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UV/IR mixing and the Goldstone theorem in noncommutative field theory

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Abstract

Noncommutative IR singularities and UV/IR mixing in relation with the Goldstone theorem for complex scalar field theory are investigated. The classical model has two coupling constants, λ_1 and λ_2, associated to the two noncommutative extensions $\phi^* \star \phi \star \phi^* \star \phi$ and $\phi^* \star \phi^* \star \phi \star \phi$ of the interaction term $|\phi|^4$ on commutative spacetime. It is shown that the symmetric phase is one-loop renormalizable for all λ_1 and λ_2 compatible with perturbation theory, whereas the broken phase is proved to exist at one loop only if $\lambda_2 = 0$, a condition required by the Ward identities for global $U(1)$ invariance. Explicit expressions for the noncommutative IR singularities in the 1PI Green functions of both phases are given. They show that UV/IR duality does not hold for any of the phases and that the broken phase is free of quadratic noncommutative IR singularities. More remarkably, the pion selfenergy does not have noncommutative IR singularities at all, which proves essential to formulate the Goldstone theorem at one loop for all values of the spacetime noncommutativity parameter θ.

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1. Introduction

As is well known, in noncommutative field theory [1] the nonplanar parts of 1PI Green functions become singular when the noncommutativity spacetime parameter θ approaches zero [2]. The corresponding singularities are called noncommutative IR divergences and,

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for the theories usually considered, are quadratic, linear or logarithmic in $1/\theta$. They arise from the contribution of large loop-momenta to nonplanar one-loop Feynman integrals which, being finite for nonvanishing θ, become divergent if $\theta \to 0$. This simple but deep observation, first made in Ref. [2], is known as UV/IR mixing and for $\lambda\phi^4$ and gauge theories [2–6] takes a much stronger form, which we will refer to as strong UV/IR duality. Strong UV/IR duality states that the logarithmic noncommutative IR singularities in the nonplanar part of a 1PI Green function and the logarithmic UV divergences in its planar part are in one-to-one correspondence. UV/IR duality in this strong form seems not to be an artifact of perturbation theory, since in many instances it has been reobtained by taking the infinite tension limit of a suitable string amplitude for an open bosonic string on a magnetic B-field [7].

Noncommutative IR singularities pose serious problems for the existence of noncommutative field theories beyond one loop. They threaten renormalizability at higher loops (since locality of UV counterterms may be spoiled[1]) and may introduce tachyonic states [4,6,10] (associated to quadratic noncommutative IR singularities in 1PI two-point functions). In noncommutative gauge theories [11], quadratic and linear noncommutative IR singularities can be eliminated by introducing supersymmetry [4,6]. Indeed, in supersymmetric gauge theories, the supersymmetric partners of the gauge field provide nonplanar contributions which cancel the quadratic and linear noncommutative IR singularities in the nonsupersymmetric theories [4,6]. The supersymmetric theories thus become free of tachyonic instabilities and are left with the milder noncommutative logarithmic IR singularities. Furthermore, the results in Ref. [6] imply that supersymmetric $N = 1$ $U(1)$ gauge theory in the Yennie gauge becomes free of *all* noncommutative IR singularities at one loop.

The purpose of this paper is to study the noncommutative IR singularities of $U(1)$ complex scalar field theory, to investigate whether they satisfy UV/IR duality in the strong sense mentioned above, to explore spontaneous symmetry breaking as a mechanism to eliminate noncommutative IR singularities and to analyze how this enters the Goldstone theorem. To carry this investigation, we must first understand the UV renormalizability of the model. Although the latter should be by now well established, in our analysis we have found issues that have gone unnoticed in the literature and that are essential to understand the model's spontaneous symmetry breaking and its $U(1)$ global invariance at the quantum level. We also report on them.

To be more explicit, consider complex scalar field theory on noncommutative Minkowski spacetime, defined classically by the action

$$S_{\rm sym} = \int d^4x \big[\big(\partial_\mu \phi^*\big)\big(\partial^\mu \phi\big) - V_{\rm sym}\big(M, \lambda, \phi, \phi^*\big)\big], \tag{1.1}$$

where ϕ is a complex scalar field and the potential $V_{\rm sym}(M, \lambda_1, \lambda_2, \phi, \phi^*)$ has the form

$$V_{\rm sym}\big(M, \lambda_1, \lambda_2, \phi, \phi^*\big) = M^2|\phi|^2 + \frac{\lambda_1}{4}\, \phi^* \star \phi \star \phi^* \star \phi + \frac{\lambda_2}{4}\, \phi^* \star \phi^* \star \phi \star \phi, \tag{1.2}$$

[1] As of today, the question of higher-loop renormalizability has been addressed mainly for $\lambda\phi^4$ [8] and the Wess–Zumino model [9].

with λ_1 and λ_2 two different coupling constants. Note that in the action one must allow for the two inequivalent noncommutative extensions $\phi^* \star \phi \star \phi^* \star \phi$ and $\phi^* \star \phi^* \star \phi \star \phi$ of the commutative interaction term $|\phi|^4$. The symbol $\star$ denotes the Moyal product, defined for functions $f(x)$ and $g(x)$ as

$$(f \star g)(x) = f(x) \exp\left(\frac{i}{2} \overleftarrow{\partial_\mu} \theta^{\mu\nu} \overrightarrow{\partial_\nu}\right) g(x),$$

where $\theta^{\mu\nu}$ is a constant real antisymmetric matrix and our metric convention is $g_{\mu\nu} = \mathrm{diag}(+,-,-,-)$. We will restrict ourselves to magnetic-like matrices[2] $\theta^{\mu\nu}$, i.e., such that $\theta^{0i} = 0$ for $i = 1, 2, 3$. For $M^2 > 0$, the only field configuration that minimizes the energy is $\phi_0 = 0$ and the action (1.1) with potential (1.2) defines the symmetric phase of the classical theory. The global $U(1)$ gauge transformations that leave invariant the action take the form $\phi \to e^{i\alpha}\phi$, with α an arbitrary real constant. By contrast, for $M^2 < 0$, any field configuration ϕ_0 such that $|\phi_0|^2 = v^2$, with

$$v = \sqrt{\frac{-2M^2}{\lambda_1 + \lambda_2}},$$

minimizes the energy and classical spontaneous symmetry breaking takes place. Indeed, choosing $\phi_0 = v$ and expanding ϕ about it as

$$\phi = \frac{1}{\sqrt{2}}(\pi + i\sigma) + iv, \tag{1.3}$$

the action can be written as

$$S_{\mathrm{br}} = \int d^4x \left[\frac{1}{2}(\partial_\mu \pi)\big(\partial^\mu \pi\big) + \frac{1}{2}(\partial_\mu \sigma)\big(\partial^\mu \sigma\big) - V_{\mathrm{br}}(M, \lambda_1, \lambda_2, \pi, \sigma)\right], \tag{1.4}$$

where the potential $V_{\mathrm{br}}(M, \lambda_1, \lambda_2, \pi, \sigma)$ has the form

$$\begin{aligned} V_{\mathrm{br}}(M, \lambda_1, \lambda_2, \pi, \sigma) = {} & \frac{1}{2}\big(2M^2\big)\sigma^2 + \frac{v(\lambda_1 + \lambda_2)}{2\sqrt{2}}(\pi \star \pi \star \sigma + \sigma \star \sigma \star \sigma) \\ & + \frac{\lambda_1}{4} \pi \star \pi \star \sigma \star \sigma - \frac{\lambda_1 - \lambda_2}{8} \pi \star \sigma \star \pi \star \sigma \\ & + \frac{\lambda_1 + \lambda_2}{16}(\pi \star \pi \star \pi \star \pi + \sigma \star \sigma \star \sigma \star \sigma) \end{aligned} \tag{1.5}$$

and M^2 has been replaced with $-M^2$, so as to work with a positive M^2. The action (1.4) with potential (1.5) defines the nonsymmetric or broken phase of the classical theory. The global $U(1)$ transformations that leave S_{br} invariant are obtained from $\phi \to e^{i\alpha}\phi$ and Eq. (1.3); they read

$$\delta\pi = -\alpha\big(\sigma + \sqrt{2}\, v\big) \qquad \delta\sigma = \alpha\pi. \tag{1.6}$$

As stated, we want to study the noncommutative IR singularities and their mixing with UV divergences in both phases.

[2] In this way we do not run into problems with unitarity [12].

Our main results and the organization of the paper are as follows. In Section 2, we consider the symmetric phase and show that it is one-loop renormalizable for arbitrary λ_1 and λ_2 compatible with perturbation theory, being not necessary to take $\lambda_2 = 0$. We also give explicit expressions for the noncommutative IR singularities in the 1PI Green functions and prove that UV/IR duality in its strong form does not hold. Sections 3 to 5 are dedicated to study the broken phase. In particular, in Section 3, we demonstrate that one-loop UV renormalization for the broken phase is consistent with the Ward identities only if $\lambda_2 = 0$. In Section 4 we rederive the same result by analyzing the consistency of the nonplanar sector of the theory with the Ward identities. Section 5 presents explicit expressions for the noncommutative IR singularities in the 1PI Green functions of the broken phase. The expressions given there show that in the broken phase there are no quadratic noncommutative IR singularities, that the selfenergy for the pion field π is free of *all* noncommutative IR singularities and that the strong version of UV/IR duality does not hold. Also in Section 5 we show that the pion mass, defined as the zero of the selfenergy, remains zero after one-loop radiative corrections, thus ensuring that the Goldstone theorem holds true at one loop for arbitrary magnetic $\theta^{\mu\nu}$. Section 6 contains our conclusions.

Several related problems have been addressed in the literature. In Ref. [13] the broken phase of the noncommutative global $U(N)$ model, with $N > 1$ and $\lambda_2 = 0$, is considered and it is shown that the pion selfenergy vanishes for vanishing external momentum. Ref. [14] assumes $\lambda_2 = 0$ and proves that global $O(2)$ scalar field theory is one-loop renormalizable. Whereas these papers deal with the case $\lambda_2 = 0$, we focus on the case $\lambda_2 \neq 0$ and on noncommutative IR singularities and their implications. As concerns local models, Ref. [15] proves the consistency of UV renormalization with the BRS identities for the local $U(1)$ model and calculates the beta functions. In turn, the one-loop renormalizability of the local $U(2)$ and $U(1) \times U(1)$ models is shown in Ref. [16]. It is worth noting that in the local models λ_2 is excluded classically, since $\phi^* \star \phi^* \star \phi \star \phi$ is not invariant under local gauge transformations, while our analysis here shows that in the global model $\lambda_2 = 0$ follows from the symmetry requirements at the quantum level.

2. The symmetric phase: renormalization and noncommutative IR singularities

We first consider the symmetric phase, with classical action given by Eq. (1.1) and (1.2). At one loop, the only 1PI Green functions with UV divergences are the field selfenergy $\Sigma(p)$ and the vertex $\Gamma(p_1, p_2, p_3, p_4)$. To regularize the theory and to account for the counterterms that will be necessary to subtract the UV divergences, we introduce an invariant cutoff Λ by considering the 'bare' action

$$S_{\Lambda,0} = \int d^4x \left[\left(\partial_\mu \phi_0^*\right) \left(1 + \frac{\partial^2}{\Lambda^2}\right)^n \left(\partial^\mu \phi_0\right) - V_{\rm sym}\left(M_0, \lambda_{10}, \lambda_{20}, \phi_0, \phi_0^*\right)\right] \qquad n \geqslant 2. \tag{2.1}$$

Note that the quadratic UV divergences in the one-loop tadpole are not regularized if $n = 1$, so we must take $n \geqslant 2$. The potential $V_{\rm sym}(M_0, \lambda_{1,0}, \lambda_{2,0}, \phi_0)$ is as in Eq. (1.2) but with the renormalized quantities $M, \lambda_1, \lambda_2, \phi, \phi^*$ replaced with bare quantities

$M_0, \lambda_{10}, \lambda_{20}, \phi_0, \phi_0^*$, defined by

$$\phi_0 = Z_\phi^{1/2}\phi, \tag{2.2}$$

$$Z_\phi M_0^2 = Z_{M^2} M^2, \qquad Z_\phi^2 \lambda_{10} = \lambda_1 + \delta\lambda_1, \qquad Z_\phi^2 \lambda_{20} = \lambda_2 + \delta\lambda_2. \tag{2.3}$$

The renormalization constants Z_ϕ and Z_{M^2} have the form

$$Z_\phi = 1 + \delta z_\phi, \qquad Z_{M^2} = 1 + \frac{\delta M^2}{M^2},$$

with $\delta\lambda_1, \delta\lambda_2, \delta z_\phi$ and δM^2 collecting all terms of order one or higher in $\hbar$. The action $S_{\Lambda,0}$ can be recast as

$$S_{\Lambda,0} = S_{\Lambda,\mathrm{sym}} + S_{\mathrm{ct,sym}},$$

where $S_{\Lambda,\mathrm{sym}}$ is given by

$$S_{\Lambda,\mathrm{sym}} = \int d^4x \left[(\partial_\mu \phi^*)\left(1 + \frac{\partial^2}{\Lambda^2}\right)^n (\partial^\mu \phi) - V_{\mathrm{sym}}(M, \lambda_1, \lambda_2, \phi, \phi^*) \right] \tag{2.4}$$

and the counterterms $S_{\mathrm{ct,sym}}$ read

$$\begin{aligned} S_{\mathrm{ct,sym}} = \int d^4x \bigg[&\delta z_\phi (\partial_\mu \phi^*)(\partial^\mu \phi) - \delta M^2 \phi^*\phi - \frac{\delta\lambda_1}{4} \phi^* \star \phi \star \phi^* \star \phi \\ &- \frac{\delta\lambda_2}{4} \phi^* \star \phi^* \star \phi \star \phi \bigg]. \end{aligned} \tag{2.5}$$

It is important to emphasize that λ_1 and λ_2 are different coupling constants, so there is no reason for them to have the same running. In other words, $\delta\lambda_1$ and $\delta\lambda_2$ may be different. Eqs. (2.4) and (2.5) provide the Feynman rules depicted in Fig. 1, where we have used the

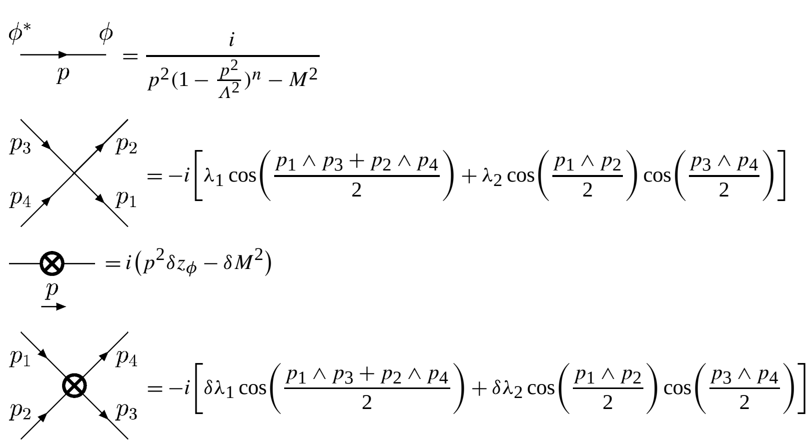


Fig. 1. Feynman rules for $S_{\Lambda,\mathrm{sym}}$ and $S_{\mathrm{ct,sym}}$.

notation

$$\tilde{p} = \theta^{\mu\nu} p_\nu, \qquad p \wedge q = \theta^{\mu\nu} p_\mu q_\nu, \qquad p \circ p = -\theta^{\mu\nu}\theta_\mu{}^\tau p_\nu p_\tau.$$

Introducing sources J_0 and J_0^* for the fields ϕ_0^* and ϕ_0, we consider the generating functional

$$\begin{aligned} Z\big[J_0, J_0^*\big] &= e^{G_c[J_0, J_0^*]} \\ &= \int \big[d\phi_0\big]\big[d\phi_0^*\big] \exp\bigg[i S_{\Lambda,\mathrm{sym}} + i S_{\mathrm{ct,sym}} + i \int d^4x \,\big(J_0^*\phi_0 + J_0\phi_0^*\big)\bigg]. \end{aligned} \tag{2.6}$$

For J_0 we write $J_0 = Z_\phi^{-1/2} J$, so that $J_0\phi_0^* = J\phi^*$ and similarly for J_0^*. To find the Ward identity associated to the $U(1)$ global symmetry, we follow the standard procedure: change variables $\phi \to e^{i\alpha}\phi$ in the integral in Eq. (2.6), take into account that under this change $S_{\Lambda,0}$ remains invariant and define the effective action $\Gamma[\phi,\phi^*]$ as the Legendre transform of $W[J, J^*]$. This leads to the Ward identity

$$\int d^4x \left(\phi \frac{\delta\Gamma}{\delta\phi} - \phi^* \frac{\delta\Gamma}{\delta\phi^*}\right) = 0. \tag{2.7}$$

Using for the effective action its expansion

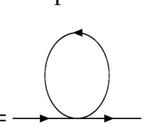
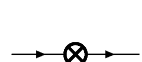
$$\begin{aligned} \Gamma[\phi,\phi^*] = \sum_{n=1}^{\infty} \frac{1}{2n!} \int & d^4x_1 \cdots d^4x_n \, d^4y_1 \cdots d^4y_n \\ & \times \phi(x_1)\cdots\phi(x_n)\phi^*(y_1)\cdots\phi^*(y_m)\Gamma^{(n,m)}(x_1,\ldots,x_n; y_1,\ldots,y_m) \end{aligned}$$

in fields, where $\Gamma^{(n)}(x_1,\ldots,x_n; y_1,\ldots,y_n)$ denotes the Green function of n ϕ-fields and n ϕ^*-fields, and going to momentum space, we obtain the following set of Ward identities for the 1PI Green functions:

$$\Gamma^{(n)}(p_1,\ldots,p_n; q_1,\ldots,q_n) = \Gamma^{(n)}(q_1,\ldots,q_n; p_1,\ldots,p_n). \tag{2.8}$$

The quantum theory is defined by the $\Lambda \to \infty$ limit of $Z[J_0, J_0^*]$, or equivalently of $\Gamma[\phi,\phi^*]$. Hence, for the symmetric phase of the quantum theory to exist, the large Λ limit must be well defined. This means that, while preserving the Ward identities, it must be possible to choose order by order in perturbation theory the counterterms so as to cancel the divergences that appear in the 1PI Green functions when $\Lambda \to \infty$. We are going to show that this is the case at one loop for all values of λ_1 and λ_2 compatible with perturbation theory.

As already mentioned, the only 1PI Green functions with UV divergences at one loop are the field selfenergy $\Sigma(p)$ and the vertex $\Gamma(p_1,p_2;p_3,p_4)$. Let us first worry about the selfenergy. Its one-loop contribution is given by

$-i\Sigma_1(p) =$  $+$  $= -i\Sigma_{\mathrm{reg}}(p) + i\big(p^2\delta z_\phi - \delta M^2\big),$ (2.9) where the regularized selfenergy $-i\,\Sigma_{\text{reg}}(p)$ reads

$$-i\,\Sigma_{\text{reg}}(p)=\int\frac{d^4k}{(2\pi)^4}\,\frac{\lambda_1+\frac{\lambda_2}{2}(1+e^{ik\wedge p})}{k^2(1-\frac{k^2}{\Lambda^2})^n-M^2}.$$

The $\theta^{\mu\nu}$-independent part of this integral gives the one-loop planar contribution $-i\,\Sigma_{\text{P}}(p)$ to the field selfenergy, while the $\theta^{\mu\nu}$-dependent part defines the nonplanar contribution $-i\,\Sigma_{\text{NP}}(p)$. Computing their limit $\Lambda\to\infty$ (see the appendix for details), we have

$$\begin{aligned}-i\,\Sigma_{\text{P}}(p)&=\left(\lambda_1+\frac{\lambda_2}{2}\right)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2(1-\frac{k^2}{\Lambda^2})^n-M^2}\\&\underset{\Lambda\to\infty}{\longrightarrow}-\frac{i}{16\pi^2}\left(\lambda_1+\frac{\lambda_2}{2}\right)\left[\frac{\Lambda^2}{n-1}-M^2\ln\!\left(\frac{\Lambda^2}{M^2}\right)+M^2 f_0\right]\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}-i\,\Sigma_{\text{NP}}(p)&=\frac{\lambda_2}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{e^{ik\wedge p}}{k^2(1-\frac{k^2}{\Lambda^2})^n-M^2}\\&\underset{\Lambda\to\infty}{\longrightarrow}-\frac{i\lambda_2M^2}{8\pi^2}\,\frac{K_1(\sqrt{p\circ pM^2})}{\sqrt{p\circ pM^2}},\end{aligned}\tag{2.11}$$

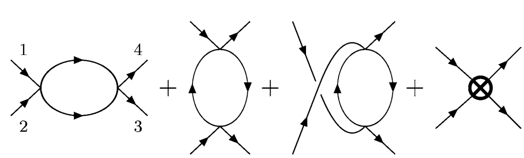
where

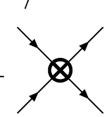
$$f_0=\sum_{r=1}^{n-1}\frac{1}{r}+\sum_{r=1}^{n}\binom{n}{r}\frac{\Gamma(r)\Gamma(2n-r)}{\Gamma(2n)}\tag{2.12}$$

and $K_\nu(\cdot)$ is the third Bessel function of order ν. Note that, when $\Lambda\to\infty$, the planar contribution diverges quadratically and the nonplanar contribution remains finite provided $p\circ p\neq 0$. To cancel the UV divergences in $-i\,\Sigma_{\text{P}}(p)$ and thus render $-i\,\Sigma_1(p)$ UV finite, we adopt an MS type scheme and take for δz_ϕ and δM^2

$$\delta z_\phi=0,\qquad \delta M^2=-\frac{1}{16\pi^2}\left(\lambda_1+\frac{\lambda_2}{2}\right)\left[\frac{\Lambda^2}{n-1}-M^2\ln\!\left(\frac{\Lambda^2}{M^2}\right)\right].\tag{2.13}$$

For the one-loop correction to the 4-vertex

$-i\,\Gamma_1(p_1,p_2;p_3,p_4)=$ 

$=-i\,\Gamma_{\text{reg}}(p_1,p_2;p_3,p_4)+$ 

we proceed similarly. The regularized contribution $-i\,\Gamma_{\text{reg}}(p_1,p_2,p_3,p_4)$ is the sum of the first three diagrams and can be decomposed

$$-i\,\Gamma_{\text{reg}}(p_1,p_2;p_3,p_4)=-i\,\Gamma_{\text{P}}(p_1,p_2;p_3,p_4)-i\,\Gamma_{\text{NP}}(p_1,p_2;p_3,p_4)\tag{2.14}$$

in a planar part $-i\Gamma_{\mathrm{P}}(p_1, p_2; p_3, p_4)$ and a nonplanar part $-i\Gamma_{\mathrm{NP}}(p_1, p_2; p_3, p_4)$. The planar contribution contains all the divergences that arise when $\Lambda \to \infty$, while the nonplanar contribution is well defined for $\Lambda \to \infty$ and $\theta^{\mu\nu} \neq 0$. After some calculations, for the planar contribution we obtain

$$\begin{aligned}
&-i\Gamma_{\mathrm{P}}(p_1, p_2; p_3, p_4) \\
&\quad \underset{\Lambda\to\infty}{\longrightarrow} \frac{i}{16\pi^2}\left[\left(\lambda_1^2 + \frac{\lambda_2^2}{4}\right)\cos\left(\frac{p_1\wedge p_3 + p_2\wedge p_4}{2}\right)\right. \\
&\qquad\quad \left. + \lambda_2\left(\lambda_1 + \frac{\lambda_2}{4}\right)\cos\left(\frac{p_1\wedge p_2}{2}\right)\cos\left(\frac{p_3\wedge p_4}{2}\right)\right]\ln\left(\frac{\Lambda^2}{M^2}\right) + \text{f.c.}, \qquad (2.15)
\end{aligned}$$

where "f.c." collects finite, regular contributions for nonexceptional configurations of external momenta. In the MS type scheme that we have adopted, cancellation of UV divergences requires taking $\delta\lambda_1$ and $\delta\lambda_2$ as

$$\begin{aligned}
\delta\lambda_1 &= \frac{1}{16\pi^2}\left(\lambda_1^2 + \frac{\lambda_2^2}{4}\right)\ln\left(\frac{\Lambda^2}{M^2}\right), \\
\delta\lambda_2 &= \frac{1}{16\pi^2}\lambda_2\left(\lambda_1 + \frac{\lambda_2}{4}\right)\ln\left(\frac{\Lambda^2}{M^2}\right). \qquad (2.16)
\end{aligned}$$

Thus far we have that the counterterms $S_{\mathrm{ct,sym}}$ with δz_ϕ, δM^2, $\delta\lambda_1$ and $\delta\lambda_2$ as in Eqs. (2.13) and (2.16) cancel the divergences that occur in the one-loop 1PI diagrams generated by $S_{\Lambda,\mathrm{sym}}$ when $\Lambda \to \infty$, thus ensuring that the $\Lambda \to \infty$ limit of $\Gamma[\phi, \phi^*]$ exists at one loop. Furthermore, since by construction $\Gamma[\phi, \phi^*]$ satisfies the Ward identity (2.7) for all Λ, λ_1 and λ_2, and since the divergent contributions for $\Lambda \to \infty$ in the 1PI Green functions, given by Eqs. (2.10) and (2.15), satisfy the Ward identities (2.8), the limit $\Lambda \to \infty$ preserves the Ward identities. Hence the symmetric phase of the quantum theory exists at one loop. We stress that the symmetric phase is renormalizable at one loop for all values of λ_1 and λ_2 compatible with perturbation theory, and that there is no need to assume $\lambda_2 = 0$. In other words, if one writes

$$\begin{aligned}
\lambda_1 + \delta\lambda_1 &= Z_{11}\lambda_1 + Z_{12}\lambda_2, \\
\lambda_2 + \delta\lambda_2 &= Z_{21}\lambda_1 + Z_{22}\lambda_2,
\end{aligned}$$

the fact that λ_1 and λ_2 are different coupling constants means that there are no requisites on the Z_{ij} other than those arising from the Ward identities, and we have seen that these do not impose any. From Eqs. (2.16) we obtain

$$\begin{aligned}
Z_{11} &= 1 + \frac{1}{16\pi^2}\lambda_1 \ln\left(\frac{\Lambda^2}{M^2}\right), & Z_{12} &= \frac{1}{16\pi^2}\frac{\lambda_2}{4}\ln\left(\frac{\Lambda^2}{M^2}\right), \\
Z_{21} &= \frac{1}{16\pi^2}\frac{\lambda_2}{2}\ln\left(\frac{\Lambda^2}{M^2}\right), & Z_{22} &= 1 + \frac{1}{16\pi^2}\frac{2\lambda_1 + \lambda_2}{4}\ln\left(\frac{\Lambda^2}{M^2}\right),
\end{aligned}$$

which are different among themselves. In Sections 3 and 4, we will see that for the broken phase the Ward identities require $\lambda_2 = 0$.

Once we know that the symmetric phase of the theory exists at one loop, we move on to study the noncommutative IR singularities in the 1PI Green functions. The one-loop nonplanar contribution (2.11) to the selfenergy is well defined for $\theta^{\mu\nu} \neq 0$. For $\theta^{\mu\nu} \to 0$, however, it becomes singular. In fact, sending $\theta^{\mu\nu} \to 0$ in Eq. (2.11) and using the results in the appendix, we have

$$\lim_{\theta^{\mu\nu}\to 0} \lim_{\Lambda\to\infty} [-i\,\Sigma_{\rm NP}(\theta, p)] = -\frac{i\lambda_2}{8\pi^2}\left\{\frac{1}{p\circ p} + \frac{M^2}{4}\left[\ln\bigl(p\circ p M^2\bigr) - 2\ln 2 + 2\gamma - 1\right]\right\}. \tag{2.17}$$

The origin of these noncommutative IR singularities can be understood by looking at the integral expression for $-i\,\Sigma^{\rm NP}(p)$ in Eq. (2.11). At $\Lambda \to \infty$, the integral is well defined if $\theta^{\mu\nu} \neq 0$, but diverges quadratically if $\theta^{\mu\nu} = 0$. The contribution to the integral from arbitrarily high momenta k^μ is curbed by the noncommutativity of spacetime, with $1/p\circ p$ acting as a regulator. This is precisely the UV/IR mixing argument [2], that for $\lambda\phi^4$ and gauge theories [2–6] goes beyond this observation for nonplanar integrals and states that the logarithmic noncommutative IR singularities in the nonplanar part of a 1PI Green function and the logarithmic UV divergences in its planar part can be obtained from each other by replacing $p_i \circ p_i \leftrightarrow 1/\Lambda^2$ for all the external momenta p_i. This stronger form of UV/IR mixing does not hold here, since the planar part $-i\,\Sigma_{\rm P}(p)$ of the selfenergy has UV logarithmic divergences proportional to λ_1, whereas the nonplanar part $-i\,\Sigma_{\rm NP}(p)$ does not have contributions proportional to λ_1 [see Eqs. (2.10) and (2.11)]. Without loss of generality, we can take a reference frame in which all the components of $\theta^{\mu\nu}$ vanish except for

$$\theta^{12} = -\theta^{21} \equiv \theta. \tag{2.18}$$

In this frame, and using the notation $p^\mu = (p^0, \vec{p}_\perp, p^3)$ and $\vec{p}_\perp = (p^1, p^2)$, Eq. (2.17) takes the form

$$-i\,\Sigma_{\rm NP}(p) \approx -\frac{i\lambda_2}{8\pi^2}\left[\frac{1}{\theta^2\vec{p}_\perp^{\,2}} + \frac{M^2}{2}\ln\bigl(\theta M^2\bigr)\right],$$

where the symbol $\approx$ denotes that the limit $\theta^{\mu\nu} \to 0$, $\Lambda \to \infty$ has been taken and that all finite contributions have been dropped. Besides the selfenergy, the four-vertex is the only other 1PI Green function that may develop noncommutative IR singularities in its nonplanar part. After some calculus, for the singular behaviour at $\theta \to 0$ of its nonplanar part, we obtain in the frame (2.18)

$$\begin{aligned} -i\,\Gamma_{\rm NP}(p_1, p_2, p_3, p_4) &\approx -\frac{i}{16\pi^2}\left[\lambda_2\left(\lambda_1 + \frac{3}{8}\lambda_2\right)\cos\left(\frac{p_1\wedge p_3 + p_2\wedge p_4}{2}\right)\right. \\ &\quad \left. + \left(\frac{3}{4}\lambda_1^2 + \lambda_1\lambda_2 + \frac{5}{8}\lambda_2^2\right)\cos\left(\frac{p_1\wedge p_2}{2}\right)\cos\left(\frac{p_3\wedge p_4}{2}\right)\right]\ln\bigl(\theta M^2\bigr). \end{aligned} \tag{2.19}$$

It is clear that the replacement $\theta^2 M^2 \leftrightarrow 1/\Lambda^2$ does not relate the noncommutative IR singularities in this equation with the UV divergences in the planar part given in Eq. (2.15). We conclude that UV/IR duality in its strong form does not hold.

To finish our discussion of noncommutative IR divergences, we study if these introduce perturbative tachyonic instabilities as in nonsupersymmetric gauge theories. The dispersion relation up to one loop reads

$$p^2 - M^2 - \Sigma_1(p) = 0.$$

For external momenta p^μ such that $\lambda_2/p \circ pM^2 \ll 1$, where perturbation theory is valid, the dominant part of $\Sigma_1(p)$ is the first term in Eq. (2.17), so we write

$$p^2 = M^2 + \frac{\lambda_2}{8\pi^2 p \circ p} + \text{subleading terms}.$$

Since $p \circ p$ is positive definite, there are no perturbative tachyonic instabilities.

3. The broken phase: UV counterterms

We start writing an action analogous to $S_{\Lambda,0}$ for the symmetric phase which (i) generates through perturbation theory finite Green functions at $\Lambda \to \infty$, and (ii) is symmetric under global $U(1)$ transformations. To this end, we combine Eqs. (2.2) and (1.3) so that

$$\phi_0 = Z_\phi^{1/2}\left[\frac{1}{\sqrt{2}}(\pi + i\sigma) + iv\right], \tag{3.1}$$

substitute the latter in Eq. (1.1), use Eq. (2.3) and replace M^2 with $-M^2$. This yields for $S_{\Lambda,0}$

$$S_{\Lambda,0} = S_{\Lambda,\mathrm{br}} + S_{\mathrm{ct,br}},$$

where $S_{\Lambda,\mathrm{br}}$ is given by

$$S_{\Lambda,\mathrm{br}} = \int d^4x\left[\frac{1}{2}(\partial_\mu\pi)\left(1 + \frac{\partial^2}{\Lambda^2}\right)^n(\partial^\mu\pi) + \frac{1}{2}(\partial_\mu\sigma)\left(1 + \frac{\partial^2}{\Lambda^2}\right)^n(\partial^\mu\sigma) - V_{\mathrm{br}}(M, \lambda_1, \lambda_2, \pi, \sigma)\right], \tag{3.2}$$

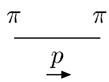
the counterterms $S_{\mathrm{ct,br}}$ read

$$\begin{aligned} S_{\mathrm{ct,br}} = \int d^4x \bigg\{ & \frac{\delta z_\phi}{2}(\partial_\mu\pi)(\partial^\mu\pi) + \frac{\delta z_\phi}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) - \sqrt{2}\,v\delta_1\sigma - \frac{\delta_1}{2}\pi^2 - \frac{\delta_2}{2}\sigma^2 \\ & - \frac{v(\delta\lambda_1 + \delta\lambda_2)}{2\sqrt{2}}(\pi\star\pi\star\sigma + \sigma\star\sigma\star\sigma) - \frac{\delta\lambda_1}{4}\pi\star\pi\star\sigma\star\sigma \\ & + \frac{\delta\lambda_1 - \delta\lambda_2}{8}\pi\star\sigma\star\pi\star\sigma - \frac{\delta\lambda_1 + \delta\lambda_2}{16}(\pi\star\pi\star\pi\star\pi + \sigma\star\sigma\star\sigma\star\sigma)\bigg\} \end{aligned} \tag{3.3}$$

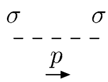
and δ_1 and δ_2 take the form

$$\delta_1 = \delta M^2 + \frac{v^2}{2}(\delta\lambda_1 + \delta\lambda_2), \tag{3.4}$$

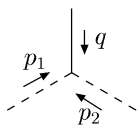
$$\delta_2 = \delta M^2 + \frac{3}{2}v^2(\delta\lambda_1 + \delta\lambda_2). \tag{3.5}$$



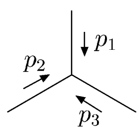
$$= \frac{i}{p^2(1-\frac{p^2}{\Lambda^2})^n}$$



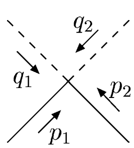
$$= \frac{i}{p^2(1-\frac{p^2}{\Lambda^2})^n - 2M^2}$$



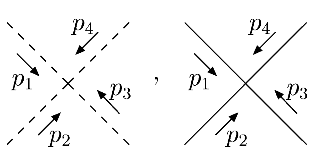
$$= -\frac{iv}{\sqrt{2}}(\lambda_1+\lambda_2)\cos\left(\frac{p_1\wedge p_2}{2}\right)$$



$$= -\frac{3iv}{\sqrt{2}}(\lambda_1+\lambda_2)\cos\left(\frac{p_1\wedge p_2}{2}\right)$$



$$= \frac{i}{2}(\lambda_1+\lambda_2)\cos\left(\frac{p_1\wedge q_2+p_2\wedge q_1}{2}\right) - i\lambda\cos\left(\frac{p_1\wedge p_2}{2}\right)\cos\left(\frac{q_1\wedge q_2}{2}\right)$$



$$\begin{aligned} = -\frac{i}{2}(\lambda_1+\lambda_2)\Bigg[&\cos\left(\frac{p_1\wedge p_2+p_1\wedge p_3+p_2\wedge p_3}{2}\right) \\ &+\cos\left(\frac{p_1\wedge p_2+p_1\wedge p_3-p_2\wedge p_3}{2}\right) \\ &+\cos\left(\frac{p_1\wedge p_2-p_1\wedge p_3-p_2\wedge p_3}{2}\right)\Bigg] \end{aligned}$$

Fig. 2. Feynman rules for $S_{\Lambda,\mathrm{br}}$.

It is straightforward to check that $S_{\Lambda,\mathrm{br}}$ and $S_{\mathrm{ct,br}}$ are both invariant under the $U(1)$ global transformations (1.6) and that their Feynman rules are those shown in Figs. 2 and 3.

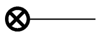
Introducing real sources J_π and J_σ for the fields π and σ through

$$J = \frac{1}{\sqrt{2}}(J_\pi + iJ_\sigma), \qquad J^* = \frac{1}{\sqrt{2}}(J_\pi - iJ_\sigma)$$

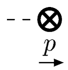
and substituting in Eq. (2.6), we have for the generating functional for the Green functions of the fields π and σ

$$\begin{aligned} Z[J_\pi, J_\sigma] &= e^{G_c[J_\pi, J_\sigma]} \\ &= \int [d\pi][d\sigma]\exp\left\{iS_{\Lambda,\mathrm{br}} + iS_{\mathrm{ct,br}} + i\int d^4x\left[J_\pi\pi - J_\sigma\left(\sigma+\sqrt{2}\,v\right)\right]\right\}. \end{aligned}$$

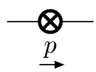
To obtain the Ward identity that controls the global $U(1)$ symmetry at the quantum level, we follow the usual method: make the change (1.6) in the integral that defines $Z[J_\pi, J_\sigma]$,



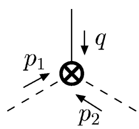
$$= -i\sqrt{2}\,v\delta_1$$



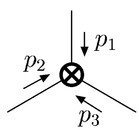
$$= i(p^2\delta z_\phi - \delta_1)$$



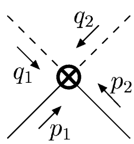
$$= i(p^2\delta z_\phi - \delta_2)$$



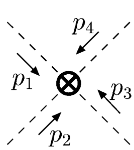
$$= -\frac{iv}{\sqrt{2}}(\delta\lambda_1 + \delta\lambda_2)\cos\left(\frac{p_1 \wedge p_2}{2}\right)$$



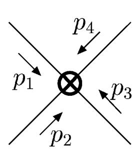
$$= -\frac{3iv}{\sqrt{2}}(\delta\lambda_1 + \delta\lambda_2)\cos\left(\frac{p_1 \wedge p_2}{2}\right)$$



$$= \frac{i}{2}(\delta\lambda_1 + \delta\lambda_2)\cos\left(\frac{p_1 \wedge q_2 + p_2 \wedge q_1}{2}\right) - i\delta\lambda_1 \cos\left(\frac{p_1 \wedge p_2}{2}\right)\cos\left(\frac{q_1 \wedge q_2}{2}\right)$$



,



$$= -\frac{i}{2}(\delta\lambda_1 + \delta\lambda_2)\left[\cos\left(\frac{p_1 \wedge p_2 + p_1 \wedge p_3 + p_2 \wedge p_3}{2}\right) + \cos\left(\frac{p_1 \wedge p_2 + p_1 \wedge p_3 - p_2 \wedge p_3}{2}\right) + \cos\left(\frac{p_1 \wedge p_2 - p_1 \wedge p_3 - p_2 \wedge p_3}{2}\right)\right]$$

Fig. 3. Feynman rules for $S_{\mathrm{ct,br}}$.

note that $S_{\Lambda,\mathrm{br}}$ and $S_{\mathrm{ct,br}}$ remain invariant under such a change and define the effective action $\Gamma[\pi, \sigma]$ as the Legendre transform of $W[J_\pi, J_\sigma]$. This yields the identity

$$\int d^4x \left(\sigma \frac{\delta\Gamma}{\delta\pi} - \pi \frac{\delta\Gamma}{\delta\sigma}\right) = -\sqrt{2}\int d^4x \frac{\delta\Gamma}{\delta\pi}. \tag{3.6}$$

If we denote by $\Gamma^{(n,m)}(x_1, \ldots, x_n; y_1, \ldots, y_m)$ the 1PI Green function of n π-fields and m σ-fields, the effective action can be written as

$$\Gamma[\pi, \sigma] = \sum_{n,m=1}^{\infty} \frac{1}{n!m!} \int d^4x_1 \cdots d^4x_n\, d^4y_1 \cdots d^4y_m \times \pi(x_1) \cdots \pi(x_n)\sigma(y_1) \cdots \sigma(y_m)\Gamma^{(n,m)}(x_1, \ldots, x_n; y_1, \ldots, y_m).$$

Substituting this in Eq. (3.6) and going to momentum space, we obtain the following set of Ward identities for the 1PI Green functions:

$$
\begin{aligned}
& m\Gamma^{(n+1,m-1)}(p_1,\dots,p_n,q_m;q_1,\dots,q_{m-1}) \\
&\quad - n\Gamma^{(n-1,m+1)}(p_1,\dots,p_{n-1};q_1,\dots,q_m,q_{m+1}) \\
&\quad = -\sqrt{2}\,v\Gamma^{(n+1,m-1)}(p_1,\dots,p_n,0;q_1,\dots,q_m).
\end{aligned}
\tag{3.7}
$$

It is important to note the zero momentum insertion on r.h.s. of the identities, since it will play a key part in our analysis in Section 4. The same comments made for the symmetric phase concerning the quantum theory apply here. Namely, for the broken phase of the quantum theory to exist, one must make sure that it is possible to take order by order in perturbation theory the counterterms so as to render the limit $\Lambda \to \infty$ of all 1PI Green functions finite, while preserving the Ward identities. In this section we show that this is possible at one loop only if $\lambda_2 = 0$.

The 1PI Green functions with UV divergences for $\Lambda \to \infty$ in their planar parts are, in the notation introduced above,

$$
\begin{array}{ll}
\Gamma^{(0,1)}(0), & \\
\Gamma^{(2,0)}(p), & \\
\Gamma^{(0,2)}(q), & \\
\Gamma^{(2,1)}(p_1,p_2;q), & p_1+p_2+q=0, \\
\Gamma^{(0,3)}(q_1,q_2,q_3), & q_1+q_2+q_3=0, \\
\Gamma^{(4,0)}(p_1,p_2,p_3,p_4), & p_1+p_2+p_3+p_4=0, \\
\Gamma^{(2,2)}(p_1,p_2;q_1,q_2), & p_1+p_2+q_1+q_2=0, \\
\Gamma^{(0,4)}(q_1,q_2,q_3,q_4), & q_1+q_2+q_3+q_4=0.
\end{array}
\tag{3.8}
$$

By the UV/IR mixing argument, these are also the only 1PI the Green functions whose nonplanar parts may develop singularities at $\Lambda \to \infty$ when $\theta^{\mu\nu} \to 0$. According to Eq. (3.7), these functions satisfy the Ward identities

$$\Gamma^{(0,1)}(0) = \sqrt{2}\,v\Gamma^{(2,0)}(0), \tag{3.9}$$

$$\Gamma^{(2,0)}(p) - \Gamma^{(0,2)}(p) = -\sqrt{2}\,v\Gamma^{(2,1)}(p,0;-p), \tag{3.10}$$

$$2\Gamma^{(2,1)}(p,q_1;q_2) - \Gamma^{(0,3)}(p,q_1,q_2) = -\sqrt{2}\,v\Gamma^{(2,2)}(p,0;q_1,q_2), \tag{3.11}$$

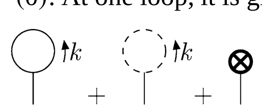
$$3\Gamma^{(2,1)}(p_1,p_2;p_3) = \sqrt{2}\,v\Gamma^{(4,0)}(p_1,p_2,p_3,0), \tag{3.12}$$

$$3\Gamma^{(2,2)}(p_1,q_1;q_2,q_3) - \Gamma^{(0,4)}(p_1,q_1,q_2,q_3) = -\sqrt{2}\,v\Gamma^{(2,3)}(p_1,0;q_1,q_2,q_3), \tag{3.13}$$

$$\Gamma^{(4,0)}(p_1,p_2,p_3,q) - 3\Gamma^{(2,2)}(p_1,p_2;p_3,q) = -\sqrt{2}\,v\Gamma^{(4,1)}(p_1,p_2,p_3,0;q). \tag{3.14}$$

We first look at $\Gamma^{(0,1)}(0)$. At one loop, it is given by

$-i\Gamma_1^{(0,1)}(0) =$



$$= -i\Gamma^{(0,1)}_{\mathrm{reg}}(0) - i\sqrt{2}\,v\delta_1, \tag{3.15}$$

where the regularized contribution $-i\Gamma^{(0,1)}_{\rm reg}(0)$ has the form

$$-i\Gamma^{(0,1)}_{\rm reg}(0) = -i\Gamma^{(0,1)}_{\rm P}(0) = \frac{v(\lambda_1+\lambda_2)}{2\sqrt{2}} \int \frac{d^4k}{(2\pi)^4}\left[\frac{3}{D_\sigma(k)} + \frac{1}{D_\pi(k)}\right] \tag{3.16}$$

and we have defined

$$D_\pi(k) = k^2\left(1-\frac{k^2}{\Lambda^2}\right)^n, \tag{3.17}$$

$$D_\sigma(k) = k^2\left(1-\frac{k^2}{\Lambda^2}\right)^n - 2M^2. \tag{3.18}$$

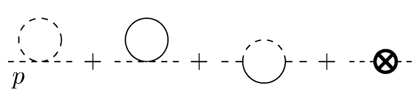
Note that $i\Gamma^{(0,1)}_{\rm reg}(0)$ is purely planar and is quadratically divergent at $\Lambda\to\infty$. To compute its large Λ limit we use Eqs. (A.2) and (A.3) in the appendix and obtain

$$\begin{aligned} -i\Gamma^{(0,1)}_{\rm reg}(0) &= -i\Gamma^{(0,1)}_{\rm P}(0) \\ &\xrightarrow[\Lambda\to\infty]{} -\frac{i}{16\pi^2}\sqrt{2}\,v(\lambda_1+\lambda_2)\left\{\frac{\Lambda^2}{n-1} - \frac{3}{2}M^2\left[\ln\left(\frac{\Lambda^2}{2M^2}\right) - f_0\right]\right\}, \end{aligned} \tag{3.19}$$

with f_0 as in Eq. (2.12). It follows that, for $\Gamma^{(0,1)}(0)$ to be finite, δ_1 must be modulo finite terms

$$\delta_1 = -\frac{\lambda_1+\lambda_2}{16\pi^2}\left[\frac{\Lambda^2}{n-1} - \frac{3}{2}M^2\ln\left(\frac{\Lambda^2}{2M^2}\right)\right]. \tag{3.20}$$

Next we consider the pion selfenergy $\Gamma^{(2,0)}(p)$. At one loop, it receives contributions from the following 1PI diagrams

$-i\Gamma^{(2,0)}_1(p) =$ 

$$= -i\Gamma^{(2,0)}_{\rm reg}(p) + i\left(p^2\delta z_\phi - \delta_1\right). \tag{3.21}$$

Using the Feynman rules, the contribution $-i\Gamma^{(2,0)}_{\rm reg}(p)$ can be written as the sum

$$-i\Gamma^{(2,0)}_{\rm reg}(p) = -i\Gamma^{(2,0)}_{\rm P}(p) - i\Gamma^{(2,0)}_{\rm NP}(p) \tag{3.22}$$

of a planar part $-i\Gamma^{(2,0)}_{\rm P}(p)$ and a nonplanar part $-i\Gamma^{(2,0)}_{\rm NP}(p)$, given by

$$\begin{aligned} \begin{Bmatrix} -i\Gamma^{(2,0)}_{\rm P}(p) \\ -i\Gamma^{(2,0)}_{\rm NP}(p) \end{Bmatrix} = \frac{1}{4}\int \frac{d^4k}{(2\pi)^4}\Bigg[& \frac{\lambda_1+\lambda_2}{D_\pi(k)} \begin{Bmatrix} 2 \\ \cos(p\wedge k) \end{Bmatrix} \\ & + \frac{1}{D_\sigma(k)} \begin{Bmatrix} 2\lambda_1 \\ (\lambda_2-\lambda_1)\cos(p\wedge k) \end{Bmatrix} \\ & + \frac{2M^2(\lambda_1+\lambda_2)}{D_\pi(k+p)D_\sigma(k)} \begin{Bmatrix} 1 \\ \cos(p\wedge k) \end{Bmatrix}\Bigg]. \end{aligned} \tag{3.23}$$

At $\Lambda\to\infty$, the planar contribution diverges, while the nonplanar contribution remains finite if $\theta^{\mu\nu}\neq 0$ [3]. From Eqs. (A.2), (A.3) and (A.4) we obtain for the large Λ limit of the planar contribution

$$
\begin{aligned}
-i\Gamma_{\rm P}^{(2,0)}(p) \underset{\Lambda\to\infty}{\longrightarrow} -\frac{i}{16\pi^2}\bigg\{ & \left(\lambda_1+\frac{\lambda_2}{2}\right)\frac{\Lambda^2}{n-1} \\
& -\frac{3\lambda_1+\lambda_2}{2}M^2\left[\ln\left(\frac{\Lambda^2}{2M^2}\right)-f_0\right]+\frac{\lambda_1+\lambda_2}{2}M^2 f\big(p^2\big)\bigg\},
\end{aligned}
\tag{3.24}
$$

where $f(p^2)$ has the form

$$
f(p^2)=1-\left(1-\frac{2M^2}{p^2}\right)\ln\left(1-\frac{p^2}{2M^2}\right). \tag{3.25}
$$

Since the UV divergences in the pion selfenergy are those in its planar part and are given by Eq. (3.24), for the selfenergy to be finite, δz_ϕ and δ_1 must be modulo finite terms

$$
\delta z_\phi = 0, \tag{3.26}
$$

$$
\delta_1 = -\frac{i}{16\pi^2}\left[\left(\lambda_1+\frac{\lambda_2}{2}\right)\frac{\Lambda^2}{n-1}-\frac{3\lambda_1+\lambda_2}{2}M^2\ln\left(\frac{\Lambda^2}{2M^2}\right)\right]. \tag{3.27}
$$

Eqs. (3.20) and (3.27) imply

$$
\lambda_2 = 0. \tag{3.28}
$$

In other words, if $\lambda_2 \neq 0$, there are no counterterms that consistently subtract the UV divergences in $-i\Gamma_1^{(0,1)}(0)$ and $-i\Gamma_1^{(2,0)}(p)$. Note that the structure of the counterterms in $S_{\rm ct,br}$, and in particular of those for $-i\Gamma^{(0,1)}(0)$ and $-i\Gamma^{(2,0)}(p)$, results from demanding global $U(1)$ invariance, so the condition $\lambda_2 = 0$ is a requirement of global $U(1)$ invariance.

We now set $\lambda_2 = 0$ and compute the UV divergences in the other Green functions on the list (3.8). Every 1PI Green function $\Gamma^{(m,n)}$ on this list is at one loop the sum

$$
\Gamma^{(m,n)} = \Gamma_{\rm P}^{(m,n)} + \Gamma_{\rm NP}^{(m,n)} + \Gamma_{\rm ct}^{(m,n)} \tag{3.29}
$$

of three terms. The terms $\Gamma_{\rm P}^{(m,n)}$ and $\Gamma_{\rm NP}^{(m,n)}$ collect the planar and nonplanar contributions of the corresponding 1PI diagrams formed with the Feynman rules for $S_{\Lambda,\rm br}$, while the term $\Gamma_{\rm ct}^{(m,n)}$ is the counterterm contribution provided by $S_{\rm ct,br}$. At nonvanishing external momenta, only the planar part $\Gamma_{\rm P}^{(m,n)}$ becomes divergent for $\Lambda\to\infty$. Computing these divergences and summing to them the counterterm contribution we obtain

$$
\begin{aligned}
-i\Gamma_1^{(0,2)}(q) \underset{\Lambda\to\infty}{\longrightarrow} & -\frac{i\lambda_1}{16\pi^2}\left[\frac{\Lambda^2}{n-1}-\frac{7}{2}M^2\ln\left(\frac{\Lambda^2}{2M^2}\right)\right] \\
& + i\big(q^2\delta z_\phi-\delta_2\big)+\text{f.c.},
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
& -i\Gamma_1^{(2,1)}(p_1,p_2;q) \\
& \quad \underset{\Lambda\to\infty}{\longrightarrow} \frac{iv}{\sqrt{2}}\cos\left(\frac{p_1\wedge p_2}{2}\right)\left[\frac{\lambda_1^2}{16\pi^2}\ln\left(\frac{\Lambda^2}{2M^2}\right)-(\delta\lambda_1+\delta\lambda_2)\right]+\text{f.c.},
\end{aligned}
\tag{3.31}
$$

$$
-i\Gamma_1^{(0,3)}(q_1,q_2,q_3)
$$

$$\xrightarrow[\Lambda\to\infty]{} \frac{3iv}{\sqrt{2}}\cos\left(\frac{q_1\wedge q_2}{2}\right)\left[\frac{\lambda_1^2}{16\pi^2}\ln\left(\frac{\Lambda^2}{2M^2}\right)-(\delta\lambda_1+\delta\lambda_2)\right]+\text{f.c.}, \tag{3.32}$$

$$-i\Gamma_1^{(4,0)}(p_1,p_2,p_3,p_4),\ -i\Gamma_1^{(0,4)}(p_1,p_2,p_3,p_4)$$

$$\xrightarrow[\Lambda\to\infty]{} \frac{i}{2}t_\theta(p_1,p_2,p_3)\left[\frac{\lambda_1^2}{16\pi^2}\ln\left(\frac{\Lambda^2}{2M^2}\right)-(\delta\lambda_1+\delta\lambda_2)\right]+\text{f.c.}, \tag{3.33}$$

$$-i\Gamma_1^{(2,2)}(p_1,p_2;q_1,q_2)$$

$$\begin{aligned}&\xrightarrow[\Lambda\to\infty]{} -\frac{i}{2}\cos\left(\frac{p_1\wedge q_2+p_2\wedge q_1}{2}\right)\left[\frac{\lambda_1^2}{16\pi^2}\ln\left(\frac{\Lambda^2}{2M^2}\right)-(\delta\lambda_1-\delta\lambda_2)\right]\\ &\quad+i\cos\left(\frac{p_1\wedge p_2}{2}\right)\cos\left(\frac{q_1\wedge q_2}{2}\right)\left[\frac{\lambda_1^2}{16\pi^2}\ln\left(\frac{\Lambda^2}{2M^2}\right)-\delta\lambda_1\right]+\text{f. c.},\end{aligned} \tag{3.34}$$

where $t_\theta(p_1,p_2,p_3)$ stands for

$$\begin{aligned}t_\theta(p_1,p_2,p_3)&=\cos\left(\frac{p_1\wedge p_2+p_1\wedge p_3+p_2\wedge p_3}{2}\right)\\ &\quad+\cos\left(\frac{p_1\wedge p_2+p_1\wedge p_3-p_2\wedge p_3}{2}\right)\\ &\quad+\cos\left(\frac{p_1\wedge p_2-p_1\wedge p_3-p_2\wedge p_3}{2}\right).\end{aligned} \tag{3.35}$$

Note that to calculate the UV divergences of the Green functions above, among all the 1PI one-loop diagrams that contribute to a given Green function, we only need to consider those with at most two internal lines. The reason is that 1PI one-loop diagrams with three or more internal lines contain at least three propagators and thus their planar contributions are finite by power counting at $\Lambda\to\infty$.

For $-i\Gamma_1^{(0,2)}(q)$ in Eq. (3.30) to be finite, δz_ϕ and δ_2 must be given, modulo finite terms, by $\delta z_\phi=0$ and

$$\delta_2=-\frac{\lambda_1}{16\pi^2}\left[\frac{\Lambda^2}{n-1}-\frac{7}{2}M^2\ln\left(\frac{\Lambda^2}{2M^2}\right)\right]. \tag{3.36}$$

In turn, modulo finite terms, Eqs. (3.4), (3.5), (3.20) and (3.36) yield for δM^2 and $\delta\lambda_1+\delta\lambda_2$

$$\begin{aligned}\delta M^2&=-\frac{\lambda_1}{16\pi^2}\left[\frac{\Lambda^2}{n-1}-\frac{M^2}{2}\ln\left(\frac{\Lambda^2}{2M^2}\right)\right],\\ \delta\lambda_1+\delta\lambda_2&=\frac{\lambda_1^2}{16\pi^2}\ln\left(\frac{\Lambda^2}{2M^2}\right).\end{aligned} \tag{3.37}$$

The latter equation and (3.34) imply

$$\delta\lambda_2=0, \tag{3.38}$$

$$\delta\lambda_1=\frac{\lambda_1^2}{16\pi^2}\ln\left(\frac{\Lambda^2}{2M^2}\right). \tag{3.39}$$

To determine the finite terms in δM^2, δz_ϕ and $\delta\lambda_1$, three renormalization conditions should be specified.

4. The broken phase II: Ward identities

In this section we rederive the condition $\lambda_2 = 0$ from the Ward identities (3.9)–(3.14). So let us assume that $\lambda_2 \neq 0$ and recall that the identities hold for all $\Lambda, \lambda_1, \lambda_2$ and all $\delta_1, \delta_2, \delta\lambda_1, \delta\lambda_2$. With this in mind we look at the identity (3.9). Using the expressions for $\Gamma_1^{(0,1)}(0)$ and $\Gamma_1^{(2,0)}(p)$ in Eqs. (3.15) and (3.21), the terms with δ_1 cancel and we are left with

$$\Gamma_{\mathrm{P}}^{(0,1)}(0) = \sqrt{2}\, v\big[\Gamma_{\mathrm{P}}^{(2,0)}(0) + \Gamma_{\mathrm{NP}}^{(2,0)}(0)\big]. \tag{4.1}$$

The contribution $\Gamma_{\mathrm{P}}^{(0,1)}(0)$ on the l.h.s. is given in Eq. (3.16), while for $\Gamma_{\mathrm{P}}^{(2,0)}(0)$ and $\Gamma_{\mathrm{NP}}^{(2,0)}(0)$ on the r.h.s. we have from Eq. (3.23) that

$$-i\,\Gamma_{\mathrm{P}}^{(2,0)}(0) = \frac{1}{4}\int \frac{d^4k}{(2\pi)^4}\left[\frac{\lambda_1+\lambda_2}{D_\pi(k)} + \frac{3\lambda_1+\lambda_2}{D_\sigma(k)}\right], \tag{4.2}$$

$$-i\,\Gamma_{\mathrm{NP}}^{(2,0)}(0) = \frac{\lambda_2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{D_\sigma(k)}. \tag{4.3}$$

It is clear from Eqs. (3.16), (4.2) and (4.3) that Eq. (4.1) is satisfied. This is no surprise since, as stated, the Ward identities hold for arbitrary $\Lambda, \lambda_1, \lambda_2$. The key point is that the identity (4.1) holds because there is a contribution $\Gamma_{\mathrm{NP}}^{(2,0)}(0)$ to the r.h.s. which is proportional to λ_2, diverges at $\Lambda \to \infty$ and is nonplanar. This indicates a mismatching in the planar $\Lambda \to \infty$ divergent contributions to both sides of the identity, or equivalently a mismatching in the UV divergences.[3] To subtract the UV divergences, we then have to add different counterterms to the right- and left-hand sides, in contradiction with the statement that the counterterms satisfy the Ward identities for arbitrary λ_1 and λ_2. Hence, to have a consistent subtraction procedure, we must get rid of the unwanted divergent contribution $\Gamma_{\mathrm{NP}}^{(2,0)}(0)$, and this implies taking $\lambda_2 = 0$. Note that after setting $\lambda_2 = 0$ we are left with

$$\lim_{\Lambda\to\infty} \Gamma_{\mathrm{NP}}^{(2,0)}(0) = 0.$$

The argument just given generalizes to the other identities as follows. The invariance for arbitrary $\delta_1, \delta_2, \delta\lambda_1$ and $\delta\lambda_2$ of $S_{\mathrm{ct,br}}$ under global $U(1)$ transformations implies that the counterterms in Fig. 3 satisfy the Ward identities. This means that the counterterm contributions to both sides of the identities cancel, so the identities become relations among planar and nonplanar parts of Green functions like that in Eq. (4.1). As $\Lambda \to \infty$, the planar contributions to the l.h.s. of these relations become singular, while the nonplanar contributions remain finite. Thus, the divergences that arise for $\Lambda \to \infty$ on the l.h.s. are of planar type. These divergences must be matched by only planar divergences on the r.h.s.; otherwise the UV divergences on the l.h.s. would not be balanced by the UV divergences on the r.h.s. and their subtraction would require different counterterms for each side. If all the $\Lambda \to \infty$ divergent contributions to the r.h.s. are to be planar, the nonplanar

[3] This mismatching was calculated explicitly in Section 3 [see Eqs. (3.19) and (3.24)]. The argument given here precisely avoids computing it.

contributions to this side should remain finite for $\Lambda \to \infty$. This, however, is not granted, since on the r.h.s. of the identities one of the external momenta, say p_e, vanishes and the nonplanar contributions given by nonplanar Feynman integrals with nonplanarity factor $e^{ik\wedge p_e}$ become divergent at $\Lambda \to \infty$ if $p_e = 0$. Hence, we must find conditions that rid the r.h.s. of the Ward identities of nonplanar contributions which for $p_e = 0$ become divergent at $\Lambda \to \infty$. Note that what we have precisely proved in our analysis above of the identity (3.9) is that the condition $\lambda_2 = 0$ ensures the finiteness of $\Gamma_{\rm NP}^{(2,0)}(0)$ at $\Lambda \to \infty$. Setting $\lambda_2 = 0$, we have checked that all the nonplanar contributions to the r.h.s. of the identities (3.10)–(3.14) are finite for arbitrary λ_1, so no further condition is required.

The quantum theory being defined as the large Λ limit of the theory at finite Λ implies that the Green function $\Gamma_1^{(0,1)}(0)$ on the l.h.s. of the Ward identity (3.9) must be computed at $\Lambda \to \infty$ and the function $\Gamma_1^{(2,0)}(p)$ on the r.h.s. at $\Lambda \to \infty$, $p \to 0$. In our analysis above of the identity (3.9), for the r.h.s. we have first set $p = 0$ in $\Gamma_1^{(2,0)}(p)$ and then sent $\Lambda \to \infty$. Setting $p = 0$ led to Eqs. (4.2) and (4.3), and sending $\Lambda \to \infty$ to the discussion that follows them. There is, however, one other way to compute the renormalized Green function $\Gamma_1^{(2,0)}(p)$ at $\Lambda \to \infty$, $p \to 0$; namely, to take $\Lambda \to \infty$ at nonvanishing p and then send p to zero. For the quantum theory to be well defined, both procedures must yield the same result. Let us see that this is the case. To this end we consider again the Ward identity (3.9) and take first $\Lambda \to \infty$ and then $p \to 0$. The only contribution to the l.h.s. of the identity is the p-independent planar piece $\Gamma_1^{(0,1)}(0)$, whose large Λ limit gives a divergent contribution (which will be canceled by a suitable counterterm). The r.h.s., in turn, receives contributions from $\Gamma_{\rm P}^{(2,0)}(p)$ and $\Gamma_{\rm NP}^{(2,0)}(p)$. Taking $\Lambda \to \infty$ at nonvanishing p in the planar contribution $\Gamma_{\rm P}^{(2,0)}(p)$ gives Eq. (3.24), and sending in it p to zero yields a Λ-divergent p-independent contribution. Proceeding similarly with the nonplanar contribution $\Gamma_{\rm NP}^{(2,0)}(p)$, and after using the results in the appendix, we obtain

$$\lim_{p\to 0}\lim_{\Lambda\to\infty}\Gamma_{\rm NP}^{(2,0)}(p) = -\frac{i\lambda_2}{16\pi^2}\left\{\frac{2}{p\circ p} + M^2\left[\ln\bigl(2M^2 p\circ p\bigr) - \ln 2 + \gamma - \frac{1}{2}\right]\right\}. \tag{4.4}$$

Summing the planar and nonplanar contributions to the r.h.s. of the identity, we get a Λ-divergent p-independent term (which will eventually be canceled by the appropriate counterterm) and a singular p-dependent piece $1/p\circ p$ which will not be cancelled by a counterterm and is not on the l.h.s., since the l.h.s does not depend on p. To avoid this mismatching of singular p-dependent contributions so that the Ward identity holds, we must eliminate such p-dependence from the r.h.s., hence we must take $\lambda_2 = 0$. Furthermore, only after setting $\lambda_2 = 0$, the planar contributions to both sides of the identity, given by Eqs. (3.24) and (3.19), match and the counterterm is the same for both sides of the identity (see Section 3). Thus, sending $\Lambda \to \infty$ in $\Gamma_{\rm NP}^{(2,0)}(p)$ and then $p \to 0$ leads to $\lambda_2 = 0$ and gives

$$\lim_{p\to 0}\lim_{\Lambda\to\infty}\Gamma_{\rm NP}^{(2,0)}(p) = 0.$$

We have repeated this analysis for the other Ward identities (3.10)–(3.14) and checked that, for $\lambda_2 = 0$ and arbitrary λ_1, the Green functions on the r.h.s. are free of divergences in p_e and yield the same nonplanar contributions as if one first sets $p_e = 0$ and then sends $\Lambda \to \infty$, p_e denoting the vanishing external momentum.

The difference with the Ward identities for the symmetric phase is the zero momentum insertion on the r.h.s. of the identities. At $\Lambda \to \infty$, the zero momentum insertion produces UV divergences proportional to λ_2 that, being nonplanar, cannot be locally subtracted. The condition $\lambda_2 = 0$ sets such divergences to zero. Noting that

(1) the Ward identities hold for all Λ, λ_1 and that λ_2, and
(2) the only breakings at $\Lambda \to \infty$ may arise from divergent contributions, and these preserve the identities if $\lambda_2 = 0$,

we conclude that the Ward identities hold for $\Lambda \to \infty$ if $\lambda_2 = 0$. This ensures the one-loop existence of the quantum broken phase for $\lambda_2 = 0$.

5. The Broken phase III: noncommutative IR divergences and the Goldstone theorem

Here we give explicit expressions for the behaviour of the nonplanar parts of the Green functions in Eq. (3.8) at small $\theta^{\mu\nu}$ and show that there is no UV/IR duality in the strong sense. From Section 3 we know that $\Gamma_1^{(0,1)}(0)$ does not have nonplanar contributions,

$$\Gamma_{\rm NP}^{(0,1)}(0) = 0.$$

The nonplanar part of $\Gamma_1^{(2,0)}(p)$ is given by Eq. (3.23) with $\lambda_2 = 0$. Using formulas (A.5), (A.9) and (A.11) to calculate its behaviour for large Λ and small $\theta^{\mu\nu}$, we obtain

$$\lim_{\theta^{\mu\nu}\to 0}\lim_{\Lambda\to\infty}\big[-i\Gamma_{\rm NP}^{(2,0)}(p)\big] = \frac{i\lambda_1}{16\pi^2}\frac{M^2}{2} f\big(p^2\big), \tag{5.1}$$

with $f(p^2)$ as in Eq. (3.25). For the nonplanar parts of the other Green functions on the list (3.8), after some calculations and using the results in the appendix, we have

$$-i\Gamma_{\rm NP}^{(0,2)}(q) \approx -\frac{i\lambda_1}{16\pi^2} 6M^2 \ln\big(\theta M^2\big), \tag{5.2}$$

$$-i\Gamma_{\rm NP}^{(2,1)}(p_1,p_2;q) \approx -\frac{iv}{\sqrt{2}}\cos\left(\frac{p_1\wedge p_2}{2}\right)\frac{\lambda_1^2}{16\pi^2} 3\ln\big(\theta M^2\big), \tag{5.3}$$

$$-i\Gamma_{\rm NP}^{(0,3)}(q_1,q_2,q_3) \approx -\frac{3iv}{\sqrt{2}}\cos\left(\frac{q_1\wedge q_2}{2}\right)\frac{\lambda_1^2}{16\pi^2} 3\ln\big(\theta M^2\big), \tag{5.4}$$

$$\begin{aligned}-i\Gamma_{\rm NP}^{(4,0)}(p_1,p_2,p_3,p_4) &\approx -i\Gamma_{\rm NP}^{(0,4)}(p_1,p_2,p_3,p_4)\\ &\approx -\frac{i}{2} t_\theta(p_1,p_2,p_3)\frac{\lambda_1^2}{16\pi^2} 3\ln\big(\theta M^2\big),\end{aligned} \tag{5.5}$$

$$-i\Gamma_{\rm NP}^{(2,2)}(p_1,p_2;p_3,p_4) \approx -\frac{i}{2}\cos\left(\frac{p_1\wedge p_4 + p_2\wedge p_3}{2}\right)\frac{\lambda_1^2}{16\pi^2} 3\ln\big(\theta M^2\big), \tag{5.6}$$

where $t_\theta(p_1, p_2, p_3)$ is as in Eq. (3.35).

Comparing these expressions with Eqs. (3.19), (3.24) and (3.30)–(3.34), we see that the noncommutative IR singularities and the UV divergences cannot be obtained from each other by replacing $\theta^2 M^2 \leftrightarrow 1/\Lambda^2$, thus showing that there is no UV/IR duality in the strong sense. We also note that, unlike the symmetric phase, there are no quadratic noncommutative IR divergences. Indeed, the selfenergy of the σ-field only contains logarithmic noncommutative IR divergences, and the selfenergy of the π-field does not develop any noncommutative IR singularity at all. It is also clear from the equations above that the noncommutative IR singularities satisfy the Ward identities. This is no surprise, since we know from Section 4 that the Ward identities hold and taking $\theta^{\mu\nu} \to 0$ amounts to setting $\tilde{p}_i \to 0$ as external momentum configuration.

We finally want to study if the Goldstone theorem holds at one loop. To do this, we need the renormalized pion selfenergy. As is usual in the commutative case, we take as one of the renormalization conditions that the vacuum expectation value of the field σ remains equal to its classical value, i.e., $\langle\sigma\rangle = v$. This is equivalent to $-i\Gamma_1^{(1,0)} = 0$, which together with Eqs. (3.15) and (3.19) completely specifies δ_1 as

$$\delta_1 = -\frac{\lambda_1}{16\pi^2}\left\{\frac{\Lambda^2}{n-1} - \frac{3}{2}M^2\left[\ln\left(\frac{\Lambda^2}{2M^2}\right) - f_0\right]\right\}.$$

Substituting this in Eq. (3.21), using (3.22)–(3.24) and summing the tree-level and one-loop contributions, we obtain for the renormalized pion selfenergy

$$\Gamma_{\mathrm{R}}^{(2,0)}(p) = p^2 - \frac{\lambda_1}{16\pi^2}\frac{M^2}{2} f\left(p^2\right) - \Gamma_{\mathrm{NP}}(p), \tag{5.7}$$

where $f(p^2)$ is as in Eq. (3.25) and

$$\Gamma_{\mathrm{NP}}(p) = \lim_{\Lambda\to\infty} \Gamma_{\mathrm{NP}}^{(2,0)}(p)$$

is the large Λ limit of the nonplanar contribution $\Gamma_{\mathrm{NP}}^{(2,0)}(p)$ to the pion selfenergy. To compute $\Gamma_{\mathrm{NP}}(p)$, we use Eqs. (A.5), (A.6) and (A.7) in the appendix for the three terms in (3.23) and obtain

$$\begin{aligned}\Gamma_{\mathrm{NP}}(p) = {} & \frac{\lambda_1}{16\pi^2}\left[\frac{1}{p\circ p} - 2M^2\frac{K_1(\sqrt{2p\circ pM^2})}{\sqrt{2p\circ pM^2}}\right] \\ & - \frac{\lambda_1}{16\pi^2}\frac{M^2}{2}\int_0^\infty \frac{dt}{t}\int_0^1 d\alpha\, \exp\left\{t\alpha(1-\alpha)p^2 - 2t\alpha M^2 - \frac{p\circ p}{4t}\right\}.\end{aligned} \tag{5.8}$$

If we define the mass squared as the value of p^2 for which the selfenergy vanishes, to find the pion mass, we must solve the equation $\Gamma_{\mathrm{R}}^{(2,0)}(p) = 0$. Note in this regard that the renormalized pion selfenergy is a regular function of p^2, M^2 and $p\circ p$, so the equation $\Gamma_{\mathrm{R}}^{(2,0)}(p) = 0$ may in principle have $\theta^{\mu\nu}$-dependent solutions with $\theta^{\mu\nu} \neq 0$ and $p^\mu \neq 0$. To solve $\Gamma_{\mathrm{R}}^{(2,0)}(p) = 0$, we proceed by iteration and, since at tree level the solution is $p^2 = 0$, we write $p^2 = \lambda_1 \delta p^2 + O(\lambda_1^2)$. Substituting this in Eq. (5.7) and noting that $f(p^2) \to 0$

for $p^2 \to 0$, we are left with

$$\delta p^2 = \Gamma_{\mathrm{NP}}\big(p^2 = 0\big). \tag{5.9}$$

Setting $p^2 = 0$ in the second line in Eq. (5.8) and performing the integral, it is straightforward to see that the two lines in Eq. (5.8) cancel each other, so that $\Gamma_{\mathrm{NP}}(p^2 = 0) = 0$. The solution to $\Gamma_{\mathrm{R}}^{(2,0)}(p) = 0$, up to order λ_1, is then $p^2 = 0$ and the Goldstone theorem is preserved by one-loop radiative corrections. Note that the fact that the renormalized pion selfenergy is free of noncommutative IR singularities is essential for the Goldstone theorem to hold at one loop. Had the selfenergy developed noncommutative logarithmic IR singularities, these would have entered the mass as $\ln(\theta M^2)$, making it ill defined for small θ.

6. Conclusion and discussion

We have studied the one-loop renormalizability, the noncommutative IR singularities and the UV/IR mixing in both the symmetric and the broken phases of noncommutative global $U(1)$ scalar field theory. We have considered the general case of two interaction terms in the classical action, $\lambda_1 \phi^* \star \phi \star \phi^* \star \phi$ and $\lambda_2 \phi^* \star \phi^* \star \phi \star \phi$, and used as regulator an invariant cutoff Λ. For the symmetric phase, we have shown that the quantum theory exists at one loop for all values of the coupling constants λ_1 and λ_2 compatible with perturbation theory, and that there is no need to take $\lambda_2 = 0$. We have also given explicit expressions for the noncommutative IR singularities and checked that UV/IR duality does not hold in its strong form.

As concerns the broken phase, we have seen that the Ward identities imply that the quantum theory exists at one loop only if λ_2 vanishes. This is so because the Ward identities have a zero-momentum insertion term that for large Λ yields UV divergent contributions proportional to λ_2 that cannot be locally subtracted. To have a renormalizable theory, one must get rid of such contributions, and this requires $\lambda_2 = 0$. We have also given explicit expressions for the noncommutative IR singularities in the 1PI Green functions of the broken phase and shown that there is no strong UV/IR duality. The situation as concerns noncommutative IR singularities, UV/IR duality and the Ward identities is different to those cases previously studied in the literature. Consider for example $U(1)$ gauge theory: since UV/IR duality holds and the UV divergences are consistent with the Ward identities, the logarithmic noncommutative IR singularities satisfy the Ward identities. For the case at hand, however, the UV divergences satisfy the Ward identities, there is no UV/IR duality and, yet, the noncommutative IR singularities satisfy the Ward identities.

Comparing the symmetric and the broken phases, we have seen that after spontaneous symmetry breaking the theory does not have quadratic noncommutative IR divergences. Furthermore, the pion selfenergy is free of noncommutative IR singularities of any type, which makes possible to formulate the Goldstone theorem for all $\theta^{\mu\nu}$. Had UV/IR hold, the pion selfenergy would have contained noncommutative logarithmic IR singularities $\ln(\theta M^2)$ and these would have spoiled the theorem. Since the interaction term $\phi^* \star \phi \star \phi^* \star \phi$ for which the broken phase makes sense at one loop is also invariant under local $U(1)$ gauge transformations, it would be interesting to investigate the implications of noncommutative IR singularities and UV/IR mixing for the Goldstone theorem in local models [15,16].

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Appendix A

In the computations we have performed in Sections 2 to 5 we have encountered the following integrals:

$$\begin{aligned} I_\pi(q) &= \int \frac{d^4k}{(2\pi)^4}\frac{e^{iq\wedge k}}{D_\pi(k)}, \\ I_\sigma(q,M) &= \int \frac{d^4k}{(2\pi)^4}\frac{e^{iq\wedge k}}{D_\sigma(k)}, \\ I_{\pi\pi}(q,p) &= \int \frac{d^4k}{(2\pi)^4}\frac{e^{iq\wedge k}}{D_\pi(k)D_\pi(k+p)}, \\ I_{\sigma\pi}(q,p,M) &= \int \frac{d^4k}{(2\pi)^4}\frac{e^{iq\wedge k}}{D_\sigma(k)D_\pi(k+p)}, \\ I_{\sigma\sigma}(q,p,M) &= \int \frac{d^4k}{(2\pi)^4}\frac{e^{iq\wedge k}}{D_\sigma(k)D_\sigma(k+p)}. \end{aligned} \tag{A.1}$$

We are interested in their large Λ limit. To compute it, we proceed as follows. We first Wick rotate to euclidean space, make the change $k \to k\Lambda$ and define $\hat{p}^\mu \equiv p^\mu/\Lambda$ and $\widehat{M} \equiv M/\Lambda$. The integrals above then become functions of the dimensionless variables $\tilde{q}^\mu\Lambda$, $\hat{p}^\mu$ and $\widehat{M}$. Next we use algebraic identities like

$$\frac{1}{1+(k+\hat{p})^2} = \frac{1}{1+k^2}\left[1 - \frac{\hat{p}^2+2\hat{p}k}{1+(k+\hat{p})^2}\right]$$

or

$$\frac{1}{k^2(1+k^2)^n+2\widehat{M}^2} = \frac{1}{k^2+2\widehat{M}^2}\left[1 - \sum_{r=1}^{n}\binom{n}{r}\frac{(k^2)^{r+1}}{k^2(1+k^2)^n+2\widehat{M}^2}\right]$$

to decompose every integral in a sum of integrals, whose limit $\Lambda \to \infty$ we study employing Lebesgue's dominated convergence theorem. Finally we use Schwinger parameters to compute the integrals that give nonvanishing contributions at $\Lambda \to \infty$ and rotate back to Minkowski spacetime. Following this procedure we obtain for $q = 0$

$$I_\pi(0) \underset{\Lambda\to\infty}{\longrightarrow} -\frac{i}{16\pi^2}\frac{\Lambda^2}{n-1}, \tag{A.2}$$

$$I_\sigma(0,M) \underset{\Lambda\to\infty}{\longrightarrow} -\frac{i}{16\pi^2}\left\{\frac{\Lambda^2}{n-1} - 2M^2\left[\ln\left(\frac{\Lambda^2}{2M^2}\right) - f_0\right]\right\}, \tag{A.3}$$

$$\begin{aligned}
I_{\pi\pi}(0,p) &\underset{\Lambda\to\infty}{\longrightarrow} \frac{i}{16\pi^2}\left[-\ln\left(-\frac{p^2}{\Lambda^2}\right) + 1 - \sum_{r=1}^{2n-1}\frac{1}{\Gamma(r)}\right],\\
I_{\sigma\pi}(0,p,M) &\underset{\Lambda\to\infty}{\longrightarrow} \frac{i}{16\pi^2}\left[\ln\left(\frac{\Lambda^2}{2M^2}\right) + f\left(p^2\right) - f_0\right],\\
I_{\sigma\sigma}(0,p,M) &\underset{\Lambda\to\infty}{\longrightarrow} \frac{i}{16\pi^2}\left[\ln\left(\frac{\Lambda^2}{2M^2}\right) - g\left(p^2\right) - f_0\right],
\end{aligned} \tag{A.4}$$

where f_0 and $f(p^2)$ are as in Eqs. (2.12) and (3.25) and $g(p^2)$ reads

$$g\left(p^2\right) = 2 - \sqrt{1-8M^2/p^2}\,\ln\left(\frac{\sqrt{1-8M^2/p^2}+1}{\sqrt{1-8M^2/p^2}-1}\right).$$

For $q \neq 0$, the results for I_π and I_σ are relatively simple,

$$I_\pi(q) \underset{\Lambda\to\infty}{\longrightarrow} -\frac{i}{4\pi^2}\frac{1}{q\circ q}, \tag{A.5}$$

$$I_\sigma(q,M) \underset{\Lambda\to\infty}{\longrightarrow} \frac{iM^2}{2\pi^2}\frac{K_1(\sqrt{2q\circ qM^2})}{\sqrt{2q\circ qM^2}}, \tag{A.6}$$

whereas for $I_{\pi\pi}$, $I_{\sigma\pi}$ and $I_{\sigma\sigma}$ we have

$$\left.\begin{array}{r} I_{\pi\pi}(q,p)\\ I_{\sigma\pi}(q,p,M)\\ I_{\sigma\sigma}(q,p,M)\end{array}\right\} \underset{\Lambda\to\infty}{\longrightarrow} \frac{i}{16\pi^2}\int_0^\infty\frac{dt}{t}\int_0^1 d\alpha\,\exp\left[t\alpha(1-\alpha)p^2 - 2t\epsilon M^2 - \frac{q\circ q}{4t} - i\alpha q\wedge p\right], \tag{A.7}$$

with

$$\epsilon = \begin{cases} 0 & \text{for } I_{\pi\pi},\\ \alpha & \text{for } I_{\sigma\pi},\\ 1 & \text{for } I_{\sigma\sigma}.\end{cases} \tag{A.8}$$

To study the noncommutative IR singularities in the 1PI Green functions we need only the expressions at $\Lambda\to\infty$, $\theta^{\mu\nu}\to 0$. They can be easily computed and turn out to be

$$\lim_{q\to 0}\lim_{\Lambda\to\infty} I_\sigma(q,M) = \frac{iM^2}{8\pi^2}\left[\frac{1}{q\circ qM^2} - \frac{1}{2}\ln\left(2q\circ qM^2\right) + \gamma - \ln 2 - \frac{1}{2}\right], \tag{A.9}$$

$$\lim_{q\to 0}\lim_{\Lambda\to\infty} I_{\pi\pi}(q,p) = -\frac{i}{16\pi^2}\left[\ln\left(-q\circ qp^2\right) + 2\ln 2 - 2\gamma + 1\right], \tag{A.10}$$

$$\begin{aligned}
\lim_{q\to 0}\lim_{\Lambda\to\infty} I_{\sigma\pi}(q,p,M) &= -\frac{i}{16\pi^2}\left[\ln\left(2q\circ qM^2\right) - f\left(p^2\right) - 2(\ln 2 - \gamma) - 1\right],\\
\lim_{q\to 0}\lim_{\Lambda\to\infty} I_{\sigma\sigma}(q,p,M) &= -\frac{i}{16\pi^2}\left[\ln\left(2q\circ qM^2\right) - g(p) - \ln 2 + \gamma - 1\right].
\end{aligned} \tag{A.11}$$

Note, e.g., that substitution of Eqs. (A.5), (A.9) and (A.11) in (3.23) yields Eq. (4.4).

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