SYMPLECTIC INVARIANTS OF SEMITORIC SYSTEMS AND THE INVERSE PROBLEM FOR QUANTUM SYSTEMS

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In memory of Professor Johannes (Hans) J. Duistermaat (1942-2010)

Abstract. Simple semitoric systems were classified about ten years ago in terms of a collection of invariants, essentially given by a convex polygon with some marked points corresponding to focus--focus singularities. Each marked point is endowed with labels which are symplectic invariants of the system. We will review the construction of these invariants, and explain how they have been generalized or applied in different contexts. One of these applications concerns quantum integrable systems and the corresponding inverse problem, which asks how much information of the associated classical system can be found in the spectrum. An approach to this problem has been to try to compute invariants in the spectrum. We will explain how this has been recently achieved for some of the invariants of semitoric systems, and discuss an open question in this direction.

1. Introduction

An integrable system on a symplectic manifold (M, ω) of dimension $2n$ is given by n smooth functions

$$f_1, \ldots, f_n \colon (M, \omega) \to \mathbb{R}$$

which are independent and in involution, with respect to the symplectic form ω. For example, the spherical pendulum, the coupled angular momenta, and the Jaynes-Cummings coupled spin-oscillator model, are all integrable systems.

Much effort has been made in recent years to develop a general symplectic theory for integrable systems, by constructing as many symplectic invariants of these systems as possible. In a few cases a complete set of invariants has been found, which classifies all integrable systems in a certain class. Two such classes are the class of integrable systems of toric type (on compact manifolds of any dimension), and the class of integrable systems of semitoric type (on 4-manifolds, compact or not), under some conditions. The first class concerns systems for which all the functions $f_1, \ldots, f_n$ generate periodic flows of the same period, while the second class concerns systems f_1, f_2 for which f_1 generates a periodic flow but there is no requirement on f_2.

In this paper we will highlight some classical results about toric integrable systems and a few recent results about semitoric integrable systems. In order to do this we first introduce the language of symplectic geometry and integrable systems, and the main concepts one needs to understand the aforementioned results concerning toric and semitoric integrable systems. Then we will discuss quantum integrable systems. These are given by a collection of semiclassical operators

$$P^1 = (P^1_\hbar)_{\hbar \in I}, \ldots, P^n = (P^n_\hbar)_{\hbar \in I}$$

on a sequence of Hilbert spaces $(\mathcal{H}_\hbar)_{\hbar \in I}$, whose principal symbols form a classical integrable system in the sense described above. Here I is a subset of $(0, 1]$ which accumulates at 0.

We will concentrate on quantum integrable systems of toric and semitoric type, that is, those whose principal symbols form an integrable system of toric and semitoric type, respectively. We will explain why understanding the symplectic geometry of their classical counterparts plays a key role in the study of inverse problems for quantum integrable systems.

In this direction, we will discuss progress on the inverse spectral conjecture for semitoric systems [92, Conjecture 9.1] from about ten years ago. The conjecture essentially says that one can construct, from the data given by the semiclassical joint spectrum of a quantum semitoric integrable system P_1, P_2 (given for each fixed value of the $\hbar$ by the support of the joint spectral measure), the associated classical integrable system given by the principal symbols f_1, f_2 of the semiclassical operators P_1, P_2. This can be illustrated with a diagram, where the answer to the question on the second arrow is *yes* for quantum (simple) semitoric systems according to the conjecture:

$$\text{Quantum system} \rightsquigarrow \text{Semiclassical Spectrum} \underbrace{\rightsquigarrow}_{\text{possible?}} \text{Classical system}$$

The question of how much information is lost in the step of going from a quantum integrable system to its semiclassical spectrum has received significant attention in recent years. The answer to the question in the second arrow of the diagram has been shown to be *yes* in some cases, for example for quantum toric systems [22]. In the case of quantum semitoric systems, under some conditions, one can read off from the spectrum some properties of the classical system [68]. In order to do this one needs to understand the symplectic invariants of semitoric systems, and then how to compute them in the spectrum (using, for example, microlocal analysis). We will discuss these results and an open question in the last section of the paper, while the previous sections are devoted to reviewing some the main concepts of symplectic geometry of integrable systems which we need for the last section. The open question concerns spectral theory of non-simple semitoric systems; their classical counterparts have been recently classified [79]. Throughout the paper we often do not present the most general definitions or results, and try to convey the main ideas instead.

2. Symplectic geometry of classical integrable systems

The term *symplectic* was introduced in Weyl's book [114] as an analogue to *complex*. Symplectic geometry has its roots in classical mechanics. One can trace the first steps to the seventeenth century, to the works of Galileo Galilei, Christiaan Huygens, and Isaac Newton. The phase space of a mechanical system is modeled by a symplectic manifold. The first example of a symplectic manifold was given in 1808 by Joseph-Louis Lagrange in his works on motions of planets [64, 65].

After Lagrange, two precursors of symplectic geometry were the works of Carl Gustav Jacob Jacobi and William Rowan Hamilton. Hamilton gave a deep formulation of Lagrangian mechanics around 1835. Their methods and ideas were influential in the modern view point, which starts with important contributions by many authors around the early 1970s, see [85, 112] and the references therein. After these developments several aspects of the subject became subjects on their own right. One these aspects concerned symplectic geometry of finite dimensional integrable systems. Next we introduce these systems and discuss some of their properties.

2.1. Phase space and symplectic forms.

A *symplectic manifold* is a pair (M, ω) where M is a smooth manifold and ω is a non-degenerate closed 2-form $\omega \in \Omega^2(M)$, called a *symplectic form*. The non-degeneracy condition on ω means that, at every $m \in M$, the skew-symmetric bilinear form $\omega_m \colon \mathrm{T}_m M \times \mathrm{T}_m M \to \mathbb{R}$ is non-degenerate. This is a linear algebra condition which implies that M is even-dimensional and also orientable, since ω^n defines a volume form on M. The closedness of ω implies that if M is compact the cohomology class $[\omega^k]$, $k \in \{1, \ldots, n\}$, is non trivial, so the even dimensional de Rham cohomology groups $\mathrm{H}^{2n}_{\mathrm{dR}}(M)$ of M, $k \in \{1, \ldots, n\}$, are non trivial.

Putting these restrictions together we conclude that the only sphere S^n which admits a symplectic form (in fact, many) is S^2. Other typical examples of symplectic manifolds are $(\mathbb{R}^{2n}, \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i)$, where $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are coordinates on $\mathbb{R}^{2n}$, the cotangent bundles $(\mathrm{T}^*X, \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}\xi_i)$ where $(x_1, \ldots, x_n)$ are coordinates on an n-dimensional compact manifold X and $(\xi_1, \ldots, \xi_n)$ are the usual cotangent conjugate coordinates, or (S, ω) where S is a surface and ω an area form on it.

In fact, $\mathbb{R}^{2n}$ is the local model for all symplectic manifolds of dimension $2n$: Darboux proved [29] that they are all locally diffeomorphic to $(\mathbb{R}^{2n}, \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i)$, so other than its dimension, symplectic manifolds have no local invariants.

Unless otherwise stated, in this paper *all symplectic manifolds are assumed to be connected.*

2.2. **Integrability of Hamiltonians.** Given a smooth function $f\colon (M,\omega) \to \mathbb{R}$ on a symplectic manifold there exists a unique smooth vector field $\mathcal{X}_f$ such that $\omega(\mathcal{X}_f, \cdot) = -\mathrm{d}f$; the vector field $\mathcal{X}_f$ is usually called the *Hamiltonian vector field induced by* f. Also often one refers to f as a *Hamiltonian function*, or simply a *Hamiltonian*.

If (M,ω) has dimension $2n$, a *classical integrable system on* M is given by a collection of n real-valued smooth functions $f_1, \ldots, f_n$ on M such that: (1) $f_1, \ldots, f_n$ are in involution, that is, $\{f_i, f_j\} := \omega(\mathcal{X}_{f_i}, \mathcal{X}_{f_j}) = 0$ for all i, j (another way to say this is that f_i is constant along the flow of $\mathcal{X}_{f_j}$ for all i, j) and, (2) $f_1, \ldots, f_n$ are independent, that is, $\mathcal{X}_{f_1}, \ldots, \mathcal{X}_{f_n}$ are linearly independent almost everywhere in M.

The term "integrable system" comes from considering only the Hamiltonian f_1, and then looking for the maximal possible number of "integrals of f_1" (i.e. functions f_j with $\{f_1, f_j\} = 0$) which are independent. If M has dimension $2n$, one can have at most $n-1$ such integrals $f_2, \ldots, f_n$, and then the modern view point has been to call the joint map $F = (f_1, \ldots, f_n)\colon M \to \mathbb{R}^n$ the *momentum map* of the integrable system, or even more often the *integrable system* itself. This terminology is in part motivated by Hamiltonian group actions where F is the momentum of the action [63, 102].

2.3. **Example of an integrable system: the spherical pendulum.** The spherical pendulum, already studied by Huygens in the seventeenth century, is a famous example of integrable system. In the language of symplectic geometry it is described by the cotangent bundle of S^2. Let (θ, φ) be the standard spherical angles, where φ is the rotation angle around the vertical axis and θ is the angle from the North Pole. Let $(\xi_\theta, \xi_\varphi)$ be the cotangent conjugate variables on T^*S^2. Then the cotangent bundle $(\mathrm{T}^*S^2, \omega_{\mathrm{T}^*S^2})$ endowed with the canonical cotangent bundle symplectic form and the Hamiltonian function

$$f_1(\underbrace{\theta, \varphi}_{\text{sphere}}, \underbrace{\xi_\theta, \xi_\varphi}_{\text{fiber}}) = \underbrace{\frac{1}{2}\left((\xi_\theta)^2 + \frac{(\xi_\varphi)^2}{\sin^2\theta}\right)}_{\text{kinetic energy}} + \underbrace{\cos\theta}_{\text{potential}} .$$

is an integrable system by considering the vertical angular momentum $f_2(\theta, \varphi, \xi_\theta, \xi_\varphi) = \xi_\varphi$. So in our modern language, $F = (f_1, f_2)\colon \mathrm{T}^*S^2 \to \mathbb{R}^2$ is an integrable system. Note that the Hamiltonian function f_1 is smooth on the cotangent bundle T^*S^2 (the apparent singularity given by $1/\sin^2\theta$ is simply an artifact of the choice of spherical coordinates).

2.4. **The topology of the fibers.** A point m at which $\mathcal{X}_{f_1}(m), \ldots, \mathcal{X}_{f_n}(m)$ of T_mM are linearly dependent is a *singularity* of F. If m is not a singularity, it is *regular*. If a fiber $F^{-1}(c)$ contains some singularity we call it *singular*. If $F^{-1}(c) \neq \varnothing$ contains no singularity we call it *regular*.

It follows from the definition of integrable system, by simply following the flows of the Hamiltonian vector fields of its components, that if C is a connected component of a regular fiber $F^{-1}(c)$, and if in addition the Hamiltonian vector fields $\mathcal{X}_{f_1}, \ldots, \mathcal{X}_{f_n}$ are complete on $F^{-1}(c)$, then C is diffeomorphic to $\mathbb{R}^{n-k} \times \mathbb{T}^k$, where $\mathbb{T}^k := (S^1)^k$ is a k-dimensional torus.

If the regular fiber $F^{-1}(c)$ is compact, then $\mathcal{X}_{f_1}, \ldots, \mathcal{X}_{f_n}$ are complete on $F^{-1}(c)$, and C is diffeomorphic to the n-dimensional torus $\mathbb{T}^n$. If $F^{-1}(c)$ is both compact and connected, it is diffeomorphic to $\mathbb{T}^n$. Moreover, ω vanishes along it, so $F^{-1}(c)$ is a Lagrangian submanifold (i.e. a submanifold on which ω vanishes); it has been traditionally called a *Liouville torus*. One often refers to $F\colon M \to \mathbb{R}^n$ as a singular Lagrangian fibration by Liouville tori, where "singular" emphasizes that F may have singular fibers of various kinds: for example tori of dimension $m \in \{0, \ldots, n-1\}$, but also with more complicated topology, such as wedges of 2-spheres, as in Figure 1.

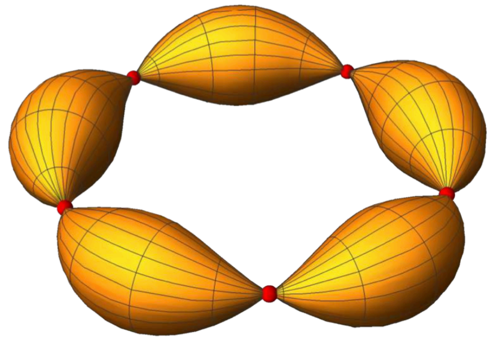

FIGURE 1. A wedge of 5 spheres, or equivalently, a torus with 5 pinches. Integrable systems $F\colon M \to \mathbb{R}^2$ can have singular fibers like these (generally, fibers with any number of pinched points exist). The symplectic geometry of this type of fibers has been understood only recently, that is, that foliated neighborhoods of the fiber have been classified up to symplectomorphisms which preserve the foliation structure. These are typical singular fibers of non-simple semitoric systems (in Section 4.6).

A useful tool for proving results about integrable systems is Morse theory for the components f_i of F. However, at least from the point of view of symplectic geometry, the applications of Morse theory are limited unless M is compact or f_i is proper, meaning that the preimages under f_i of compact subsets of $\mathbb{R}$ are compact. Furthermore, most of the global *symplectic* classifications of integrable systems known to date require that F is at least a proper map (which is automatic if at least one of the f_i is proper). For these reasons, unless otherwise stated, in this paper *integrable systems F are assumed to be proper maps into $\mathbb{R}^n$.* In particular this implies that all fibers of F are compact.

A different question is under which conditions the fibers of F are connected. The question of whether $F^{-1}(c)$ is connected can be formulated as the question of whether the solution set to the system of equations $f_1 = c_1, \ldots, f_n = c_n$ is or not connected, where $c = (c_1, \ldots, c_n)$. Even if $M = \mathbb{R}^{2n}$ and the formulas defining the f_i are simple, say quadratic formulas or polynomials, the question can be challenging (it has relations to real algebraic geometry). At least in symplectic geometry, most approaches to this question have employed some version of Morse theory [7, 87, 107].

2.5. **Action-angle coordinates.** Finite dimensional integrable systems have been studied from different view points and the literature, both in physics and mathematics, is extensive. Nonetheless, at least from the point of view of symplectic geometry, our knowledge is limited.

One of the few general theorems about the symplectic geometry of integrable systems is the existence of action-angle coordinates [5, 74] which says that each regular fiber, in addition to being diffeomorphic to $\mathbb{T}^n$ (recall that we are assuming throughout that F is proper), sits inside of the cotangent bundle $T^*\mathbb{T}^n$ as the zero section, and in a neighborhood of the fiber the integrable system has the normal form $F\colon T^*\mathbb{T}^n \to \mathbb{R}^n$, where

$$F\,(\underbrace{x_1, \ldots, x_n}_{\text{angle}}, \underbrace{\xi_1, \ldots, \xi_n}_{\text{action}}) = \underbrace{(\xi_1, \ldots, \xi_n)}_{\text{action}}.$$

To be more precise, we are assuming in this statement that we have restricted ourselves to an adequate invariant open set, so we can always assume that the fibers of F are connected.

The existence of *global* action-angle coordinates was analyzed by Duistermaat in [34], a paper which may be considered to mark the beginning of the symplectic global theory of finite dimensional integrable systems. Since then this global problem has been analyzed in different contexts, see for example the case of non-commutative integrable systems [42].

2.6. **Linearization of non-degenerate singularities.** In this paper we always assume that the singularities of $F\colon M \to \mathbb{R}^n$ are *non-degenerate*, we refer to [93] for the precise definition. This notion is a vector-valued extension of the condition of being Morse non-degenerate for real valued functions $M \to \mathbb{R}$ but it is more technical to describe. The condition is satisfied by examples such as the Jaynes-Cummings model and the coupled angular momenta.

Under the condition of non-degeneracy, Eliasson described all local models for these singularities, in what is considered one of the most influential results of the subject. His result says that non--degenerate singularities are linearizable [40, 41, 109] in the sense that there exist coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ near the singular point m in which $m = (0, \ldots, 0)$, the symplectic form ω has the expression $\omega = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}\xi_i$, and there exist functions $q_1, \ldots, q_n$ of $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ such that the integrable system $F = (f_1, \ldots, f_n)$ satisfies the Poisson bracket equation $\{f_j, q_i\} = 0$, for all indices i, j, where q_i is one of the following possibilities:

(1) elliptic type: $q_i = \frac{x_i^2+\xi_i^2}{2}$;
(2) hyperbolic type: $q_i = x_i\xi_i$;
(3) real type: $q_i = \xi_i$;
(4) focus-focus type: $q_i = x_i\xi_{i+1} - x_{i+1}\xi_i$ followed by $q_{i+1} = x_i\xi_i + x_{i+1}\xi_{i+1}$.

If there are no components of hyperbolic type the Poisson bracket equation can be written as

$$(F - F(m)) \circ \varphi = g \circ (q_1, q_2, \ldots, q_n),$$

where $\varphi = (x_1, \ldots, x_n, \xi_1, \ldots, x_n, \xi_n)^{-1}$ and g is a diffeomorphism from a small neighborhood of $(0, \ldots, 0)$ into another such neighborhood such that $g(0, \ldots, 0) = (0, \ldots, 0)$. For simplicity, usually we assume that $F(m)$ is the origin. With few exceptions, in this paper we will restrict the discussion to singularities with no components of type (2). In this case Eliasson's theorem says that near any singularity there are symplectic coordinates in which F has the normal form $(q_1, \ldots, q_n)$ up to (translations and) composition by a local diffeomorphism. One reason to rule out hyperbolic components is that their appearance makes it difficult to construct global symplectic invariants.

2.7. **Singularities in dimension** 4. Dimension 4 is the dimension (other than 2) for which we have the best understanding of the symplectic geometry of integrable systems. The possibilities for q_1 and q_2 are:

- m is regular (rank 2): $q_1 = \xi_1$ and $q_2 = \xi_2$;
- m is transversally-elliptic (rank 1): $q_1 = \frac{x_1^2+\xi_1^2}{2}$ and $q_2 = \xi_2$;
- m is elliptic-elliptic (rank 0): $q_1 = \frac{x_1^2+\xi_1^2}{2}$ and $q_2 = \frac{x_2^2+\xi_2^2}{2}$;
- m is focus-focus (rank 0): $q_1 = x_1\xi_2 - x_2\xi_1$ and $q_2 = x_1\xi_1 + x_2\xi_2$;
- m is elliptic-hyperbolic (rank 0): $q_1 = \frac{x_1^2+\xi_1^2}{2}$ and $q_2 = x_2\xi_2$;
- m is hyperbolic-hyperbolic (rank 0): $q_1 = x_1\xi_1$ and $q_2 = x_2\xi_2$;
- m is transversally-hyperbolic (rank 1): $q_1 = x_1\xi_1$ and $q_2 = \xi_2$.

If we rule out hyperbolic components (last three possibilities), in coordinates (x_1, x_2, ξ_1, ξ_2) for which $\omega = \mathrm{d}x_1 \wedge \mathrm{d}\xi_1 + \mathrm{d}x_2 \wedge \mathrm{d}\xi_2$, the system F has the form (q_1, q_2) up to some local diffeomorphism.

2.8. **Classifications and isomorphisms.** A leading goal of much of the recent research in symplectic geometry of integrable systems has been to construct objects (numbers, functions, polytopes, etc) which are invariant by isomorphisms, in terms of which a class of integrable systems can be classified up to these isomorphisms. Examples of such classes could be: those which take place on a 2-dimensional phase space, those whose fibers are submanifolds, those for which the Hamiltonian vector field associated to each component generates a periodic flow, etc.

Here an *isomorphism* between integrable systems $F\colon (M,\omega) \to \mathbb{R}^n$ and $F\colon (M',\omega') \to \mathbb{R}^n$ usually refers to a diffeomorphism of phase space $f\colon M \to M'$ which preserves the foliation structure by leaves induced by the system viewed as a fibration (essentially meaning that $f^*F' = g \circ F$ for some smooth function g) and the symplectic structure (that is, $f^*\omega = \omega'$).

This type of classification of integrable systems up to this notion of isomorphism is often referred to as a *symplectic classification*, to emphasize the contrast with other classifications in which one is interested in a notion of isomorphism which does not necessarily have to preserve the symplectic structure. Such classifications may be of a differentiable or topological nature instead. There are many works in these and related directions, see for instance [11, 93, 117] and the references therein.

At least from the point of view of spectral theory and quantization of integrable systems, the most useful classifications need to be symplectic. In dimension two there is a complete classification due to to Dufour-Molino-Toulet [33]. In dimensions higher than two little is known, even less classifications, with two notable exceptions: toric integrable sytems on compact manifolds of any dimension, and semitoric integrable systems (on compact or noncompact manifolds) but only in dimension 4. We will discuss these classifications in the upcoming sections.

Since symplectic classifications of integrable systems have only been achieved in a few cases, often the specifics of the notion of isomorphism have been tailored to each case, to obtain the most optimal form of a classification. The notion above, depending on the context, may not be the most suitable. For example, if we consider systems $F = (f_1, f_2)\colon M \to \mathbb{R}^2$ in which the flow of $\mathcal{X}_{f_1}$ is periodic, the equation "$f^*F' = g \circ F$" should be replaced by the more specific equation

$$f^*(f_1', f_2') = (f_1, h(f_1', f_2'))$$

for some smooth function h with $\frac{\partial h}{\partial f_2} > 0$ as in [91]. For simplicity we will not dwell on this; a discussion appears in [93, Sections 5 and 6]. In Section 4 we will see a classification (up to this notion of isomorphism) of systems of this type, called *semitoric*, under some conditions.

3. Toric integrable systems

One of the branches of symplectic geometry which underwent a significant growth in the 1980s was the study of Hamiltonian torus actions. Effective Hamiltonian actions of tori of dimension n on compact connected symplectic manifolds of dimension $2n$ were classified in [30]. Such an action can be viewed as an integrable system on a compact manifold for which all of its components generate periodic flows of the same period, say 2π; these systems are called *toric*.

3.1. The periodicity condition on the Hamiltonians.

We say that an integrable system

$$F = \underbrace{(f_1, \dots, f_n)}_{\text{induce action of } \mathbb{T}^n} : M \to \mathbb{R}^n$$

on a symplectic $2n$-dimensional manifold is *toric* if the Hamiltonian vector fields $\mathcal{X}_{f_1}, \dots, \mathcal{X}_{f_n}$ generate periodic flows of the same period, say 2π, and the action of $\mathbb{T}^n$ on M produced by concatenating these flows is effective.

Unless otherwise stated, in this paper we will only consider *toric integrable systems on compact connected manifolds*. Indeed compactness is a crucial condition for the results in this section and for the applications in Section 5.

The periodicity implies that the singularities of toric integrable systems cannot have focus-focus or hyperbolic type components, that is, if $m = (0, \dots, 0)$ and $\omega = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}\xi_i$, locally in a neighborhood of m the integrable system must have the form

$$F(x_1,\dots,x_1,\xi_1,\dots,\xi_n)=\Big(\underbrace{\frac{x_1^2+\xi_1^2}{2},\dots,\frac{x_k^2+\xi_k^2}{2}}_{\text{elliptic type}},\overbrace{\xi_{k+1},\dots,\xi_n}^{\text{real type}}\Big).$$

Toric integrable systems have connected fibers, a fact known as Atiyah's connectivity [7] in the more general context of Hamiltonian $\mathbb{T}^m$-actions, $m\in\{1,\dots,n\}$, on compact connected symplectic $2n$-dimensional manifolds (there are also extensions of this to certain infinite dimensional settings, see for instance [8, 53]). This fact is closely related to the classification of toric integrable systems which we discuss in Section 3.3.

All fibers of F are diffeomorphic to tori of varying dimensions $\mathbb{T}^k$, $k\in\{0,\dots,n\}$, which is not the case for general integrable systems as we will see in a moment (earlier we say a typical singular fiber in Figure 1). For example, in dimension 4 the possibilities are:

$$F(x_1,x_2,\xi_1,\xi_2)=\Big(\frac{x_1^2+\xi_1^2}{2},\frac{x_2^2+\xi_2^2}{2}\Big)\Longrightarrow F^{-1}(F(m))=\underbrace{\{m\}}_{\text{elliptic-elliptic singularity}}\quad;$$

$$F(x_1,x_2,\xi_1,\xi_2)=\Big(\frac{x_1^2+\xi_1^2}{2},\xi_2\Big)\Longrightarrow\underbrace{F^{-1}(F(m))\simeq S^1}_{\text{transversally-elliptic singularities}}\quad;$$

$$F(x_1,x_2,\xi_1,\xi_2)=(\xi_1,\xi_2)\Longrightarrow\underbrace{F^{-1}(F(m))\simeq\mathbb{T}^2}_{\text{regular points}}.$$

The fibers corresponding to these local models are illustrated in Figure 2, in terms of the image under F of the singularity.

3.2. **Example of a toric integrable system: rotation on complex projective space.** Consider on S^2 the standard area form $\omega=\mathrm{d}\theta\wedge\mathrm{d}h$, where θ is the angle and h is the height of a point in S^2. The height function $F(\theta,h)=h$ is a Hamiltonian on S^2, which defines a toric integrable system. This system corresponds to the rotational S^1-action on S^2 about the z-axis, which has momentum map precisely equal to F. Clearly $F(S^2)=[-1,1]$.

As a generalization to higher dimensions, consider the complex projective space $\mathbb{C}P^n$ endowed with a λ-multiple ($\lambda>0$) of the Fubini-Study form. Then $F\colon\mathbb{CP}^n\to\mathbb{R}^n$ given by

$$F([z_0:z_1:\ldots:z_n])=\Big(\frac{\lambda|z_1|^2}{\sum_{i=0}^n|z_i|^2},\dots,\frac{\lambda|z_n|^2}{\sum_{i=0}^n|z_i|^2}\Big).$$

is a toric integrable system, induced by the Hamiltonian $\mathbb{T}^n$-action by rotations defined by the formula $(\mathrm{e}^{\mathrm{i}\theta_1},\dots\mathrm{e}^{\mathrm{i}\theta_n})[z_0:z_1,\dots:z_n]=[z_0:\mathrm{e}^{\mathrm{i}\theta_1}z_1:\dots\mathrm{e}^{\mathrm{i}\theta_n}z_n]$. If $\mathrm{e}_1=(1,0,\dots,0),\dots,\mathrm{e}_n=(0,\dots,0,1)$, are the standard basis vectors in $\mathbb{R}^n$, then the image of the integrable system is the set $F(\mathbb{C}P^n)=$ convex hull $\{0,\lambda\mathrm{e}_1,\dots,\lambda\mathrm{e}_n\}$. Notice that F has an interesting property: its image, which is a subset of $\mathbb{R}^n$, is a convex polytope. Moreover, this convex polytope is the convex hull of the fixed point set of the $\mathbb{T}^n$-action on $\mathbb{C}P^n$. This property turns out to be a general property of toric integrable systems, as we explain next.

3.3. **Classification.** One of the fundamental theorems of equivariant symplectic geometry, due to Atiyah [7] and Guillemin-Sternberg [47] says that, if M is compact and connected, the image $F(M)$ is a convex polytope in $\mathbb{R}^n$, in fact, obtained by a recipe: it is the convex hull of the images of fixed points of the Hamiltonian action of the n-torus on M induced by concatenating the flows of the f_i. This polytope has the property that if two toric integrable systems are isomorphic then their associated images coincide (up to translations and composition with a matrix in $\mathrm{GL}(n,\mathbb{Z})$).

In fact, the Atiyah-Guillemin-Sternberg convexity theorem[1], as is usually referred this result, applies to general Hamiltonian m-dimensional torus actions, for any $m \in \{1, \ldots, n\}$, where $2n$ is the dimension of M. In this case F would be the action momentum map and has m components. Of course only in the case when $m = n$ this represents simultaneously a Hamiltonian torus action and an integrable system on M. It is in this case when their result can be used as a stepping stone for obtaining a classification of toric integrable systems on compact manifolds.

Indeed, shortly after this result, Delzant showed that the polytopes obtained as images of toric integrable systems are of a special type: simple, rational, and smooth. Now these polytopes are called *Delzant polytopes*; the essential condition for a Delzant polytope in $\mathbb{R}^n$ is that there are precisely n edges meeting at each vertex and the normal vectors to the facets meeting at the vertex form a basis of the integral lattice.

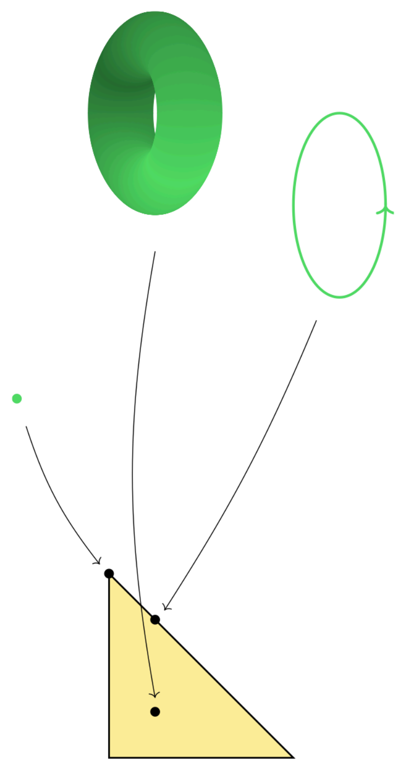

FIGURE 2. The information of a toric integrable system $F\colon (M,\omega) \to \mathbb{R}^n$ on a compact connected manifold can be read off from the polytope $F(M) = \Delta$. In particular the topology of a fiber over a point $p \in \Delta$ is encoded by the dimension of the face of Δ containing p. The fiber over a vertex of Δ is a point, while the fiber over a point in an edge of Δ is a circle. If p is any point in the interior of the polygon, the fiber is a torus.

Moreover, Delzant proved that such polytopes are in bijective correspondence with toric integrable systems on compact manifolds:

$$\text{toric system} \quad \underbrace{\rightsquigarrow}_{\text{information lost?}} \quad \overbrace{\text{its image } \Delta}^{\text{Delzant polytope}} \quad \underbrace{\rightsquigarrow}_{\text{Delzant's Theorem}} \quad \text{toric system}$$

[1]Convexity and related properties of Hamiltonian actions were later studied in different contexts, for example see for instance [1, 45, 62, 77, 113], to name a few.

Here by "bijective correspondence" we mean:

- the *uniqueness* statement that two systems $F\colon (M,\omega) \to \mathbb{R}^n$ and $F'\colon (M',\omega') \to \mathbb{R}^n$ are isomorphic if and only if they have the same convex polytope as image (up to translations and $\mathrm{GL}(n,\mathbb{Z})$ transformations), so the answer to the question in the first arrow of the diagram is *no* (up to isomorphisms), and,
- the *existence* statement which says that starting from any Delzant polytope Δ in $\mathbb{R}^n$ one can construct a toric integrable system $F\colon (M,\omega) \to \mathbb{R}^n$ on a compact connected symplectic $2n$-dimensional manifold whose image is Δ. This construction, which corresponds to the second arrow of the diagram, can be achieved via the method of *symplectic reduction.*

Since Δ classifies $F\colon (M,\omega) \to \mathbb{R}^n$, one can know everything about F, up to isomorphisms, from Δ. In particular the fiber structrure of F can be read off from the polytope Δ: the fiber of F over $p \in \Delta$ is diffeomorphic to an k-dimensional torus, where k is the dimension of the lowest dimensional face of Δ such that $p \in \Delta$. For example, in the case of $\mathbb{CP}^2$ discussed earlier, Δ is a 2-dimensional simplex with boundary a triangle. The fibers of F over the vertices of the triangle are points (elliptic-elliptic singularities), over the edges of the triangle are diffeomorphic to S^1 (transversally-elliptic singularities), and over any point in the interior of the triangle they are 2-tori (regular points); this is depicted in Figure 2.

Delzant's classification has been a precursor for the study of more general integrable systems. Since then, there have been classifications of systems of toric type in a variety contexts. For instance, on noncompact manifolds [61], and on log-symplectic manifolds [50, 72]. Delzant's result has been extended to general symplectic torus actions in some cases [9, 37, 82]. There is also a natural connection between the study of toric integrable systems and toric varieties in algebraic geometry, see [38] for a detailed study.

4. Semitoric integrable systems

Semitoric integrable systems form a class of integrable systems which generalizes the class of toric integrable systems. They have been the focus of intense research activity in the past ten years or so. They were classified in dimension 4 under some conditions in [91, 92]. The main property of a semitoric system is that all the functions that define it, but one, generate periodic flows. So toric systems are semitoric. At least from the point of view of symplectic geometry, the main difference with toric systems is that semitoric systems can have focus-focus singularities and the fibers containing them are not diffeomorphic to tori, for instance as in Figure 1 (while the fibers of toric systems are diffeomorphic to tori of varying dimensions, as in Figure 2). As we will see, the appearance of this type of singular fiber makes the symplectic geometry of these systems very rich.

The Jaynes-Cummings model is a well known semitoric system. Its fiber structure looks like that of a toric system, with the exception that it has one fiber containing a focus-focus singularity, which is a torus pinched at exactly one point. The coupled angular momenta is another example.

4.1. The periodicity condition on all but one Hamiltonian.

An integrable system

$$F = (\underbrace{f_1, \ldots, f_{n-1}}_{\text{induce action of } \mathbb{T}^{n-1}}, f_n)\colon M \to \mathbb{R}^n$$

is *semitoric* if the Hamiltonian vector fields $\mathcal{X}_{f_1}, \ldots, \mathcal{X}_{f_{n-1}}$ generate periodic flows of the same period, say 2π, and the action of $\mathbb{T}^{n-1}$ on M produced by concatenating these flows is effective. As earlier, we also require that the singularities of F are non-degenerate and do not have hyperbolic components. In the current theory there is also the assumption that $f_1, \ldots, f_{n-1}$ are all proper.

If $n = 2$ these conditions mean (by Section 2.7) that $f_1\colon M \to \mathbb{R}$ is a proper momentum map for an effective Hamiltonian S^1-action, and the local models of $F = (f_1, f_2)$ near a point m are, in some coordinates (x_1, x_2, ξ_1, ξ_2) in which $m = (0,0,0,0)$ and $\omega = \mathrm{d}x_1 \wedge \mathrm{d}\xi_1 + \mathrm{d}x_2 \wedge \mathrm{d}\xi_2$, one of the following:

$$F(x_1,x_2,\xi_1,\xi_2) = \Big(\frac{x_1^2+\xi_1^2}{2}, \frac{x_2^2+\xi_2^2}{2}\Big) \Longrightarrow F^{-1}(F(m)) = \underbrace{\{m\}}_{\text{elliptic-elliptic singularity}} ;$$

$$F(x_1,x_2,\xi_1,\xi_2) = \Big(\frac{x_1^2+\xi_1^2}{2}, \xi_2\Big) \Longrightarrow \underbrace{F^{-1}(F(m)) \simeq S^1}_{\text{transversally-elliptic singularities}} ;$$

$$F(x_1,x_2,\xi_1,\xi_2) = (\xi_1, \xi_2) \Longrightarrow \underbrace{F^{-1}(F(m)) \simeq \mathbb{T}^2;}_{\text{regular points}}$$

$$F(x_1,x_2,\xi_1,\xi_2) = (x_1\xi_2 - x_2\xi_1, x_1\xi_1 + x_2\xi_2) \Longrightarrow \underbrace{F^{-1}(F(m)) \simeq \text{pinched } \mathbb{T}^2}_{\text{a few focus-focus singularities inside}} .$$

The singular fibers, which are connected in this case and also under slightly weaker conditions [87, 107], are either points, circles, or tori pinched at a finite amount of points (i.e. wedges of 2-spheres as in Figure 1). This last type of fiber does not appear in toric integrable systems.

If $n = 2$ we have a complete understanding of the global symplectic geometry of semitoric systems under the assumptions above: they were classified about ten years ago in [91, 92] under the extra requirement that each fiber of f_1, and hence of $F = (f_1, f_2)$, contains at most one focus-focus singularity; this is often called *simplicity*. So each singular fiber of F could be pinched at most once. This requirement was recently removed in [79], hence in particular allowing fibers of the form shown in Figure 1. We will discuss this recent extension in Section 4.6, but before that section we assume that all systems satisfy this simplicity condition.

4.2. **Example of a simple semitoric integrable system: the Jaynes-Cummings model.** A well known example of semitoric system, which is also one of the few for which the symplectic invariants have been explicitly computed, is the Jaynes-Cummings model [57, 27] from physics, also referred to as the coupled spin-oscillator. This is a system which models simple physical phenomena, and is given by

$$F(x,y,z,u,v) := \Big(\underbrace{f_1 = \frac{u^2+v^2}{2} + z}_{\text{periodic flow}}, f_2 = \frac{ux+vy}{2}\Big)$$

in coordinates $(x,y,z,u,v) \in S^2 \times \mathbb{R}^2$, where S^2 inherits the coordinates from the usual inclusion $S^2 \subset \mathbb{R}^3$. Note that f_1 corresponds to the momentum map for the Hamiltonian S^1-action by rotations on the plane about the origin. All the singularities of F with the exception of $m = (0,0,1,0,0)$ are elliptic or transversally elliptic, while m itself is a focus-focus singularity [94]. Hence the singular fiber containing m is a torus pinched at m.

There are other famous systems which are semitoric, notably the coupled angular-momenta. An example which is not semitoric strictly speaking, but for which one component still generates a periodic flow is the spherical pendulum in Section 2.3. In this case the system fails to satisfy our definition of semitoric system because the Hamiltonian generating a periodic flow is not proper, even though the joint map F is.

While there is no general symplectic classification for examples like the spherical pendulum in which the periodic component is not proper, some first steps were taken in [87, 88] to construct a polygonal invariant out of the image of $F(M)$ which resembled the convex polygon of Atiyah-Guillemin-Sternberg [7, 47] and its later generalization to semitoric systems [91, 107].

4.3. **Symplectic invariants and uniqueness of simple semitoric systems in dimension four.** In dimension four, a simple semitoric system $F = (f_1, f_2)$ is determined [91, Theorem 6.2] up to isomorphisms by a convex polygon Δ with marked points $p_1, \ldots, p_n \in \Delta$, each of which comes with a label assigned

$$p_\ell \rightsquigarrow \Big(k, \sum_{i,j} a_{ij} x^i y^j\Big) \in \mathbb{Z} \times \mathbb{R}[x, y],$$

where k is an integer and $\sum_{i,j} a_{ij} x^i y^j$ is a formal Taylor series on two variables. These invariants are depicted in Figure 3.

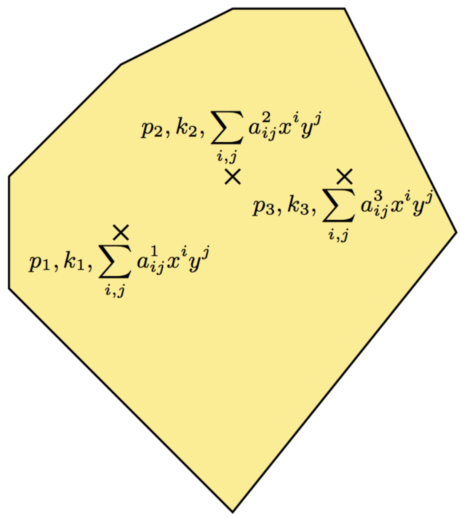

FIGURE 3. The figure shows a polygon Δ in the plane, with several points inside of it. Each point comes endowed with a label consisting of an integer and a formal Taylor series on variables x, y. This is the generic picture which captures the symplectic invariants of a simple semitoric integrable system on a 4-manifold. In this sense, the figure is a *complete invariant*. That is, all of the symplectic geometry of the simple semitoric system $F\colon (M, \omega) \to \mathbb{R}^2$ can be read off from this figure. The uniqueness theorem for semitoric systems explained in Section 4.3 says that any two simple semitoric systems with the same complete invariant must be isomorphic. However, this result does not tell us which polygons can appear, and whether there are restrictions on the positions of the points, the coefficients of the Taylor series, or the integers k. This is a separate, existence problem, discussed in Section 4.5.

The main tools for proving this are the existence of action-angle coordinates [5, 34] and the results on linearization of non-degenerate singularities in [40, 41, 75, 109]. Note that the k and the coefficients a_{ij} of the Taylor series will vary, in general, for each marked point. Notice that this is a *uniqueness result*, while it does give no information on how many simple semitoric systems there are or how to construct them, which would be an *existence result*. We well discuss this existence question shortly (in Section 4.5).

So the list of symplectic invariants of simple semitoric systems consists of a polygon, decorated by n points and n labels[2]. The marked points here correspond to the images of the focus-focus

[2]The number n of marked points and their exact positions inside of Δ (more precisely along the vertical line which contains them) are both symplectic invariants. With this point of view, one could say that there are five invariants: the polygon Δ itself, the number of points inside of it, the position of each point inside of Δ, the n-tuple of integer

singularities of F (there are finitely many of them as shown in [107]). Hence if the system is toric, then there are no marked points and hence no labels, so we are left with a convex polygon, and if M is compact this recovers the classification result of Delzant in Section 3.3. Next we briefly explain the meaning of these invariants:

- The convex polygon Δ is equal to $F(M)$ if M is compact and the system is toric. In general, Δ does not equal $F(M)$, instead it is obtained from $F(M)$ by making vertical cuts along the focus-focus values [107], hence unfolding the singular affine structure induced by F. These vertical cuts can be made either upwards, or downwards, resulting in a family of polygons and the actual polygon invariant is an orbit of such polygons by the action which takes into consideration the cut directions; it is called the *semitoric polygon invariant*, the precise notion is given in [91, Section 4.3]. In fact, Δ satisfies some specific rationality properties as explained in [92, Section 4.1], which generalize those of the Delzant polygons in the classification of toric integrable systems on compact connected manifolds.
- The Taylor series $\sum_{i,j} a_{ij} x^i y^j$ encodes the dynamical behavior of the Hamiltonian vector fields $\mathcal{X}_{f_1}$ and $\mathcal{X}_{f_2}$ as they approach the focus-focus singularities of F, and symplectically determines the semiglobal normal form of the fibration by Liouville tori near the singular fiber containing the focus-focus singularity corresponding to the point in the polygon with the label $(k, \sum_{i,j} a_{ij} x^i y^j)$.
- The integer k, called the *twisting index*, encodes the topology of F viewed as a singular Lagrangian fibration near the singular fiber containing the focus-focus singularity corresponding to the point in the polygon with the label $(k, \sum_{i,j} a_{ij} x^i y^j)$. This index can be thought of as encoding the way in which the semiglobal normal form determined by $\sum_{i,j} a_{ij} x^i y^j$ is glued into the integrable system, relative to the global integral affine structure induced by F. The construction of k is subtle and beyond the scope of this paper, we refer to [91, Section 5.2] for those interested in learning about it. The actual *twisting index invariant* is given as an orbit of the tuple of all twisting indices, say $(k_1, \ldots, k_n) \in \mathbb{Z}^n$, corresponding to the points $p_1, \ldots, p_n \in \Delta$ under some natural group action on them.

To get a better idea of the result, we explain how to construct $\sum_{i,j} a_{ij} x^i y^j$. Let m be a focus--focus singularity of $F\colon M \to \mathbb{R}^2$. By Eliasson's theorem there are coordinates (x_1, x_2, ξ_1, ξ_2) near m in which $m = (0, 0, 0, 0)$, $\omega = \mathrm{d}x_1 \wedge \mathrm{d}\xi_1 + \mathrm{d}x_2 \wedge \mathrm{d}\xi_2$ and a local diffeomorphism h such that $h \circ F = (x_1\xi_2 - x_2\xi_1, x_1\xi_1 + x_2\xi_2)$. This normalization extends to a neighborhood of $F^{-1}(F(m))$, we call it (H_1, H_2). Now we are going to construct two period maps. For each regular value c close to $F(m)$ choose $A \in F^{-1}(c)$ and define $\tau_2(c)$ to be the time it takes the Hamiltonian flow associated to $\mathcal{X}_{H_2}$ leaving from A to meet the Hamiltonian flow associated to H_1 which passes through A, and let $\tau_1(c) \in \mathbb{R}/2\pi\mathbb{Z}$ be the time that it takes to go from this intersection point back to A along the Hamiltonian flow line of $\mathcal{X}_{H_1}$, in this way closing the trajectory. The behavior of $\mathcal{X}_{H_2}$ field as we approach c becomes singular. We remove the singular behavior from the period maps, which is of a logarithmic nature, by defining $\sigma_1(c) = \tau_1(c) - \mathrm{Im}(\log(c))$ and $\sigma_2(c) = \tau_2(c) + \mathrm{Re}(\log(c))$, which are well-defined and smooth, and from them we construct the 1-form $\sigma_1 \mathrm{d}c_1 + \sigma_1 \mathrm{d}c_2$ which is closed near m and hence exact: $\sigma_1 \mathrm{d}c_1 + \sigma_2 \mathrm{d}c_2 = \mathrm{d}S$. The invariant $\sum a_{ij} x^i y^j$ is the Taylor series of S (up to a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action, as in [101]).

The invariant $\sum a_{ij} x^i y^j$ is a semiglobal invariant [106], that is, it determines up to isomorphisms a neighborhood of $F^{-1}(F(m))$, meaning two integrable systems are isomorphic in a neighborhood of a fiber with a focus-focus singularity if and only if they have the same Taylor series associated to them. In fact, essentially any Taylor series of two variables (with a small restriction on a coefficient)

---

labels k, and the n-tuple of Taylor series, one per point. These were the five invariants introduced in [91] in terms of which semitoric systems are classified in [91, 92]; our formulation in the present paper is equivalent.

can appear as a symplectic invariant, and hence $\sum a_{ij}x^iy^j$ provides a symplectic classification a neighborhood of the fiber containing m.

In the case that there are $\lambda > 1$ focus-focus singularities in the same singular fiber, it has been recently shown [90] that the symplectic invariant of a neighborhood of the singular fiber containing the λ focus-focus singularities consists of λ Taylor series: $\sum_{i,j} a^1_{ij}x^iy^j, \ldots, \sum_{i,j} a^\lambda_{ij}x^iy^j$, which again encode the singular dynamics of the Hamiltonian vector fields $\mathcal{X}_{f_1}$ and $\mathcal{X}_{f_2}$ as they approach these λ singularities.

4.4. **Calculations of symplectic invariants of certain semitoric systems.** The Taylor series invariant has been calculated in some examples; for the case of the Jaynes-Cummings system the Taylor series at the only focus-focus singularity was calculated in [94] to be

$$\sum_{i,j} a_{ij}x^iy^j = \frac{\pi}{2}x + 5\log 2y + \mathcal{O}(2).$$

A detailed study of the invariants of this system was later given in [2], see [4] for a survey of results in this direction.

Let's briefly discuss another example: the case of the coupled angular momenta [99]. Let $R_2 > R_1 > 0$. On $M = S^2 \times S^2$ with coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$ we consider:

$$\begin{cases} f_1 := R_1z_1 + R_2z_2 \\ \\ f_{2,t} := (1-t)z_1 + t(x_1x_2 + y_1y_2 + z_1z_2) \end{cases} \qquad \forall t \in [0,1]$$

with symplectic form $-(R_1\omega_{S^2} \oplus R_2\omega_{S^2})$, where ω_{S^2} is the standard symplectic form on S^2. In this case there are two values $t^- < \frac{1}{2} < t^+$ such that $(0,0,1,0,0,-1)$ is a singularity of

$$F_t := (f_1, f_{2,t})\colon S^2 \times S^2 \to \mathbb{R}^2$$

which is elliptic-elliptic between 0 and t^-, degenerate at t^-, focus-focus between t^- and t^+, degenerate at t^+, and elliptic-elliptic again from t^+ to 1.

The integrable system F_t is toric up to local diffeomorphisms (this is often called being of *toric type*) for $t < t^-$ and $t > t^+$, and semitoric for values of t in between t^- and t^+. In [69] the polygon invariant was computed for $t = 1/2$, and it was also shown that there is a focus-focus singularity with label $k = 0$, and if $R_1 = 1, R_2 = 5/2$ the Taylor series is

$$\sum_{i,j} a_{ij}x^iy^j = \arctan\left(\frac{9}{13}\right)x + \left(\frac{7}{2}\log 2 + 3\log 3 - \frac{3}{2}\log 5\right)y + \mathcal{O}(2).$$

For a complete study of the symplectic invariants of the coupled angular momenta see [3]. Further generalizations and other families of deformations were studied in [54, 67].

4.5. **The construction (i.e. existence) of simple semitoric systems in dimension** 4. It was shown in [92] that for each semitoric polygon Δ with interior marked points $p_1, \ldots, p_n$, each labelled by a pair $(k, \sum_{i,j} a_{ij}x^iy^j)$, there exists a semitoric integrable system $F\colon M \to \mathbb{R}^2$ which has Δ and $p_1, \ldots, p_n$, each with a label $(k, \sum_{i,j} a_{ij}x^iy^j)$, as its symplectic invariants, as defined in Section 4.3 (this is depicted in Figure 3). Together with the uniqueness result in Section 4.3, this gives a symplectic classification of simple semitoric integrable systems. (The precise formulation of the classification in [92, Theorem 4.7] involves considering certain natural group actions on the space of marked semitoric polygons, to avoid redundancies and making sure the invariants are defined without ambiguity). We may illustrate this with the following diagram, while keeping in mind that the first arrow was discussed in Section 4.3, while the second arrow is the purpose of this section:

$$\text{simple semitoric syst.} \underbrace{\rightsquigarrow}_{\text{lost?}} \overbrace{\Delta, p_1, k_1, \sum_{i,j} a^1_{ij} x^i y^j, \ldots, p_n, k_n, \sum_{i,j} a^n_{ij} x^i y^j}^{\text{symplectic invariants}} \underbrace{\rightsquigarrow}_{\S 4.5} \text{simple semitoric syst.}$$

By Section 4.3 the answer to the question in the first arrow of the diagram, as to whether any data is lost in the process of finding the invariants, is *no* (up to isomorphisms). For brevity and to keep the discussion nontechnical we have omitted what type of symplectic invariants can appear as an abstract list of objects defined without reference to symplectic manifolds or integrable systems. In the toric case, recall this list consisted of all Delzant polytopes in $\mathbb{R}^n$; in the semitoric case, one needs to allow more flexibility on the polytopes that can appear, as well as have other conditions imposed on the abstract spaces where the remaining symplectic invariants are: space of formal Taylor series, $\mathbb{Z}$, etc.

The idea of proof of this existence result, which shows an explicit construction, is as follows. Basically one needs to glue the "regular" part of the system (given by action-angle coordinates) with the "singular" part, near the focus-focus fibers (which involves Eliasson's coordinates, as employed for the construction of the Taylor series invariant in Section 4.3):

- In the first step Δ is covered by an appropriate collection of sets $(\Delta_\alpha)_\alpha \subset \mathbb{R}^2$ and one defines symplectic manifolds over them $(M_\alpha, F_\alpha : M_\alpha \to \Delta_\alpha)$ using action-angle coordinates. The second step is devoted to glueing these models, using a symplectic glueing method which we describe below, in order to obtain a continuous map

$$F\colon M - (\text{neighborhoods of the focus-focus fibers}) \longrightarrow \mathbb{R}^2,$$

  where by a focus-focus fiber we mean a singular fiber containing one focus-focus singularity. Next we attach the neighborhoods of the focus-focus fibers, determined by the Taylor series. This glueing is delicate and requires an $\epsilon - \delta$ analysis near focus-focus fibers. There is remaining freedom when glueing in these fibers which is controlled by the twisting index invariant.
- In this way, one obtains a continuous map $F\colon M \to \mathbb{R}^2$ which is "essentially" the semitoric integrable system we want to construct, but has the problem that it is singular. By singular we mean that it is non smooth (so we cannot yet speak of an integrable system), near the overlaps of the neighborhoods of the focus-focus fibers (described by Eliasson's normal form) with the neighborhoods of the regular fibers (described by action-angle coordinates).
- To overcome this smoothness problem of F and produce a semitoric integrable system, one needs to suitably modify F to make it smooth everywhere. This is a subtle analytic problem which takes the last step, and a significant part of the paper [92].

To conclude, we discuss how to do the symplectic glueing mentioned in the second step above. The point is that this glueing works for continuous maps on symplectic manifolds, and hence why above there needed to be a final step, to smoothen the continuous map which the next result gives. The result can be formulated quite generally as follows (see Section [92, Section 3.4] for a detailed formulation and proof). Let us suppose that $(M_\alpha)_{\alpha\in A}$ are symplectic manifolds and $F_\alpha : M_\alpha \to V_\alpha \subset \mathbb{R}^n$ are proper continuous maps where V_α is open. For each $\alpha, \beta \in A$ assume that $\varphi_{\alpha\beta} : F_\alpha^{-1}(D_{\alpha\beta}) \to F_\beta^{-1}(D_{\alpha\beta})$ is a symplectomorphism such that $\varphi^*_{\alpha\beta} F_\beta = F_\alpha$ and whenever $V_\alpha \cap V_\beta \cap V_\gamma \neq \emptyset$, one has that $\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$. Then one can prove that the smooth manifold M obtained by glueing the symplectic manifolds $(M_\alpha)_{\alpha\in A}$ by means of the transformations $(\varphi_{\alpha\beta})$ is Hausdorff, paracompact, and symplectic, and that there exists a proper continuous map $F : M \to \bigcup_{\alpha\in A} V_\alpha \subset \mathbb{R}^n$ such that $F_\alpha = F \circ y_\alpha$, where $y_\alpha : M_\alpha \hookrightarrow M$, $\alpha \in A$, are the natural inclusion symplectic embeddings.

4.6. **Semitoric systems with multiply pinched focus-focus fibers in dimension** 4. The classification of simple semitoric systems in Sections 4.3 and 4.5 has been extended in [79] to non-simple systems. Hence the same singular fiber of J, and hence of F, may contain any finite number $\lambda \in \mathbb{Z}^+$ of focus-focus singularities; topologically such a fiber is a torus with λ pinched points, that is, a wedge of λ spheres as shown in Figure 1.

In this situation, a classification which extends the one explained earlier can be given in terms of the natural extensions of the invariants we have described. The complete symplectic invariant is a polygon (which may not be convex due to a slight difference in the way it is constructed) with n marked points $p_1, \ldots, p_n$ inside of it, and for each such point p_ℓ there is assigned a label

$$p_\ell \rightsquigarrow \Big(k, \overbrace{\underbrace{\sum_{i,j} a^1_{ij} x^i y^j, \ldots, \sum_{i,j} a^\lambda_{ij} x^i y^j}_{\text{One Taylor series per pinch in } F^{-1}(p_\ell)}}^{=(\sum_{i,j} a^s_{ij} x^i y^j)_{s=1}^{\lambda}}\Big) \in \mathbb{Z} \times (\mathbb{R}[x,y])^\lambda,$$

where there are as many Taylor series corresponding to a particular point p_ℓ as there are pinched points in the fiber $F^{-1}(p_\ell)$. In a diagram:

$$\text{semitoric system} \underbrace{\rightsquigarrow}_{\text{lost?}} \overbrace{\Delta, p_1, k_1, (\sum_{i,j} a^{1s}_{ij} x^i y^j)_{s=1}^{\lambda_1} \ldots p_n, k_n, (\sum_{i,j} a^{ns}_{ij} x^i y^j)_{s=1}^{\lambda_n}}^{\text{symplectic invariants}} \underbrace{\rightsquigarrow}_{\S 4.6} \text{semitoric system}$$

The same way as for simple semitoric systems, the answer to the question in the first arrow of the diagram (whether data is lost in the process of finding the invariants) is *no* (up to isomorphisms).

These Taylor series were constructed in [90] as a generalization of the case when $\lambda = 1$ from [106]. In the same way as in Section 4.3, the integer k and the Taylor series associated to each point will vary with the point.

The precise classification statement [79, Theorem 4.10] is more involved because the twisting index invariant and the Taylor series invariants at the focus-focus singularities are deeply connected if there exists at least one ℓ_0 for which the number of focus-focus singularities in $F^{-1}(p_{\ell_0})$ is 2 or higher, and should not be considered as separate invariants. Instead a significant part of [79] is devoted to describing a single invariant which incorporates the information of all the Taylor series and all the twisting indices simultaneously.

Concrete examples of non simple semitoric systems appear in [31, 54].

4.7. **Moduli spaces of simple semitoric systems.**

4.7.1. *Minimal models of toric and simple semitoric systems.* A toric (respectively semitoric) integrable system $F\colon M \to \mathbb{R}^2$ is *minimal* if there is no toric (respectively semitoric) integrable system $F'\colon M' \to \mathbb{R}^2$ such that F can be obtained from F' by a blow up respecting the toric (respectively semitoric) structure.

There are three minimal models for toric integrable systems: their associated fan corresponds to a square, a triangle, or a Hirzebruch trapezoid ([76, Theorem 8.2] and [46]). The symplectic geometry of minimal models of simple semitoric integrable systems was studied in [59]. In this case there is an explicit list of minimal models in terms of a generalization of the fan; this generalization is called the *helix*, and can be considered a symplectic analogue of the fan of a nonsingular complete toric variety in algebraic geometry, that takes into account the effects of the monodromy near focus-focus singularities. By [59, Theorem 1.3] there are seven minimal models for simple semitoric integrable systems, corresponding to seven inequivalent helices.

4.7.2. *A metric and a topology on the moduli spaces of toric and simple semitoric systems.* One of the motivations to introduce the helix, and the other invariants of integrable systems (Taylor series, twisting index, semitoric polygon, etc), as well as the minimal models, is to shed light on the structure of the moduli spaces of the corresponding integrable systems which have these invariants.

The simplest cases to consider are probably the case of toric and simple semitoric integrable systems. A first step in this direction was given in [86], where the case of toric integrable systems on compact manifolds is studied, and a topology is defined on the corresponding moduli space $\mathcal{M}_{\mathrm{t}}$.

The topology is constructed indirectly by defining a distance on the moduli space, and then considering the topology induced by this distance. In this paper the authors study a variety of topological properties of this space, in particular, path-connectedness, compactness, and completeness. The construction of the distance is related to the Duistermaat-Heckman measure [36]. This construction was generalized in [78] to simple semitoric integrable systems, where the author defines a metric on the moduli space of simple semitoric systems $\mathcal{M}_{\mathrm{s}}$, and studies several of its properties. The connectivity properties of this moduli space were studied in [58, Section 6.3].

4.7.3. *Functions on moduli spaces of toric and simple semitoric systems.* One of the useful consequences of having topologies on these moduli spaces is that now one is able to quantify the variation of certain natural functions $\mathcal{M}_{\mathrm{t}} \to \mathbb{R}$ and $\mathcal{M}_{\mathrm{s}} \to \mathbb{R}$ defined on them. For instance, on $\mathcal{M}_{\mathrm{t}}$ with the topology mentioned earlier, one can define the so called *toric packing capacity* $\mathcal{M}_{\mathrm{t}} \to \mathbb{R}$ as follows.

First, let $\mathrm{B}^{2n}(r) \subset \mathbb{R}^{2n} \simeq \mathbb{C}^n$ be the ball of radius r centered at the origin. It comes endowed with a natural rotational action of the n-torus $\mathbb{T}^n$ component by component of $\mathbb{C}^n$. With this notation in mind, a *ball packing* P of (M, ω) is any disjoint union of symplectically embedded balls $\mathrm{B}^{2n}(r) \hookrightarrow M$ into M, of any possibly varying radii $r > 0$. A *toric ball packing* [80, 81] of a toric system $F\colon M \to \mathbb{R}^n$ is a ball packing of (M, ω) such that the symplectic embeddings of balls are *equivariant* with respect to the natural n-torus action on $\mathrm{B}^{2n}(r)$ and the n-torus action induced on M by F. The *volume of* P is defined to be the sum of the volumes of the balls with respect to the volume form ω^n; we denote it by $\mathrm{vol}(P)$. With these definitions in place we may define a function $c\colon \mathcal{M}_{\mathrm{t}} \to \mathbb{R}$ by assigning to each toric integrable system the number

$$c(F) = \Big(\frac{1}{\mathrm{vol}(\mathrm{B}^{2n})} \sup\{\mathrm{vol}(P) \mid P \text{ is a toric ball packing of } F\}\Big)^{\frac{1}{2n}}.$$

An analogous function can be defined on $\mathcal{M}_{\mathrm{s}}$. These functions are examples of what are called in [44] *G-equivariant symplectic capacities*, which generalize to the equivariant setting (here G is any Lie group) the usual notion of symplectic capacity.

The continuity of these (and closely related) functions was studied in [43] and then in [44]. Indeed, by [43, Theorem A] and [44, Theorem 1.2] the function $c\colon \mathcal{M}_{\mathrm{t}} \to \mathbb{R}$ is everywhere discontinuous and the restriction to the subspace of toric integrable systems with exactly N points fixed by the induced n-torus action is continuous for any choice of $N > 0$. Also in [44, Theorem 1.2] an analogous continuity statement is given for semitoric integrable systems.

4.8. **Other classifications or related results.** The symplectic theory of toric integrable systems discussed in Section 3, and the classifications of the Fomenko school [11], were two of the motivations for pursuing the global symplectic classification of simple semitoric systems and its generalizations which we have discussed in Sections 4.3, 4.5, and 4.6.

The classifications we have presented have relations to almost toric systems and systems with semitoric features as in [56, 71, 103, 110, 111], and also the theory of Hamiltonian S^1-spaces [55, 60].

There is also recent work on the so called toric-focus systems [98] and systems which have semitoric features but also include singularities with hyperbolic components [39]. In the article [104] there is an application of the ideas in the proof of this classification to study symplectic forms on noncompact manifolds.

5. Inverse spectral geometry of quantum toric or semitoric systems

A quantum integrable system is given by a collection of semiclassical commuting self-adjoint operators $P_1, \ldots, P_n$ whose principal symbols form a classical integrable system. Some well known examples of quantum integrable systems are the quantum spherical pendulum, discussed by Cushman and Duistermaat in [28] and the "Champagne bottle" [23]. Quantum integrable systems given by Berezin-Toeplitz quantization are common in the physics literature. An example is the coupled angular momenta [69, 99]; see also [89, Section 8.3] for a proof that it is a Berezin-Toeplitz system.

The study of quantum integrable systems such as these examples fits into a general framework of ideas, which we briefly discuss next, which uses a combination of techniques from symplectic geometry and microlocal analysis to go back and forth between classical and quantum systems.

5.1. **The inverse spectral problem for quantum integrable systems.** The inverse problem we are going to discuss next belongs to a class of semiclassical inverse spectral questions which has been the focus of intense attention in recent years [26, 51, 52, 66, 84, 89, 108]. The problem goes back to pioneer works of Bérard [10], Brüning-Heintze [15], Colin de Verdière [24, 25], Duistermaat-Guillemin [35], and Guillemin-Sternberg [48], in the 1970s/1980s, and is closely related to inverse problems that are not directly semiclassical but use similar microlocal methods [115, 116].

The following statement which corresponds to [93, Conjecture 9.1], concerns only simple semitoric systems; we keep the original formulation, including the notation used therein: "a semitoric system J, H is determined up to symplectic equivalence by its semiclassical joint spectrum as $\hbar \to 0$. From any such spectrum one can construct explicitly the associated semitoric system, i.e. the set of points in $\mathbb{R}^2$ where on the x-basis we have the eigenvalues of $\hat{J}$, and on the vertical axis the eigenvalues of $\hat{H}$ restricted to the λ-eigenspace of $\hat{J}$". A motivation to develop the symplectic geometric results of the previous sections has been to shed light on this conjecture.

The proof strategy, outlined in [95], consists of detecting the symplectic invariants (Section 4.3) in the joint spectrum, and use them to construct the associated classical system. The semiclassical inverse spectral problem can be posed more generally as the question of how much information about the associated classical system can be obtained from the knowledge of the semiclassical spectrum of a quantum integrable system:

$$\overbrace{(P^1_\hbar)_{\hbar\in I}, \ldots, (P^n_\hbar)_{\hbar\in I}}^{\text{quantum integrable system}} \rightsquigarrow \overbrace{(X_\hbar)_{\hbar\in I} \subset \mathbb{R}^n}^{\text{one joint spectrum for each } \hbar} \underbrace{\rightsquigarrow}_{\text{possible?}} \overbrace{f_1, \ldots, f_n}^{\text{principal symbols of } (P^1_\hbar)_{\hbar\in I},\ldots,(P^n_\hbar)_{\hbar\in I}},$$

where $f_1, \ldots, f_n$ stand for the principal symbols of the operators on the left hand side.

5.2. **Progress towards the inverse spectral problem in the simple semitoric case.** Next we will briefly discuss how the general approach we mentioned earlier can be implemented to shed light on the inverse spectral conjecture for simple semitoric systems. We start by presenting the basic language of spectral geometry.

5.2.1. *The joint spectrum.* Let (M, ω) be a connected symplectic manifold of dimension 4 and let I be any subset of $(0, 1]$ which has as an accumulation point at 0. For any complex Hilbert space $\mathcal{H}$ we denote by $\mathcal{L}(\mathcal{H})$ the set of linear, possibly unbounded, self-adjoint operators on $\mathcal{H}$ with a dense domain.

Consider a sequence of Hilbert spaces $(\mathcal{H}_\hbar)_{\hbar\in I}$. A space Ψ of *semiclassical operators* is a subspace of $\prod_{\hbar\in I} \mathcal{L}(\mathcal{H}_\hbar)$, which contains the identity and is equipped with a weakly positive principal symbol map, which is a (normalized) $\mathbb{R}$-linear map $\sigma\colon \Psi \to \mathcal{C}^\infty(M, \mathbb{R})$ satisfying a *product formula* (if P, Q are in Ψ and if $P \circ Q$ is well defined and is in the space Ψ then we have an equality $\sigma(P \circ Q) = \sigma(P)\sigma(Q)$) and a *weak positivity condition* (if $\sigma(P) \geqslant 0$ there is a function $\hbar \mapsto \epsilon(\hbar)$ tending to zero as $\hbar \to 0$ and such that one has $P \geqslant -\epsilon(\hbar)$ for all $\hbar \in I$).

Two famous examples of such semiclassical operators are semiclassical pseudodifferential operators [32, 118] and semiclassical (or Berezin-)Toeplitz operators [14, 13, 17, 19, 20, 21, 73, 100].

If $P = (P_\hbar)_{\hbar\in I} \in \Psi$, the image $\sigma(P)$ is called the *principal symbol of* P. The principal symbol plays a fundamental role in the formulation of the inverse spectral problem for quantum integrable systems which we will give shortly.

Let $P = (P_\hbar)_{\hbar\in I}$ and $Q = (Q_\hbar)_{\hbar\in I}$ be semiclassical operators on $(\mathcal{H}_\hbar)_{\hbar\in I}$. We say that P and Q *commute* if for each $\hbar \in I$ the operators $P_\hbar$ and $Q_\hbar$ commute (note that in the case of unbounded self-adjoint operators, by definition this means that their projector-valued spectral measures commute). In this case one can define for each fixed value of the parameter $\hbar$ the so called *joint spectrum* of $(P_\hbar, Q_\hbar)$ as the support of the joint spectral measure. We denote it by $\mathrm{JointSpec}(P_\hbar,\, Q_\hbar)$.

If the Hilbert space $\mathcal{H}_\hbar$ is finite dimensional, or more generally, if the joint spectrum is discrete,

$$\mathrm{JointSpec}(P_\hbar, Q_\hbar) = \Big\{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \exists v \neq 0,\ P_\hbar v = \lambda_1 v,\ Q_\hbar v = \lambda_2 v\Big\}.$$

The *joint spectrum* of (P, Q) is, by definition, the collection of all $\mathrm{JointSpec}(P_\hbar, Q_\hbar), \hbar \in I$, and it is denoted by $\mathrm{JointSpec}(P,\, Q)$. For convenience we also view $\mathrm{JointSpec}(P,\, Q)$ as a set depending on $\hbar$.

All of these definitions extend easily to $2n$-dimensional manifolds and collections of commuting semiclassical operators $P_1, \ldots, P_n$.

5.2.2. *Bohr-Sommerfeld rules.* The abstract inverse spectral result for simple semitoric systems we are going to present in the next section requires that the operators follow some well known properties, known as the *Bohr-Sommerfeld rules.* These are easy to state, and we are going to formulate them next. They are also known to hold for integrable systems of pseudodifferential operators [16, 105], where the manifold M is a cotangent bundle, and for integrable systems of Berezin-Toeplitz operators on prequantizable compact symplectic manifolds [18].

Let $F = (f_1, f_2)\colon (M, \omega) \to \mathbb{R}^2$ be an integrable system on a connected symplectic manifold of dimension 4. Let P and Q be commuting semiclassical operators with principal symbols $f_1, f_2\colon M \to \mathbb{R}$. We say that $\mathrm{JointSpec}(P,\, Q)$ *satisfies the Bohr-Sommerfeld rules* if for every regular value c of F we can find a ball $\mathrm{B}(c, \epsilon_c)$ centered at c (of some radius as small as necessary), such that, on $\mathrm{B}(c, \epsilon_c)$, we have

$$\mathrm{JointSpec}(P, Q) = g_\hbar(2\pi\hbar\mathbb{Z}^2 \cap D) + \mathcal{O}(\hbar^2)$$

with $g_\hbar = g_0 + \hbar g_1$, where g_0, g_1 are smooth maps defined on a bounded open set $D \subset \mathbb{R}^2$, g_0 is a diffeomorphism into its image, $c \in g_0(D)$ and the components of $g_0^{-1} = (\mathcal{A}_1,\, \mathcal{A}_2)$ are such that $(\mathcal{A}_1 \circ F, \mathcal{A}_2 \circ F)$ form a basis of action variables in the sense that the two Hamiltonians in action-angle coordinates correspond to (ξ_1, ξ_2) in Section 2.5.

The displayed equation is a precise statement if one uses the Hausdorff distance d_H to compare how far a set is from another; with this in mind, if for instance $(A_\hbar)_{\hbar\in I}$ and $(B_\hbar)_{\hbar\in I}$ are sequences of uniformly bounded subsets of $\mathbb{R}^2$, then the notation $A_\hbar = B_\hbar + \mathcal{O}(\hbar^N)$ means that there is a constant $C > 0$ such that $\mathrm{d}_H(A_\hbar,\, B_\hbar) \leqslant C\hbar^N$ for all $\hbar \in I$, see [68, Sections 2.6 and 2.7].

5.2.3. *Results for all invariants with the exception of the twisting indices.* A *quantum semitoric integrable system* (P, Q) is given by two semiclassical commuting self-adjoint operators whose principal symbols form a classical semitoric integrable system. The quantum semitoric integrable system (P, Q) is *simple* if the corresponding classical semitoric integrable system is also simple.

The main result known for simple quantum semitoric systems [68, Theorem A] can be stated as follows. Let (P, Q) be a simple quantum semitoric system on M for which the Bohr-Sommerfeld rules hold. Then from the knowledge of $\mathrm{JointSpec}(P, Q) + \mathcal{O}(\hbar^2)$, one can recover all of the invariants of the symplectic classification in Sections 4.3 and 4.5, except for the twisting index k, namely:

- the convex polygon,

- the marked points $p_1, \ldots, p_n$ in the interior of the polygon, and
- for each point above the Taylor series $\sum_{i,j} a_{ij}\, x^i y^j$.

One needs to combine microlocal analytic techniques and symplectic geometry in order to recover from the joint spectrum the polygon, the interior marked points $p_1, \ldots, p_n$ and for each such point the label $\sum_{i,j} a_{ij}\, x^i y^j$. For this the crucial part is that we have a semiclassical spectrum, so we know the variation with respect to $\hbar$ for a sequence of values converging to 0. This is analyzed with the help of the Bohr-Sommerfeld rules.

The image of the joint map of principal symbols $(f_1, f_2)(M)$ is the limit as $\hbar \to 0$ of the joint spectrum. To recover the polygon invariant and the marked points is a more subtle task because here it matters the position of the images of the focus-focus singularities inside of $F(M)$, reflected in how and where the joint eigenvalues (the elements of the spectrum) concentrate, as $\hbar \to 0$. This step consists essentially of recovering the singular affine structure induced by the fibration given by the integrable system near the focus-focus singularities. For this we need to use Eliasson's linearization theorem and the work on singular affine structures induced by semitoric systems [91, 92, 107], among other ingredients. In this way one recovers the integral affine structure from the spectrum modulo $\mathcal{O}(\hbar^2)$.

In order to recover $\sum_{i,j} a_{i,j} x^i y^j$ at the focus-focus singularities one needs to use Bohr-Sommerfeld rules, semiclassical Fourier transforms and the lemma of non stationary phase. To recover the explicit form of $\sum_{i,j} a_{i,j} x^i y^j$ is essential to understand its construction in terms of the period maps τ_1 and τ_2 introduced in Section 4.3.

5.3. **Solution to the inverse spectral problem in the compact toric case.** The inverse spectral problem problem was solved in the positive in [22], for quantum toric integrable systems given by Berezing-Toeplitz operators on compact manifolds (prior to the developments in the semitoric case in Section 5.2.3). Recall from Section 3 that we always assume that toric integrable systems take place on compact connected manifolds, but the dimension of the manifold can be arbitrary.

In analogy with the semitoric case which we have just discussed, by a *quantum toric integrable system* we mean a quantum integrable system $P_1, \ldots, P_n$ on a symplectic $2n$-dimensional manifold (M, ω) such that the principal symbols $f_1, \ldots, f_n$ of $P_1, \ldots, P_n$ form a toric integrable system.

The proof strategy in this case is the same as in the semitoric case, with the simplification that there is only one symplectic invariant, the image $F(M) \subset \mathbb{R}^n$, as shown in Section 3.3 (see Figure 2).

The paper [22] contains a detailed analysis of the semiclassical spectral theory of toric integrable systems, and as a consequence of it, it is shown the semiclassical joint spectrum of $P_1, \ldots, P_n$ converges to the Delzant polytope $\Delta = F(M)$. This is why in this case the dimension of M is not essential (as it was in the semitoric case, where our results have been restricted to dimension 4); what is essential is that M is compact (there is a recent extension of Delzant's theorem to noncompact manifolds [61] which may be useful in generalizing [22] to the noncompact case). This result was motivated and preceded by [24, 25].

As a consequence of the semiclassical spectral theory developed therein it was shown that all toric integrable systems on compact connected manifolds can be quantized [22, Theorem 1.4]. That is, if $F := (f_1, \ldots, f_n) \colon M \to \mathbb{R}^n$ is a toric integrable system on a compact manifold then there exists a quantum toric integrable system $\psi_1, \ldots, \psi_n$ with associated principal symbols $f_1, \ldots, f_n$.

5.4. **Other results.** The result in Section 5.2.3 gives no information on the twisting index k for any of the points. Nonetheless it sheds light on the inverse spectral conjecture.

An inverse spectral result was given in [96] for a neighborhood of a singular fiber of an integrable system containing exactly one focus-focus singularity m. It was shown therein that the joint spectrum of the quantum integrable system near the singular fiber determines the Taylor series invariant at m, and hence determines symplectically a neighborhood of the fiber up to isomorphisms of integrable systems. This result was needed for the global result described in Section 5.2.3.

For general classes of self adjoint pseudodifferential operators and for Berezin-Toeplitz operators on compact manifolds it was shown in [97] that

$$\lim_{\hbar\to 0} \mathrm{JointSpec}(P_1, \ldots, P_d) = F(M) \subset \mathbb{R}^d,$$

where F is the joint map formed by the principal symbols $P_1, \ldots, P_d$. This convergence is illustrated in Figure 4 for the quantum spherical pendulum which shows, as originally illustrated in [97, Figure 1], that as the parameter $\hbar$ approaches the accumulation point 0 the semiclassical joint spectrum fills the inside of the curve in red color. This curve is the boundary of the image of the joint map of principal symbols $F = (f_1, f_2)\colon M \to \mathbb{R}^2$.

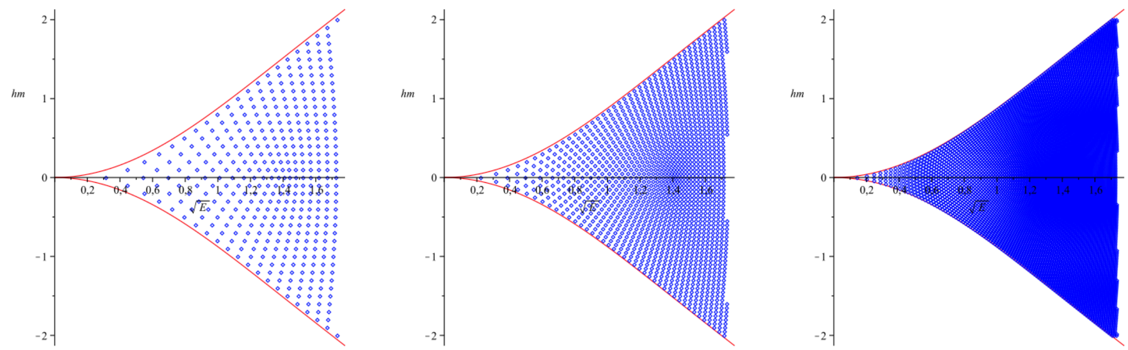

FIGURE 4. Joint spectrum of normalized quantum spherical pendulum as $\hbar \to 0$.

There are several related results in [89] which illustrate the general phenomena of convergence of the semiclassical spectrum, in the semiclassical limit, to the joint image of the principal symbols in $\mathbb{R}^n$ (which is often called the *classical spectrum*). These results were extended to the case of unitary operators in [70].

The articles [12, 95] contain a number of open questions and problems about classical and quantum integrable systems as well as some strategies to approach them.

5.5. **A question.** If the inverse spectral conjecture [93, Conjecture 9.1] holds as stated for simple semitoric systems, it would be interesting to see if a similar statement can be given for non-simple semitoric systems (in Section 4.6), and we can ask the question (essentially from [79, Section 1]):

Question. *Are there two non-simple quantum semitoric integrable systems* (P, Q) *and* (P', Q') *with the same joint spectrum (modulo* $\mathcal{O}(\hbar^2)$*) and such that the associated classical systems of principal symbols* (f_1, f_2) *and* (f_1', f_2') *are not isomorphic?*

It would be interesting to study this question for the integrable systems in [31].

A solution to this question would be helpful in understanding to what extent the structure or composition of the singular set of a classical integrable system (f_1, f_2) can be read off from the spectrum the quantum integrable system (P, Q) with principal symbols given by (f_1, f_2). We saw in Section 5.3 that all of the symplectic geometry of toric integrable systems on compact connected symplectic manifolds can be read off from the spectrum, up to isomorphisms.

Acknowledgements. I am very grateful to the Departamento de Álgebra, Geometría y Topología at the Universidad Complutense de Madrid for the excellent hospitality during a visit in December of 2019 and January 2020, when part of this paper was written. I would also like to thank Yohann Le Floch, Joseph Palmer, and Xiudi Tang, for comments on a preliminary version of this paper, and Xiudi Tang for help drawing the figures.

This paper is dedicated to Professor J.J. Duistermaat. I was fortunate to collaborate with him on four papers on symplectic group actions. His work on symplectic geometry and analysis continues to influence my own research on symplectic geometry of integrable systems and group actions. The articles [49, 83] discuss some of Professor Duistermaat's contributions.

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