Principal eigenvalue, maximum principles and linear stability for nonlocal diffusion equations in metric measure spaces

Aníbal Rodríguez-Bernal *

March 10, 2022

Departamento de Análisis Matemático y Matemática Aplicada
Universidad Complutense de Madrid,
Madrid 28040, SPAIN
and
Instituto de Ciencias Matemáticas
CSIC-UAM-UC3M-UCM†, Spain

Key words: nonlocal equations, dispersal, principal eigenvalues, maximum principle, linear stability
Mathematical Subject Classification 2010: 45Axx, 45Cxx, 45M10, 45M20.
ABSTRACT: We study principal eigenvalues and maximum principles for stationary nonlocal operators in spaces of integrable functions defined on general metric measure spaces under minimal assumptions on the kernels. Several characterizations of the principal eigenvalue are given as well as several conditions guaranteeing existence. Characterization of the (strong) maximum principle is also given. For evolution problems we prove the strong maximum principle and characterize stability in terms of the sign of the principal eigenvalue. We recover, extend and improve all previously known results, obtained for smooth open sets in euclidean space under continuity assumptions on the data.

1 Introduction

Diffusion is an ubiquitous process in nature that is modeled with different tools in smooth and in rough media. In smooth media (e.g. an open set in euclidean space or a regular manifold) diffusion operators naturally involve differential operators which apply to smooth functions. A prototype example in this situation is the Laplacian. In contrast, in rough media, different operators must be considered and a large family of such operators are the so called nonlocal diffusion ones (or dispersal) as we now introduce. Observe that nonlocal diffusion operators can be considered in metric measure spaces defined as follows

Definition 1.1 *A* **metric measure space** *is a metric space* (Ω, d) *with a* σ*-finite, regular, and complete Borel measure* dx *in* Ω*, and that associates a finite positive measure to the balls of* Ω*.*

*Partially supported by Projects MTM2016-75465 and PID2019-103860GB-I00, MINECO, Spain.
†Partially supported by ICMAT Severo Ochoa Grant CEX2019-000904-S funded by MCIN/AEI/ 10.13039/501100011033

Thus, if Ω is a measure metric space, assume $u(x,t)$ is the density of some population at the point $x \in \Omega$ at time t, and $J(x,y)$ is a positive function defined in $\Omega \times \Omega$ that represents the fraction of the population jumping from a location y to location x, per unit time. Then $\int_\Omega J(x,y)u(y,t)\,dy$ is the rate at which the individuals arrive to location x from all other locations $y \in \Omega$. Analogously, $\int_\Omega J(y,x)u(x,t)\,dy = u(x,t)\int_\Omega J(y,x)\,dy$ is the rate at which individuals leave from location x to any other place in Ω. Hence the evolution in time of the population can be written as

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)u(y,t)\,dy - h_*(x)u(x,t), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where u_0 is the initial distribution density of the population and $h_*(x) = \int_\Omega J(y,x)\,dy$. Observe that in a symmetric media the rate at which individuals arrive to location x from location y is the same to the rate at which they arrive to location y from location x. Hence, we have $J(x,y) = J(y,x)$ and (1.1) can be recast as

$$u_t(x,t) = \int_\Omega J(x,y)\Big(u(y,t) - u(x,t)\Big)\,dy, \quad x \in \Omega,\ t > 0.$$

The general linear version of (1.1) that we consider in this paper reads

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)u(y,t)\,dy - h(x)u(x,t), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where h and u_0 are suitable given functions in Ω.

This setting allows to study diffusion processes in very different types of media like, for example, graphs (which are used to model complicated structures in chemistry, molecular biology or electronics, or they can also represent basic electric circuits in digital computers), compact manifolds, multi-structures composed by several compact sets with different dimensions (for example, a dumbbell domain), or even some fractal sets such as the Sierpinski gasket, see [26] for some details. The case when Ω is an open set of euclidean space, (1.2) and variations of it have been consider thoroughly in the literature, see e.g. [3] and references therein and some more references at the end of this introduction. Other approaches to diffusion in non-smooth media can be found in [5, 21, 29, 9].

Hereafter we consider the following general notations. Let Ω be a metric measure space as defined above and let J be a nonnegative kernel defined as $J : \Omega \times \Omega \to \mathbb{R}$ and considered as a mapping

$$\Omega \ni x \mapsto J(x,\cdot) \geq 0 \quad \text{(a nonnegative function defined in } \Omega\text{)}.$$

Then the associated nonlocal diffusion operator is given by

$$Ku(x) = \int_\Omega J(x,y)u(y)\,dy \quad x \in \Omega \tag{1.3}$$

for suitable functions defined in Ω.

We will set (1.2) in several function spaces X (see Assumptions 1.3 below) depending on the properties of the kernel J, and we study several properties of the nonlocal diffusion operator

$$Ku - hu, \quad u \in X \tag{1.4}$$

and the corresponding evolution problem (1.2) with $u_0 \in X$. In particular, we are interested in studying the existence of a principal eigenvalue, that is an eigenvalue with an associated positive eigenfunction and the validity of the maximum principle, that is, wether $Ku - hu \leq 0$ implies $u \geq 0$ in Ω. We will also explore the impact of these properties in the stability of (1.2), that is, wether solutions of (1.2) converge exponentially to zero or may grow exponentially as time evolves.

Notice that maximum principles and principal eigenvalues are very powerful tools to deal with diffusion problems modeled with second order elliptic or parabolic equations. In particular, they provide tools for the analysis of nonlinear elliptic or parabolic diffusion problems such as sub and supersolutions, apriori estimates, comparison etc. In such cases, the smoothing/compactness properties of the associated resolvent operators imply, at least in the case of regular bounded domains, that the spectrum of the elliptic diffusion operator is discrete and, typically through Krein-Rutman type arguments (strongly related to the maximum principle), one can prove that the first eigenvalue is the principal eigenvalue, in the sense defined above. On the other hand, again the smoothing properties of the resolvent of differential operators, imply that eigenvalues and eigenfunctions are quite independent of the space in which a particular problem is posed (e.g. $X = L^2(\Omega)$ with $\Omega \subset \mathbb{R}^N$ smooth and bounded). Also, this implies that solutions of parabolic problems regularize in time. Finally, the principal eigenvalue captures the exponential growth or decay of the corresponding linear parabolic problem.

For nonlocal problems, the situation is a little more involved since, (1.4) has essential spectrum and then the spectral behaviour is more complicated for both the stationary and evolution problems. Also, the resolvent of nonlocal diffusion operators enjoy no smoothing/compactness properties and hence, even the spectrum can be rather space dependent for, say, spaces of integrable or continuous functions. Also, solutions of the parabolic problem do not regularize and remain as smooth as the initial data. So, most of the difficulties dealing with nonlocal problems stem from this lack of smoothing/compactness. However, since the kernel J is nonnegative, the operator K has nice positivity and compactness properties that can be exploited.

For differential diffusion operators, the simplicity of principal eigenvalues is linked to the connectedness of the domain. For nonlocal problems in this paper, connectedness can be relaxed to R-connectedness as we now define.

Definition 1.2 *If $R > 0$, we say that Ω is* R**-connected** *if for all $x, y \in \Omega$, there exist $N \in \mathbb{N}$ and a finite set of points $\{x_0, \ldots, x_N\}$ in Ω such that $x_0 = x$, $x_N = y$ and $d(x_{i-1}, x_i) < R$, for all $i = 1, \ldots, N$.*

Hence, Ω can have several connected components at a distance less than R from some other.

Also, the usual boundedness assumption for Ω for differential operators, which is linked to the compactness of the resolvent of the diffusion operator, can be relaxed here to a finite measure assumption on Ω.

Now we describe in detail the main results in this paper. At the end of this Introduction we collect references to previous related results in the literature. For most of the results here we will assume the following standing hypotheses and notations. First, observe that we use $|\cdot|$ to denote the measure of a set; for example $|\Omega|$ is the measure of Ω.

Assumptions 1.3
- *Ω is R–connected, $|\Omega| < \infty$, the measure of the balls of radius R do not degenerate, i.e.*

$$|B(x,R)| \geq m_0 > 0, \quad \textit{for all} \quad x \in \Omega, \tag{1.5}$$

and h is s bounded measurable function in Ω.

- *J is locally strictly positive in the sense that*

$$J(x,y) > J_0 > 0 \quad \textit{for all } x,\, y \in \Omega, \textit{ such that } \ d(x,y) < R \tag{1.6}$$

and J satisfies for some $1 \leq p_0 \leq \infty$ either

$$J \in L^\infty(\Omega, L^{p_0'}(\Omega)) \quad \textit{or} \quad J \in BUC(\Omega, L^{p_0'}(\Omega)) \tag{1.7}$$

(where p_0' denotes the conjugate exponent to p_0, i.e. $\frac{1}{p_0} + \frac{1}{p_0'} = 1$).

- *Accordingly, if $J \in L^\infty(\Omega, L^{p_0'}(\Omega))$ we will consider anyone of the following* admissible *spaces*

$$X = L^p(\Omega), \quad \textit{for } p_0 \leq p < \infty, \textit{ if } h \in L^\infty(\Omega), \tag{1.8}$$

while if $J \in BUC(\Omega, L^{p_0'}(\Omega))$ we will consider

$$X = L^p(\Omega), \quad \textit{for } p_0 \leq p \leq \infty, \textit{ if } h \in L^\infty(\Omega), \textit{ or } \quad X = C_b(\Omega) \quad \textit{if } h \in C_b(\Omega). \tag{1.9}$$

Assumptions (1.7) imply that for the spaces X as in (1.8) or (1.9), $K \in \mathcal{L}(X, L^\infty(\Omega))$ and moreover $K \in \mathcal{L}(X, X)$ is compact, see Corollary 2.2. This implies that $K - hI$ has an essential spectrum

$$\sigma_{ess}(K - hI) = R(-h)$$

where $R(-h)$ is the essential range of the function $-h$, see (2.2), and a (possibly empty) discrete point spectrum $\sigma_p(K - hI) = \{\mu_n\}_{n=1}^M$, $M \in \mathbb{N} \cup \{\infty\}$. If $M = \infty$, then $\{\mu_n\}_{n=1}^\infty$ accumulates in $R(-h)$. See Theorem 2.3.

Note that the essential spectrum above is independent of the space X and are the points λ in the spectrum such that $K - (h + \lambda)I$ is not a Fredholm operator of index zero. On the other hand, the point spectrum $\sigma_p(K - hI) = \{\mu_n\}_{n=1}^M$ is potentially dependent of the space X. However, condition (1.7) guarantees that the point spectrum, hence the whole spectrum $\sigma(K - hI)$, is independent of X, see Proposition 2.4. In fact one proves that eigenfunctions belong to all spaces X.

Then for $K - hI$ we study questions related to principal eigenvalues and maximum principles as we now define. Observe first that the spaces X in (1.8) or (1.9) have naturally defined order relation"$\leq$" understood as a pointwise a.e. inequality w.r.t. the measure in Ω. Also, we use below the notion of an essentially positive (measurable) function, which we denote $\phi > 0$, see (2.4). Finally, we use below the essential infimum and supremum of functions in X, see (2.3), which, by simplicity, we denote inf and sup instead of ess inf and ess sup respectively.

Definition 1.4 *Assume that J, h and X are as above.*
i) We say $\mu \in \mathbb{R}$ is a **principal eigenvalue** *of $K - hI$ in X iff there exists $0 < \phi \in X$ such that*

$$K\phi - h\phi = \mu\phi \quad \textit{in } \Omega.$$

ii) We say that for $\lambda \in \mathbb{R}$ the **maximum principle** *is satisfied if $u \in X$ with*

$$Ku - (h + \lambda)u \leq 0 \textit{ in } \Omega, \quad \textit{implies} \quad u \geq 0 \textit{ in } \Omega.$$

iii) We say that for $\lambda \in \mathbb{R}$ the **strong maximum principle** *is satisfied if $u \in X$ with*

$$Ku - (h+\lambda)u \le 0 \text{ in } \Omega, \quad \text{implies either } u = 0 \text{ or } u \ge \alpha > 0$$

for some constant $\alpha > 0$.

One of our main result reads as follows, see Theorem 3.3.

Theorem 1.5 *Assume Ω is R–connected, $|\Omega| < \infty$ and J, h and X are as in the standing Assumptions 1.3.*

Then we define

$$\Lambda := \inf_{0<\varphi\in X} \sup_{\Omega} \frac{K\varphi - h\varphi}{\varphi}$$

and we have the following results.

i)

$$-\inf_{\Omega} h \le \Lambda = \sup Re(\sigma_X(K - hI)) \le \sup_{\Omega}(h_0 - h)$$

where $h_0(x) = \int_{\Omega} J(x,y)\, dy$.

In particular, Λ is the only possible principal eigenvalue of $K - hI$ in X.

ii) If

$$\Lambda > -\inf_{\Omega} h.$$

then Λ is the principal eigenvalue of $K - hI$ in X. In such a case Λ is a simple isolated eigenvalue of $K - hI$ in X with bounded eigenfunction. If moreover $J \in BUC(\Omega, L^{p_0'}(\Omega))$ and $h \in C_b(\Omega)$, the eigenfunction is continuous.

iii) The maximum principle is satisfied for $\lambda > \Lambda$ and is not satisfied for $\lambda < \Lambda$ nor for $\lambda = \Lambda > -\inf_{\Omega} h$.

iv) If $\lambda > \Lambda$ then the strong maximum principle is satisfied.

Also, we have the following characterization of Λ, see Proposition 3.8,

$$\Lambda = \sup_{0<\varphi\in X} \inf_{\Omega} \frac{K\varphi - h\varphi}{\varphi}. \tag{1.10}$$

Then, denoting $h_m = \inf_{\Omega} h$, we develop some criteria to guarantee that $\Lambda > -h_m$, hence Λ is the principal eigenvalue of $K - hI$. These include either one of the following conditions, see Corollary 3.10 and Corollary 3.11,

$$|\{h = h_m\}| > 0, \tag{1.11}$$

or, there exists $x_0 \in \Omega$ and $r \le R$ as in (1.6) such that

$$\int_{B(x_0,r)} \frac{dx}{h(x) - h_m} = \infty, \tag{1.12}$$

or,

$$osc_{\Omega}(h) := \sup_{\Omega} h - \inf_{\Omega} h < \inf_{\Omega} h_0. \tag{1.13}$$

When $|\{h = h_m\}| = 0$, condition (1.12) can be obtained through measure–geometric properties of the set $\{h = h_m\}$ (related to its fractal dimension) and the way h approaches its infimum, see Lemma 3.12 and Remark 3.13.

In particular, when $h = 0$ we prove the operator K is of Krein–Rutman type in X in the sense that we have the following result, see Proposition 3.14.

Proposition 1.6 *Assume Ω compact and connected, (1.5) holds and for some $1 \leq p_0 \leq \infty$*

$$J \in BUC(\Omega, L^{p_0'}(\Omega))$$

and denote $X = L^p(\Omega)$ for $p_0 \leq p \leq \infty$ or $X = C_b(\Omega)$. Then

$$0 < \Lambda(K) = r(K) := \sup\{|\lambda|, \ \lambda \in \sigma(K)\}$$

that is, the spectral radius of K in X, is the unique principal eigenvalue of K and is simple.

In particular, we get

$$0 < \Lambda(K) \leq \inf_{p_0 \leq p \leq \infty} \|J\|_{L^p(\Omega, L^{p'}(\Omega))} \leq \mu(\Omega)^{\frac{1}{p_0}} \|J\|_{L^\infty(\Omega, L^{p_0'}(\Omega))}.$$

When $\Lambda > -\inf_\Omega h$ we prove in Theorem 3.28 the following characterization of the principal eigenvalue in the same line as for second order elliptic differential operators, see [7, 24, 1, 2, 25].

Theorem 1.7 *With the notations above, assume $\Lambda > -\inf_\Omega h$. Then, the following statements are equivalent*
i) $\Lambda < 0$.
ii) $K - hI$ satisfies the maximum principle in X.
iii) $K - hI$ satisfies the strong maximim principle in X.
iv) There exists $0 < \xi \in X$ such that $K\xi - h\xi \not\leq 0$.

We also analyse in detail the case $\Lambda = -\inf_\Omega h$. Notice that in this case $\Lambda \in \sigma_{ess}(K - hI)$, the essential spectrum, and therefore one expects the properties of $K - (h + \Lambda)I$ to be quite different from the case where Λ is in the point spectrum. For this we assume, for some $s \geq 1$,

$$0 \leq \frac{1}{h - h_m} \in L^s(\Omega). \tag{1.14}$$

Then if $s < p_0$ then $\Lambda = -h_m$ is not a principal eigenvalue in any of the spaces X, see Proposition 3.17 and Corollary 3.18. On the other hand, if $p_0 \leq s < \infty$ define $1 \leq p_0 < q_0$ by

$$0 \leq \frac{1}{q_0} := \frac{1}{p_0} - \frac{1}{s}.$$

Then we analyse the kernel of the operator $K - (h - h_m)I$ and prove that in some spaces $\Lambda = -h_m$ can still have an associated positive eigenfunction, whereas in other it can not, see Propositions 3.17 and 3.20. Moreover, if assume the set $\mathscr{C} := \{h = h_m\}$ is non empty (but of zero measure by (1.11)) and $\Lambda = -h_m$ does not have an associated positive eigenfunction, there exists other singular "measure eigenfunctions" as in the following result, see Proposition 3.21 for a more refined statement.

Proposition 1.8 *With the notations above, assume furthermore that*

$$J \in L^q(\Omega, C_b(\Omega)) \quad \textit{for some } q_0 \leq q < \infty.$$

and $\Lambda = -h_m$ does not have a positive eigenfunction in $X = L^p(\Omega)$ for some (or any) $p_0 \leq p \leq s$.

Then for each regular Borel measure with support in $\mathscr{C}$, that is, $d\sigma \in \mathcal{M}(\mathscr{C})$, there exists a unique w such that

$$u = d\sigma + w$$

is a singular solution of $Ku = (h - h_m)u$. *Moreover,* $w \in L^p(\Omega)$ *for all* $p_0 \leq p < p_* = \frac{sq}{s+q} \leq s$.

If $d\sigma$ *is a positive measure then* $w \geq \frac{\alpha}{h-h_m} > 0$ *for some* $\alpha > 0$ *and* u *is a positive singular solution of* $Ku = (h - h_m)u$.

Notice that this results says that the set of eigenfunctions for $\lambda = \Lambda = -h_m$ can be identified with the set of regular Borel measures concentrated in the set $\mathscr{C} := \{h = h_m\}$.

We also get the following result on the maximum principle for the operator $K - (h - h_m)I$, see Proposition 3.22.

Proposition 1.9 *With the assumptions above, assume* $\Lambda = -h_m$ *does not have a positive eigenfunction in* $X = L^p(\Omega)$ *for some (or any)* $p_0 \leq p \leq s$. *Then we have the following results.*
i) For $X = C_b(\Omega)$ *or* $X = L^p(\Omega)$ *with* $q_0 \leq p \leq \infty$ *and* $s < p$ *then for every nontrivial* $u \in X$ *we have that*

$$Ku - (h - h_m)u \quad \textit{changes sign in } \Omega,$$

that is, any nonzero function in the range $R(K - hI)$, *changes sign in* Ω.
ii) Assume $2p_0 \leq s$ *so that* $q_0 \leq s$. *Then the operator* $K-(h-h_m)I$ *satisfies the strong maximum principle in* $X = L^p(\Omega)$ *with* $q_0 \leq p \leq s$. *Moreover, if* $u \in X$ *and* $Ku - (h - h_m)u \leq 0$ *then* $u \geq \frac{\alpha}{h-h_m}$ *for some* $\alpha > 0$.

Observe that the case $X = L^p(\Omega)$ with $p_0 \leq p < q_0$ remains open.

We also study continuous dependence of the spectrum of $K - hI$, see Proposition 2.6, and of Λ with respect to h, J or Ω, see Propositions 3.2 and 3.26. In case J is symmetric, a variational characterization of Λ is obtained in Proposition 3.7. Finally in Proposition 3.25 and Corollary 3.27 we derive some criteria for the sign of Λ. Observe that the latter result sets the function h_* in (1.1) as a natural threshold for the sign of Λ. This will be very useful when studying stability properties of the parabolic problem (1.2), see Proposition 1.11 below.

Turning now to the linear parabolic problem (1.2) we prove the following results. First we prove the strong maximum principle, see Proposition 4.2.

Proposition 1.10 (Parabolic strong maximum principle)

With the assumptions above, for every $u_0 \in X$, *nonnegative and not identically zero, the solution* $u(t, u_0)$ *of (1.2) satisfies*

$$\inf_{\Omega} u(t, u_0) > 0, \quad t > 0.$$

We also prove that Λ gives upper and lower bounds for the solutions in (1.2). In particular, we obtain stability when $\Lambda < 0$ and instability when $\Lambda > 0$, see Proposition 4.6.

Proposition 1.11 *Under the assumptions above fix any* $\tilde{\lambda}, \lambda$ *such that*

$$\tilde{\lambda} < \Lambda < \lambda.$$

Then
i) Any solution of (1.2) with $u_0 \in X$ *satisfies*

$$\|u(t)\|_X \leq Me^{\lambda t}\|u_0\|_X, \quad t \geq 0.$$

ii) Assume either $\Lambda > -\inf_\Omega h$ *or* $J \in BUC(\Omega, L^{p'}(\Omega))$. *Also, by Proposition 1.10, assume without loss of generality that* $0 \leq u_0 \in X$ *is such that* $u_0 \geq \alpha > 0$.

Then there exists a positive bounded function $\tilde{\varphi}$ *in* Ω *such that*

$$0 < e^{\tilde{\lambda} t} \tilde{\varphi}(x) \leq u(x,t), \quad x \in \Omega, \ t > 0.$$

iii) For any solution of (1.2) with $u_0 \in L^\infty(\Omega)$ *there exists a positive function* $\varphi \in X$ *such that*

$$|u(x,t,u_0)| \leq e^{\lambda t} \varphi(x) \quad x \in \Omega, \ t > 0.$$

Both parts ii) and iii) hold true for $\lambda = \tilde{\lambda} = \Lambda$ *provided* $\Lambda > -\inf_\Omega h$.

In particular, if $\Lambda < 0$ *all solutions of (1.2) converge to* 0 *in* X *as* $t \to \infty$*. Moreover, if* $u_0 \in L^\infty(\Omega)$ *then* $u(t) \to 0$ *uniformly in* Ω *as* $t \to \infty$*.*

On the other hand, if $\Lambda > 0$ *then all positive solutions of (1.2) converge pointwise to* ∞ *as* $t \to \infty$*.*

Some results in this paper have appeared before in somewhat more restrictive situations. To the best of our knowledge, in all cases Ω is an open connected set of $\mathbb{R}^N$ and the kernel $J(x,y)$ is assumed to be continuous. In many cases J has some special structure as for example symmetric kernels $J(x,y) = J(y,x)$ or convolution type kernels $J(x,y) = J_0(x-y)$. Also h is assumed to be a continuous function and the problems are set in the space of continuous functions $X = C(\overline{\Omega})$. This setting allows to use the Krein–Rutman theorem. Notice that these assumptions on J fit in our setting by taking $p_0 = 1$ in Assumptions 1.3.

For example, restricting to the cases of bounded domains, [18] deals with a one dimensional finite interval Ω, with a continuous positive symmetric kernel and a Lipschitz h. In their setting, they prove that the principal eigenvalue exists working in $L^2(\Omega)$. In [4] the higher dimensional situation was considered with a everywhere positive symmetric kernel and a continuous h. Working in $X = C(\overline{\Omega})$, via the Krein-Rutman theorem, they prove the existence of the principal eigenvalue given as the largest real part in the spectrum of $K - h$ and prove some monotonicity w.r.t. h. In [16] the authors consider the convolution symmetric smooth case with $h = 1$. They prove the existence of the principal eigenvalue in $X = C(\overline{\Omega})$, characterize it through variational properties in $L^2(\Omega)$ and study monotonicity and regular dependence w.r.t. the domain. In the same setting, [15] studied maximum and antimaximum principles for the stationary problem. For a certain class of continuous non symmetric kernels, a continuous h and working in $X = C(\overline{\Omega})$, in [10] they prove that condition (1.12) (hence (1.11) as well) guarantees that Λ is the principal eigenvalue, i.e. $\Lambda > -\inf_\Omega h$. They also study monotonicity properties of Λ with respect to J, h and the domain and they prove the maximum principle holds for $\lambda \geq \Lambda$. For the latter they had to add a sign condition on the boundary (in $\mathbb{R}^N$) of Ω. Also, [11] with continuous J and h, proved the existence of singular eigenfunctions when $\Lambda = -\inf_\Omega h$, assuming $s = 1$ in (1.14). In [28] the authors consider a convolution type, C^1 smooth kernel and continuous h and prove that condition (1.13) implies $\Lambda > -\inf_\Omega h$. They also studied dependence on parameters. Reference [6] deals with continuous J and h and prove different characterizations of Λ like the ones in part iii) of Theorem 1.5 or in Proposition 3.7. Reference [23] deals with continuous J and h and they give criteria, in the line of the results above, for the existence of the principal eigenvalue, characterize it as in (1.10) and studied the maximum principle and the sign of Λ in $X = C(\overline{\Omega})$; see Remark 3.23 for a more technical comparison with the results in this paper. Notice that most of the references above also deal with some nonlinear version of the parabolic problem (1.2) and some consider also the case Ω is unbounded. See also, [19], [12], [17].

Therefore we prove here that neither the underlying euclidean structure of the set Ω in the references above, nor the continuity assumptions on the data are essential and our results are obtained in spaces of integrable functions in metric measure spaces with the minimal regularity in J, h presented above. This extends the range of applicability to wider classes of non-smooth media. Also, we avoid any use of the Krein–Rutman theorem and hence our results apply in spaces of integrable functions as in (1.8) or (1.9). Although we use several ideas and approaches of the references quoted above, our proofs become shortcuts for the proofs in them. Also, the references above use in an essential way the continuity of J and h, so they arguments can not be just adapted to the minimal regularity considered here.

Finally, observe that in the setting of the references above, our results provide some version of the strong maximum principle both for stationary and evolution nonlocal diffusion problems.

2 Preliminaries on linear stationary operators

In this section we analyse several properties of the nonlocal diffusion operator (1.4).

2.1 Kernels and nonlocal operators

The next results state regularity and compactness properties of the nonlocal operator K in (1.3) derived from properties of the kernel J. For more details, see [26].

Proposition 2.1 *i) Assume* $1 \leq p, q \leq \infty$ *and* $J \in L^q(\Omega, L^{p'}(\Omega))$. *Then* $K \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$, *the mapping* $J \mapsto K$ *is linear and continuous, and*

$$\|K\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \leq \|J\|_{L^q(\Omega, L^{p'}(\Omega))}.$$

Moreover, if $q < \infty$ *then* $K \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ *is compact.*
ii) Assume $1 \leq p \leq \infty$, $J \in L^\infty(\Omega, L^{p'}(\Omega))$ *and for any measurable set* $D \subset \Omega$

$$\lim_{x \to x_0} \int_D J(x,y)\, dy = \int_D J(x_0, y)\, dy, \qquad \forall x_0 \in \Omega.$$

Then $K \in \mathcal{L}(L^p(\Omega), C_b(\Omega))$, *the mapping* $J \mapsto K$ *is linear and continuous, and*

$$\|K\|_{\mathcal{L}(L^p(\Omega), C_b(\Omega))} \leq \|J\|_{L^\infty(\Omega,\, L^{p'}(\Omega))}.$$

In particular, if $J \in C_b(\Omega, L^{p'}(\Omega))$ *then* $K \in \mathcal{L}(L^p(\Omega), C_b(\Omega))$ *and*

$$\|K\|_{\mathcal{L}(L^p(\Omega), C_b(\Omega))} \leq \|J\|_{C_b(\Omega,\, L^{p'}(\Omega))}.$$

Moreover, if $J \in BUC(\Omega, L^{p'}(\Omega))$ *then* $K \in \mathcal{L}(L^p(\Omega), C_b(\Omega))$ *is compact. In particular,* $K \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$ *is compact.*

In particular notice that if $J \in L^\infty(\Omega, L^1(\Omega))$ then $K \in \mathcal{L}(L^\infty(\Omega), L^\infty(\Omega))$ and the function

$$h_0(x) = \int_\Omega J(x,y)\, dy \tag{2.1}$$

satisfies $h_0 \in L^\infty(\Omega)$. If moreover $J \in BUC(\Omega, L^1(\Omega))$ then $h_0 \in C_b(\Omega)$.

As a direct consequence of the previous Proposition, the following result collects cases in which $K \in \mathcal{L}(X, X)$, for $X = L^p(\Omega)$ or $X = C_b(\Omega)$.

Corollary 2.2 *i) If for some* $1 \le p \le \infty$, $J \in L^p(\Omega, L^{p'}(\Omega))$ *then* $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$. *Moreover if* $p < \infty$ *then* $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ *is compact.*
ii) If $J \in C_b(\Omega, L^1(\Omega))$ *then* $K \in \mathcal{L}(C_b(\Omega), C_b(\Omega))$. *Moreover if* $J \in BUC(\Omega, L^1(\Omega))$, *then* $K \in \mathcal{L}(L^\infty(\Omega), C_b(\Omega))$ *is compact. In particular,* $K \in \mathcal{L}(L^\infty(\Omega), L^\infty(\Omega))$ *is compact and* $K \in \mathcal{L}(C_b(\Omega), C_b(\Omega))$ *is compact.*
iii) If $|\Omega| < \infty$ *and* $J \in L^\infty(\Omega, L^{p_0'}(\Omega))$ *for some* $1 \le p_0 \le \infty$, *then* $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, *for all* $p_0 \le p < \infty$ *and is compact. If moreover,* $J \in BUC(\Omega, L^{p_0'}, (\Omega))$ *then* $K \in \mathcal{L}(L^\infty(\Omega), L^\infty(\Omega))$ *and* $K \in \mathcal{L}(C_b(\Omega), C_b(\Omega))$ *are compact.*

Note that for a measurable function $g : \Omega \to \mathbb{R}$ we define the **essential range** of g (range for short) as

$$R(g) = \{s \in \mathbb{R},\ |\{x,\ |g(x) - s| < \varepsilon\}| > 0 \text{ for all } \varepsilon > 0\} \tag{2.2}$$

which coincides with the set of $s \in \mathbb{R}$ such that $\frac{1}{g(x)-s} \notin L^\infty(\Omega)$. Also, if g is continuous this coincides with the image set of g.

In a standard way, we will also make use of the essential infimum and supremum of a measurable function, which we will denote infimum and supremum for short, defined as

$$\inf_\Omega g = \sup\{\alpha \in \mathbb{R},\ |\{g \le \alpha\}| = 0\}, \quad \sup_\Omega g = \inf\{\alpha \in \mathbb{R},\ |\{g \ge \alpha\}| = 0\}. \tag{2.3}$$

Note that both $\inf_\Omega g$ and $\sup_\Omega g$ belong to the (essential) range $R(g)$ if they are finite.

The following result was obtained in [26, Theorem 3.24], treating $K - hI$ as a compact perturbation of the multiplication operator hI in X.

Theorem 2.3 *If* $J \in L^p(\Omega, L^{p'}(\Omega))$ *for some* $1 \le p < \infty$, *denote* $X = L^p(\Omega)$ *while if* $J \in BUC(\Omega, L^1(\Omega))$ *denote* $X = L^\infty(\Omega)$ *or* $X = C_b(\Omega)$. *If* $X = L^p(\Omega)$, *with* $1 \le p \le \infty$, *we will assume* $h \in L^\infty(\Omega)$ *while if* $X = C_b(\Omega)$, *we will assume* $h \in C_b(\Omega)$.

Then the spectrum of $K - hI$ *satisfies* $\sigma(K - hI) = \sigma_{ess} \cup \sigma_p$ *where the essential range is*

$$\sigma_{ess} = R(-h)$$

and the (possibly empty) discrete point spectrum

$$\sigma_p = \{\mu_n\}_{n=1}^M,\ M \in \mathbb{N} \cup \{\infty\}.$$

If $M = \infty$, *then* $\{\mu_n\}_{n=1}^\infty$ *accumulates in* $R(-h)$.

Note that the essential spectrum above is independent of X and are the points in the spectrum such that $K - (h + \lambda)I$ is not a Fredholm operator of index zero. Also note that the point spectrum, if nonempty, is potentially dependent of the space X. Hence, the following result, taken from [26, Proposition 3.25], guarantees that the point spectrum, hence the whole spectrum $\sigma(K - hI)$, is independent of X.

Proposition 2.4 *Assume* $|\Omega| < \infty$ *and* $J \in L^\infty(\Omega, L^{p_0'}(\Omega))$ *for some* $1 \le p_0 \le \infty$ *and* $h \in L^\infty(\Omega)$ *then for all* $p_0 \le p < \infty$, $K - hI \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, *and* $\sigma_{L^p(\Omega)}(K - hI)$ *is independent of* p.

If moreover $J \in BUC(\Omega, L^{p_0'}(\Omega))$, *the spectrum above coincides also with* $\sigma_{L^\infty(\Omega)}(K - hI)$. *If, additionally,* $h \in C_b(\Omega)$, *the spectrum above coincides also with* $\sigma_{C_b(\Omega)}(K - hI)$.

Notice that indeed one proves that eigenfunctions belong to all spaces X.

2.2 Continuous dependence

Now we present some results on the continuous dependence of the operators $K - hI$ and their spectrum, with respect to the kernel J and the function h.

Proposition 2.5 *i) Assume for some* $1 \leq p \leq \infty$, $\{J_n\}_n \subset L^p(\Omega, L^{p'}(\Omega))$ *and* $J_n \to J$ *in that space and* $h_n \to h$ *in* $L^\infty(\Omega)$. *Then*

$$K_n - h_n I \to K - hI \quad \textit{in } \mathcal{L}(L^p(\Omega), L^p(\Omega)).$$

ii) Assume $\{J_n\}_n \subset BUC(\Omega, L^1(\Omega))$ *and* $J_n \to J$ *in that space and* $h_n \to h$ *in* $C_b(\Omega)$. *Then*

$$K_n - h_n I \to K - hI \quad \textit{in } \mathcal{L}(C_b(\Omega), C_b(\Omega)).$$

iii) Assume $|\Omega| < \infty$ *and for some* $1 \leq p_0 \leq \infty$ *we have* $\{J_n\}_n \subset L^\infty(\Omega, L^{p_0'}(\Omega))$ *and* $J_n \to J$ *in that space and* $h_n \to h$ *in* $L^\infty(\Omega)$. *Then for all* $p_0 \leq p \leq \infty$,

$$K_n - h_n I \to K - hI \quad \textit{in } \mathcal{L}(L^p(\Omega), L^p(\Omega)).$$

If additionally, $\{J_n\}_n \subset BUC(\Omega, L^{p_0'}(\Omega))$ *and* $J_n \to J$ *in that space and* $h_n \to h$ *in* $C_b(\Omega)$, *then we also get*

$$K_n - h_n I \to K - hI \quad \textit{in } \mathcal{L}(C_b(\Omega), C_b(\Omega)).$$

Proof. Denoting $X = L^p(\Omega)$ with $1 \leq p \leq \infty$ or $X = C_b(\Omega)$, parts i) and ii) are direct consequences of Corollary 2.2 and the estimates on K in terms of the kernel J in Proposition 2.1, combined with the convergence of the multiplication operators $h_n I \to hI$ in $\mathcal{L}(X, X)$.

For part iii) observe first that with the assumptions we get $J_n \to J$ in $L^p(\Omega, L^{p'}(\Omega))$ for any $1 \leq p_0 \leq p \leq \infty$ and then use part i). Finally note that the additionally assumption on J_n implies $J_n \to J$ in $BUC(\Omega, L^1(\Omega))$ and then we use part ii). ∎

Now combining this result with Theorem 2.3 and Proposition 2.4 we get the following result about the continuity of the spectrum.

Proposition 2.6 *Assume either one of the following cases.*
i) For some $1 \leq p < \infty$, $\{J_n\}_n \subset L^p(\Omega, L^{p'}(\Omega))$ *and* $J_n \to J$ *in that space and* $h_n \to h$ *in* $L^\infty(\Omega)$. *Then denote* $X = L^p(\Omega)$.
ii) $\{J_n\}_n \subset BUC(\Omega, L^1(\Omega))$ *and* $J_n \to J$ *in that space and* $h_n \to h$ *in* $L^\infty(\Omega)$. *Then denote* $X = L^\infty(\Omega)$.

If moreover $h_n \to h$ *in* $C_b(\Omega)$ *then denote* $X = L^\infty(\Omega)$ *or* $X = C_b(\Omega)$.
iii) $|\Omega| < \infty$ *and for some* $1 \leq p_0 \leq \infty$ *we have* $\{J_n\}_n \subset L^\infty(\Omega, L^{p_0'}(\Omega))$ *and* $J_n \to J$ *in that space and* $h_n \to h$ *in* $L^\infty(\Omega)$. *Then denote* $X = L^p(\Omega)$ *for any* $p_0 \leq p < \infty$.
iv) $|\Omega| < \infty$ *and for some* $1 \leq p_0 \leq \infty$ *we have* $\{J_n\}_n \subset BUC(\Omega, L^{p_0'}(\Omega))$ *and* $J_n \to J$ *in that space and* $h_n \to h$ *in* $L^\infty(\Omega)$. *Then denote* $X = L^p(\Omega)$ *for any* $p_0 \leq p \leq \infty$.

If moreover $h_n \to h$ *in* $C_b(\Omega)$ *then denote* $X = L^p(\Omega)$ *for any* $p_0 \leq p \leq \infty$ *or* $X = C_b(\Omega)$.

Then, in each of the cases above we have

$$K_n - h_n I \to K - hI \quad \textit{in } \mathcal{L}(X, X),$$

the spectrum of $K_n - h_n I$ and $K - hI$ are independent of X and

a) For each $\lambda \in \rho(K - hI)$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\lambda \in \rho(K_n - h_n I)$.

b) For $\lambda = \mu \in \sigma_p(K - hI)$, with multiplicity N, there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ in the ball $B(\lambda, \varepsilon) \subset \mathbb{C}$, the operator $K_n - h_n I$ has exactly N eigenvalues, counting multiplicities.

c) For $\lambda \in R(-h)$ there exists $\lambda_n \in R(-h_n)$ such that $\lambda_n \to \lambda$.

Proof. Part a) follows from [20], Chapter 4, Section 3, Theorem 3.1, pag 208.

Part b) follows from [20], Chapter 4, Section 3, Theorem 3.16, page 212.

Finally, for part c) note that if $\lambda \in R(-h)$ and there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ we had $d(\lambda, R(-h_n)) \geq \delta > 0$ the we would get $|h_n(x) + \lambda| \geq \delta > 0$ for all $x \in \Omega$ and $n \in \mathbb{N}$. From this we get $|h(x) + \lambda| \geq \delta > 0$ for all $x \in \Omega$ and therefore $\frac{1}{h+\lambda} \in L^\infty(\Omega)$, which contradicts $\lambda \in R(-h)$. ■

2.3 Positivity properties

Now we define the *essential support* of a nonnegative measurable function.

Definition 2.7 *Let z be a nonnegative measurable function $z : \Omega \to \mathbb{R}$. We define the* **essential support** *of z (support for short) as:*

$$\mathrm{supp}(z) = \big\{x \in \Omega \ : \ \forall \delta > 0, \ |\{z > 0\} \cap B(x, \delta)| > 0\big\},$$

where $B(x, \delta)$ is the ball centered in x, with radius δ.

Observe that for a measurable nonnegative function $z : \Omega \to \mathbb{R}$

$$\mathrm{supp}(z) = \Omega \text{ if and only if } z > 0 \text{ a.e. in } \Omega. \tag{2.4}$$

In such a case we say that z is **essentially positive** (positive for short), and write it $z > 0$.

Given two measurable functions $w, z : \Omega \to \mathbb{R}$ we will say that w is (essentially) strictly above z and write $w > z$, if $w - z > 0$ in the sense above.

The following result gives that under certain positivity of the kernel J, the operator K strictly increases the support of a nonnegative function, see [26, Proposition 3.15]

Proposition 2.8 *Assume $J \geq 0$ satisfies that, for some $R > 0$,*

$$J(x,y) > 0 \text{ for all } x,\ y \in \Omega, \text{ such that } d(x,y) < R. \tag{2.5}$$

If $z \geq 0$ is a nontrivial measurable function defined in Ω then,

$$\mathrm{supp}(K(z)) \supset B(\mathrm{supp}(z), R) := \bigcup_{y \in \mathrm{supp}(z)} B(y, R).$$

In particular, assume Ω is R–connected as in Definition 1.2. Then either $\mathrm{supp}(z) = \Omega$ or $\mathrm{supp}(K(z))$ is strictly larger than $\mathrm{supp}(z)$. Indeed if $\mathrm{supp}(z) \subsetneq \Omega$ but $\mathrm{supp}(K(z)) = \mathrm{supp}(z)$ then $B(\mathrm{supp}(z), R) \setminus \mathrm{supp}(z) = \emptyset$ then every point in $\Omega \setminus \mathrm{supp}(z)$ is at a distance larger than R from $\mathrm{supp}(z)$, which contradicts that Ω is R–connected.

As a consequence of Proposition 2.8, successive iterations of K will increase the support of z up to cover any compact set in Ω. In particular, if Ω is compact and connected (hence R–connected for any $R > 0$), then in a fixed number of iterations, independent of z, $\mathrm{supp}(K^n(z)) = \Omega$; see [26, Proposition 3.15] for full details.

The following result gives a more quantitative measure of the "positivization" properties of K provided a stronger positivity condition than (2.5) on J is assumed.

Proposition 2.9 *Assume Ω has finite measure, $|\Omega| < \infty$ and the balls in Ω with fixed radius $R > 0$, do not degenerate in measure, that is, for some $m_0 > 0$*

$$|B(x,R)| \geq m_0 > 0, \quad \text{for all} \quad x \in \Omega. \tag{2.6}$$

Assume also that $J \geq 0$ and satisfies that there exist $J_0 > 0$ such that

$$J(x,y) > J_0 > 0 \quad \text{for all } x,\, y \in \Omega, \text{ such that } \ d(x,y) < R. \tag{2.7}$$

Then if $z > 0$, there exists $\alpha = \alpha(z) > 0$ such that

$$K(z)(x) \geq \alpha > 0, \quad \text{a.e. } x \in \Omega.$$

In particular, for $z = 1$ we get that the function in (2.1) satisfies

$$h_0(x) \geq J_0 m_0 > 0, \quad \text{a.e. } x \in \Omega.$$

Proof. Assume first $\inf_\Omega z > \delta > 0$. Then for any $x \in \Omega$

$$K(z)(x) = \int_\Omega J(x,y) z(y)\, dy \geq \int_{B(x,R)} J(x,y) z(y)\, dy \geq J_0 |B(x,R)| \delta \geq J_0 m_0 \delta > 0.$$

Assume now $\inf_\Omega z = 0$. Then, since $|\Omega| < \infty$ we have $|\{z \leq \delta\}| \to 0$ as $\delta \to 0$. Then for δ small enough we can assume that for each $x \in \Omega$, $|B(x,R) \cap \{z > \delta\}| > m_0/2$.

Hence for $x \in \Omega$

$$K(z)(x) \geq \int_{B(x,R)\cap\{z>\delta\}} J(x,y) z(y)\, dy \geq J_0 |B(x,R) \cap \{z > \delta\}| \delta \geq J_0 \frac{m_0}{2} \delta > 0.$$

The result for $h_0 = K(1)$ is immediate. Observe that h_0 could be unbounded above unless we asume $J \in L^\infty(\Omega, L^1(\Omega))$. ■

Remark 2.10 *Observe that even if $|\Omega| = \infty$ the result above remains true for $z > 0$ such that $\inf_\Omega z = 0$ and, as $\delta \to 0$,*

$$\sup_{x\in\Omega} |B(x,R) \cap \{z \leq \delta\}| \to 0.$$

3 Principal eigenvalues and maximum principle

Now we discuss properties related to principal eigenvalues and maximum principles as in Definition 1.4. For this assume that J, h and X are as in Theorem 2.3.

Then following ideas in [7], [10], [6, 23] and references therein we define

Definition 3.1 *With the notations above we define*

$$\Lambda = \inf_{0<\varphi\in X} \sup_{\Omega} \frac{K\varphi - h\varphi}{\varphi}.$$

That is, Λ is the infimum of all $\lambda \in \mathbb{R}$ such that there exists $0 < \varphi \in X$ such that

$$K\varphi - h\varphi \le \lambda\varphi \quad \text{in } \Omega. \tag{3.1}$$

Some immediate consequences are the following

Proposition 3.2 *i) The set of λ for which (3.1) holds for some $0 < \varphi \in X$ is a half line, that is, if $\lambda' > \lambda$ then (3.1) also holds for λ' (and the same φ). If $\lambda > \Lambda$ then we can always assume in (3.1) that $K\varphi - h\varphi < \lambda\varphi$ in Ω.*
ii) If J satisfies (2.5) then

$$\Lambda \ge -\inf_{\Omega} h.$$

iii) If $|\Omega| < \infty$ and $J \in L^\infty(\Omega, L^1(\Omega))$ so h_0 defined in (2.1) satisfies $h_0 \in L^\infty(\Omega)$, then Λ is well defined and

$$\Lambda \le \sup_{\Omega}(h_0 - h).$$

iv) If μ is a principal eigenvalue of $K - hI$ in X as in Definition 1.4 then

$$\mu \ge \Lambda.$$

v) If Ω is R–connected, J satisfies (2.5), $\lambda > -\inf_\Omega h$ and $0 \le \varphi \in X$ is nontrivial and satisfies

$$K\varphi - h\varphi \le \lambda\varphi$$

then $0 < \varphi \in X$ and $\lambda \ge \Lambda$. If, additionally, $|\Omega| < \infty$, the measure of the balls satisfies (2.6) and J satisfies (2.7), then there exists $\alpha > 0$ such that $\varphi \ge \alpha > 0$ in Ω.
vi) As a function of J and h, $\Lambda = \Lambda(J, h)$ is increasing in J and decreasing in h. Moreover, if for some $\delta > 0$ we have $h_1 + \delta \le h \le h_2 - \delta$ in Ω, then

$$\Lambda(h_1) - \delta \ge \Lambda(h) \ge \Lambda(h_2) + \delta.$$

vii) If $\Omega' \subset \Omega$, and we denote by $X(\Omega')$ the spaces $L^p(\Omega')$, with $1 \le p \le \infty$, or $C_b(\Omega')$ respectively and for $\varphi \in X(\Omega')$ we define

$$K_{\Omega'}\varphi(x) = \int_{\Omega'} J(x, y)\varphi(y)\, dy, \quad x \in \Omega'$$

then

$$\Lambda(\Omega') := \inf_{0<\varphi\in X(\Omega')} \sup_{\Omega'} \frac{K_{\Omega'}\varphi - h\varphi}{\varphi} \le \Lambda = \Lambda(\Omega).$$

Proof. The first part in i) is immediate. Also, if $\lambda > \Lambda$, choose $\lambda > \lambda' > \Lambda$. Since for λ', (3.1) is also satisfied for some function $0 < \varphi \in X$ then $K\varphi - h\varphi \le \lambda'\varphi < \lambda\varphi$.
ii) If J satisfies (2.5) and (3.1) is satisfied for $0 < \varphi \in X$ then

$$0 < K\varphi \le (h + \lambda)\varphi$$

and therefore $h+\lambda>0$ in Ω. Thus $\lambda \geq -\inf_\Omega h$ and then $\Lambda \geq -\inf_\Omega h$.
iii) Since $h_0 \in L^\infty(\Omega)$, take $\varphi = 1 \in X$ then

$$\frac{K\varphi - h\varphi}{\varphi} = h_0 - h \leq \lambda := \sup_\Omega \big(h_0 - h\big).$$

Hence, the set of λ such that (3.1) holds for some $0<\varphi\in X$ is non empty and Λ is well defined.
iv) If μ is a principal eigenvalue, by definition, (3.1) holds for $\lambda=\mu$ and then, by part i), $\mu \geq \Lambda$.
v) If $\mathrm{supp}(\varphi) \subsetneq \Omega$, since Ω is R–connected, by Proposition 2.8 the support of $K\varphi$ is larger than the support of φ and we reach a contradiction with $K\varphi \leq (h+\lambda)\varphi$ since $h+\lambda>0$. Hence $0<\varphi\in X$ and (3.1) is satisfied, whence $\lambda \geq \Lambda$. The rest follows from Proposition 2.9 since $K\varphi > \alpha' > 0$ implies $\varphi \geq \frac{\alpha'}{h+\lambda} > \alpha > 0$ in Ω.
vi) The first part is immediate from the definition of Λ. For the rest observe that by Definition 3.1 we get $\Lambda(h+c) = \Lambda(h) - c$ for $c \in \mathbb{R}$.
vii) Observe that if $0<\varphi\in X$ then the restriction to Ω', $\phi = \varphi_{|\Omega'}$ satisfies $0<\phi\in X(\Omega')$ and for $x\in\Omega'$, $K_{\Omega'}\phi(x) \leq K\varphi(x)$. Hence

$$\Lambda(\Omega') \leq \sup_{\Omega'} \frac{K_{\Omega'}\phi}{\phi} - h \leq \sup_{\Omega'} \frac{K\varphi}{\varphi} - h \leq \sup_{\Omega} \frac{K\varphi}{\varphi} - h.$$

Minimising in φ we get the result. ■

Now our goal here is to prove the following main result.

Theorem 3.3 *Assume Ω is R–connected, $|\Omega|<\infty$ and J, h and X are as in the standing Assumptions 1.3. Then*
i)

$$-\inf_\Omega h \leq \Lambda = \sup Re(\sigma_X(K-hI)) \leq \sup_\Omega \big(h_0 - h\big).$$

In particular, Λ is the only possible principal eigenvalue of $K-hI$ in X.
ii) If

$$\Lambda > -\inf_\Omega h.$$

then Λ is the principal eigenvalue of $K-hI$ in X. In such a case Λ is a simple isolated eigenvalue of $K-hI$ in X with bounded eigenfunction. If moreover $J \in BUC(\Omega, L^{p_0'}(\Omega))$ and $h \in C_b(\Omega)$, the eigenfunction is continuous.
iii) The maximum principle is satisfied for $\lambda>\Lambda$ and is not satisfied for $\lambda<\Lambda$ nor for $\lambda=\Lambda>-\inf_\Omega h$.
iv) If $\lambda>\Lambda$ then the strong maximum principle is satisfied.

In order to prove Theorem 3.3 we make use of the following Lemma. Notice it does not use that K is compact.

Lemma 3.4 *Assume Ω is R–connected, $|\Omega|<\infty$, h and X are as above, (2.6) holds true and J satisfies (2.7) and $J \in L^\infty(\Omega, L^{p'}(\Omega))$ so $K \in \mathcal{L}(XL^\infty(\Omega))$, $\lambda > -\inf_\Omega h$, and there exist $0<\varphi\in X$ and $\phi\in X$ such that*

$$K\varphi - h\varphi \leq \lambda\varphi \quad in\ \Omega.$$

and

$$K\phi \geq (h+\lambda)\phi \quad \text{in } \Omega.$$

Then either $\phi \leq 0$ in Ω or $\phi > 0$ in Ω and is a multiple of φ, $K\varphi - h\varphi = \lambda\varphi$ and λ is a principal eigenvalue of $K - hI$ in X.

Proof. Assume $\phi > 0$ in a set of positive measure. By part v) in Proposition 3.2 we have $\varphi \geq \alpha > 0$ in Ω and $\phi \leq \frac{K\phi}{h+\lambda} \in L^\infty(\Omega)$, hence ϕ is bounded above.

Thus, let $s > 0$ be the largest number such that $\varphi \geq s\phi$, that is, $\frac{1}{s} = \sup_\Omega \left(\frac{\phi}{\varphi}\right) > 0$. Since

$$K(\varphi - s\phi) \leq (h+\lambda)(\varphi - s\phi),$$

if $\varphi - s\phi \geq 0$ is nontrivial, then from part v) in Proposition 3.2 we get $\varphi - s\phi > \varepsilon > 0$ in Ω for some $\varepsilon > 0$. But then there exist some $\tilde{s} > 0$ such that

$$\varphi \geq \varepsilon + s\phi > (\tilde{s} + s)\phi$$

which is a contradiction with the definition of s. Therefore $\varphi = s\phi$ and the result follows. ■

Now we can prove Theorem 3.3.

Proof of Theorem 3.3. We proceed in different steps.
Step1. We prove now that if $\lambda > \Lambda \geq -\inf_\Omega h$ then $\lambda \in \rho_X(K - hI)$. This and part iv) in Proposition 3.2 implies that Λ is the only possible principal eigenvalue.

In fact, since $\lambda > -\inf_\Omega h$ then the multiplication operator $(h + \lambda I) : X \to X$ is invertible and $K : X \to X$ is compact, then it is enough to prove that $K - (h+\lambda)I$ is injective. If we assume otherwise then there exists a nontrivial $\phi \in X$ such that $K\phi = (h+\lambda)\phi$. Since $\lambda > \Lambda$ then by part ii) in Proposition 3.2 there exists $0 < \varphi \in X$ such that $K\varphi - h\varphi < \lambda\varphi$. Since either ϕ or $-\phi$ are positive in a set of positive measure in Ω, then Lemma 3.4 implies ϕ is a multiple of φ and $K\varphi - h\varphi = \lambda\varphi$, which is a contradiction.
Step 2. We now prove that if $\Lambda > -\inf_\Omega h$ then Λ is a principal eigenvalue. Moreover, in such a case, Theorem 2.3 implies $\Lambda > -\inf_\Omega h$ is an isolated eigenvalue of $K - hI$ and Lemma 3.4 implies it is simple. Finally, if $0 < \varphi \in X$ is an associated eigenfunction, then $\varphi = \frac{K\varphi}{h+\lambda} \in L^\infty(\Omega)$ and is even continuous under the additional assumption on J and h. Thus we conclude the proof of part ii) in the theorem.

To prove the claim, observe that for any $\varepsilon > 0$ and $\lambda > \Lambda$, from Step 1, there exists a unique solution $\phi_{\varepsilon,\lambda} \in X$ of

$$K\phi - (h+\lambda)\phi = -\varepsilon.$$

Also, since $\lambda > \Lambda$, there exists $0 < \varphi_\lambda \in X$ such that $K\varphi_\lambda - h\varphi_\lambda \leq \lambda\varphi_\lambda$. Then Lemma 3.4 applied to φ_λ and $-\phi_{\varepsilon,\lambda}$ implies $\phi_{\varepsilon,\lambda} \geq 0$, since λ is not an eigenvalue, by Step 1.

But then, by part v) in Proposition 3.2, we have $\phi_{\varepsilon,\lambda} > \alpha > 0$ for some $\alpha = \alpha(\varepsilon, \lambda)$.

Now, if for some $\varepsilon > 0$, $\{\phi_{\varepsilon,\lambda}\}_{\Lambda<\lambda<\Lambda+1}$ is bounded in X, since $K \in \mathcal{L}(X, X)$ is compact, we can assume that for some sequence $\lambda_n \to \Lambda$, $\varphi_n = \phi_{\varepsilon,\lambda_n}$ is such that $0 < K(\varphi_n)$ converges in X to some $0 \leq \xi \in X$. Then

$$0 < \varphi_n = \frac{\varepsilon + K(\varphi_n)}{h+\lambda_n} \to \frac{\varepsilon + \xi}{h+\Lambda} = \varphi \quad \text{in } X.$$

Then $\varphi > 0$, $\xi = K(\varphi) \in L^\infty(\Omega)$ and $K\varphi - (h+\Lambda)\varphi = -\varepsilon$ and then

$$\sup_\Omega \frac{K\varphi - h\varphi}{\varphi} = \Lambda - \varepsilon$$

which contradicts that Λ is the infimum.

Therefore, we can take $\varepsilon_n \to 0$ and $\lambda_n \in [\Lambda, \Lambda+1]$ such that $\|\phi_{\varepsilon_n,\lambda_n}\|_X \to \infty$, $\lambda_n \to \lambda_* \in [\Lambda, \Lambda+1]$ and such that, taking, $0 < \varphi_n = \frac{\phi_{\varepsilon_n,\lambda_n}}{\|\phi_{\varepsilon_n,\lambda_n}\|_X}$, $0 < K(\varphi_n)$ converges in X to some $0 \leq \xi \in X$. Since φ_n satisfies

$$K\varphi_n - (h+\lambda_n)\varphi_n = \tilde{\varepsilon}_n = \frac{\varepsilon_n}{\|\phi_{\varepsilon_n,\lambda_n}\|_X} \to 0,$$

then

$$0 < \varphi_n = \frac{\tilde{\varepsilon}_n + K(\varphi_n)}{h+\lambda_n} \to \frac{\xi}{h+\lambda_*} = \varphi \quad \text{in } X.$$

Then $\|\varphi\|_X = 1$, $\varphi \geq 0$, $\xi = K(\varphi) \in L^\infty(\Omega)$ and $K\varphi - (h+\lambda_*)\varphi = 0$. Hence, λ_* is an eigenvalue and thus $\lambda_* = \Lambda$ by Step 1. Again, by part v) in Proposition 3.2, we have $\varphi > 0$ and Λ is a principal eigenvalue as claimed.

Step 3. Observe that from Steps 1 and 2 above, if either $\Lambda > -\inf_\Omega h$ or $\Lambda = -\inf_\Omega h$, we conclude that

$$\Lambda = \sup\Big(\sigma_X(K-hI) \cap \mathbb{R}\Big) \leq \sup Re(\sigma_X(K-hI)).$$

The reverse inequality will be obtained below, using properties of the linear evolution equation, see the end of Section 4, below Proposition 4.5.

Now we turn to prove iii).

Step 4. Now we prove the maximum principle for $\lambda > \Lambda$. Indeed if $u \in X$ satisfies

$$Ku - (h+\lambda)u \leq 0 \text{ in } \Omega$$

then $\phi = -u$ satisfies $K\phi \geq (h+\lambda)\phi$ and by Lemma 3.4, since λ is not an eigenvalue, we get $u \geq 0$ in Ω.

Step 5. If $\lambda = \Lambda > -\inf_\Omega h$ then Λ is the principal eigenvalue and the positive eigenfunction contradicts the maximum principle.

Step 6. Assume $\lambda < \Lambda$ and satisfies the maximum principle.

First, if $\Lambda > -\inf_\Omega h$ and ϕ is a positive eigenfunction of Λ then $K(-\phi) - (h+\lambda)(-\phi) = (\Lambda - \lambda)(-\phi) \leq 0$ in Ω and thus the maximum principle implies $\phi \leq 0$ which is a contradiction.

Second, if $\Lambda = -\inf_\Omega h$ then $A = \{h + \lambda < 0\}$ has positive measure $|A| > 0$ and then the characteristic function $\phi = -\mathcal{X}_A \leq 0$ satisfies $K\phi - (h+\lambda)\phi \leq 0$ in Ω and this contradicts the maximum principle. Notice that if $X = C_b(\Omega)$ we use that A can be approximated by compact sets from the interior and we can take a positive continuous function ϕ with compact support in a compact subset of A. With such a ϕ we reach a contradiction as above.

Step 7. Now we prove v). For this, if $\lambda > \Lambda \geq -\inf_\Omega h$ and $u \in X$ satisfies $Ku - (h+\lambda)u \leq 0$ in Ω, from Step 4 we know $u \geq 0$. Since $Ku \leq (h+\lambda)u$ and Ω is R–connected, by part v) in Proposition 3.2, if $u \neq 0$ we get $Ku \geq \alpha > 0$ and then $u \geq \frac{\alpha}{h+\lambda} \geq \alpha' > 0$ in Ω. ■

Remark 3.5 *The possibility that* $\Lambda = -\inf_\Omega h$ *has positive eigenfunctions will be analysed further below in Section 3.2.*

In particular, the reciprocal of part ii) will be discussed in Corollary 3.18.

Using Proposition 2.6 and the characterization in part i) of Theorem 3.3 we obtain the following.

Corollary 3.6 *Assume* Ω, J, h *and* $\{J_n\}_n, \{h_n\}_n$ *satisfy the assumptions in Theorem 3.3 and the assumptions in parts iii) or iv) in Proposition 2.6. Then we have*

$$\lim_{n\to\infty} \Lambda(K_n - h_n I) = \Lambda(K - hI).$$

Proof. Note that since any $\lambda \in \mathbb{C}$ such that $Re(\lambda) > \Lambda := \Lambda(K - hI)$ satisfies $\lambda \in \rho(K - hI)$, by part a) in Proposition 2.6 we get $\lambda \in \rho(K_n - h_n I)$ for all sufficiently large n. Hence, denoting $\Lambda(n) := \Lambda(K_n - h_n I)$,

$$\limsup_n \Lambda(n) \leq \Lambda.$$

If $\Lambda > -\inf_\Omega h$ then it is a simple eigenvalue of $K - hI$ and by part b) in Proposition 2.6 there is a sequence of simple eigenvalues of $K_n - h_n I$, denoted $\{\mu_n\}_n$ such that $\mu_n \to \Lambda$. Hence $Re(\mu_n) \leq \Lambda(n)$ and then

$$\Lambda = \lim_n Re(\mu_n) \leq \liminf_n \Lambda(n).$$

If $\Lambda = -\inf_\Omega h$, since $h_n \to h$ uniformly in Ω, then $\inf_\Omega h_n \to \inf_\Omega h$ and then $\Lambda(n) \geq -\inf_\Omega h_n$ which gives

$$\Lambda = -\inf_\Omega h \leq \liminf_n \Lambda(n).$$

■

The following result, obtained in [27] for the case of a symmetric kernel, gives an alternative description of Λ using the variational properties in $L^2(\Omega)$.

Proposition 3.7 *Assume in* Ω, J *and* h *are as in Theorem 3.3 and assume furthermore that in (1.7) we have* $1 \leq p_0 \leq 2$ *and*

$$J(x, y) = J(y, x). \tag{3.2}$$

Then the spectrum of $K - hI$ *is real, independent of* X *and*

$$\Lambda = \sup_{\substack{\varphi \in L^2(\Omega) \\ \|\varphi\|_{L^2(\Omega)} = 1}} E(\varphi)$$

where

$$E(\varphi) = -\frac{1}{2} \int_\Omega \int_\Omega J(x, y)(\varphi(y) - \varphi(x))^2 \, dy \, dx - \int_\Omega (h(x) - h_0(x)) \, \varphi^2(x) \, dx$$

with h_0 *as in (2.1).*

The proof is based on the observation that, using Fubini, from (3.2) one gets that $K - hI$ is selfadjoint in $L^2(\Omega)$. Therefore its spectrum in $L^2(\Omega)$ is real and bounded above by

$$\sup_{\substack{\varphi \in L^2(\Omega) \\ \|\varphi\|_{L^2(\Omega)} = 1}} \langle (K - hI)\varphi, \varphi \rangle_{L^2(\Omega)}$$

which belongs to the spectrum, [8, p. 165]. Suitable integration gives that $\langle (K-hI)\varphi,\varphi\rangle_{L^2(\Omega)} = E(\varphi)$. Since $1 \leq p_0 \leq 2$ then $X = L^2(\Omega)$ is one of the admissible spaces and by Proposition 2.4 the spectrum is independent of X.

We can also give the following alternative characterization of Λ in Definition 3.1, see [7] for the case of second order differential equations and [6, 23, 13] and references therein for the case of nonlocal operators in open sets of the euclidean space $\mathbb{R}^N$.

Proposition 3.8 *We have*

$$\Lambda = \sup_{0<\varphi\in X} \inf_{\Omega} \frac{K\varphi - h\varphi}{\varphi}$$

Proof. Indeed denoting

$$\tilde{\Lambda} = \sup_{0<\varphi\in X} \inf_{\Omega} \frac{K\varphi - h\varphi}{\varphi}$$

notice first that $\tilde{\Lambda} \leq \Lambda$ since, otherwise there would exists $\lambda > -\inf_\Omega h$, and $0 < \varphi, \tilde{\varphi} \in X$ such that

$$\sup_{\Omega} \frac{K\varphi - h\varphi}{\varphi} < \lambda < \inf_{\Omega} \frac{K\tilde{\varphi} - h\tilde{\varphi}}{\tilde{\varphi}}$$

which would contradict Lemma 3.4.

For the converse, first if $\Lambda > -\inf_\Omega h$, from part ii) in Theorem 3.3, so taking a positive eigenfunction φ, we get $\tilde{\Lambda} \geq \inf_\Omega \frac{K\varphi - h\varphi}{\varphi} = \Lambda$.

Second, if $\Lambda = -\inf_\Omega h$, denote $h_m = \inf_\Omega h$, take $0 < \varepsilon_n \to 0$ and consider the set of positive measure $\Omega_n = \{x \in \Omega,\ h(x) > h_m + \varepsilon_n\}$ and define $h_n(x) = \begin{cases} h(x) & x \in \Omega_n \\ h_m & x \in \Omega \setminus \Omega_n \end{cases}$ which is bounded, $h_n \leq h$, $\inf_\Omega h_n = h_m$ and $\|h - h_n\|_\infty \leq \varepsilon_n$.

Since $h - \varepsilon_n \leq h_n \leq h$ we get

$$\tilde{\Lambda} \leq \tilde{\Lambda}(K - h_n I) \leq \tilde{\Lambda} + \varepsilon_n.$$

On the other hand, since $\Omega \setminus \Omega_n$ has positive measure, part i) of Corollary 3.10 below implies $\Lambda(K - h_n I) > -h_m$ and therefore $\tilde{\Lambda}(K - h_n I) = \Lambda(K - h_n I) \to \Lambda$ as $n \to \infty$, by Corollary 3.6 and we get the result. ■

3.1 Criteria for $\Lambda > -\inf_\Omega h$

Now we derive some necessary conditions for Λ.

Proposition 3.9 *With the notations above, assume $|\Omega| < \infty$, (2.6) holds true and J satisfies (2.7). Then*
i) For any $x_0 \in \Omega$ and R as in (2.7) we have

$$\frac{1}{h + \Lambda} \in L^1(B(x_0, R)).$$

In particular, $\frac{1}{h+\Lambda} \in L^1_{loc}(\Omega)$.

Also, if J is lower semicontinuous in its second variable, then for any compact set $A \subset \Omega$ we have

$$\inf_{y\in A} \Big(\int_A \frac{J(x,y)}{h(x)+\Lambda}\, dx \Big) \le 1.$$

ii)

$$\inf_\Omega h_0 - \sup_\Omega h \le \inf_\Omega (h_0 - h) \le \Lambda.$$

iii) For any $x_0 \in \Omega$ and R as in (2.7) we have

$$J_0 m_0 - \sup_{\Omega\cap B(x_0,R)} h \le \Lambda.$$

Proof. Take $\lambda > \Lambda$ and $0 < \varphi \in X$ such that $K\varphi \le (h+\lambda)\varphi$.

First, integrating

$$\frac{\int_\Omega J(x,y)\varphi(y)\, dy}{h(x)+\lambda} \le \varphi(x) \tag{3.3}$$

in an arbitrary measurable set $A \subset \Omega$ we get

$$\int_A \int_A \frac{J(x,y)}{h(x)+\lambda}\varphi(y)\, dy\, dx \le \int_A \int_\Omega \frac{J(x,y)}{h(x)+\lambda}\varphi(y)\, dy\, dx \le \int_A \varphi(x)\, dx.$$

Using Fubini, the left hand side gives

$$\int_A \Big(\int_A \frac{J(x,y)}{h(x)+\lambda}\, dx \Big)\varphi(y)\, dy \ge \inf_{y\in A} \Big(\int_A \frac{J(x,y)}{h(x)+\lambda}\, dx \Big) \int_A \varphi(y)\, dy,$$

whence $\inf_{y\in A} \Big(\int_A \frac{J(x,y)}{h(x)+\lambda}\, dx \Big) \le 1$.

In particular, iff we take $A = B(x_0,R)$ with $x_0 \in \Omega$ and R as in (2.7)

$$J_0 \int_{B(x_0,R)} \frac{dx}{h(x)+\lambda} \le 1.$$

Taking a sequence $\lambda_n \to \Lambda$ and using Fatou's lemma we get $\frac{1}{h+\Lambda} \in L^1(B(x_0,R))$.

Also, if J is lower semicontinuous in its second variable and A is compact, then for each $\lambda > \Lambda$ and $\varepsilon > 0$ there exists $y_{\lambda,\varepsilon} \in A$ such that $\int_A \frac{J(x,y_{\lambda,\varepsilon})}{h(x)+\lambda}\, dx \le 1+\varepsilon$.

Hence, as $\lambda \to \Lambda$ and $\varepsilon \to 0$ we can assume $y_{\lambda,\varepsilon} \to y_0 \in A$ and using that J is lower semicontinuous and Fatou's lemma we get

$$\int_A \frac{J(x,y_0)}{h(x)+\Lambda}\, dx \le 1$$

and we get i).

For ii), again from (3.3) and using part v) in Proposition 3.2,

$$\varphi(x) \ge \frac{\int_\Omega J(x,y)\varphi(y)\, dy}{h(x)+\lambda} \ge \frac{h_0(x)}{h(x)+\lambda} \inf_\Omega \varphi$$

and we get $h_0(x)\frac{\inf_\Omega \varphi}{\varphi(x)} - h(x) \le \lambda, \quad x \in \Omega$.

Taking a minimising sequence $\{x_n\}_n \subset \Omega$ for φ, we get

$$\inf_\Omega h_0 - \sup_\Omega h \le \inf_\Omega (h_0 - h) \le \lambda.$$

Taking $\lambda \to \Lambda$ we get ii).

For iii), if instead of minimising φ in Ω in (3.3), we work with $x \in B(x_0, R)$ with $x_0 \in \Omega$ and R as in (2.7) we get from (3.3)

$$\varphi(x) \ge \frac{\int_{B(x_0,R)} J(x,y)\varphi(y)\,dy}{h(x)+\lambda} \ge \frac{J_0 m_0}{h(x)+\lambda} \inf_{\Omega\cap B(x_0,R)} \varphi, \qquad x \in B(x_0,R)$$

and we get $J_0 m_0 \frac{\inf_{\Omega\cap B(x_0,R)}\varphi}{\varphi(x)} - h(x) \le \lambda, \quad x \in \Omega \cap B(x_0,R)$.

Taking a minimising sequence $\{x_n\}_n \subset \Omega \cap B(x_0,R)$ for φ, we get

$$J_0 m_0 - \sup_{\Omega\cap B(x_0,R)} h \le \lambda.$$

Taking $\lambda \to \Lambda$ we get iii). ■

From this we derive the following criteria to guarantee that $\Lambda > -\inf_\Omega h$ and hence, Λ is the principal eigenvalue.

Corollary 3.10 *With the notations above, assume $|\Omega| < \infty$, (2.6) holds true and J satisfies (2.7). Denote $h_m = \inf_\Omega h$ and assume either one of the following conditions*
i)

$$|\{h = h_m\}| > 0,$$

ii) There exists $x_0 \in \Omega$ and $r \le R$ as in (2.7) such that

$$\int_{B(x_0,r)} \frac{dx}{h(x) - h_m} = \infty,$$

iii) There exists a compact set $C \subset \Omega$ such that

$$\int_C \frac{dx}{h(x) - h_m} = \infty.$$

iv) If J is lower semicontinuous in its second variable, we assume there exists a compact set $A \subset \Omega$ such that

$$\inf_{y\in A} \int_A \frac{J(x,y)}{h(x) - h_m}\,dx > 1.$$

v)

$$osc_\Omega(h) := \sup_\Omega h - \inf_\Omega h < \inf_\Omega h_0, \quad \text{or} \quad \inf_\Omega (h_0 - h) > -\inf_\Omega h.$$

iv) For any $x_0 \in \Omega$ and R as in (2.7) we have

$$\sup_{\Omega\cap B(x_0,R)} h - m < J_0 m_0 \le \inf_\Omega h_0.$$

Then, in each of the cases above we have

$$\Lambda > -\inf_\Omega h.$$

On the other hand we also have the following criteria.

Corollary 3.11 *Assume Ω, J, h are as in Theorem 3.3, $h_m = \inf_\Omega h$ and*

$$\frac{1}{h-h_m} \in X = L^p(\Omega) \quad \text{with } p_0 \le p < \infty.$$

i) If

$$\sup_{x\in\Omega} \int_\Omega \frac{J(x,y)}{h(y)-h_m}\, dy \le 1$$

then $\Lambda = -\inf_\Omega h$.
ii) If

$$\inf_{x\in\Omega} \int_\Omega \frac{J(x,y)}{h(y)-h_m}\, dy > 1$$

then $\Lambda > -\inf_\Omega h$.

Proof. Take $h_m = \inf_\Omega h$ and $0 < \varphi = \frac{1}{h-h_m} \in X$. Then one readily gets

$$K\varphi(x) - (h(x)-h_m)\varphi(x) = \int_\Omega \frac{J(x,y)}{h(y)-h_m}\, dy - 1.$$

In case i), from Definition 3.1 we get $\Lambda \le \sup_\Omega \frac{K\varphi - h\varphi}{\varphi} \le -h_m$, while $\Lambda \ge -h_m$ by Proposition 3.2.

In case ii) we get $\inf_\Omega \frac{K\varphi-h\varphi}{\varphi} > -h_m$ and by Proposition 3.8 we get $\Lambda > -h_m$. ∎

When $|\{h = h_m\}| = 0$ we give in the Lemma below a simple condition for part ii) in Corollary 3.10 which takes into account the measure geometric properties of the set $\{h = h_m\}$ and the shape of the function h near this set. The second part of the Lemma gives a simple condition for the assumption in Corollary 3.11 and will also be useful in Section 3.2.

Lemma 3.12 *Let $\emptyset \ne K_0 \subset \{h = h_m\}$ and assume that for $t > 0$ small*

$$|\{x,\ \mathrm{dist}(x, K_0) \le t\}| = O(t^\beta), \quad \beta > 0.$$

Then
i) Assume for x in a neighbourhood $\mathcal{O}$ of K_0 we have $h_m \le h(x) \le h_m + H(\mathrm{dist}(x,K_0))$ with $H : \mathbb{R}^+ \to \mathbb{R}^+$ and $H(0) = 0$ and $H(t) \le At^k$ near $t = 0$ for some $A, k > 0$.

Then

$$\int_{\mathcal{O}} \frac{dx}{h(x)-h_m} = \infty,$$

provided

$$\beta \le k.$$

ii) Assume $K_0 = \{h = h_m\}$ and for $x \in \Omega$ we have $h(x) \ge h_m + H(\mathrm{dist}(x,K_0))$ with $H : \mathbb{R}^+ \to \mathbb{R}^+$ and $H(0) = 0$ and $H(t) \ge At^k$ near $t = 0$ for some $A, k > 0$.

Then for $1 \le q < \infty$

$$\frac{1}{h-h_m} \in L^q(\Omega)$$

provided

$$1 \le q < \frac{\beta}{k}.$$

Proof. i) Observe that

$$\int_{\mathcal{O}} \frac{dx}{h(x)-h_m} = \int_0^\infty |\{\frac{1}{h-h_m} \geq s\} \cap \mathcal{O}|\, ds$$

and

$$\int_1^\infty |\{\frac{1}{h-h_m} \geq s\} \cap \mathcal{O}|\, ds = \int_1^\infty |\{h \leq h_m + \frac{1}{s}\} \cap \mathcal{O}|\, ds = \int_0^1 |\{h \leq h_m + r\} \cap \mathcal{O}|\, \frac{dr}{r^2}.$$

From the assumptions in H, for r small,

$$|\{h \leq h_m + r\} \cap \mathcal{O}| \geq |\{x,\ dist(x, K_0) \leq Br^{1/k}\}| \geq Cr^{\beta/k}$$

and therefore the integral is infinite provided $2 - \beta/k \geq 1$, i.e. $\beta \leq k$.
ii) Analogously

$$\int_\Omega \frac{dx}{(h(x)-h_m)^q} = \int_0^\infty |\{\frac{1}{(h-h_m)^q} \geq s\}|\, ds$$

and

$$\int_1^\infty |\{\frac{1}{(h-h_m)^q} \geq s\}|\, ds = \int_1^\infty |\{h \leq h_m + \frac{1}{s^{\frac{1}{q}}}\}|\, ds = \int_0^1 |\{h \leq h_m + r\}|\, \frac{dr}{r^{q+1}}.$$

From the assumptions in H, for r small,

$$|\{h \leq h_m + r\}| \leq |\{x,\ dist(x, K_0) \leq Br^{1/k}\}| \leq Cr^{\beta/k}$$

and therefore the integral is finite provided $q + 1 - \beta/k < 1$, i.e. $1 \leq q < \frac{\beta}{k}$. ■

Remark 3.13 *As a very particular case of Lemma 3.12, observe that if $\Omega \subset \mathbb{R}^N$ is an open set then if K_0 has fractal dimension $0 \leq d_F(K_0) \leq N$, then for $t > 0$ small*

$$|\{x,\ dist(x, K_0) \leq t\}| = O(t^{N-d_F(K_0)}),$$

see Proposition 3.2, chapter 3 in [14], and then the condition in part i) of the lemma reads

$$N - k \leq d_F(K_0).$$

In particular, if K_0 is an isolated point $x_0 \in \Omega$, since $d_F(K_0) = 0$, we obtain the condition $k \geq N$ which, for a sufficiently smooth function h leads to

$$D^j h(x_0) = 0, \quad \text{for } j = 1, \ldots, N-1$$

and we recover the result in [10].

Also, part ii) of the lemma reads

$$d_F(K_0) < N - kq.$$

In particular, if K_0 is formed by a finite number of isolated points then $d_F(K_0) = 0$ and the condition reads $1 \leq q < \frac{N}{k}$. In the case of a smooth h this requieres that at each point $x_0 \in K_0$

$$D^j h(x_0) = 0, \quad \text{for } j = 1, \ldots, j_0 \leq N-2, \quad D^{j_0+1} h(x_0) \neq 0$$

and then $1 \leq q < \frac{N}{j_0+1}$.

We end up this section with the following result on the operator K. Notice that below we take $h = 0 \in C_b(\Omega)$.

Proposition 3.14 *Assume Ω compact and connected, (2.6) holds and for some $1 \leq p_0 \leq \infty$*

$$J \in BUC(\Omega, L^{p_0'}(\Omega))$$

and denote $X = L^p(\Omega)$ for $p_0 \leq p \leq \infty$ or $X = C_b(\Omega)$.

Then

$$0 < \Lambda(K) = r(K) := \sup\{|\lambda|, \ \lambda \in \sigma(K)\}$$

that is, the spectral radius of K in X, is the unique principal eigenvalue of K and is simple.

In particular, we get

$$0 < \Lambda(K) \leq \inf_{p_0 \leq p \leq \infty} \|J\|_{L^p(\Omega, L^{p'}(\Omega))} \leq |\Omega|^{\frac{1}{p_0}} \|J\|_{L^\infty(\Omega, L^{p_0'}(\Omega))}.$$

We say that K is of Krein–Rutman type in X.

Proof. From Theorem 2.3, with $h = 0$, we know the spectrum of K is made up of $\{0\}$ and a sequence of eigenvalues converging to 0. From Proposition 2.4 we know that the spectrum of K does not depend on X. From Theorem 3.3, with $h = 0$ we know that $0 < \Lambda(K)$ and is the unique principal eigenvalue of K and is simple.

On the other hand, choosing $X = C_b(\Omega)$, since Ω is compact and connected and (2.6) holds, then we can use the Krein–Rutman theorem, [22, 30], to obtain that $\Lambda(K)$ is the spectral radius, $r(K)$, in $X = C_b(\Omega)$, see Proposition 3.19 in [26].

Since the spectrum of K does not depend on X, we get the result while the bound on $\Lambda(K)$ comes form the estimate $r(K) \leq \|K\|_{\mathcal{L}(L^p(\Omega), L^p(\Omega))}$ for $p_0 \leq p \leq \infty$, and by the estimates in Proposition 2.1 we have $\|K\|_{\mathcal{L}(L^p(\Omega), L^p(\Omega))} \leq \|J\|_{L^p(\Omega, L^{p'}(\Omega))}$. Using Hölder's inequality and minimising in p we get the estimate. ■

3.2 The case $\Lambda = -\inf_\Omega h$

Now we analyse in detail the case $\Lambda = -\inf_\Omega h = -h_m$. Our goal is to determine whether or not Λ is still a principal eigenvalue and if $K - (h - h_m)I$ satisfies the maximum principle.

Notice that in this case $\Lambda \in \sigma_{ess}(K - hI)$ and therefore one expects the properties of $K - (h + \Lambda)I$ to be quite different from the case where $\Lambda > -\inf_\Omega h$ and then Λ is in the point spectrum.

According to part ii) in Corollary 3.10 and Corollary 3.11 the integrability of $\frac{1}{h - h_m}$ plays a role in the analysis. Hence, we will assume that for some $s \geq 1$ we have

$$0 \leq \frac{1}{h - h_m} \in L^s(\Omega) \tag{3.4}$$

see Lemma 3.12, and define the set of integrable exponents for the function $\frac{1}{h - h_m}$

$$I = I(\frac{1}{h - h_m}) := \{s \geq 1, \ \frac{1}{h - h_m} \in L^s(\Omega)\}, \quad s_0 := \sup I(\frac{1}{h - h_m}) < \infty. \tag{3.5}$$

Therefore $I \subset [1, s_0]$ with the same endpoints.

Remark 3.15 *If $\Omega = B(0,1) \subset \mathbb{R}^N$ and $\alpha < N$, the function $g(x) = \frac{1}{|x|^\alpha}$ has $I(g) = [1, \frac{N}{\alpha})$, while the function $g(x) = \frac{1}{|x|^\alpha \log(\frac{A}{|x|})}$, with $A > 1$, has $I(g) = [1, \frac{N}{\alpha}]$.*

We now introduce the following notations.

Definition 3.16 *For $1 \leq p \leq \infty$,*
i) We say $1 \leq p \preccurlyeq s_0$ iff $\frac{1}{h-h_m} \in L^p(\Omega)$, that is, $p \leq s_0$ if $s_0 \in I$, or $p < s_0$ if $s_0 \notin I$.
ii) We say $s_0 \preccurlyeq p$ iff $\frac{1}{h-h_m} \notin L^p(\Omega)$, that is, $s_0 < p$ if $s_0 \in I$, or $s_0 \leq p$ if $s_0 \notin I$.

To begin with, we observe that in some spaces $\Lambda = -h_m$ could still be the principal eigenvalue and in some others it can not be.

Proposition 3.17 *Assume Ω, J, h and X are as in Theorem 3.3.*
i) If $\Lambda = -\inf_\Omega h$ has a positive eigenfunction in X then $X = L^p(\Omega)$ with

$$p \preccurlyeq s_0.$$

ii) If

$$p \preccurlyeq s_0$$

then there exists some (constant) kernel J such that $\Lambda = -h_m$ has a positive eigenfunction in $X = L^p(\Omega)$ given by $0 < \varphi = \frac{1}{h-h_m} \in X$.

Proof. i) Assume $\Lambda = -h_m$ has a positive eigenfunction in X, i.e. $0 < \varphi \in X$ satisfies $K\varphi = (h - h_m)\varphi$. Then by Proposition 2.9 we get $K\varphi > \alpha > 0$ and then $0 < \frac{\alpha}{h-h_m} \leq \varphi$ hence $\frac{1}{h-h_m} \in X$.
ii) Take $0 < \varphi = \frac{1}{h-h_m} \in X$ and $J = \frac{1}{\int_\Omega \frac{1}{h-h_m}}$. Then one readily gets $K\varphi - (h - h_m)\varphi = 0$ and $\Lambda = -h_m$ has $\varphi \in X$ as a positive eigenfunction. ■

With this we get the following reciprocal of part ii) of Theorem 3.3. This reciprocal holds in [23] since they work in $C(\overline{\Omega})$ for $\Omega \subset \mathbb{R}^N$.

Corollary 3.18 *i) If $X = C_b(\Omega)$ or $X = L^p(\Omega)$ with with $p_0 \leq p \leq \infty$ such that*

$$s_0 \preccurlyeq p$$

then Λ is the principal eigenvalue in X if and only if

$$\Lambda > -\inf_\Omega h.$$

ii) If $s_0 \preccurlyeq p_0$ then $\Lambda = -h_m$ is not a principal eigenvalue in any of the admissible spaces in Assumptions 1.3.

Therefore, we assume from now on that

$$p_0 \preccurlyeq s_0. \tag{3.6}$$

To analyse in detail the operator $K-(h-h_m)I$ and, in particular, its eventual eigenfunctions, we first sketch our driving argument: if $\lambda \geq -\inf_\Omega h$, $u \in X$ and we want to compute $f =$

$Ku-(h+\lambda)u$ we define $v=(h+\lambda)u \in X$ and then $f=\tilde{K}_\lambda v - v$ where $\tilde{K}_\lambda$ is the linear diffusion operator as in (1.3) associated to the kernel

$$\tilde{J}_\lambda(x,y) := \frac{J(x,y)}{h(y)+\lambda}.$$

Obviously for $\lambda > -h_m$, $\tilde{J}_\lambda$ and J have the same integrability properties, which are lost as $\lambda \to -h_m$ since $\frac{1}{h+\lambda}$ becomes unbounded. This singular limit, changes drastically the functional properties of $\tilde{K}_{-h_m}$. Under certain assumptions however, the equation $f=\tilde{K}_{-h_m}v - v$ can still be set in a larger space $Y \supseteq X$. In particular notice that since the standing Assumptions 1.3 $J(x,\cdot) \in L^{p_0'}(\Omega)$ then $J_{-h_m}(x,\cdot) \in L^{q'}(\Omega)$ for $\frac{1}{q'} = \frac{1}{p_0'} + \frac{1}{s}$ for all $s \preccurlyeq s_0$. Then we define q_0 as

$$0 \leq \frac{1}{q_0} := \frac{1}{p_0} - \frac{1}{s_0} \leq 1$$

if $s_0 \in I$ or

$$0 \leq \frac{1}{q_0} := (\frac{1}{p_0} - \frac{1}{s_0})_- \leq 1$$

otherwise, where $(x)_-$ denotes any number smaller than x but as close as we want to x. Observe that $1 \leq p_0 < q_0 \leq \infty$.

Now we analyse the kernels $\tilde{J}_\lambda$ and the nonlocal operators $\tilde{K}_\lambda$.

Proposition 3.19 *With the notations above and the standing Assumptions 1.3 we have the following results.*
i) For $\lambda > -h_m$, if $J \in L^\infty(\Omega, L^{p_0'}(\Omega))$ then $\tilde{J}_\lambda \in L^\infty(\Omega, L^{q_0'}(\Omega))$ while if $J \in BUC(\Omega, L^{p_0'}(\Omega))$ then $\tilde{J}_\lambda \in BUC(\Omega, L^{q_0'}(\Omega))$.
ii) If $J \in L^\infty(\Omega, L^{p_0'}(\Omega))$ then $\tilde{J}_{-h_m} \in L^\infty(\Omega, L^{q_0'}(\Omega))$ and

$$\tilde{J}_\lambda \to \tilde{J}_{-h_m}, \quad \text{in } L^\infty(\Omega, L^{q_0'}(\Omega)) \text{ as } \lambda \to -h_m.$$

If moreover $J \in BUC(\Omega, L^{p_0'}(\Omega))$ then $\tilde{J}_{-h_m} \in BUC(\Omega, L^{q_0'}(\Omega))$.
iii) If $J \in L^\infty(\Omega, L^{p_0'}(\Omega))$ denote $Y = L^q(\Omega)$ for $q_0 \leq q < \infty$, while if $J \in BUC(\Omega, L^{p_0'}(\Omega))$ denote $Y = L^q(\Omega)$ for $q_0 \leq q \leq \infty$ or $Y = C_b(\Omega)$. Then $\tilde{K}_\lambda, \tilde{K}_{-h_m} \in \mathcal{L}(Y, L^\infty(\Omega))$, $\tilde{K}_\lambda, \tilde{K}_{-h_m} \in \mathcal{L}(Y,Y)$ are compact, the spectrum of $\tilde{K}_\lambda$ and $\tilde{K}_{-h_m}$ are independent of Y and

$$\tilde{K}_\lambda \to \tilde{K}_{-h_m} \quad \text{in } \mathcal{L}(Y,Y) \text{ as } \lambda \to -h_m. \tag{3.7}$$

iv) For $\lambda \geq -h_m$ the operators $\tilde{K}_\lambda$ satisfy the assumptions in Theorem 3.3 in the spaces Y in part iii) with $\tilde{h} = 0$. In particular, for $\lambda > -h_m$, in the space Y we have

$$0 < \tilde{\Lambda}(\lambda) < 1,$$

and

$$\tilde{\Lambda}(\lambda) \to \tilde{\Lambda}(-h_m) \leq 1, \quad \text{as } \lambda \to -h_m$$

and they are the principal eigenvalues and also the spectral radiuses of $\tilde{K}_\lambda$ and $\tilde{K}_{-h_m}$ respectively.

Proof. i) This is immediate since for fixed $\lambda > -h_m$ we have

$$0 < c \leq \frac{1}{h(x)+\lambda} \leq C, \quad x \in \Omega$$

for some positive constants c, C and $q_0' \leq p_0'$.
ii) Observe that for $\lambda > -h_m$ we have for some $c > 0$ and any $s \preccurlyeq s_0$,

$$0 < c \leq \frac{1}{h+\lambda} \nearrow \frac{1}{h-h_m}, \quad \text{monotonically in } L^s(\Omega) \text{ as } \lambda \to -h_m$$

which in turn, using Hölder's inequality, implies that for each $x \in \Omega$

$$\tilde{J}_\lambda(x,\cdot) := \frac{J(x,\cdot)}{h(\cdot)+\lambda} \nearrow \tilde{J}_{-h_m}(x,\cdot) := \frac{J(x,\cdot)}{h(\cdot)-h_m}, \quad \text{monotonically in } L^{q_0'}(\Omega) \text{ as } \lambda \to -h_m$$

and then $\tilde{J}_{-h_m} \in L^\infty(\Omega, L^{q_0'}(\Omega))$ or $\tilde{J}_{-h_m} \in BUC(\Omega, L^{p_0'}(\Omega))$ respectively and

$$\tilde{J}_\lambda \nearrow \tilde{J}_{-h_m}, \quad \text{in } L^\infty(\Omega, L^{q_0'}(\Omega)) \text{ as } \lambda \to -h_m.$$

iii) This follows from Proposition 2.6.
iv) Observe first that for $\lambda \geq -h_m$ the kernels $\tilde{J}_\lambda$ satisfy condition (2.7) and $\tilde{J}_\lambda \in L^\infty(\Omega, L^{q_0'}(\Omega))$. Hence the operators $\tilde{K}_\lambda$ satisfy the assumptions in Theorem 3.3 in Y with $\tilde{h} = 0$ and then, by Proposition 3.14, we get $0 < \tilde{\Lambda}(\lambda)$ and it is the principal eigenvalue and the spectral radius of $\tilde{K}_\lambda$. Also, Corollary 3.6 implies

$$\tilde{\Lambda}(\lambda) \to \tilde{\Lambda}(-h_m), \quad \text{as } \lambda \to -h_m.$$

Now, for $\lambda > -h_m$ let us show that $\tilde{K}_\lambda - I$ satisfies the maximum principle in $Y = L^q(\Omega)$ for $q \geq q_0 > p_0$. For this assume $v \in Y$ satisfies $\tilde{K}_\lambda v - v \leq 0$ in Ω. Then $u = \frac{v}{h+\lambda} \in X := L^q(\Omega)$ and $Ku - (h+\lambda)u \leq 0$ in Ω. Since $\lambda > \Lambda = -h_m$, and by Proposition 2.4 the spectrum of $K - hI$ is independent of X we know from Theorem 3.3 that the maximum principle is satisfied for $K - (h+\lambda)$ in $X = L^q(\Omega)$, whence $u \geq 0$ and then $v \geq 0$.

Then part iii) in Theorem 3.3 implies $\tilde{\Lambda}(\lambda) \leq 1$. But indeed, if $\tilde{\Lambda}(\lambda) = 1$, since it is the principal eigenvalue in $Y = L^q(\Omega)$ with $q \geq q_0$, then there exists $0 < \tilde{\phi} \in Y$ such that $\tilde{K}_\lambda \tilde{\phi} = \tilde{\phi}$. But then $0 < \phi = \frac{\tilde{\phi}}{h+\lambda} \in X := L^q(\Omega)$ is such that $K\phi - (h+\lambda)\phi = 0$ but this implies that $\lambda > \Lambda = -h_m$ is a principal eigenvalue in X, which is a contradiction with part i) in Theorem 3.3.

Therefore $0 < \tilde{\Lambda}(\lambda) < 1$ and then $0 < \tilde{\Lambda}(-h_m) \leq 1$. ∎

Now we prove that the condition $\tilde{\Lambda}(-h_m) = 1$ characterizes the situation in which $\Lambda = -h_m$ has a positive eigenfunction and so it is a principal eigenvalue under assumption (3.6).

Proposition 3.20 *Under the above notations and assumptions, including (3.6), they are equivalent*
i) $\tilde{\Lambda}(-h_m) = 1$
ii) $\Lambda = -h_m$ *has a positive eigenfunction in* $X = L^p(\Omega)$ *for any* $p_0 \leq p \preccurlyeq s_0$.
iii) $\Lambda = -h_m$ *has a positive eigenfunction in* $X = L^p(\Omega)$ *for some* $p_0 \leq p \preccurlyeq s_0$.
In such a case, all positive eigenfunctions belong to the same one dimensional subspace.

Proof. First, if $\tilde{\Lambda}(-h_m) = 1$ since it is the principal eigenvalue and is simple, there exists $0 < \tilde{\phi} \in Y$, such that $\tilde{\phi} = \tilde{K}_{-h_m}\tilde{\phi} \in L^\infty(\Omega)$. But then $0 < \phi = \frac{\tilde{\phi}}{h-h_m} \in X := L^p(\Omega)$ for any $p \preccurlyeq s_0$ and is such that $K\phi - (h-h_m)\phi = 0$. Hence i) $\implies$ ii) $\implies$ iii).

Conversely, if $\Lambda = -h_m$ has a positive eigenfunction in $X = L^p(\Omega)$ for some $p_0 \leq p \preccurlyeq s_0$ then there exist $0 < \phi \in L^p(\Omega)$ such that $(h-h_m)\phi = K\phi \in L^\infty(\Omega)$. Then $0 < \tilde{\phi} = (h-h_m)\phi \in L^\infty(\Omega) \subset Y$ satisfies $\tilde{K}_{-h_m}\tilde{\phi} = \tilde{\phi}$. Hence $\tilde{\Lambda}(-h_m) = 1$. ■

Assume now that the set $\mathscr{C} := \{h = h_m\}$ is non empty (but of zero measure by (3.4)). Then we analyse below the kernel of the operator $K - (h-h_m)I$ and show that if $\Lambda = -h_m$ does not have a positive eigenfunction, then there are other singular "measure eigenfunctions" of $K - hI$ for $\lambda = -h_m$ some of which are positive. Moreover, the set of solutions of

$$Ku = (h - h_m)u \tag{3.8}$$

can be identified with the set of regular Borel measures concentrated in the set $\mathscr{C} := \{h = h_m\}$.

For this suitable additional regularity assumptions on J are required. More precisely we have the following.

Proposition 3.21 *With the notations above, assume furthermore that*

$$J \in L^q(\Omega, C_b(\Omega)) \quad \textit{for some } q_0 \leq q < \infty.$$

and $\Lambda = -h_m$ does not have a positive eigenfunction in $X = L^p(\Omega)$ for some (or any) $p_0 \leq p \preccurlyeq s_0$.

Then for each regular Borel measure with support in $\mathscr{C}$, that is, $d\sigma \in \mathcal{M}(\mathscr{C})$, there exists a unique w such

$$u = d\sigma + w$$

is a singular solution of (3.8). Moreover, $w \in L^p(\Omega)$ for all $p_0 \leq p < p_* = \frac{s_0 q}{s_0+q} \preccurlyeq s_0$.

If $d\sigma$ is a positive measure then $w \geq \frac{\alpha}{h-h_m} > 0$ for some $\alpha > 0$ and u is a positive singular solution of (3.8).

Proof. Using that for each $x \in \Omega$, $J(x,\cdot) \in C_b(\Omega)$ observe that u as in the statement satisfies $Ku = (h-h_m)u$ iff $v = (h-h_m)w$ satisfies $\tilde{K}_{-h_m}v - v = -H$ with

$$H(x) = \int_{\mathscr{C}} J(x,y)\, d\sigma(y), \quad x \in \Omega.$$

Because of the assumption on J in the statement we have $H \in Y = L^q(\Omega)$ and from the assumptions and Proposition 3.20 we get $\tilde{\Lambda}(-h_m) < 1$. Thus we have a unique solution $v \in Y$ of the equation $\tilde{K}_{-h_m}v - v = -H$. Therefore $w = \frac{v}{h-h_m} \in X = L^p(\Omega)$ for p given by

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{s} \tag{3.9}$$

for any $1 \leq s \preccurlyeq s_0$. In particular $\frac{1}{p} \geq \frac{1}{s}$ and then $p \preccurlyeq s_0$. The largest value of p is obtained as $s \to s_0$ so $\frac{1}{p} \geq \frac{1}{p_*} = \frac{1}{q} + \frac{1}{s_0} \leq \frac{1}{q_0} + \frac{1}{s_0} = \frac{1}{p_0}$ thus $p_* \geq p_0$.

If $d\sigma$ is a positive measure then $H \geq 0$ and is non trivial. Since $\tilde{\Lambda}(-h_m) < 1$ then $\tilde{K}_{-h_m} - I$ satisfies the strong maximum principle in Y, we get $v \geq \alpha > 0$ and then $w \geq \frac{\alpha}{h-h_m} \geq \alpha' > 0$ in Ω. ■

Observe that a related result was obtained in [11] assuming $\frac{1}{h-h_m} \in L^1(\Omega)$, working in $X = C(\overline{\Omega})$, $\Omega \subset \mathbb{R}^N$.

Now we present the following result on the maximum principle for the operator $K-(h-h_m)I$.

Proposition 3.22 *With the assumptions in Proposition 3.19 assume $\Lambda = -h_m$ does not have a positive eigenfunction in $X = L^p(\Omega)$ for some (or any) $p_0 \leq p \preccurlyeq s_0$.*

Then we have the following results.

i) For $X = C_b(\Omega)$ or $X = L^p(\Omega)$ with $q_0 \leq p \leq \infty$ and $s_0 \preccurlyeq p$ then for every nontrivial $u \in X$ we have that

$$Ku - (h - h_m)u \quad \text{changes sign in } \Omega,$$

that is, any nonzero function in the range $R(K - hI)$, changes sign in Ω. In particular, $K - (h - h_m)I$ satisfies the maximum principle.

ii) Assume $2p_0 \preccurlyeq s_0$ so that $q_0 \leq s_0$. Then the operator $K - (h - h_m)I$ satisfies the strong maximum principle in $X = L^p(\Omega)$ with $q_0 \leq p \preccurlyeq s_0$. Moreover, if $u \in X$ and $Ku-(h-h_m)u \leq 0$ then $u \geq \frac{\alpha}{h-h_m}$ for some $\alpha > 0$.

Proof. For $X = C_b(\Omega)$ or $X = L^p(\Omega)$ with $q_0 \leq p \leq \infty$, assume $u \in X$ is nonzero and is such that $Ku - (h - h_m)u \leq 0$. Then $v = (h - h_m)u \in Y = L^p(\Omega)$ satisfies $\tilde{K}_{-h_m}v - v \leq 0$.

Since from the assumption and Proposition 3.20 we get $\tilde{\Lambda}(-h_m) < 1$, then by part iii) in Theorem 3.3 the strong maximum principle holds for $\tilde{K}_{-h_m} - I$ in $Y = L^p(\Omega)$, hence either $v = 0$ or $v \geq \alpha > 0$. But $v = 0$ implies $\mathrm{supp}(u) \subset \{h = h_m\}$ which has zero measure and then $u = 0$ which is absurd. Therefore $v \geq \alpha > 0$ and then $\frac{\alpha}{h-h_m} \leq u \in X$.

If $s_0 \preccurlyeq p$ this is a contradiction and then $Ku - (h - h_m)u$ must change sign in Ω. Hence we prove part i).

On the other hand, if $q_0 \leq p \preccurlyeq s_0$, which requires $2p_0 \preccurlyeq s_0$, from $u \geq \frac{\alpha}{h-h_m}$ in Ω, we get $u \geq \alpha' > 0$ in Ω, which proves part ii). ■

Remark 3.23 *i) Observe that the case $X = L^p(\Omega)$ with $p_0 \leq p < q_0$ remains open.*

ii) Reference [23] treats the maximum principle for $K - (h - h_m)I$ in the case $\frac{1}{h-h_m} \in L^1(\Omega)$ working in $X = C(\overline{\Omega})$ and $\Omega \subset \mathbb{R}^N$ with continuous kernel J. In fact in their setting they prove that the maximum principle holds for $\Lambda = -h_m$.

Their assumptions correspond here to $p_0 = 1$, $s_0 = 1$ and $q_0 = \infty$ hence $X = L^\infty(\Omega)$ or $X = C_b(\Omega)$ and we are in case i) of the Proposition.

3.3 Criteria for the sign of Λ

In this section we use all the previous results to derive some criteria to establish the sign of Λ. For this we will assume the standing Assumptions 1.3. We will also denote $h_m = \inf_\Omega h$.

First two general remarks:

Lemma 3.24 *i) If there exist $0 < \xi \in X$ such that $K\xi - h\xi \leq 0$ then $\Lambda \leq 0$.*

ii) If there exist $\eta \in X$ that changes sign in Ω and such that $K\eta - h\eta \leq 0$ then $\Lambda \geq 0$.

Proof. i) This follows by Definition 3.1 and (3.1) with $\lambda = 0$.

ii) If $\Lambda < 0$ then by part iii) in Theorem 3.3 for $\lambda = 0$, the maximum principle would be satisfied which contradicts the existence of η in the statement. ■

Then we have the following result.

Proposition 3.25 *i) If* $h_m < 0$ *then* $\Lambda > 0$.
ii) If $h_m = 0$*, and either* $|\{h = 0\}| > 0$ *or* $\frac{1}{h} \notin L^1_{loc}(\Omega)$ *or* $\sup_\Omega h < \inf_\Omega h_0$ *or* $h + \delta \leq h_0$ *for some* $\delta > 0$*, then* $\Lambda > 0$ *and is the principal eigenvalue.*
iii) If $h_m > 0$*, assume there exists* $0 < \xi \in X$ *such that* $K\xi - h\xi \nleq 0$*, then* $\Lambda < 0$.
iv) If $h_m > 0$*, assume there exists* $\eta \in X$ *that changes sign in* Ω *such that* $K\eta - h\eta \leq 0$ *then* $\Lambda > 0$.

Proof. Part i) is immediate since from part i) in Proposition 3.2, $\Lambda \geq -h_m > 0$.

Part ii) follows from Corollary 3.10 since either condition implies $\Lambda > -h_m = 0$.

Part iii) is true because by i) in Lemma 3.24 we get $\Lambda \leq 0$. But if $\Lambda = 0 > -h_m$, so it is a principal eigenvalue, then there exists $0 < \phi \in X$ such that $K\phi - h\phi = 0$. Taking $\varphi = \xi$ and ϕ this contradicts Lemma 3.4 with $\lambda = 0$ because $K\xi - h\xi \neq 0$.

Part iv) is true because by ii) in Lemma 3.24 we get $\Lambda \geq 0$. But if $\Lambda = 0 > -h_m$, so it is a principal eigenvalue, then there exists $0 < \xi \in X$ such that $K\xi - h\xi = 0$. Taking $\varphi = \xi$ and $\phi = -\eta$, this contradicts Lemma 3.4 with $\lambda = 0$ since η changes sign. ■

Now we can derive the following improvement of the monotonicity properties of Λ in part vi) of Proposition 3.2.

Proposition 3.26 *Assume in* Ω*,* J *and* h *are as in Theorem 3.3 and are such that* $\Lambda = \Lambda(J, h)$ *is the principal eigenvalue of* $K - hI$.
i) Assume $J_1 \nleq J \leq J_2$ *in* $\Omega \times \Omega$ *have the same regularity than* J *in Theorem 3.3 and* J_1 *satisfies (2.7) (with may be a different constant* J_0*). Assume moreover that for some* $\delta > 0$ *we have*

$$J(x, y) < J_2(x, y) - \delta \quad \text{for all } x,\, y \in \Omega, \text{ such that } d(x, y) < R.$$

Then

$$\Lambda(J_1) < \Lambda(J) < \Lambda(J_2).$$

ii) Assume $h \nleq \tilde{h}$*. Then*

$$\Lambda(h) > \Lambda(\tilde{h}).$$

Proof. Take $0 < \xi \in X$ a positive and bounded eigenfunction associated to Λ.
i) Denote $K_i = K_{J_i}$ as in (1.3). Then we have $K_1\xi - h\xi = K\xi - h\xi + (K_1 - K)\xi = \Lambda\xi + (K_1 - K)\xi$. Now observe that $(K_1 - K) = K_{J_1 - J} = -K_{J - J_1}$ and then $(K_1 - K)\xi \nleq 0$. Hence

$$K_1\xi - (h + \Lambda)\xi \nleq 0$$

and by Lemma 3.24, $\Lambda(J_1, h + \Lambda) \leq 0$. But since by assumption $\Lambda > -\inf_\Omega h$, by iii) in Proposition 3.25 we get $\Lambda(J_1, h + \Lambda) = \Lambda(J_1, h) - \Lambda < 0$.

On the other hand, by Proposition 3.8 we get

$$\Lambda(J_2) \geq \inf_\Omega \big(\frac{K_2\xi}{\xi} - h\big) = \inf_\Omega \big(\frac{K\xi}{\xi} - h + \frac{(K_2 - K)\xi}{\xi}\big) = \Lambda + \inf_\Omega \frac{(K_2 - K)\xi}{\xi}.$$

Again $K_2 - K = K_{J_2 - J}$ and $J_2 - J$ satisfies (2.7) with constant δ. Then by Proposition 2.9 we have $K_{J_2 - J}\xi \geq \alpha > 0$ in Ω. Therefore $\Lambda(J_2) > \Lambda$.
ii) We have $K\xi - \tilde{h}\xi = K\xi - h\xi + (h - \tilde{h})\xi = \Lambda\xi + (h - \tilde{h})\xi$. Hence

$$K\xi - (\tilde{h} + \Lambda)\xi = (h - \tilde{h})\xi \nleq 0$$

and by Lemma 3.24, $\Lambda(\tilde{h}+\Lambda) \leq 0$. Also we know $\Lambda > -\inf_\Omega h \geq -\inf_\Omega \tilde{h}$, i.e. $\inf_\Omega(\tilde{h}+\Lambda) > 0$ and we are as in iii) in Proposition 3.25. Hence $\Lambda(\tilde{h}+\Lambda) = \Lambda(\tilde{h}) - \Lambda < 0$. ■

In particular we get the following result that sets the function h_0 in (2.1) as a threshold for the sign of Λ.

Corollary 3.27 *i) If $h = h_0$ then $\Lambda = 0$ with constant eigenfunction.*
ii) If $h_0 \nleq h$ then $\Lambda < 0$.
iii) If for some $\delta > 0$, $h + \delta \leq h_0$ then $\Lambda > 0$.
iv) In the symmetric case and $1 \leq p_0 \leq 2$ as in Proposition 3.7,

$$\int_\Omega h < \int_\Omega h_0 \quad \textit{then } \Lambda > 0.$$

Proof. i) Just observe that $K1 = h_0 - h = 0$, and then $\lambda = 0$ is the principal eigenvalue.
ii) This is because Proposition 3.26.
iii) This comes from part vi) in Proposition 3.2.
iv) From Proposition 3.7 we get that for $\varphi = \frac{1}{|\Omega|^{1/2}}$

$$\Lambda \geq E(\varphi) = \frac{1}{|\Omega|} \int_\Omega (h_0 - h) > 0.$$

■

Finally, note that when $\Lambda > -\inf_\Omega h$ the results above can be arranged to provide following characterization of the principal eigenvalue in the same line as for second order elliptic differential operators, see [7, 24, 1, 2, 25].

Theorem 3.28 *With the notations above, assume $\Lambda > -\inf_\Omega h$. Then, the following statements are equivalent*
i) $\Lambda < 0$.
ii) $K - hI$ satisfies the maximum principle in X.
iii) $K - hI$ satisfies the strong maximim principle in X.
iv) There exists $0 < \xi \in X$ such that $K\xi - h\xi \nleq 0$.

Proof. *i)* $\iff$ *ii)* is by part iii) in Theorem 3.3 with $\lambda = 0$.
i) $\Longrightarrow$ *iii)* is by part iv) in Theorem 3.3 with $\lambda = 0$ while *iii)* $\Longrightarrow$ *ii)* is obvious.
i) $\Longrightarrow$ *iv)* is by Definition 3.1 and (3.1) with $\lambda = 0$.
Finally *iv)* $\Longrightarrow$ *i)* because we are as in part iii) of Proposition 3.25. ■

4 Linear evolutionary problems

Now we turn to the linear evolution equation (1.2). First we recall the following results, c.f. [26, Theorem 4.3].

Proposition 4.1 *Assume $J \in L^p(\Omega, L^{p'}(\Omega))$ with $1 \leq p \leq \infty$ and then denote $X = L^p(\Omega)$ and assume $h \in L^\infty(\Omega)$. Alternatively, assume $J \in C_b(\Omega, L^1(\Omega))$ and then denote $X = C_b(\Omega)$ and assume $h \in C_b(\Omega)$.*

i) Then $L = K - hI \in \mathcal{L}(X, X)$ is a linear operator that generates a group $e^{Lt} \in \mathcal{L}(X, X)$ for $t \in \mathbb{R}$, and the solutions of the initial value problem

$$\begin{cases} u_t(x,t) &= (K - hI)u(x,t) \quad x \in \Omega, \\ u(x,0) &= u_0(x), \qquad\qquad x \in \Omega, \end{cases} \tag{4.1}$$

are given by $u(t) = e^{Lt}u_0$. Finally if $\sup Re(\sigma_X(L)) < \delta$ then

$$\|e^{Lt}\|_{\mathcal{L}(X)} \le Me^{\delta t}.$$

ii) For each $u_0 \in X$ and $t \in \mathbb{R}$ we have

$$u(x,t) = e^{Lt}u_0(x) = e^{-h(x)t}u_0(x) + \int_0^t e^{-h(x)(t-s)}K(u)(x,s)ds. \tag{4.2}$$

iii) **(Parabolic maximum principle)** *If $J \ge 0$, then for every nonnegative $u_0 \in X$, the solution of problem (4.1), $u(t) = e^{Lt}u_0$, is nonnegative for all $t \ge 0$, and it is nontrivial if $u_0 \not\equiv 0$.*

Moreover, if J satisfies (2.5) and Ω is R-connected as in Definition 1.2, then for every $u_0 \in X$, nonnegative and not identically zero,

$$\text{supp}(e^{Lt}u_0) = \Omega, \text{ for all } t > 0,$$

that is, the solution of (4.1) is strictly positive in Ω, for all $t > 0$.

Now we can improve the positivity properties of the solutions of (4.1) as follows.

Proposition 4.2 (Parabolic strong maximum principle)

With the notations in Proposition 4.1, assume Ω is R–connected, $|\Omega| < \infty$ and the measure satisfies (2.6) and J satisfies (2.7).

Then for every $u_0 \in X$, nonnegative and not identically zero,

$$\inf_\Omega e^{Lt}u_0 > 0, \quad t > 0.$$

Proof. Note that denoting $v(t) = e^{h(x)t}u(t)$ we have $v_t(t) = e^{h(x)t}K(u(t)) \ge 0$ for $t > 0$ which implies that for any $t > s > 0$ we have $u(t) \ge e^{-h(x)(t-s)}u(s)$. Also, from this, as in (4.2) we get

$$u(t) = e^{-h(x)(t-s)}u(s) + \int_s^t e^{-h(x)(t-r)}K(u)(r)\,dr \ge e^{-h(x)(t-s)}u(s) + \beta(t-s)K(u(s))$$

with $\beta(\tau) = \tau e^{-\|h\|_\infty \tau}$; hence $\beta(\tau) > 0$ for $\tau > 0$ and $\beta(0) = 0$.

From part iii) in Proposition 4.1 we have $u(s) > 0$ in Ω and then from Proposition 2.9 we get $\inf_\Omega K(u(s)) \ge \alpha(s) > 0$. Thus, for any $t > s$

$$u(t) \ge \beta(t-s)K(u(s)) \ge \beta(t-s)\alpha(s) > 0.$$

■

Observe now that from (4.2), for every $\omega_0 \in X$ and $T > 0$, we consider the mapping $\mathcal{F}_{\omega_0} : C([0,T], X) \to C([0,T], X)$ defined as

$$\mathcal{F}_{\omega_0}(\omega)(x,t) = e^{-h(x)t}\omega_0(x) + \int_0^t e^{-h(x)(t-s)}K(\omega)(x,s)\,ds, \ x \in \Omega, \ 0 \le t \le T.$$

Then we have the following immediate result.

Lemma 4.3 *If $\omega_0,\ z_0 \in X$, and $\omega,\ z \in Y_T = C([0,T],X)$ with the sup norm that we denote $|||\cdot|||$, then*

$$|||\mathcal{F}_{\omega_0}(\omega) - \mathcal{F}_{z_0}(z)||| \le C_1(T)\|\omega_0 - z_0\|_X + C_2(T)|||\omega - z|||, \tag{4.3}$$

where $C_1(T) = e^{\|h_-\|_{L^\infty(\Omega)}T}$, $C_2(T) = CTe^{\|h_-\|_{L^\infty(\Omega)}T}$, are increasing and continuous, and $C_2(T) \to 0$, as $T \to 0$.

Definition 4.4 *We say that $\overline{u} \in C([a,b],X)$ is a* **supersolution** *to (4.1) in $[a,b]$, if for any $t \ge s$, with $s,t \in [a,\, b]$*

$$\overline{u}(t) \ge e^{-h(x)(t-s)}\overline{u}(s) + \int_s^t e^{-h(x)(t-r)}K(\overline{u})(r)\,dr. \tag{4.4}$$

We say that $\underline{u}$ is a **subsolution** *if the reverse inequality holds.*

Clearly if $\overline{u} \in C([a,b],X)$ is differentiable and satisfies that

$$\overline{u}_t(t) \ge (K-h)\overline{u}(t) \quad \text{for} \quad t \in (a,\, b) \tag{4.5}$$

then $\overline{u}$ is a supersolution in the sense of (4.4). The same happens for subsolutions if the reverse inequality holds.

With this and the standard Picard's iterations, we can prove the following.

Proposition 4.5 *With the assumptions of Proposition 4.1, let $u(t,\, u_0) = e^{Lt}u_0$ be the solution to (4.1) with initial data $u_0 \in X$, and let $\overline{u}(t)$ be a supersolution to (4.1) in $[0,T]$.*

If $\overline{u}(0) \ge u_0$, then

$$\overline{u}(t) \ge u(t,\, u_0), \qquad \text{for} \ \ t \in [0,T].$$

The same is true for subsolutions with reversed inequality.

Proof. Since from (4.2) the solution of (4.1) can be written as $u(t) = \mathcal{F}(u)(t)$, with $\mathcal{F} = \mathcal{F}_{u_0}$, we choose $\tau < T$ small enough such that in Lemma 4.3 we have $C_2(\tau) < 1$. Hence, by (4.3) we have that $\mathcal{F}(\cdot)$ is a contraction in $V = C\big([0,\tau],X\big)$. Then we consider the sequence of Picard iterations,

$$u_{n+1}(t) = \mathcal{F}(u_n)(t), \quad n = 1,2,\ldots, \quad 0 \le t \le \tau$$

with $u_1(t) = \overline{u}(t)$ for $t \in [0,\tau]$, which converges to u in V.

Then we get by definition and using $\overline{u}(0) \ge u_0$

$$\overline{u}(t) \ge \mathcal{F}(\overline{u})(t) = u_2(t), \quad t \in [0,\tau].$$

From here, since $\mathcal{F}$ is increasing, $\overline{u}(t) \ge \mathcal{F}(\overline{u})(t) \ge \mathcal{F}(u_2)(t) = u_3(t)$ for $t \in [0,\tau]$ and by induction $\overline{u}(t) \ge u_n(t)$ for all $n = 1,2\ldots$ and $t \in [0,\tau]$. Hence

$$\overline{u}(t) \ge u(t,u_0), \quad t \in [0,\tau].$$

Now, we take $\widetilde{\tau} = \min\{2\tau, T\}$ and (4.1) with initial data $\tilde{u}_0(\tau) = u(\cdot,\tau)$. Then in $[\tau,\tilde{\tau}]$ $u(t)$ is the unique fixed point of

$$\mathcal{F}(u)(t) = e^{-h(x)(t-\tau)}\tilde{u}_0 + \int_\tau^t e^{-h(x)(t-s)}K(u)(s)\,ds \quad t \in [\tau,\widetilde{\tau}]$$

in $V = C([\tau, \widetilde{\tau}], X)$, and the supersolution satisfies by definition and $\overline{u}(\tau) \geq \tilde{u}_0$ that $\overline{u}(t) \geq \mathcal{F}(\overline{u}(t))$ in $[\tau, \widetilde{\tau}]$. Following the same argument as above, we obtain that the supersolution, $\overline{u}$, and the solution, u, are ordered in $[\tau, \widetilde{\tau}]$. From this we get $\overline{u}(t) \geq u(t, u_0)$, for $t \in [0, \widetilde{\tau}]$. Repeating this process we get the result in $[0, T]$. ■

Now we can end up the proof of part i) in Theorem 3.3.

End of the proof of i) in Theorem 3.3. Assume $\lambda > \Lambda$ and $0 < \varphi \in X$ is such that $K\varphi - h\varphi \leq \lambda\varphi$. Then from part v) in Proposition 3.2, we know $\varphi \geq \alpha > 0$ and $U(t) = e^{\lambda t}\varphi > 0$ satisfies

$$U_t \geq KU - hU, \quad t \geq 0$$

and $U(0) = \varphi$, that is, U is a positive supersolution to (4.1).

On the other hand, take $\mu \in \sigma_X(K - hI)$ and we want to show that $\Lambda \geq Re(\mu)$. From Theorem 2.3 we just need to consider the case when μ is an eigenvalue and from Step 1 of the proof of Theorem 3.3 we just need to consider the case when μ is complex. Let ϕ be a complex eigenfunction associated to μ. Then $u(t) = e^{\mu t}\phi$ is a complex solution of (4.1) and its real and imaginary parts, $v(t) = Re(u(t))$ and $w(t) = Im(u(t))$, are real solutions of (4.1).

Since $K \in \mathcal{L}(X, L^\infty(\Omega))$ then $\phi = v_0 + iw_0 = \frac{K\phi}{h+\lambda} \in L^\infty(\Omega)$ and therefore there exists $a < 0 < b$ such that $a\varphi \leq v_0, w_0 \leq b\varphi$. Then, from Proposition 4.5 we get

$$aU(t) \leq v(t), w(t) \leq bU(t), \quad t \geq 0$$

and from this, $Re(\mu) \leq \lambda$. Taking $\lambda \to \Lambda$, we get the result. ■

Then we have the following result.

Proposition 4.6 *Assume Ω is R–connected, $|\Omega| < \infty$ and J, h and X are as in the standing Assumptions 1.3. Also, fix any*

$$\tilde{\lambda} < \Lambda < \lambda.$$

Then
i) Any solution of (4.1) with $u_0 \in X$ satisfies

$$\|u(t)\|_X \leq Me^{\lambda t}\|u_0\|_X, \quad t \geq 0.$$

ii) Assume either $\Lambda > -\inf_\Omega h$ or $J \in BUC(\Omega, L^{p'}(\Omega))$. Also, by Proposition 4.2, assume without loss of generality that $0 \leq u_0 \in X$ is such that $u_0 \geq \alpha > 0$.

Then there exists a positive bounded function $\tilde{\varphi}$ in Ω such that

$$0 < e^{\tilde{\lambda} t}\tilde{\varphi}(x) \leq u(x, t) \quad x \in \Omega, \ t > 0.$$

iii) For any solution of (4.1) with $u_0 \in L^\infty(\Omega)$ there exists a positive function $\varphi \in X$ such that

$$|u(x, t, u_0)| \leq e^{\lambda t}\varphi(x) \quad x \in \Omega, \ t > 0.$$

Both parts ii) and iii) hold true for $\lambda = \tilde{\lambda} = \Lambda$ provided $\Lambda > -\inf_\Omega h$.

In particular, if $\Lambda < 0$ all solutions of (4.1) converge to 0 in X as $t \to \infty$. Moreover, if $u_0 \in L^\infty(\Omega)$ then $u(t) \to 0$ uniformly in Ω as $t \to \infty$.

On the other hand, if $\Lambda > 0$ then all positive solutions of (4.1) converge pointwise to ∞ as $t \to \infty$.

Proof. Part i) follows from part i) in Proposition 4.1 and part i) in Theorem 3.3.

From Proposition 3.8, for $\tilde{\lambda}$ and λ as in the statement, there exists $0 < \varphi, \tilde{\varphi} \in X$ such that

$$\tilde{\lambda} < \inf_{\Omega} \frac{K\tilde{\varphi} - h\tilde{\varphi}}{\tilde{\varphi}} \leq \Lambda \leq \sup_{\Omega} \frac{K\varphi - h\varphi}{\varphi} < \lambda.$$

Then $\underline{u}(x,t) = e^{\tilde{\lambda}t}\tilde{\varphi} > 0$ and $\overline{U}(t) = e^{\lambda t}\varphi > 0$ satisfy respectively

$$\underline{u}_t \leq K\underline{u} - h\underline{u}, \qquad \overline{U}_t \geq K\overline{U} - h\overline{U}, \quad t \geq 0$$

i.e. they are positive sub and super solutions of (4.1).

Also, from part v) of Proposition 3.2 we get $\varphi \geq \beta > 0$. On the other hand, if $\Lambda > -\inf_{\Omega} h$ we can assume $\tilde{\lambda} > -\inf_{\Omega} h$ and then we have $\tilde{\varphi} < \frac{K\tilde{\varphi}}{h+\tilde{\lambda}} \in L^{\infty}(\Omega)$ and we get $\tilde{\varphi} \leq \alpha$. If, otherwise $\Lambda = -\inf_{\Omega} h$, since $J \in BUC(\Omega, L^{p'}(\Omega))$ then from Proposition 2.4 we know the spectrum in X coincides with the spectrum in $L^{\infty}(\Omega)$, hence by Proposition 3.8, $\Lambda = \sup_{0<\varphi\in L^{\infty}(\Omega)} \inf_{\Omega} \frac{K\varphi - h\varphi}{\varphi}$ and then we can assume $\tilde{\varphi} \in L^{\infty}(\Omega)$ as well.

Now let $u(x,t)$ be a positive solution of (4.1). From Proposition 4.2 we can assume $\inf_{\Omega} u(x,0) > \alpha_0 > 0$. We can also assume $\tilde{\varphi}$ above satisfies $0 < \tilde{\varphi} < \alpha_0$. Then Proposition 4.5 implies $\underline{u}(x,t) = e^{\tilde{\lambda}t}\tilde{\varphi} \leq u(x,t)$ and we get ii).

Finally, let $u(x,t)$ be a solution of (4.1) such that $u(x,0)$ is bounded above. Then we can assume φ above satisfies $u(x,0) \leq \varphi$. Then Proposition 4.5 implies $u(x,t) \leq \overline{U}(x,t) = e^{\lambda t}\varphi$.

Hence, if $u_0 \in L^{\infty}(\Omega)$ the above argument applies to the solution of (4.1) with initial data $|u_0|$ and then

$$|u(t,u_0)| \leq u(t,|u_0|) \leq e^{\lambda t}\varphi.$$

If $\Lambda > -\inf_{\Omega} h$ then we can take $\lambda = \tilde{\lambda} = \Lambda$ the principal eigenvalue and $\varphi = \tilde{\varphi}$ and associated positive and bounded principal eigenfunction. ■

Recall that in Section 3.3 we derived criteria to determine the sign of Λ.

5 Conclusion

In this paper we have developed a comprehensive theory for principal eigenvalues for nonlocal linear operators in general metric mesure spaces. The nonlocal operators are defined through kernels with minimal regularity assumptions and are set in Lebesgue spaces of integrable functions as well as in the space of continuous functions.

The connection between principal eigenvalues and maximum principles is also addressed and the results show a strong similarity with the well known analogues for second order elliptic PDEs.

For evolution problems we prove the strong maximum principle, develop the technique of sub and super solutions and prove the connection between the sign of the principal eigenvalue and the stability or instability of the solutions. Again results resemble strongly the known results for linear second order parabolic equation.

References

[1] H. Amann. Maximum principles and principal eigenvalues. In *Ten mathematical essays on approximation in analysis and topology*, pages 1–60. Elsevier B. V., Amsterdam, 2005.

[2] H. Amann and J. López-Gómez. A priori bounds and multiple solutions for superlinear indefinite elliptic problems. *J. Differential Equations*, 146(2):336–374, 1998.

[3] F. Andreu-Vaillo, J.M. Mazón, J. D. Rossi, and J. Toledo-Melero. *Nonlocal diffusion problems*, volume 165 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[4] P. W. Bates and G. Zhao. Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal. *J. Math. Anal. Appl.*, 332(1):428–440, 2007.

[5] D. ben Avraham and S. Havlin. *Diffusion and reactions in fractals and disordered systems.* Cambridge University Press, Cambridge, 2000.

[6] H. Berestycki, J. Coville, and H. Vo. On the definition and the properties of the principal eigenvalue of some nonlocal operators. *J. Funct. Anal.*, 271(10):2701–2751, 2016.

[7] H. Berestycki, L. Nirenberg, and S. R. S. Varadhan. The principal eigenvalue and maximum principle for second-order elliptic operators in general domains. *Comm. Pure Appl. Math.*, 47(1):47–92, 1994.

[8] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[9] E. Christensen, C. Ivan, and M. L. Lapidus. Dirac operators and spectral triples for some fractal sets built on curves. *Adv. Math.*, 217(1):42–78, 2008.

[10] J. Coville. On a simple criterion for the existence of a principal eigenfunction of some nonlocal operators. *J. Differential Equations*, 249(11):2921–2953, 2010.

[11] J. Coville. Singular measure as principal eigenfunctions of some nonlocal operators. *Appl. Math. Lett.*, 26(8):831–835, 2013.

[12] J. Coville, J. Dávila, and S. Martínez. Pulsating fronts for nonlocal dispersion and KPP nonlinearity. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 30(2):179–223, 2013.

[13] J. Coville and F Hamel. On generalized principal eigenvalues of nonlocal operators with a drift. *Nonlinear Anal.*, 193:111569, 20, 2020.

[14] K. Falconer. *Fractal geometry.* John Wiley & Sons Inc., Hoboken, NJ, second edition, 2003. Mathematical foundations and applications.

[15] J. García-Melián and J. D. Rossi. Maximum and antimaximum principles for some nonlocal diffusion operators. *Nonlinear Anal.*, 71(12):6116–6121, 2009.

[16] J. García-Melián and J. D. Rossi. On the principal eigenvalue of some nonlocal diffusion problems. *J. Differential Equations*, 246(1):21–38, 2009.

[17] G. Hetzer, W. Shen, and A. Zhang. Effects of spatial variations and dispersal strategies on principal eigenvalues of dispersal operators and spreading speeds of monostable equations. *Rocky Mountain J. Math.*, 43(2):489–513, 2013.

[18] V. Hutson, S. Martinez, K. Mischaikow, and G. T. Vickers. The evolution of dispersal. *J. Math. Biol.*, 47(6):483–517, 2003.

[19] C.Y. Kao, Y. Lou, and W. Shen. Random dispersal vs. non-local dispersal. *Discrete Contin. Dyn. Syst.*, 26(2):551–596, 2010.

[20] T. Kato. *Perturbation theory for linear operators.* Springer Verlag, 1976.

[21] J. Kigami. Measurable Riemannian geometry on the Sierpinski gasket: the Kusuoka measure and the Gaussian heat kernel estimate. *Math. Ann.*, 340(4):781–804, 2008.

[22] M. G. Kreĭn and M. A. Rutman. Linear operators leaving invariant a cone in a Banach space. *Amer. Math. Soc. Translation*, 1950(26):128, 1950.

[23] F. Li, J. Coville, and X. Wang. On eigenvalue problems arising from nonlocal diffusion models. *Discrete Contin. Dyn. Syst.*, 37(2):879–903, 2017.

[24] J. López-Gómez. The strong maximum principle. In *Mathematical analysis on the self-organization and self-similarity*, RIMS Kôkyûroku Bessatsu, B15, pages 113–123. Res. Inst. Math. Sci. (RIMS), Kyoto, 2009.

[25] J. López-Gómez. *Linear second order elliptic operators.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[26] A. Rodríguez-Bernal and S. Sastre. Linear nonlocal diffusion problems in metric measure spaces. *Proceedings of the Royal Society of Edinburg*, 146:833–863, 2016.

[27] A. Rodríguez-Bernal and S. Sastre. Stability and instability results for equilibria of nonlocal reaction-diffusionproblems. In preparation, 2025.

[28] W. Shen and X. Xie. On principal spectrum points/principal eigenvalues of nonlocal dispersal operators and applications. *Discrete Contin. Dyn. Syst.*, 35(4):1665–1696, 2015.

[29] R. S. Strichartz. *Differential equations on fractals.* Princeton University Press, Princeton, NJ, 2006. A tutorial.

[30] P. Takáč. A short elementary proof of the Kreĭn-Rutman theorem. *Houston J. Math.*, 20(1):93–98, 1994.