(M)$ is $T_p\Sigma$ ([7], 1(a)), thus $X \in \mathfrak{X}_\Sigma$ is tangent to Σ if and only if $II^R(X, R) = 0$.

Because of the properties of \square , the restriction of II^R to vectorfields in $\mathfrak{X}(\Sigma)$ is a well-defined $(0, 2)$ symmetric tensor field $II_\Sigma^R \in S^2(\Sigma)$. Furthermore, since $\square_X Y$ is a one-form on M and the radical is one-dimensional, the condition $II_\Sigma^R = 0$ does not depend on the radical vectorfield R . A *transverse Riemann-Lorentz manifold* is said to be *II-flat* if $II_\Sigma^R = 0$, for some (and thus, for any) radical vectorfield R . It turns out ([7] for transverse, [1] for tangent radical) that M is *II-flat* if and only if all covariant derivatives $\nabla_X Y$, for $X, Y \in \mathfrak{X}(M)$ tangent to Σ , smoothly extend to M . Moreover,

in that case, $\nabla_X Y|_\Sigma$ only depends on $X|_\Sigma$ and $Y|_\Sigma$, thus we obtain another well-defined map

$$III^R : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \rightarrow C^\infty(\Sigma), (X, Y) \mapsto II^R(\nabla_X Y, R)$$

which is a $(0, 2)$ symmetric tensorfield on Σ . A *transverse Riemann-Lorentz II-flat metric* is said to be *III-flat* if $III^R = 0$.

If the radical is tangent, $\nabla_R R$ becomes transverse ([1]); therefore, in order that a II-flat metric becomes III-flat, the radical must be transverse. And we have the following result ([7]), concerning the extendibility of curvature tensors:

Theorem 1 *The covariant curvature K smoothly extends to M if and only if the radical is transverse and g is II-flat, while the Ricci tensor Ric smoothly extends to M if and only if the radical is transverse and g is III-flat.*

2 A Gauss formula for Transverse Riemann-Lorentz Manifolds

Let (M, g) be a transverse Riemann-Lorentz manifold with transverse radical.

Lemma 2 *There exists a unique (canonically defined) radical vectorfield R such that $II^R(R, R) = 1$.*

Proof: Given a radical vectorfield U , consider $R = (II^U(U, U))^{-\frac{1}{3}} \cdot U$, which is a well-defined radical vectorfield (since the radical is transverse). Thus $II^R(R, R) = 1$. Furthermore, if $Z = fR$ is another radical vectorfield such that $II^Z(Z, Z) = 1$, then $1 = II^Z(Z, Z) = f^3 II^R(R, R) = f^3$, and consequently $f = 1$ ♣

Suppose that (M, g) is II-flat. As we said before, given $X, Y \in \mathfrak{X}(\Sigma)$, $\nabla_X Y$ is well-defined. Moreover, $\tan(\nabla_X Y) := \nabla_X Y - III^R(X, Y) \cdot R$ is indeed tangent to Σ , since

$$II^R(R, \tan(\nabla_X Y)) = III^R(X, Y) - III^R(X, Y) II^R(R, R) = 0$$

Lemma 3 *If $X, Y \in \mathfrak{X}(\Sigma)$ and ∇^Σ is the Levi-Civita connection of (Σ, g_Σ) , it holds*

$$\nabla_X Y = \nabla_X^\Sigma Y + III^R(X, Y) \cdot R$$

Proof: Let be $Z \in \mathfrak{X}(\Sigma)$. Since (M, g) is II -flat, $\nabla_X Y$ is well defined and it must hold $\square_X Y(Z) = g(\nabla_X Y, Z) = g_\Sigma(\tan(\nabla_X Y), Z)$. On the other hand, \square has always a good restriction $\square : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \rightarrow \mathfrak{X}^*(\Sigma)$, which must coincide with \square^Σ , the unique torsion-free metric dual connection on (Σ, g_Σ) . Since (Σ, g_Σ) is riemannian, it must hold $\square_X^\Sigma Y(Z) = g_\Sigma(\nabla_X^\Sigma Y, Z)$, and the result follows ♣

The existence of a canonical radical vectorfield leads to the following Gauss formula.

Proposition 4 *Let (M, g) be a transverse Riemann-Lorentz manifold with transverse radical and II -flat. Then Σ is "totally geodesic" in the sense that, if $X, Y, Z, T \in \mathfrak{X}(\Sigma)$ it holds*

$$K(X, Y, Z, T) = K^\Sigma(X, Y, Z, T)$$

where K^Σ is the covariant curvature of Σ .

Proof: As we said in the proof of previous lemma we have, for $X, Y, Z, T \in \mathfrak{X}(\Sigma)$: $\square_X Y(Z) = \square_X^\Sigma Y(Z)$, where \square^Σ is the dual connection of (Σ, g_Σ) . Moreover, since $\square_X R(T) = -\square_X T(R) = -II^R(X, T) = 0$, again previous lemma leads to

$$\square_X(\nabla_Y Z)(T) = \square_X(\nabla_Y^\Sigma Z + III^R(Y, Z)R)(T) = \square_X^\Sigma(\nabla_Y^\Sigma Z)(T)$$

what gives the result ♣

Corollary 5 *Let (M, g) be a transverse Riemann-Lorentz manifold with transverse radical. If (\mathbb{M}, g) is flat, then (M, g) is III -flat and Σ is flat.*

Proof: If $K = 0$ then $Ric = 0$. In particular, Ric extends to M , thus by Theorem 1, (M, g) is III -flat. By Proposition 4, Σ is flat ♣

We now restate Theorem 9 of [5] in the following terms (the flatness of Σ , being a consequence of the Corollary, needs not be included as an extra hypothesis):

Theorem 6 *Let (M, g) be a transverse Riemann-Lorentz manifold. Then, M is locally flat around Σ if and only if, around every singular point $p \in \Sigma$, there exists a coordinate system (\mathbb{U}, x) such that $g = \sum_{i=0}^{m-1} (dx^i)^2 + \tau(dx^m)^2$, where $\tau = 0$ is a local equation for Σ .*

3 Conformal geometry and the extendibility of Weyl curvature

Let us consider a transverse Riemann-Lorentz manifold (M, g) and the family $\mathcal{C} = \{e^{2f}g : f \in C^\infty(M)\}$. Take $\bar{g} = e^{2f}g \in \mathcal{C}$. Then (M, \bar{g}) is also a transverse Riemann-Lorentz manifold, and $\bar{\Sigma} = \Sigma$. Moreover, for each singular point $p \in \Sigma$ the radical subspaces are the same: $\overline{Rad}_p(M) = Rad_p(M)$. We say that (M, \mathcal{C}) is a *transverse Riemann-Lorentz conformal manifold* if some (and thus any) $g \in \mathcal{C}$ is transverse Riemann-Lorentz. Let (M, \mathcal{C}) be a transverse Riemann-Lorentz conformal manifold. We say that $g \in \mathcal{C}$ is *conformally II-flat* if $II_\Sigma^R = hg_\Sigma$, for some radical vectorfield R and some $h \in C^\infty(\Sigma)$. This definition does not depend on R and, even more, it is conformal: if $\bar{g} = e^{2f}g \in \mathcal{C}$, then it holds

$$\overline{II}_\Sigma^R = e^{2f} \{II_\Sigma^R - Rf|_\Sigma g_\Sigma\} \quad (1)$$

Thus we say that (M, \mathcal{C}) is *conformally II-flat* if some (and thus, any) metric $g \in \mathcal{C}$ is conformally II-flat.

Proposition 7 *A transverse Riemann-Lorentz conformal manifold (M, \mathcal{C}) is conformally II-flat if and only if around every singular point $p \in \Sigma$ there exist an open neighbourhood \mathbb{U} in M and a metric $g \in \mathcal{C}$ which is II-flat on \mathbb{U} , that is $II_{\Sigma \cap \mathbb{U}} = 0$.*

Proof: Let (\mathbb{U}, E) be an adapted orthonormal frame near $p \in \Sigma$ (that is, E_m is radical and (E_1, \dots, E_{m-1}) are orthonormal) and $g \in \mathcal{C}$. If \mathcal{C} is conformally II-flat, then there exists $h \in C^\infty(\Sigma)$ such that $II_\Sigma^{E_m} = hg_\Sigma$. Take $\hat{h} \in C^\infty(\mathbb{U})$ any local extension of h (shrinking \mathbb{U} if necessary). There exists $f \in C^\infty(\mathbb{U})$ (shrinking again \mathbb{U} if necessary) satisfying $E_m f = \hat{h}$ (since it is locally a first order linear equation), what gives on \mathbb{U} : $II_\Sigma^{E_m} = (E_m f)|_\Sigma g_\Sigma$. Let $\hat{f} \in C^\infty(M)$ be any extension of (possibly a restriction of) f . Applying (1) to g and $\bar{g} := e^{2\hat{f}}g \in \mathcal{C}$ we have $\overline{II}_\Sigma^{E_m} = 0$.

To show the converse we start considering $g \in \mathcal{C}$. Since conformally II-flatness is a local condition, it suffices to take an arbitrary $p \in \Sigma$ and $\bar{g} = e^{2\hat{f}}g \in \mathcal{C}$ such that \bar{g} is II-flat around p . Then, formula (1) applied to g and \bar{g} shows that $II_p^\xi = (\xi f) g_p$, where $\xi \in Rad_p(M) - \{0\}$ ♣

In what follows, we study *conformally II-flat Riemann-Lorentz conformal* structures with transverse radical. Let g and $\bar{g} = e^{2f}g \in \mathcal{C}$ be two

transverse Riemann-Lorentz metrics which are II -flat. Formula (1) shows that $(Rf)|_\Sigma = 0$. The expression of $\text{grad}_g(f)$ in an adapted orthonormal frame such that $R = E_m$ is $\text{grad}_g(f) = \sum_{i=1}^{m-1} (E_i f) E_i + \tau^{-1}(Rf) R$, thus $\text{grad}_g(f)$ extends to the whole M . Now a simple computation gives

$$\overline{III}^R = e^{2f} \{ III^R - II^R(\text{grad}_g(f), R) g_\Sigma \} \quad (2)$$

We say that $g \in \mathcal{C}$ is *conformally III -flat* if it is II -flat (in order that III^R exists) and it holds $III^R = k g_\Sigma$, for some radical vectorfield R and some $k \in C^\infty(\Sigma)$. Since II -flatness is not conformal, the above definition, although independent of R , cannot be conformal. However, it is conformal in the subset of II -flat metrics.

Definition 8 *We say that a transverse Riemann-Lorentz conformal manifold (M, \mathcal{C}) with transverse radical is conformally III -flat if it is conformally II -flat and every $g \in \mathcal{C}$ which is II -flat on some open \mathbb{U} of M is also conformally III -flat on \mathbb{U} .*

Note that there may exist no conformally III -flat metrics on a conformally III -flat manifold, simply because there may exist no II -flat metric there. However, since a conformally III -flat space is conformally II -flat, we deduce from Proposition 7 that there always exist locally II -flat metrics. Let us show that in fact there also exist locally III -flat metrics:

Proposition 9 *A transverse Riemann-Lorentz conformal manifold (M, \mathcal{C}) with transverse radical is conformally III -flat if and only if around every singular point $p \in \Sigma$ there exist an open neighbourhood \mathbb{U} in M and a metric $g \in \mathcal{C}$ which is III -flat on \mathbb{U} , that is $III_{\Sigma \cap \mathbb{U}} = 0$.*

Proof: Consider $p \in \Sigma$ and (\mathbb{U}, E) a *completely adapted orthonormal frame* (i.e., E_m is radical and (E_1, \dots, E_{m-1}) are orthonormal and tangent to Σ). If (M, \mathcal{C}) is conformally III -flat, there exist $g \in \mathcal{C}$ which is II -flat on \mathbb{U} (without loss of generality) and $k \in C^\infty(\Sigma \cap \mathbb{U})$, such that $III^{E_m} = k g_\Sigma$. Since the radical is transverse, we have $II_{mm}^{E_m} \neq 0$, thus $k_1 := \frac{k}{II_{mm}^{E_m}}$ is C^∞ on $\Sigma \cap \mathbb{U}$. As in Proposition 7 we can obtain $f \in C^\infty(\mathbb{U})$ such that $E_m f = \tau \hat{k}_1$, where $\tau = g(E_m, E_m)$ and $\hat{k}_1 \in C^\infty(\mathbb{U})$ is any local extension of k_1 . Since $(E_m f)|_\Sigma = 0$, we get $\text{grad}_g(f) \in \mathfrak{X}(\mathbb{U})$ and we have $II^{E_m}(\text{grad}_g(f), E_m) = (\tau^{-1} E_m f)_\Sigma II_{mm}^{E_m} = k$. Now, take any extension $\hat{f} \in C^\infty(M)$ of (possibly a

restriction of) f . Since g is II -flat, we deduce from (1) that $\bar{g} = e^{2\hat{f}}g \in \mathcal{C}$ is also II -flat on \mathbb{U} . We also deduce that \bar{g} is III -flat on \mathbb{U} .

To prove the converse, first observe that the hypothesis implies in particular that (M, \mathcal{C}) is conformally II -flat. Consider $p \in \Sigma$ and $g \in \mathcal{C}$, II -flat on a neighbourhood of p . By hypothesis, there exists $\bar{g} = e^{2f}g \in \mathcal{C}$ which is III -flat around p . Thus we deduce from (2) that $III^R = II^R(\text{grad}_g(f), R)g_\Sigma$, so g is conformally III -flat ♣

In what follows we shall assume that $\dim M = m \geq 4$. We now study the extendibility of the *Weyl tensor*, naturally defined on $(\mathbb{M}, \mathcal{C}_\mathbb{M})$. It is well-known that this tensor plays a main role in deciding when \mathbb{M} is (locally) conformally flat, according to *Weyl Theorem: a pseudoriemannian conformal manifold is (locally) conformally flat if and only if the Weyl tensor vanishes identically* (see for instance the preliminaries of [3]). At the end of the paper we discuss the problem of establish a modified version of Weyl Theorem for transverse Riemann-Lorentz conformal manifolds.

The *Weyl tensor* W on $(\mathbb{M}, g_\mathbb{M})$ can be defined as

$$W := K - h \bullet g \in \mathcal{I}_4^0(\mathbb{M}),$$

where $h = \frac{1}{m-2} \left\{ Ric - \frac{Sc}{2(m-1)}g \right\}$ is the *Schouten tensor*, Ric is the Ricci tensor and Sc is the scalar curvature associated to $(\mathbb{M}, g_\mathbb{M})$, and where

$$\bullet : S^2(\mathbb{M}) \times S^2(\mathbb{M}) \rightarrow \mathcal{I}_4^0(\mathbb{M})$$

is the so-called *Kulkarni-Nomizu product*, given by

$$\theta \bullet \omega(x, y, z, t) := \det \begin{pmatrix} \theta(x, z) & \omega(x, t) \\ \theta(y, z) & \omega(y, t) \end{pmatrix} + \det \begin{pmatrix} \omega(x, z) & \theta(x, t) \\ \omega(y, z) & \theta(y, t) \end{pmatrix}$$

If we pick $\bar{g} = e^{2f}g \in \mathcal{C}$, then the Weyl tensor associated to $(\mathbb{M}, \bar{g}_\mathbb{M})$ satisfies $\bar{W} = e^{2f}W$, thus the *Weyl conformal curvature* $\mathcal{W} := \uparrow_2^1 W \in \mathcal{I}_3^1(\mathbb{M})$ becomes a conformal invariant. Notice that the extendibility of W (which is equivalent to the extendibility of \mathcal{W}) is a conformal condition, therefore it should be stated in terms of the conformal structure. In fact, we prove that it is equivalent to conformal III -flatness.

Theorem 10 *Let (M, \mathcal{C}) be a transverse Riemann-Lorentz conformal manifold, with $\dim M = m \geq 4$. Then W (smoothly) extends to the whole M if and only if the radical is transverse and \mathcal{C} is conformally III -flat.*

Proof: If (M, \mathcal{C}) has transverse radical and is conformally *III*-flat, there exist (Proposition 9) a M -open covering $\{\mathbb{U}_\alpha\}$ of Σ and a family of metrics $\{g_\alpha\}$ in \mathcal{C} such that g_α is *III*-flat on \mathbb{U}_α . By Theorem 1, the covariant curvature K_α , the Ricci tensor Ric_α and the scalar curvature Sc_α associated to g_α extend to $\Sigma \cap \mathbb{U}_\alpha$, therefore the Weyl tensor W_α also extends to $\Sigma \cap \mathbb{U}_\alpha$. Since this is a conformal condition, W_α extends to $\Sigma \cap \mathbb{U}_\beta$ for all β , and thus W_α extends to the whole M .

To show the converse we start picking an adapted orthonormal frame (\mathbb{U}, E) . Then, we can express the functions $W_{abcd} = W(E_a, E_b, E_c, E_d)$ as second order polynomials in $\tau^{-1} = (g(E_m, E_m))$. Let us call $(W_{abcd})_0, (W_{abcd})_1$ and $(W_{abcd})_2$ the differentiable coefficients of the terms of order 0, 1 and 2. Since $\tau = 0$ is a local equation for Σ , W extends to \mathbb{U} if and only if the restricted functions $(W_{abcd})_2|_\Sigma$ and $(W_{abcd})_1 + \tau^{-1}(W_{abcd})_2|_\Sigma$ identically vanish.

Suppose the radical is tangent to Σ at a singular point $p \in \Sigma$. We can choose the frame such that $E_1(p), E_2(p) \in T_p M - T_p \Sigma$. But then, using that $II^{E_m}(E_m, E_m)(p) = 0$ (because the radical is tangent), we obtain $(W_{1323}(p))_2 = \frac{\varepsilon_3}{m-2} II_p^{E_m}(E_1, E_m) II_p^{E_m}(E_2, E_m)$. Since E_1 and E_2 are transverse to Σ at p , $(W_{1323}(p))_2 \neq 0$, hence W cannot be extended. Therefore the radical must be transverse to Σ .

Once we know that the radical must be always transverse to Σ (thus $II_{mm}^{E_m} \neq 0$), we can choose the orthonormal frame (\mathbb{U}, E) completely adapted. Thus, picking i, j, k different from m , with i, j different from k , and using that $II_{im}^{E_m} = 0$, we have: if $i \neq j$, then $0 = (W_{ikjk})_2|_\Sigma = -\frac{\varepsilon_k}{m-2} II_{ij}^{E_m} II_{mm}^{E_m}$. Since $II_{mm}^{E_m} \neq 0$, we get $II_{ij}^{E_m} = 0$. If $i = j$ (and using $II_{ij}^{E_m} = 0$), the $\binom{m-1}{2}$ equalities $0 = (W_{ikik})_2|_\Sigma$, suitably manipulated, give us $\varepsilon_i II_{ii}^{E_m} + \varepsilon_k II_{kk}^{E_m} = \frac{2C}{m-1}$, where $C = \sum_{l=1}^{m-1} \varepsilon_l II_{ll}^{E_m} \in C^\infty(\mathbb{U})$. Subtracting the equation for i, k from the equation for k, j , we obtain $\varepsilon_i II_{ii}^{E_m} - \varepsilon_j II_{jj}^{E_m} = 0$, thus $\varepsilon_i II_{ii}^{E_m} = \varepsilon_j II_{jj}^{E_m}$. Defining $k := \varepsilon_1 II_{11}^{E_m} \in C^\infty(\Sigma \cap \mathbb{U})$, it holds $II_{ii}^{E_m} = \varepsilon_i \varepsilon_1 II_{11}^{E_m} = k g_{ii}$ and $II_{ij}^{E_m} = 0 = k g_{ij}$ (where $i \neq j$), what means $II_\Sigma^{E_m} = k g_\Sigma$, that is, g is conformally *II*-flat on \mathbb{U} , and therefore (M, \mathcal{C}) is conformally *II*-flat.

Once we know that (M, \mathcal{C}) is conformally *II*-flat, we can choose a metric $g \in \mathcal{C}$ which is *II*-flat on \mathbb{U} (shrinking \mathbb{U} if necessary). By Theorem 1, the covariant curvature K associated to g extends to $\Sigma \cap \mathbb{U}$ and, since W also does it, necessarily $h \bullet g$ extends to $\Sigma \cap \mathbb{U}$. Picking i, j, k different from m , with i, j different from k , we get $(h \bullet g)_{ikjk} = \varepsilon_k h_{ij} + \delta_{ij} \varepsilon_i h_{kk} = A_{ijk} + \tau^{-1} B_{ijk}$,

therefore the function

$$B_{ijk} := \frac{1}{m-2} \left\{ \varepsilon_k K_{imjm} + \delta_{ij} \varepsilon_i K_{kmkm} - \frac{2\varepsilon_k \delta_{ij} \varepsilon_i}{m-1} \sum_{l=1}^{m-1} \varepsilon_l K_{lm lm} \right\}$$

must vanish on Σ . Using the same argument as before, but with the equalities $0 = B_{ijk}|_{\Sigma}$, we get $III^{E_m} = kg_{\Sigma}$, where $k := \varepsilon_1 III_{11}^{E_m} \in C^{\infty}(\Sigma \cap \mathbb{U})$, that is g is conformally III -flat on \mathbb{U} , and thus (M, \mathcal{C}) is conformally III -flat \clubsuit

Let us consider the following conjecture:

Conjecture 11 *Let (M, \mathcal{C}) be a transverse Riemann-Lorentz conformal manifold, with $\dim M = m \geq 4$. A necessary condition for being $W = 0$ is that, around every singular point $p \in \Sigma$, there exist a coordinate system (\mathbb{U}, x) and a metric $g \in \mathcal{C}$ such that $g = \sum_{i=0}^{m-1} (dx^i)^2 + \tau (dx^m)^2$, where $\tau = 0$ is a local equation for Σ .*

Using Theorem 6, it becomes obvious that the necessary condition stated in the conjecture is always sufficient for having $W = 0$ around Σ .

If the conjecture is true, Σ must be (locally) conformally flat, which is well known equivalent to either $W^{\Sigma} = 0$ (if $m > 4$) or $\nabla_X^{\Sigma} h^{\Sigma}(Y, Z) = \nabla_Y^{\Sigma} h^{\Sigma}(X, Z)$ (if $m = 4$). But the extendibility of W , equivalent (Theorem 10) to conformal III -flatness, implies (Proposition 9) the existence of a metric $g \in \mathcal{C}$ which is III -flat around Σ , thus satisfying (Proposition 4):

$$W|_{T\Sigma} = (K - h \bullet g)|_{T\Sigma} = K^{\Sigma} - h|_{T\Sigma} \bullet g_{\Sigma} = W^{\Sigma} + (h^{\Sigma} - h|_{T\Sigma}) \bullet g_{\Sigma} \quad .$$

Because conditions $W = 0$ and $W^{\Sigma} = 0$ are conformal, any counterexample (M, \mathcal{C}) to the above conjecture must admit a metric $g \in \mathcal{C}$ which is III -flat around Σ and satisfies either $h^{\Sigma} \neq h|_{T\Sigma}$ (if $m > 4$) or (Lemma 3) $\nabla_X h(Y, Z) \neq \nabla_Y h(X, Z)$, for some $X, Y, Z \in \mathfrak{X}(\Sigma)$ (if $m = 4$). Now a straightforward computation for III -flat metrics, using an orthonormal completely adapted frame, leads to the following expression in terms of extendible quantities:

$$\begin{aligned} h_{ij}^{\Sigma} - h_{ij}|_{T\Sigma} &= \frac{-1}{m-2} \left\{ \frac{K_{imjm}}{\tau} - \frac{1}{m-3} \sum_{l=1}^{m-1} K_{iljl} - \right. \\ &\quad \left. - \frac{1}{m-1} \left[\sum_{k=1}^{m-1} \frac{K_{kmkm}}{\tau} - \frac{1}{m-3} \sum_{k,l=1}^{m-1} K_{klkl} \right] \delta_{ij} \right\} |_{\Sigma} \quad , \end{aligned}$$

$(i, j = 1, \dots, m-1)$, which shows that the construction of counterexamples is not easy.

In fact, *the conjecture is true for transverse Riemann-Lorentz warped products*, as we show right now. Let us consider a m -dimensional ($m \geq 4$) transverse Riemann-Lorentz manifold (M, g) of the form $M = I \times S$, where $\dim I = 1$, $0 \in I$, and $g = f(t)^2 g_S - t dt^2$, where $f \in C^\infty(I)$, $f > 0$ and g_S is riemannian (we identify t , f and g_S with the corresponding pullbacks by the canonical projections). Thus $\Sigma = \{0\} \times S$ is homothetic to S with scale factor $f(0)$. Calling $U \in \mathfrak{X}(M)$ the (nowhere zero) lift of the vectorfield $\frac{d}{dt} \in \mathfrak{X}(I)$, one immediately sees that U is radical and transverse to Σ . It is not difficult to compute the curvature tensors on \mathbb{M} . Standar results on warped products (see [8], Chapter 7) lead to (we denote by $X, Y \in \mathfrak{X}(M)$ the lifts of corresponding vectorfields $\bar{X}, \bar{Y} \in \mathfrak{X}(S)$) $\nabla_U U = \frac{1}{2t} U$, $\nabla_U X = \nabla_X U = \frac{f'}{f} X$ and $\nabla_X Y = g(X, Y) \frac{f'}{tf} U + \nabla_X^S \bar{Y}$ (where ∇^S is the Levi-Civita connection on S and $\nabla_X^S \bar{Y}$ is the lift of the corresponding vectorfield on S) and also the following expressions for the curvature tensors:

$$\left\{ \begin{array}{l} K = f^2 K^S + \frac{f'^2 f^2}{2t} g_S \bullet g_S + \frac{f}{2} \left(\frac{f'}{t} - 2f'' \right) g_S \bullet dt^2 \\ Ric = Ric^S - \left(\frac{f}{2t} \left(\frac{f'}{t} - 2f'' \right) - (m-2) \frac{f'^2}{t} \right) g_S + \frac{m-1}{2f} \left(\frac{f'}{t} - 2f'' \right) dt^2 \\ Sc = \frac{Sc^S}{f^2} - \frac{m-1}{f^2} \left(\frac{f}{t} \left(\frac{f'}{t} - 2f'' \right) - (m-2) \frac{f'^2}{t} \right) \\ h = \frac{m-3}{m-2} h^S + \left(\frac{Sc^S}{2(m-2)^2(m-1)} + \frac{f'^2}{2t} \right) g_S + \\ \quad + \left(\frac{t Sc^S}{2(m-1)(m-2)f^2} + \frac{1}{2f} \left(\frac{f'^2}{f} + \frac{f'}{t} - 2f'' \right) \right) dt^2 \\ W = f^2 W^S + \frac{1}{(m-2)} \left(Ric^S - \frac{Sc^S}{m-1} g_S \right) \bullet \left(\frac{f^2}{m-3} g_S + t dt^2 \right) \end{array} \right.$$

$(K^S, Ric^S, Sc^S, h^S$ and W^S denote of course the pullbacks by the projection of the corresponding tensor fields on S). It follows:

Lemma 12 *The following three conditions are equivalent: (1) K extends to M , (2) $f'(0) = 0$ and (3) h extends to M . Also the following are equivalent: (1) Ric extends to M , (2) $(f'/t)(0) = 0$ and (3) Sc extends to M . Moreover, W extends to M in any case.*

The fact that W extends to M was obvious from the very beginning: the map $\Psi \equiv \psi \times id : (I - \{0\}) \times S \rightarrow \mathbb{R} \times S$, given by $T \equiv \psi(t) := \int_0^t \frac{|s|^{\frac{1}{2}} ds}{f(s)}$, is

a conformal diffeomorphism onto its (non-connected) image with the metric $\bar{g} \equiv -(dT)^2 + g_S$, thus it preserves the $\binom{1}{3}$ -Weyl tensors, and since \bar{g} is regular around $T = 0$ and $f(0) \neq 0$, \bar{W} (and therefore W) extends to the whole M . It follows from Theorem 10 that the conformal manifold $(M, [g])$ is (in any case) conformally *III*-flat.

Lemma 13 *The following four conditions are equivalent: (1) $W = 0$, (2) $W^S = 0 = Ric^S - \frac{Sc^S}{m-1}g_S$ and (3) Σ has constant (sectional) curvature.*

Proof: (1) \Leftrightarrow (2) follows from the above formula. (2) \Rightarrow (3): $Ric^S - \frac{Sc^S}{m-1}g_S = 0$ implies (Schur's lemma) $Sc^S = (m-1)(m-2)C$ (constant), thus $h^S = \frac{C}{2}g_S$; moreover $W^S = 0$ leads to $K^S = \frac{C}{2}g_S \bullet g_S$. (3) \Rightarrow (2): From $K^S = \frac{C}{2}g_S \bullet g_S$, one immediately gets $W^S = 0 = Ric^S - \frac{Sc^S}{m-1}g_S$ ♣

Proposition 14 *The Conjecture 11 is true for any transverse Riemann-Lorentz conformal manifold (M, \mathcal{C}) such that some $g \in \mathcal{C}$ is a warped product.*

Proof: Let $g = f(t)^2 g_S - t dt^2 \in \mathcal{C}$ be a transverse warped product metric on $M = I \times S$. Note that $g = f(t)^2 \left\{ g_S - \frac{t}{f(t)^2} dt^2 \right\}$. From $W = 0$ and Lemma 13 we get, around any $p \in \Sigma$, coordinates (\mathbb{V}, y) of Σ such that $f(0)^2 g_S = g_\Sigma = e^{2h} \sum_{i=1}^{m-1} (dy^i)^2$, for some $h \in C^\infty(\Sigma)$. Choosing $x^i := y^i \circ \pi$, $x^m := t$ and $\tau := \frac{-te^{-2h}}{f(t)^2}$, we get $g = e^{2h} f(t)^2 \left\{ \sum_{i=1}^{m-1} (dx^i)^2 + \tau (dx^m)^2 \right\}$, and we are finished ♣

References

- [1] E. Aguirre and J. Lafuente. *Trasverse Riemann-Lorentz metrics with tangent radical*. Diff. Geom. its App, 24, 2, 91-100, 2005.
- [2] J. B. Hartle and S. W. Hawking. *Wave Function of the Universe*. Phys. Rev., D41, 1815-34, 1990.
- [3] U. Hertrich-Jeromin. *Introduction to Möbius Differential Geometry*. Cambridge Univ. Press, 2003.
- [4] M. Kossowski. *Fold singularities in pseudoriemannian geodesic tubes*. Proc. Amer. Math. Soc., 95, 463-469, 1985.

- [5] M. Kossowski. *Pseudo-riemannian metric singularities and the extendability of parallel transport*. Proc. Amer. Math. Soc., 99, 147-154, 1987.
- [6] M. Kossowski and M. Kriele. *Transverse, type changing, pseudo riemannian metrics and the extendability of geodesics*. Proc. R. Soc. Lond. A 444, 297-306, 1994.
- [7] M. Kossowski and M. Kriele. *The volume blow-up and characteristic classes for transverse, type changing, pseudo-riemannian metrics*. Geom. Dedicata 64, 1-16. 1997.
- [8] B. O'Neill. *Semi-riemannian Geometry*. Academic Press, 1983.