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The spin Sutherland model of D_N type and its associated spin chain

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Abstract

In this paper we study the su(m) spin Sutherland (trigonometric) model of D_N type and its related spin chain of Haldane–Shastry type obtained by means of Polychronakos's freezing trick. As in the rational case recently studied by the authors, we show that these are new models, whose properties cannot be simply deduced from those of their well-known BC_N counterparts by taking a suitable limit. We identify the Weyl-invariant extended configuration space of the spin dynamical model, which turns out to be the N-dimensional generalization of a rhombic dodecahedron. This is in fact one of the reasons underlying the greater complexity of the models studied in this paper in comparison with both their rational and BC_N counterparts. By constructing a non-orthogonal basis of the Hilbert space of the spin dynamical model on which its Hamiltonian acts triangularly, we compute its spectrum in closed form. Using this result and applying the freezing trick, we derive an exact expression for the partition function of the associated Haldane–Shastry spin chain of D_N type.

Keywords: Exactly solvable spin models; Spin chains; Dunkl operators

1. Introduction

Recent studies have revealed that exactly solvable and integrable one-dimensional quantum many body systems with long-range interactions [1–8] are closely connected with a wide range of topics in modern physics as well as mathematics. In particular, this type of exactly solv-

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able systems have appeared as prototype models of various condensed matter systems exhibiting generalized exclusion statistics [8–10], quantum Hall effect [11] and quantum electric transport phenomena [12,13]. In the context of high-energy physics, the dynamics of particles or fields in the near-horizon region of black holes has been described through such integrable systems [14–16]. More recently, quantum integrable spin chains with long-range interaction have played a key role in calculating higher loop effects in the spectra of trace operators of planar $\mathcal{N}=4$ super Yang–Mills theory [17–19]. Furthermore, this type of quantum integrable systems are found to be connected with different areas of mathematics like random matrix theory [20], multivariate orthogonal polynomials [21–24], Dunkl operators [25,26], and Yangian quantum groups [27–30].

Due to such a large variety of potential applications, the construction of new quantum integrable systems with long-range interactions and the computation of their exact solutions have emerged as an important area of activity in the current literature. The study of this type of systems with dynamical degrees of freedom was pioneered by Calogero, who found the exact spectrum of an N-particle quantum system on the line with two-body interactions inversely proportional to the square of the distances and subject to a confining harmonic potential [1]. An exactly solvable trigonometric variant of this rational Calogero model was subsequently proposed by Sutherland [2,3]. The particles in this so-called Sutherland model move on a circle, with two-body interactions proportional to the inverse square of their chord distances. In a parallel development, Haldane and Shastry found an exactly solvable quantum spin-$\frac{1}{2}$ chain with long-range interactions [5,6]. The lattice sites of this su(2) Haldane–Shastry (HS) spin chain are equally spaced on a circle, all spins interacting with one another through pairwise exchange interactions inversely proportional to the square of their chord distances. A close relation between the HS chain with su(m) spin degrees of freedom and the su(m) spin generalization of the Sutherland model [31–33] was subsequently established by using the so-called "freezing trick" [7,34]. More precisely, it is found that in the strong coupling limit the particles in the spin Sutherland model "freeze" at the coordinates of the equilibrium position of the scalar part of the potential, and the dynamical and spin degrees of freedom decouple. The equilibrium coordinates coincide with the equally spaced lattice points of the HS spin chain, so that the decoupled spin degrees of freedom are governed by the Hamiltonian of the su(m) HS model. Moreover, in this freezing limit the conserved quantities of the spin Sutherland model immediately yield those of the HS spin chain, thereby explaining its complete integrability. Application of this freezing trick to the rational Calogero model with spin degrees of freedom leads to a new integrable spin chain with long-range interaction [7]. The sites of this chain—commonly known in the literature as the Polychronakos or Polychronakos–Frahm (PF) spin chain—are unequally spaced on a line, and in fact coincide with the zeros of the Hermite polynomial of degree N [35]. By applying the freezing trick, the exact partition functions of the PF and HS spin chains have also been exactly computed [36,37].

The above mentioned type of quantum integrable systems can be generalized to form a much wider class by taking advantage of their hidden mathematical structure. Indeed, Olshanetsky and Perelomov established the existence of an underlying A_N root system structure for both the spinless Calogero and Sutherland models, and constructed generalizations thereof associated with any (extended) root system [4]. Spin generalizations of the BC_N Calogero and Sutherland models have also been proposed, and various properties of the related lattice models of HS type have been studied with the help of the freezing trick [38–43]. Among the other classical root systems, the exceptional ones are comparatively less interesting, since their associated models consist of at most 8 particles. Until recently, the B_N, C_N and D_N Calogero–Sutherland models (particularly the corresponding spin models) have been largely ignored, probably due to the fact that they were believed to be simple limiting cases of their BC_N counterparts. However, in a recent paper [44] the present authors have computed the spectrum of the su(m) spin Calogero model of D_N type, thereby showing that this model is in fact a singular limit of its BC_N version. More precisely, it is well known that the Hilbert space of the spin Calogero model associated with the BC_N root system can be constructed from the Hilbert space of an auxiliary differential-difference operator by using a single projector. In contrast, it is found that two independent projectors of BC_N type with opposite "chiralities" are needed to construct the Hilbert space of the D_N-type spin Calogero model from that of the corresponding auxiliary operator [44]. Consequently, the Hilbert space of the latter model can be expressed as a direct sum of the Hilbert spaces associated with two different BC_N models with opposite chiralities. This explains why the spectrum of the D_N model cannot be obtained as the limit of its BC_N counterpart when one of the coupling constants tends to zero. In Ref. [44] we also studied the spin chain associated with the D_N-type spin Calogero model, showing that its Hamiltonian differs from the limit of its BC_N analog by a term which can be interpreted as an impurity interaction at one end of the chain. By applying Polychronakos's freezing trick we were also able to compute the chain's partition function in closed form, showing again that its spectrum markedly differs from that of its BC_N counterpart.

In this paper we study the trigonometric variant of the D_N-type spin Calogero model and its freezing limit, i.e., the D_N-type spin Sutherland model and its related spin chain. Just as in the rational case, the Hamiltonian of the D_N-type spin Sutherland model can be formally obtained as a certain limit of its BC_N counterpart. However, the relation between these models turns out to be even more subtle than in the rational case. Roughly speaking, this is due to the fact that the Weyl-invariant extended configuration space of the D_N model—which turns out to be the N-dimensional generalization of a rhombic dodecahedron—does not coincide with that of the BC_N model, which is simply a hypercube. As a consequence, the (scaled) Fourier basis of the Hilbert space of the BC_N model's auxiliary operator no longer spans a complete set of the Hilbert space of the auxiliary operator of the D_N model. This entails an additional level of difficulty (but also of interest) by comparison with the rational case, for which the auxiliary operators of the BC_N and D_N models share the same Hilbert space. On the other hand, as in the rational case, we shall still need two projectors of BC_N type with opposite chiralities in order to construct the Hilbert space of the D_N spin model from that of its corresponding auxiliary operator. Therefore, the Hilbert space of the D_N spin model actually consists of *four*—and not two, as in the rational case—different sectors, characterized by their chirality and parity under reflections of the particles' coordinates. This fundamental difference explains why the spectrum of the D_N-type spin Sutherland model is essentially different from that of its BC_N counterpart. It also accounts for the greater complexity of the partition function of the associated chain of D_N type (which we have also computed in closed form by means of the freezing trick) compared to its BC_N version studied in Ref. [42].

The paper is organized as follows. In Section 2 we introduce the Hamiltonians H, H_{sc} and $\mathcal{H}$ of the D_N-type spin Sutherland model, its scalar version, and the associated spin chain of HS type, respectively. We show that the sites of this chain, defined as the coordinates of the (unique) equilibrium point of the scalar part of the spin Hamiltonian in the principal Weyl alcove of the D_N root system, can be expressed in terms of the roots of a suitable Jacobi polynomial. Using this characterization, we prove that the Hamiltonian $\mathcal{H}$ differs from the limit of its BC_N counterpart by a spin reversing term at each end of the chain.

Section 3 is devoted to the computation of the spectrum of the Hamiltonians H and H_{sc} using an auxiliary *scalar* differential-difference operator H'. In order to improve the clarity of the exposition, we have divided it into four subsections. In the first one, we show that for any one-dimensional representation π of the D_N Weyl group $\mathfrak{W}$, the Hamiltonians H and $H_{\rm sc}$ are equivalent (isospectral) to their π-symmetric extensions to the $\mathfrak{W}$-orbit C of the configuration space. The representation π is then uniquely determined by requiring that the action of the extension of H (resp. $H_{\rm sc}$) coincide with that of $H' \otimes \mathbb{1}$ (resp. H') on the subspace of $L^2(C) \otimes \Sigma$ (resp. $L^2(C)$) of π-symmetric functions, where Σ denotes the su(m) spin space. This property, together with the fact that H' commutes with the projector onto π-symmetric functions, enable us to evaluate the spectra of H and $H_{\rm sc}$ by triangularizing the simpler operator H'. The first step in this direction is to construct a (non-orthogonal) basis of $L^2(C)$, which includes as a proper subset the limit of the basis of $L^2([-\frac{\pi}{2}, \frac{\pi}{2}])$ for the BC_N model. As a by-product of this construction, we show that the translations of C generate a tessellation of the N-dimensional Euclidean space. By expressing H' as a sum of squares of a suitable family of Dunkl operators, we show that when the above basis of $L^2(C)$ is ordered in an appropriate way the action of H' becomes triangular. Using this basis, in the last subsection we construct a corresponding (non-orthogonal) basis of the subspace of π-symmetric functions $L^2(C) \otimes \Sigma$ (resp. $L^2(C)$) in which the action of H (resp. $H_{\rm sc}$) is also triangular. This completes the calculation of the spectra of the Hamiltonians H and $H_{\rm sc}$.

In Section 4 we compute the partition function $\mathcal{Z}$ of the spin chain $\mathcal{H}$ using Polychronakos's freezing trick. More precisely, we evaluate the partition function $\mathcal{Z}$ as the large coupling constant limit of the quotient of the partition functions of H and $H_{\rm sc}$. The resulting formula exhibits a greater complexity than its rational counterpart and, in particular, cannot be expressed in a simple way in terms of the partition functions of the BC_N trigonometric spin chains. We end up this section by showing that the latter expression for the partition function is indeed a polynomial in $q \equiv \mathrm{e}^{-1/(k_{\rm B}T)}$, as should be the case for a finite system with nonnegative integer energies. Finally, a brief summary of the paper's results is presented in Section 5.

2. The models

Our starting point will be a brief review of the su(m) spin Sutherland model of BC_N type, with Hamiltonian [39,42]

$$H^{(\mathrm{B})} = -\sum_i \partial_{x_i}^2 + a\sum_{i\neq j}\big[\sin^{-2} x_{ij}^-(a+S_{ij}) + \sin^{-2} x_{ij}^+(a+\tilde{S}_{ij})\big] + b\sum_i \sin^{-2} x_i(b-\epsilon S_i) + b'\sum_i \cos^{-2} x_i\big(b'-\epsilon S_i\big). \tag{1}$$

Here the sums run from 1 to N (as always hereafter, unless otherwise stated), $a, b, b' > 1/2$, $\epsilon = \pm 1$, and $x_{ij}^{\pm} = x_i \pm x_j$. The operators S_{ij} and S_i in the latter equation act on the finite-dimensional Hilbert space

$$\Sigma = \big\langle |s_1, \ldots, s_N\rangle \;\big|\; s_i = -M, -M+1, \ldots, M\big\rangle, \quad M \equiv \frac{m-1}{2} \in \frac{\mathbb{N}}{2}, \tag{2}$$

associated to the particles' internal degrees of freedom, as follows:

$$\begin{aligned} S_{ij}|s_1, \ldots, s_i, \ldots, s_j, \ldots, s_N\rangle &= |s_1, \ldots, s_j, \ldots, s_i, \ldots, s_N\rangle, \\ S_i|s_1, \ldots, s_i, \ldots, s_N\rangle &= |s_1, \ldots, -s_i, \ldots, s_N\rangle. \end{aligned} \tag{3}$$

We have also used the customary notation $\tilde{S}_{ij} = S_i S_j S_{ij}$. Note that the spin operators S_{ij} and S_i can be expressed in terms of the fundamental su(m) spin generators J_k^α at the site k (with the normalization $\mathrm{tr}(J_k^\alpha J_k^\gamma) = \frac{1}{2}\delta^{\alpha\gamma}$) as

$$S_{ij} = \frac{1}{m} + 2\sum_{\alpha=1}^{m^2-1} J_i^\alpha J_j^\alpha, \qquad S_i = \sqrt{2m}\,J_i^1.$$

Due to the singularities at the hyperplanes $x_i \pm x_j = k\pi$, $x_i = k\pi$ and $x_i = \frac{\pi}{2} + k\pi$ (with $1 \leqslant i < j \leqslant N$ and $k \in \mathbb{Z}$), the configuration space of the Sutherland Hamiltonian (1) can be taken as the principal Weyl alcove

$$A^{(\mathrm{B})} = \left\{\mathbf{x} \in \mathbb{R}^N\colon\ 0 < x_1 < x_2 < \cdots < x_N < \frac{\pi}{2}\right\} \tag{4}$$

of the BC_N root system. The spectrum of the spin model (1) was computed in Ref. [42] by constructing a suitable basis of its Hilbert space in which the Hamiltonian $H^{(\mathrm{B})}$ is represented by a triangular matrix.

Applying the so-called freezing trick [7] to the Hamiltonian (1) with $b = \beta a$ and $b' = \beta' a$ one obtains the su(m) Haldane–Shastry (antiferromagnetic) spin chain of BC_N type, whose Hamiltonian we shall take as

$$\begin{aligned}\mathcal{H}^{(\mathrm{B})} = {} & \frac{1}{2}\sum_{i<j}\left[\sin^{-2}\theta_{ij}^-(1+S_{ij}) + \sin^{-2}\theta_{ij}^+(1+\tilde{S}_{ij})\right] \\ & + \frac{1}{4}\sum_i\left(\beta\sin^{-2}\theta_i + \beta'\cos^{-2}\theta_i\right)(1-\epsilon S_i).\end{aligned} \tag{5}$$

Here $\theta_{ij}^\pm = \theta_i \pm \theta_j$, where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_N)$ is the unique equilibrium [41] in the set $A^{(\mathrm{B})}$ of the scalar potential

$$U^{(\mathrm{B})}(\mathbf{x}) = \sum_{i\neq j}\left(\sin^{-2}x_{ij}^- + \sin^{-2}x_{ij}^+\right) + \sum_i\left(\beta^2\sin^{-2}x_i + \beta'^2\cos^{-2}x_i\right). \tag{6}$$

As shown in the latter reference, the lattice sites θ_i are related to the zeros ζ_i of the Jacobi polynomial $P_N^{(\beta-1,\beta'-1)}$ by

$$\zeta_i = \cos(2\theta_i). \tag{7}$$

The spin chain (5) was studied in Ref. [42], where its partition function was computed in closed form with the help of the freezing trick.

The Hamiltonian H of the su(m) spin Sutherland model of D_N type is defined by setting $b = b' = 0$ in Eq. (1), i.e.,

$$H = -\sum_i \partial_{x_i}^2 + a\sum_{i\neq j}\left[\sin^{-2}x_{ij}^-(a+S_{ij}) + \sin^{-2}x_{ij}^+(a+\tilde{S}_{ij})\right]. \tag{8}$$

The configuration space A of the D_N model (8) is determined by the hard-core singularities of the Hamiltonian on the hyperplanes $x_i \pm x_j = k\pi$ (with $i \neq j$ and $k \in \mathbb{Z}$). More precisely, we shall take as A the open subset of $\mathbb{R}^N$ defined by the inequalities

$$0 < x_i \pm x_j < \pi, \quad 1 \leqslant j < i \leqslant N. \tag{9}$$

If $N > 2$ (as we shall assume hereafter), it is straightforward to check that this set can be equivalently expressed as

$$A = \left\{\mathbf{x} \in \mathbb{R}^N\colon\ |x_1| < x_2 < \cdots < x_N < \pi - x_{N-1}\right\}, \tag{10}$$

which is again the principal Weyl alcove of the D_N root system

$$\frac{1}{\pi}(\pm\mathbf{e}_i \pm \mathbf{e}_j), \quad 1 \leqslant i < j \leqslant N. \tag{11}$$

The points $\mathbf{x} \in A$ clearly satisfy the inequalities

$$0 < x_2 < \cdots < x_{N-1} < \pi/2, \quad x_1 > -\pi/2,\ x_N < \pi; \tag{12}$$

note, in particular, that the set A properly contains the BC_N configuration space (4).

Similarly (cf. (5) and (6)), we define the Hamiltonian of the su(m) HS spin chain of D_N type as

$$\mathcal{H} = \frac{1}{2}\sum_{i<j}\big[\sin^{-2}\vartheta_{ij}^{-}(1+S_{ij}) + \sin^{-2}\vartheta_{ij}^{+}(1+\tilde{S}_{ij})\big], \tag{13}$$

where the lattice sites ϑ_i are the coordinates of the unique minimum $\boldsymbol{\vartheta}$ in the set A of the scalar potential

$$U(\mathbf{x}) = \sum_{i\neq j}\big(\sin^{-2}x_{ij}^{-} + \sin^{-2}x_{ij}^{+}\big). \tag{14}$$

Heuristically, the relation between the spin dynamical model (8) and its associated spin chain (13) can be explained as follows. Defining the coordinate-dependent matrix multiplication operator

$$h(\mathbf{x}) = \frac{1}{2}\sum_{i<j}\big[\sin^{-2}x_{ij}^{-}(1+S_{ij}) + \sin^{-2}x_{ij}^{+}(1+\tilde{S}_{ij})\big],$$

the spin Hamiltonian (8) can be decomposed as

$$H = H_{\mathrm{sc}} + 4a h(\mathbf{x}), \tag{15}$$

where

$$H_{\mathrm{sc}} = -\sum_i \partial_{x_i}^2 + a(a-1)U(\mathbf{x}) \tag{16}$$

is the Hamiltonian of the scalar Sutherland model of D_N type. Thus, for sufficiently large a all the eigenfunctions of H_{sc} are sharply peaked around the unique minimum $\boldsymbol{\vartheta}$ of the scalar potential U in the set A [45]. Hence, if $\varphi_i(\mathbf{x})$ is an eigenfunction of H_{sc} with energy E_i^{sc} and $|\sigma_j\rangle$ is an eigenstate of the chain $\mathcal{H}$ with eigenvalue $\mathcal{E}_j$, for $a \gg 1$ we have

$$h(\mathbf{x})\varphi_i(\mathbf{x})|\sigma_j\rangle \simeq \varphi_i(\mathbf{x})h(\boldsymbol{\vartheta})|\sigma_j\rangle \equiv \varphi_i(\mathbf{x})\mathcal{H}|\sigma_j\rangle = \mathcal{E}_j\varphi_i(\mathbf{x})|\sigma_j\rangle.$$

By Eq. (15), H is approximately diagonal in the basis with elements $\varphi_i(\mathbf{x})|\sigma_j\rangle$, and its eigenvalues E_{ij} satisfy

$$E_{ij} \simeq E_i^{\mathrm{sc}} + 4a\mathcal{E}_j, \quad a \gg 1. \tag{17}$$

It was shown in Ref. [41] that the scalar potential $U(\mathbf{x})$ has a unique minimum in the configuration space A, which coincides with the unique maximum in this set of the ground state wave function of the scalar Sutherland Hamiltonian of D_N type (16), given by

$$\rho(\mathbf{x}) = \prod_{i<j}\big|\sin x_{ij}^{-}\sin x_{ij}^{+}\big|^{a}. \tag{18}$$

The lattice sites ϑ_i of the chain (13) are thus the unique solution in A of the nonlinear system

$$\sum_{j;\,j\neq i}\bigl(\cot\vartheta_{ij}^{-}+\cot\vartheta_{ij}^{+}\bigr)=0,\quad 1\leqslant i\leqslant N. \tag{19}$$

As in the BC_N case, we define the variables ξ_i by

$$\xi_i=\cos(2\vartheta_i),\quad 1\leqslant i\leqslant N.$$

Note that, since $\boldsymbol{\vartheta}\in A$, we obviously have

$$1\geqslant\xi_1>\xi_2>\cdots>\xi_{N-1}>\xi_N\geqslant-1. \tag{20}$$

In terms of the new coordinates ξ_i, the system (19) can be written as

$$\bigl(1-\xi_i^2\bigr)\sum_{j;\,j\neq i}\frac{1}{\xi_i-\xi_j}=0,\quad 1\leqslant i\leqslant N. \tag{21}$$

Since $\xi_1-\xi_j>0$ for all $j>1$ and $\xi_N-\xi_j<0$ for all $j<N$, from Eq. (21) it immediately follows that $\xi_1^2=\xi_N^2=1$, so that $\xi_1=-\xi_N=1$ by Eq. (20). Substituting into (21) we obtain the following system for the remaining coordinates $\xi_2,\dots,\xi_{N-1}$:

$$\bigl(1-\xi_i^2\bigr)\sum_{\substack{j=2\\ j\neq i}}^{N-1}\frac{1}{\xi_i-\xi_j}=2\xi_i,\quad 2\leqslant i\leqslant N-1. \tag{22}$$

Note that the latter system is invariant under the transformation $\xi_i\mapsto-\xi_i$, so that (by uniqueness) $\xi_i=\xi_{N+1-i}$. Comparing (22) with the system

$$2\bigl(1-\zeta_i^2\bigr)\sum_{\substack{j=1\\ j\neq i}}^{N'}\frac{1}{\zeta_i-\zeta_j}=\beta-\beta'+\bigl(\beta+\beta'\bigr)\zeta_i,\quad 1\leqslant i\leqslant N', \tag{23}$$

satisfied by the zeros ζ_i ($i=1,\dots,N'$) of the Jacobi polynomial $P_{N'}^{(\beta-1,\beta'-1)}$ (cf. Ref. [46]), we conclude that the coordinates $\xi_2,\dots,\xi_{N-1}$ are the zeros of $P_{N-2}^{(1,1)}$. (Note that $P_{N-2}^{(1,1)}$ is proportional to the Gegenbauer polynomial $C_{N-2}^{(3/2)}$, cf. Ref. [47].) In terms of the original site coordinates ϑ_i we have

$$0=\vartheta_1<\vartheta_2<\cdots<\vartheta_{N-1}<\vartheta_N=\frac{\pi}{2},$$

with $P_{N-2}^{(1,1)}(\cos(2\vartheta_i))=0$ for $i=2,\dots,N-1$. Note that Eqs. (21)–(23) also yield an alternative characterization of the coordinates ξ_i as the N roots of the Jacobi polynomial $P_N^{(-1,-1)}$, which was to be expected, since the potential $U^{(\mathrm{B})}$ in Eq. (6) reduces to the D_N potential U when $\beta=\beta'=0$. The equivalence of both characterizations of the site coordinates is easily established with the help of the identity $4P_N^{(-1,-1)}(t)=(t^2-1)P_{N-2}^{(1,1)}(t)$, cf. [48].

We shall next discuss the precise relation between the D_N spin chain Hamiltonian (13) and the limit as $(\beta,\beta')\to0$ of its BC_N counterpart (5). To this end, we use the trigonometric identities

$$\sin^{-2}\theta_i=\frac{2}{1-\zeta_i},\qquad\cos^{-2}\theta_i=\frac{2}{1+\zeta_i},$$

and note that as $(\beta,\beta')\to 0$ all the roots ζ_i of the Jacobi polynomial $P_N^{(\beta-1,\beta'-1)}$ tend to the corresponding roots ξ_i of $P_N^{(-1,-1)}$. Thus all the terms in the last sum in the Hamiltonian (5) tend to zero as $(\beta,\beta')\to(0,0)$, except the first and the last one. In order to evaluate the limit of these two terms, we divide (23) by $1\pm\zeta_i$ and sum the resulting equation over i, obtaining

$$2\beta\sum_i\frac{1}{1-\zeta_i}=2\beta'\sum_i\frac{1}{1+\zeta_i}=N\big(\beta+\beta'+N-1\big).$$

Hence

$$\lim_{(\beta,\beta')\to 0}\frac{2\beta}{1-\zeta_1}=\lim_{(\beta,\beta')\to 0}\frac{2\beta'}{1+\zeta_N}=N(N-1).\tag{24}$$

Since $\zeta_i\to\xi_i$ as $(\beta,\beta')\to 0$ and $\theta_i,\vartheta_i\in A$ (recall that $A^{(B)}\subset A$), we have $\lim_{(\beta,\beta')\to 0}\theta_i=\vartheta_i$ for all $i=1,\dots,N$. From Eqs. (5), (13) and (24) it immediately follows that

$$\lim_{(\beta,\beta')\to 0}\mathcal{H}^{(\mathrm{B})}=\mathcal{H}+\frac12 N(N-1)\bigg[1-\frac{\epsilon}{2}(S_1+S_N)\bigg].\tag{25}$$

Thus the limit as $(\beta,\beta')\to 0$ of the Hamiltonian of the HS chain of BC_N type yields its D_N analog, plus an additional term which can be interpreted as an "impurity" at both ends of the latter chain.

3. Spectrum of the dynamical models

The aim of this section is to compute the spectra of the su(m) spin Sutherland model of D_N type (8) and its scalar counterpart (16). In order to facilitate this computation, we introduce the auxiliary *scalar* operator

$$H'=-\sum_i\partial_{x_i}^2+a\sum_{i\neq j}\big[\sin^{-2}x_{ij}^-(a-K_{ij})+\sin^{-2}x_{ij}^+(a-\tilde K_{ij})\big],\tag{26}$$

where K_{ij} and K_i are coordinate permutation and sign reversing operators, defined by

$$(K_{ij}f)(x_1,\dots,x_i,\dots,x_j,\dots,x_N)=f(x_1,\dots,x_j,\dots,x_i,\dots,x_N),$$
$$(K_i f)(x_1,\dots,x_i,\dots,x_N)=f(x_1,\dots,-x_i,\dots,x_N),$$

and $\tilde K_{ij}\equiv K_iK_jK_{ij}$.

3.1. Extensions of H and H_{sc}

Due to the character of their singularities, the operators H and H_{sc} are naturally defined on suitable dense subspaces of the Hilbert spaces $L^2(A)\otimes\Sigma$ and $L^2(A)$, respectively. On the other hand, the appearance in the RHS of Eq. (26) of the generators K_iK_j and K_{ij} of the Weyl group $\mathfrak{W}$ of the D_N root system entails that the auxiliary operator H' is defined instead on a dense subspace of $L^2(C)$, where $C\equiv\mathfrak{W}(A)$. One of the key ingredients of the method we shall use consists in replacing the operators H and H_{sc} by suitable equivalent (isospectral) extensions $\tilde H$ and $\tilde H_{\mathrm{sc}}$ thereof to appropriate subspaces of $L^2(C)\otimes\Sigma$ and $L^2(C)$, such that $\tilde H=H'\otimes\mathbb{1}$ and $\tilde H_{\mathrm{sc}}=H'$ in the latter subspaces.

We shall start by showing that the set C is explicitly given by

$$C = \{\mathbf{x} \in \mathbb{R}^N:\ 0 < |x_i \pm x_j| < \pi,\ 1 \leqslant i < j \leqslant N\}, \tag{27}$$

a characterization which will prove useful in what follows. Recall, to this end, that $\mathfrak{W}$ is generated by coordinate permutations and sign reversals of an *even* number of coordinates [49]. Since A is defined by the inequalities (9), it is obvious that $C \subset C^*$, where C^* denotes the RHS of Eq. (27). Thus, we need only show that $C^* \subset C$, or equivalently, that for every $\mathbf{x}^* \in C^*$ there is an element $W \in \mathfrak{W}$ such that $W\mathbf{x}^* \in A$. To this end, we shall make repeated use of the following elementary fact:

$$\mathbf{x} \in C^*, \qquad |x_1| < x_2 < \cdots < x_N \quad \Longrightarrow \quad \mathbf{x} \in A. \tag{28}$$

Suppose, then, that $\mathbf{x}^* \in C^*$. By reversing the sign of an even number of coordinates and applying a suitable permutation, we can transform $\mathbf{x}^*$ into another element $\mathbf{y} \in \mathfrak{W}(C^*) = C^*$ satisfying

$$y_1 < y_2 < \cdots < y_N \quad \text{and} \quad y_i > 0, \quad i = 2, \ldots, N.$$

If $y_1 + y_2 > 0$, then $|y_1| < y_2 < \cdots < y_N$, and hence $\mathbf{y} \in A$ by Eq. (28). Suppose, on the other hand, that $y_1 + y_2 < 0$, so that

$$y_1 < 0 < y_2 < \cdots < y_N.$$

Calling $z_1 = -y_2$, $z_2 = -y_1$ and $z_i = y_i$ for $i \geqslant 3$, we have $\mathbf{z} \in C^*$ and

$$z_1 + z_2 = -(y_1 + y_2) > 0, \quad z_1 < z_2 \quad \Longrightarrow \quad |z_1| < z_2.$$

If $z_2 < z_3$, then $\mathbf{z} \in A$ by Eq. (28). Otherwise, the inequalities $z_1 + z_3 = y_3 - y_2 > 0$ and $z_3 - z_1 = y_3 + y_2 > 0$ imply that

$$|z_1| < z_3 < \cdots < z_2 < \cdots < z_N.$$

Applying a suitable permutation to $\mathbf{z}$, we obtain a new element $\mathbf{u} \in C^*$ satisfying $|u_1| < u_2 < \cdots < u_N$, which belongs to A again by Eq. (28).

Remark 1. Note that the analogous set $C^{(\mathrm{B})}$ for the BC_N root system, which is simply the hypercube $(-\frac{\pi}{2}, \frac{\pi}{2})^N$ minus the singular hyperplanes $x_i \pm x_j = 0$, $1 \leqslant i \leqslant j \leqslant N$ (cf. Ref. [42]), is clearly contained in C by Eq. (27).

Remark 2. For $N = 3$, the set $\overline{C}$ is a *rhombic dodecahedron* (a zonohedron with 12 equal rhombic faces) centered at the origin [50], with edge length $\sqrt{3}\pi/2$.

Our next aim is to replace the operator H by an isospectral extension $\tilde{H}$ thereof acting on suitably (anti)symmetrized wave functions defined on the set C. If the extension is appropriately chosen, we shall see that $\tilde{H} = H' \otimes \mathbb{1}$ on the Hilbert space of $\tilde{H}$, so that the computation of the spectrum of H reduces to the analogous (but considerably simpler, in practice) task for H'.

Before proceeding with our construction, we need to introduce some additional notation. Given an element W of the Weyl group of D_N type $\mathfrak{W}$ and a factorized spin function $\varphi|\mathbf{s}\rangle \in L^2(C) \otimes \Sigma$, where $|\mathbf{s}\rangle \equiv |s_1, \ldots, s_N\rangle$ is an element of the canonical spin basis, we define the action of W on $\varphi|\mathbf{s}\rangle$ in the usual way:

$$W(\varphi|\mathbf{s}\rangle) = (\varphi \circ W^{-1})|W\mathbf{s}\rangle. \tag{29}$$

Extending this definition by linearity to the whole Hilbert space $L^2(C) \otimes \Sigma$, we obtain an action of $\mathfrak{W}$ (in fact, a representation) on the latter space. Let now $\pi : \mathfrak{W} \to \mathbb{C}$ denote a one-dimensional representation of $\mathfrak{W}$; of course, since $\mathfrak{W}$ is generated by reflections, we have $\pi(\mathfrak{W}) \subset \{-1, 1\}$. The *symmetrizer* Λ_π associated with π is the linear operator defined on $L^2(C) \otimes \Sigma$ by

$$\Lambda_\pi = \frac{1}{|\mathfrak{W}|} \sum_{W \in \mathfrak{W}} \pi(W)W, \tag{30}$$

where $|\mathfrak{W}| = 2^{N-1}N!$ is the order of $\mathfrak{W}$. By construction, we have

$$\Phi \in \Lambda_\pi\big(L^2(C) \otimes \Sigma\big), \qquad W \in \mathfrak{W} \quad \Longrightarrow \quad W\Phi = \pi(W)\Phi, \tag{31}$$

so that Λ_π is the projector onto states with well-defined parity $\pi(W)$ with respect to any transformation W in the Weyl group $\mathfrak{W}$.

Given a factorized spin function $\Psi = \psi|\mathbf{s}\rangle \in L^2(A) \otimes \Sigma$, we define its π*-symmetric extension* $\tilde{\Psi} \in \Lambda_\pi(L^2(C) \otimes \Sigma)$ by

$$\tilde{\Psi}(\mathbf{x}) = \pi(W_\mathbf{x})\psi\big(W_\mathbf{x}^{-1}\mathbf{x}\big)|W_\mathbf{x}\mathbf{s}\rangle, \quad \mathbf{x} \in C, \tag{32}$$

where $W_\mathbf{x}$ denotes the *unique* element of $\mathfrak{W}$ such that $W_\mathbf{x}^{-1}\mathbf{x} \in A$. As usual, the action of $\tilde{}$ is extended to $L^2(A) \otimes \Sigma$ by linearity. It is easy to see that $\tilde{\Psi}$ is the unique extension of Ψ to C which has well defined parity $\pi(W)$ under any transformation $W \in \mathfrak{W}$. In view of Eqs. (29)–(30), with a slight abuse of notation we can write

$$\tilde{\Psi} = |\mathfrak{W}| \cdot \Lambda_\pi(\Psi\chi_A), \tag{33}$$

where χ_A is the characteristic function of A.

The extension $\tilde{}: L^2(A) \otimes \Sigma \to \Lambda_\pi(L^2(C) \otimes \Sigma)$ is an invertible linear operator, its inverse being the restriction operator $\hat{}: \Lambda_\pi(L^2(C) \otimes \Sigma) \to L^2(A) \otimes \Sigma$ defined by $\hat{\Phi} = \Phi|_A$. Indeed, the linearity of the operator $\tilde{}$ is obvious. As to its invertibility, note first of all that if $\Psi \in L^2(A) \otimes \Sigma$ by Eq. (32) we have $\hat{\tilde{\Psi}} = \Psi$, since $W_\mathbf{x}$ is the identity when $\mathbf{x}$ belongs to A. On the other hand, if $\Phi \in \Lambda_\pi(L^2(C) \otimes \Sigma)$ using Eqs. (30) and (33) we obtain

$$\tilde{\hat{\Phi}} = \sum_{W \in \mathfrak{W}} \pi(W)W(\hat{\Phi}\chi_A) = \sum_{W \in \mathfrak{W}} \pi(W)W(\Phi\chi_A) = \sum_{W \in \mathfrak{W}} \pi(W)W(\Phi)\big(\chi_A \circ W^{-1}\big),$$

and hence, by Eq. (31),

$$\tilde{\hat{\Phi}}(\mathbf{x}) = \Phi(\mathbf{x}) \sum_{W \in \mathfrak{W}} \chi_A\big(W^{-1}\mathbf{x}\big) = \Phi(\mathbf{x})\chi_A\big(W_\mathbf{x}^{-1}\mathbf{x}\big) = \Phi(\mathbf{x}).$$

Given a one-dimensional representation π of $\mathfrak{W}$ and a linear operator T acting on $L^2(A) \otimes \Sigma$, it is natural to define its π*-symmetric extension* $\tilde{T}_\pi$ to the Hilbert space $\Lambda_\pi(L^2(C) \otimes \Sigma)$ by the prescription

$$\tilde{T}_\pi\Phi = (T\hat{\Phi})^{\tilde{}}, \qquad \Phi \in \Lambda_\pi\big(L^2(C) \otimes \Sigma\big). \tag{34}$$

By the invertibility of the $\tilde{}$ operator, we have

$$\tilde{T}_\pi = \tilde{} \circ T \circ (\tilde{})^{-1},$$

so that the operators T and $\tilde{T}_\pi$ are isospectral.

We now seek to find a suitable one-dimensional representation π of $\mathfrak{W}$ such that the π-symmetric extension of H coincides with the restriction of $H' \otimes \mathbb{1}$ to $\Lambda_\pi(L^2(C) \otimes \Sigma)$. In view of Eqs. (8) and (26), it suffices that π satisfy

$$K_iK_j\Lambda_\pi = S_iS_j\Lambda_\pi, \qquad K_{ij}\Lambda_\pi = -S_{ij}\Lambda_\pi.$$

Hence $\pi(W)$ should be defined as the sign of the permutation part of $W \in \mathfrak{W}$. From now on, when dealing with the spin Hamiltonian (8) we shall take π as above, and drop the subscript π from Λ_π and $\tilde{H}_\pi$.

Turning now to the scalar Hamiltonian H_{sc}, by Eqs. (14), (16), and (26) its extension $\tilde{H}_{\mathrm{sc}}$ to the space $\Lambda_\pi(L^2(C))$ will coincide with the restriction of H' to the latter space provided that

$$K_iK_j\Lambda_\pi = \Lambda_\pi, \qquad K_{ij}\Lambda_\pi = \Lambda_\pi.$$

Thus in this case $\pi(W) = 1$ for all $W \in \mathfrak{W}$, so that for the scalar model (16) $\Lambda_\pi \equiv \Lambda_{\mathrm{sc}}$ is the total symmetrizer with respect to both coordinate permutations and sign reversals of an even number of coordinates.

By the previous discussion, we have reduced the problem of evaluating the spectrum of the Hamiltonians H_{sc} and H to the analogous problem for the restrictions of the auxiliary operators H' and $H' \otimes \mathbb{1}$ to the Hilbert spaces $\Lambda_{\mathrm{sc}}(L^2(C))$ and $\Lambda(L^2(C) \otimes \Sigma)$, respectively. We shall next prove a more explicit characterization of these spaces that will be needed in the sequel. To this end, let $\Lambda^\pm$ be the projector onto states antisymmetric under particle permutations and with parity ± 1 under reversals of coordinates and spins. We shall show that

$$\Lambda\big(L^2(C) \otimes \Sigma\big) = \Lambda^+\big(L^2(C) \otimes \Sigma\big) \oplus \Lambda^-\big(L^2(C) \otimes \Sigma\big). \tag{35}$$

Indeed, let Λ_{a} denote the antisymmetrizer under particle permutations, and let $\{W_i^\pm\}_{i=1}^{2^{N-1}}$ be the set of reversals of an even $(+)$ or an odd $(-)$ number of coordinates and spins. We then have

$$\Lambda^\pm = \frac{1}{2^N}\left(\sum_{i=1}^{2^{N-1}} W_i^+ \pm \sum_{i=1}^{2^{N-1}} W_i^-\right)\Lambda_{\mathrm{a}},$$

and hence

$$\Lambda = \frac{1}{2^{N-1}}\left(\sum_{i=1}^{2^{N-1}} W_i^+\right)\Lambda_{\mathrm{a}} = \Lambda^+ + \Lambda^-,$$

which establishes (35). Similarly, if $\Lambda_{\mathrm{sc}}^\pm$ is the projector onto states symmetric under coordinate permutations and with parity ± 1 under sign reversals, it is easy to show that

$$\Lambda_{\mathrm{sc}}\big(L^2(C)\big) = \Lambda_{\mathrm{sc}}^+\big(L^2(C)\big) \oplus \Lambda_{\mathrm{sc}}^-\big(L^2(C)\big). \tag{36}$$

3.2. Basis of $L^2(C)$

Our next step is to construct suitable (non-orthogonal) bases[1] of $\Lambda(L^2(C) \otimes \Sigma)$ and $\Lambda_{\mathrm{sc}}(L^2(C))$ on which $H' \otimes \mathbb{1}$ and H', respectively, act triangularly. The decompositions (35)–(36), and the fact that $H' \otimes \mathbb{1}$ and H' clearly commute with $\Lambda^\pm$ and $\Lambda_{\mathrm{sc}}^\pm$, suggest that we first triangularize H' on $L^2(C)$. It should be noted that this problem is considerably harder than the corresponding one for the rational D_N model studied in Ref. [44], due to the fact that in the latter

[1] More precisely, a non-orthogonal basis of a (separable) Hilbert space $\mathfrak{H}$ is a Schauder basis of its underlying Banach space, i.e., a countable subset $\{v_i\colon\ i \in \mathbb{N}\} \subset \mathfrak{H}$ such that every element $v \in \mathfrak{H}$ can be expressed in a unique way as $\sum_{i=1}^\infty c_i v_i$, with $c_i \in \mathbb{C}$. In the rest of this section the term "basis" will often be used in this more general sense.

case $\overline{C} = \overline{C}^{(\mathrm{B})} = \mathbb{R}^N$. However, in the present case $\overline{C}$ does not coincide with $\overline{C}^{(\mathrm{B})} = [-\frac{\pi}{2}, \frac{\pi}{2}]^N$, so that one cannot assume that the functions

$$\rho(\mathbf{x})\mathrm{e}^{2\mathrm{i}\mathbf{n}\cdot\mathbf{x}}, \quad \mathbf{n} \in \mathbb{Z}^N, \tag{37}$$

obtained from the basis of $L^2([-\frac{\pi}{2}, \frac{\pi}{2}]^N)$ found in Ref. [42] by setting $b = b' = 0$, are a basis of $L^2(C)$. In fact, it turns out that the set (37) is *not* complete in $L^2(C)$, and must therefore be supplemented by additional functions in order to obtain a basis. This peculiarity, which is absent in the rational case, lends an additional layer of complexity to the relation between the trigonometric D_N models and their BC_N counterparts.

In this subsection we shall prove that the functions

$$\varphi_{\mathbf{n}}^{(\delta)}(\mathbf{x}) \equiv \rho(\mathbf{x})\mathrm{e}^{\mathrm{i}\sum_j(2n_j+\delta)x_j}, \qquad \mathbf{n} \equiv (n_1, \dots, n_N) \in \mathbb{Z}^N, \quad \delta \in \{0, 1\}, \tag{38}$$

form a Schauder basis of $L^2(C)$, leaving for the next subsection the proof that H' acts triangularly on the latter set when ordered appropriately. As we shall next discuss in more detail, the completeness of the functions (38) is essentially based on the fact that a complex exponential $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$ ($\mathbf{k} \in \mathbb{R}$) is periodic in $\overline{C}$ if and only if

$$\mathbf{k} = (2n_1 + \delta, \dots, 2n_N + \delta), \quad n_j \in \mathbb{Z}, \ \delta \in \{0, 1\}. \tag{39}$$

Since $\overline{C}$ is not a hypercube, we need to define more precisely what it means for a function to be periodic in this set. To this end, let

$$F_{ij}^{\epsilon\epsilon'} = \{\mathbf{x} \in \overline{C}\colon\ x_i + \epsilon x_j = \epsilon'\pi\}, \quad 1 \leqslant i < j \leqslant N,\ \epsilon, \epsilon' = \pm,$$

denote one of the $2N(N-1)$ faces of $\overline{C}$. If $T_{ij}^{\epsilon\epsilon'}$ is the translation along the vector $\epsilon'\pi(\mathbf{e}_i + \epsilon\mathbf{e}_j)$ perpendicular to the latter face (where $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ is the canonical basis of $\mathbb{R}^N$), then each $T_{ij}^{\epsilon,-\epsilon'}$ clearly sends the corresponding face $F_{ij}^{\epsilon\epsilon'}$ to its opposite $F_{ij}^{\epsilon,-\epsilon'}$. Given a point $\mathbf{x} \in F_{ij}^{\epsilon\epsilon'}$, we shall refer to $T_{ij}^{\epsilon,-\epsilon'}\mathbf{x}$ as the point opposite to $\mathbf{x}$ in the face $F_{ij}^{\epsilon,-\epsilon'}$. (Of course, a point lying on the intersection of $k > 1$ faces has k different opposites.)

We shall say that a continuous function $f : \overline{C} \to \mathbb{C}$ is *periodic* in $\overline{C}$ if

$$f(\mathbf{x}) = f\big(T_{ij}^{\epsilon,-\epsilon'}\mathbf{x}\big), \quad \forall \mathbf{x} \in F_{ij}^{\epsilon\epsilon'},\ 1 \leqslant i < j \leqslant N,\ \epsilon, \epsilon' = \pm. \tag{40}$$

In other words, f is periodic in $\overline{C}$ if it takes the same value on opposite points in any two faces of $\overline{C}$. Since the coroots of the D_N root system with the normalization (11) are the $2N(N-1)$ vectors

$$\pi(\pm\mathbf{e}_i \pm \mathbf{e}_j), \quad 1 \leqslant i < j \leqslant N,$$

the group $\mathfrak{T}$ generated[2] by the translations $T_{ij}^{\epsilon\epsilon'}$ is the translation group corresponding to the D_N coroot lattice (i.e., the $\mathbb{Z}$-linear span of the coroot vectors). As is well known, the semidirect product of $\mathfrak{T} \ltimes \mathfrak{W}$ yields the affine Weyl group of D_N type $\mathfrak{W}_\mathrm{a}$ [49]. We shall define two points $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^N$ to be *equivalent*, and shall write $\mathbf{x} \sim \mathbf{x}'$, provided that $\mathbf{x}' = T\mathbf{x}$ for some $T \in \mathfrak{T}$. Note that Eq. (40) and the previous definition imply that a function f is periodic in $\overline{C}$ if and only if

$$\mathbf{x}, \mathbf{x}' \in \partial\overline{C}, \quad \mathbf{x} \sim \mathbf{x}' \quad \Longrightarrow \quad f(\mathbf{x}) = f\big(\mathbf{x}'\big).$$

[2] In fact, since $T_{kj}^{+,\mp}T_{ik}^{+,\pm} = T_{ij}^{-,\pm}$ for $i < k < j$, the group $\mathfrak{T}$ is generated just by the translations $T_{ij}^{+,\pm}$.

We shall accordingly say that a continuous function $f : \mathbb{R}^N \to \mathbb{C}$ is $\mathfrak{T}$-periodic if it satisfies

$$f(\mathbf{x}) = f(T\mathbf{x}), \quad \forall T \in \mathfrak{T},\ \forall \mathbf{x} \in \mathbb{R}^N. \tag{41}$$

Remark 3. Since $T_{ij}^{++}T_{ij}^{-+}$ is a translation of 2π in the direction of the vector $\mathbf{e}_i$, it follows that every $\mathfrak{T}$-periodic function is 2π-periodic in each coordinate. The converse, however, is not true in general.

One of the main ingredients in the proof of the completeness of the set (38) in $L^2(C)$ is the fact that every continuous function $f : \overline{C} \to \mathbb{C}$ periodic in $\overline{C}$ can be uniquely extended to a $\mathfrak{T}$-periodic function $\bar{f} : \mathbb{R}^N \to \mathbb{C}$. This result is a direct consequence of the following fundamental facts:

(i) For each $\mathbf{x} \in \mathbb{R}^N$, there is a point $\mathbf{x}' \in \overline{C}$ such that $\mathbf{x} \sim \mathbf{x}'$.
(ii) Moreover, if $\mathbf{x} \sim \mathbf{x}'' \in \overline{C}$ and $\mathbf{x}'' \neq \mathbf{x}'$, then both $\mathbf{x}'$ and $\mathbf{x}''$ lie on (at least) a face of $\overline{C}$.

The proof of these two statements is straightforward. Indeed, it is well known [49] that $\overline{A}$ is a fundamental domain for the action of $\mathfrak{W}_{\mathrm{a}}$ in $\mathbb{R}^N$, i.e., that for every $\mathbf{x} \in \mathbb{R}^N$ there is a unique $\mathbf{a} \in \overline{A}$ and a suitable element R of $\mathfrak{W}_{\mathrm{a}}$ such that $\mathbf{x} = R\mathbf{a}$. Since $\mathfrak{W}_{\mathrm{a}}$ is the semidirect product of its subgroups $\mathfrak{T}$ and $\mathfrak{W}$, we can write $R = TW$, with $T \in \mathfrak{T}$ and $W \in \mathfrak{W}$. This shows that $\mathbf{x}$ is equivalent to $\mathbf{x}' = W\mathbf{a} \in \mathfrak{W}(\overline{A})$. Since the elements of $\mathfrak{W}$ are homeomorphisms, we have $\mathfrak{W}(\overline{A}) = \overline{\mathfrak{W}(A)} \equiv \overline{C}$, which proves the first statement.

As to the second one, suppose next that $\mathbf{x}$ is equivalent to two different points $\mathbf{x}'$ and $\mathbf{x}''$ of $\overline{C}$. It follows that $\mathbf{x}' \sim \mathbf{x}''$ or, equivalently,

$$x_i'' = x_i' + k_i\pi, \quad k_i \in \mathbb{Z},\ 1 \leqslant i \leqslant N;\ k_1 + \cdots + k_N \in 2\mathbb{Z}.$$

Since both $\mathbf{x}'$ and $\mathbf{x}''$ belong to $\overline{C} \subset [-\pi, \pi]^N$, the integers k_i can only take the values $0, \pm 1, \pm 2$. Suppose, first, that one of these integers is equal to ± 2. Without loss of generality, we may assume that $k_1 = 2$. Since $x_1', x_1'' \in [-\pi, \pi]$, this is only possible if $x_1' = -\pi = -x_1''$, from which it follows (taking into account that $\mathbf{x}', \mathbf{x}'' \in \overline{C}$) that $x_i' = x_i'' = 0$ for $i > 1$. Thus in this case $\mathbf{x}'$ belongs to the $2(N-1)$ faces $F_{1i}^{\pm,-}$ (with $2 \leqslant i \leqslant N$), and $\mathbf{x}''$ to the opposite faces $F_{1i}^{\pm,+}$. (Note, however, that in this case $\mathbf{x}''$ is *not* the opposite point of $\mathbf{x}'$ in any of these $2(N-1)$ faces.)

Assume now that $|k_i| < 2$ for all i. Since not all the integers k_i can be zero by hypothesis, and $k_1 + \cdots + k_N$ must be even, there are two of these integers, which w.l.o.g. we can take as k_1 and k_2, such that $k_1 = \epsilon k_2 = \epsilon'$ (with $\epsilon, \epsilon' = \pm$). From the equalities $x_1'' + \epsilon x_2'' = x_1' + \epsilon x_2' + 2\epsilon'\pi$ and the fact that $x_1' + \epsilon x_2', x_1'' + \epsilon x_2'' \in [-\pi, \pi]$, it follows that $x_1'' + \epsilon x_2'' = -(x_1' + \epsilon x_2') = \epsilon'\pi$, and therefore $\mathbf{x}'$ and $\mathbf{x}''$ lie on (at least) the opposite faces $F_{12}^{\epsilon\epsilon'}$ and $F_{12}^{\epsilon,-\epsilon'}$ of $\overline{C}$, respectively. This completes the proof of the second statement.

Remark 4. The previous result implies that N-dimensional Euclidean space can be tiled with translations of the set $\overline{C}$ along the D_N (co)root lattice. For $N = 3$, this is the well-known tessellation of $\mathbb{R}^3$ with rhombic dodecahedra [50].

We are now ready to prove that the set (38) is a (non-orthogonal) basis of $L^2(C)$. In fact, it suffices to show that the exponentials

$$\mathrm{e}^{\mathrm{i}\sum_j (2n_j+\delta)x_j}, \qquad \mathbf{n} \equiv (n_1, \dots, n_N) \in \mathbb{Z}, \quad \delta \in \{0, 1\} \tag{42}$$

are themselves a basis of $L^2(C)$. Indeed, if this is the case then any complex-valued function $f \in C_0(\overline{C})$ continuous in $\overline{C}$ and with compact support in C can be represented in the form

$$f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^N,\, \delta\in\{0,1\}} c_{\mathbf{n},\delta} \mathrm{e}^{\mathrm{i}\sum_j (2n_j+\delta)x_j}, \tag{43}$$

where the coefficients $c_{\mathbf{n},\delta} \in \mathbb{C}$ are by hypothesis uniquely determined by f. Our claim follows immediately from the fact that the function f/ρ is also in $C_0(\overline{C})$, and that the latter set is of course dense in $L^2(\overline{C})$.

The fact the exponentials (42) form a basis of $L^2(C)$ is essentially a consequence of the fact that the momenta (39) are the elements of the dual (or "reciprocal", in a more physical terminology) lattice of the D_N coroot lattice. However, for completeness' sake we shall next provide an elementary proof of this fact. We first note that, since the set $P(\overline{C})$ of complex-valued continuous functions periodic in $\overline{C}$ contains the dense set $C_0(\overline{C})$, to prove that (42) is a basis of $L^2(\overline{C})$ we need only show that every $f \in P(\overline{C})$ can be uniquely represented by a Fourier series of the form (43). Let, then, $f : \overline{C} \to \mathbb{C}$ be a continuous function periodic in $\overline{C}$, and denote by $\bar{f} : \mathbb{R}^N \to \mathbb{C}$ its $\mathfrak{T}$-periodic extension. Since the function $\bar{f}$ is 2π-periodic in each coordinate (cf. Remark 3), it can be developed in terms of the $L^2([-\pi,\pi]^N)$ Fourier basis $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$ ($\mathbf{k} \in \mathbb{Z}^N$) as

$$\bar{f}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^N} a_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \tag{44}$$

with $a_{\mathbf{k}} \in \mathbb{C}$ uniquely determined by f. Imposing that $\bar{f}$ satisfy Eq. (41) when T is one of the generators $T_{ij}^{\epsilon\epsilon'}$ of $\mathfrak{T}$ we easily obtain

$$a_{\mathbf{k}}\bigl(1 - \mathrm{e}^{\mathrm{i}\pi\epsilon'(k_i+\epsilon k_j)}\bigr) = 0, \quad \forall \mathbf{k} \in \mathbb{Z}^N,\ 1 \leqslant i < j \leqslant N,\ \epsilon, \epsilon' = \pm.$$

Hence $a_{\mathbf{k}}$ vanishes unless

$$k_i \pm k_j \equiv k_{ij}^{\pm} \in 2\mathbb{Z}, \quad 1 \leqslant i < j \leqslant N,$$

i.e., unless all the integers k_i have the same parity. Setting $k_i = 2n_i + \delta$ (with $\delta \in \{0, 1\}$), $a_{\mathbf{k}} = c_{\mathbf{n},\delta}$, and substituting into Eq. (44) we obtain

$$\bar{f}(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^N,\, \delta\in\{0,1\}} c_{\mathbf{n},\delta} \mathrm{e}^{\mathrm{i}\sum_j (2n_j+\delta)x_j},$$

where the equality should be understood in the sense of $L^2([-\pi,\pi]^N)$. Restricting to $L^2(\overline{C})$ (which is allowed, since $\overline{C} \subset [-\pi,\pi]^N$), and recalling that all the coefficients $a_{\mathbf{k}}$ are uniquely determined by f, we obtain the desired result.

Remark 5. In fact, since the translations of $\overline{C}$ along the D_N (co)root lattice are a tiling of R^N, the results of [51] imply that the functions (42) are mutually orthogonal. Of course, this does not imply that the functions (38) are themselves orthogonal, due to the presence of the factor $\rho(\mathbf{x})$.

3.3. Triangularization of H'

We shall next endow the set (38) with a suitable order such that the action of H' on the resulting basis is triangular. Note, first of all, that

$$L^2(C) = \mathfrak{H}^{(0)} \oplus \mathfrak{H}^{(1)}, \tag{45}$$

where $\mathfrak{H}^{(\delta)}$ is the closure of the subspace spanned by the basis functions $\varphi_{\mathbf{n}}^{(\delta)}$ with $\mathbf{n} \in \mathbb{Z}^N$. We will show that H' leaves invariant each of the subspaces $\mathfrak{H}^{(\delta)}$, so that we need only order each subbasis $\{\varphi_{\mathbf{n}}^{(\delta)}\}_{\mathbf{n}\in\mathbb{Z}^N}$ in such a way that H' is represented by a triangular matrix in $\mathfrak{H}^{(\delta)}$. To this end, given a multiindex $\mathbf{p} \equiv (p_1, \dots, p_N) \in \mathbb{Z}^N$ we define

$$[\mathbf{p}] = \big(|p_{i_1}|, \dots, |p_{i_N}|\big), \quad \text{with } |p_{i_1}| \geqslant \cdots \geqslant |p_{i_N}|.$$

If $\mathbf{p}' \in \mathbb{Z}^N$ is another multiindex, we shall write $\mathbf{p} \prec \mathbf{p}'$ provided that the first non-vanishing component of $[\mathbf{p}] - [\mathbf{p}']$ is negative. The basis functions $\{\varphi_{\mathbf{n}}^{(\delta)}\}_{\mathbf{n}\in\mathbb{Z}^N}$ should then be ordered in any way such that $\varphi_{\mathbf{n}}^{(\delta)}$ precedes $\varphi_{\mathbf{n}'}^{(\delta)}$ whenever $\boldsymbol{\nu} \prec \boldsymbol{\nu}'$, where

$$\boldsymbol{\nu} \equiv (2n_1 + \delta, \dots, 2n_N + \delta), \tag{46}$$

and similarly for $\boldsymbol{\nu}'$. For instance, $\varphi_{(3,1,0)}^{(0)}$ must precede $\varphi_{(2,-3,-1)}^{(0)}$, $\varphi_{(3,1,0)}^{(1)}$ should follow $\varphi_{(2,-3,-1)}^{(1)}$, while the relative precedence of $\varphi_{(2,-3,-1)}^{(0)}$ and $\varphi_{(1,3,-2)}^{(0)}$ can be arbitrarily assigned.

In order to compute the action of H' on the basis functions (38), it is convenient to introduce the Dunkl operators of D_N type

$$J_k = \mathrm{i}\partial_{x_k} + a \sum_{l\neq k} \big[\big(1 - \mathrm{i}\cot x_{kl}^-\big) K_{kl} + \big(1 - \mathrm{i}\cot x_{kl}^+\big) \tilde{K}_{kl}\big] - 2a \sum_{l<k} K_{kl}, \tag{47}$$

with $k = 1, \dots, N$, obtained from their BC_N counterparts in Ref. [42] by setting $b = b' = 0$. Note that the set C is invariant under *all* the generators K_{ij}, K_i of the Weyl group of BC_N type $\mathfrak{W}^{(\mathrm{B})}$, and hence the operator J_k is actually defined in a suitable dense subspace of $L^2(C)$. It can be shown that

$$H' = \sum_k J_k^2, \tag{48}$$

so that the action of H' on the basis (38) can be easily inferred from that of the Dunkl operators (47). In the following discussion, we shall label the basis functions $\varphi_{\mathbf{n}}^{(\delta)}$ simply by $\varphi_{\boldsymbol{\nu}}$, with $\boldsymbol{\nu}$ given by (46). As in Ref. [52], we shall begin by considering the action of J_k on a basis functions $\varphi_{\boldsymbol{\nu}}$ with $\boldsymbol{\nu}$ nonnegative and nonincreasing. For such a multiindex, we shall use the notation

$$\#(s) = \operatorname{card}\{i\colon \nu_i = s\}, \qquad \ell(s) = \min\{i\colon \nu_i = s\},$$

with $\ell(s) = +\infty$ if $\nu_i \neq s$ for all $i = 1, \dots, N$. For instance, if $\boldsymbol{\nu} = (8, 6, 6, 2, 2, 2)$ then $\#(2) = 3$ and $\ell(2) = 4$.

We shall next prove the formula

$$J_k \varphi_{\boldsymbol{\nu}} = \lambda_{\boldsymbol{\nu},k} \varphi_{\boldsymbol{\nu}} + \sum_{\substack{\boldsymbol{\nu}' \in \mathbb{Z}^N \\ \boldsymbol{\nu}' - \boldsymbol{\nu} \in (2\mathbb{Z})^N,\, \boldsymbol{\nu}' \prec \boldsymbol{\nu}}} c_{\boldsymbol{\nu},k}^{\boldsymbol{\nu}'} \varphi_{\boldsymbol{\nu}'}, \tag{49}$$

where $c_{\boldsymbol{\nu},k}^{\boldsymbol{\nu}'} \in \mathbb{C}$ and

$$\lambda_{\boldsymbol{\nu},k} = \begin{cases} -\nu_k + 2a\big(2\ell(\nu_k) + \#(\nu_k) - k - N - 1\big), & \nu_k > 0, \\ 2a(N - k), & \nu_k = 0, \end{cases} \tag{50}$$

which will play a fundamental role in the sequel. To begin with, a lengthy but straightforward calculation yields

$$\frac{J_k\varphi_{\boldsymbol{\nu}}}{\varphi_{\boldsymbol{\nu}}} = -\nu_k - 2a(N-1) + 2a\sum_{j<k}\frac{\alpha_{jk}^{\nu_j-\nu_k}-1}{\alpha_{jk}^2-1} + 2a\sum_{j>k}\frac{\alpha_{jk}^{\nu_j-\nu_k+2}-1}{\alpha_{jk}^2-1} + 2a\sum_{j\neq k}\frac{\beta_{jk}^{2-\nu_j-\nu_k}-1}{\beta_{jk}^2-1}, \tag{51}$$

where

$$\alpha_{jk} = z_j^{-1}z_k, \qquad \beta_{jk} = z_j z_k, \qquad z_j \equiv \mathrm{e}^{\mathrm{i}x_j}.$$

Consider now the first sum in Eq. (51). Since $j<k$, by hypothesis $\nu_j \geqslant \nu_k$. If $\nu_j = \nu_k$, the j-th term in this sum clearly vanishes. On the other hand, if $\nu_j > \nu_k$ we have

$$\varphi_{\boldsymbol{\nu}}\frac{\alpha_{jk}^{\nu_j-\nu_k}-1}{\alpha_{jk}^2-1} = \varphi_{\boldsymbol{\nu}} + \sum_{r=1}^{\frac12(\nu_j-\nu_k)-1} z_j^{-2r}z_k^{2r}\varphi_{\boldsymbol{\nu}}, \tag{52}$$

where the last sum only appears if $\nu_j - \nu_k > 2$. In this case the multiindices $\boldsymbol{\nu}'$ of the monomials in the summation symbol in Eq. (52) are of the form

$$\boldsymbol{\nu}' = (\nu_1,\dots,\nu_j-2r,\dots,\nu_k+2r,\dots,\nu_N), \quad r=1,\dots,\frac12(\nu_j-\nu_k)-1,$$

and hence $\boldsymbol{\nu}'-\boldsymbol{\nu} \in (2\mathbb{Z})^N$. Moreover, we have $0 < \max\{\nu_j-2r, \nu_k+2r\} < \nu_j$ for all $r = 1,\dots,\frac12(\nu_j-\nu_k)-1$, so that $\boldsymbol{\nu}' \prec \boldsymbol{\nu}$. Thus, the first sum in (51) contributes to $\lambda_{\boldsymbol{\nu},k}$ the quantity

$$2a\,\mathrm{card}\bigl\{j<k\colon\ \nu_j>\nu_k\bigr\} = 2a\bigl(\ell(\nu_k)-1\bigr). \tag{53}$$

It may be likewise verified that the multiindices $\boldsymbol{\nu}'$ corresponding to the monomials arising from the second sum in Eq. (51) either coincide with $\boldsymbol{\nu}$ or satisfy $\boldsymbol{\nu}' \prec \boldsymbol{\nu}$, and that this sum yields the following contribution to $\lambda_{\boldsymbol{\nu},k}$:

$$2a\,\mathrm{card}\{j>k\colon\ \nu_j=\nu_k\} = 2a\bigl(\ell(\nu_k)+\#(\nu_k)-k-1\bigr). \tag{54}$$

Consider, finally, the j-th term of the last sum in Eq. (51). This term is equal to 1 when $\nu_j = \nu_k = 0$, while for $\nu_j+\nu_k \geqslant 2$ we have

$$\frac{\beta_{jk}^{2-\nu_j-\nu_k}-1}{\beta_{jk}^2-1} = -\beta_{jk}^{2-\nu_j-\nu_k}\frac{\beta_{jk}^{\nu_j+\nu_k-2}-1}{\beta_{jk}^2-1} = -\sum_{r=1}^{\frac12(\nu_j+\nu_k)-1}\beta_{jk}^{-2r}.$$

Hence when $\nu_j+\nu_k \geqslant 2$ the multiindices corresponding to the basis functions arising from the last sum in Eq. (51) are of the form

$$\boldsymbol{\nu}' = (\nu_1,\dots,\nu_j-2r,\dots,\nu_k-2r,\dots,\nu_N), \quad r=1,\dots,\frac12(\nu_j+\nu_k)-1,$$

and thus $\boldsymbol{\nu}'-\boldsymbol{\nu} \in (2\mathbb{Z})^N$. Furthermore, since

$$-\nu_k+2 \leqslant \nu_j' \leqslant \nu_j-2, \qquad -\nu_j+2 \leqslant \nu_k' \leqslant \nu_k-2,$$

we have $\max\{|\nu_j'|, |\nu_k'|\} < \max\{\nu_j,\nu_k\}$, so that again $\boldsymbol{\nu}' \prec \boldsymbol{\nu}$. The contribution of the last sum in Eq. (51) to $\lambda_{\boldsymbol{\nu},k}$ is therefore equal to

$$2a\bigl(\#(0)-1\bigr)\delta_{\nu_k,0}. \tag{55}$$

Adding Eqs. (53)–(55) to the first two terms in the RHS of Eq. (51), and taking into account that $l(0)+\#(0)=N+1$, we easily obtain Eq. (50) for $\lambda_{\boldsymbol{\nu},k}$.

Eq. (49) does not hold in general if $\boldsymbol{\nu}$ does not belong to $[\mathbb{Z}^N]$, so that Eq. (50) does not yield the spectrum of the Dunkl operators J_k. On the other hand, for the purposes of computing the spectrum of H' we shall only need the following weaker result: if $\boldsymbol{\nu}\in\mathbb{Z}^N$ is a multiindex all of whose components have the same parity, then

$$J_k\varphi_{\boldsymbol{\nu}}=\sum_{\substack{\boldsymbol{\nu}'\in\mathbb{Z}^N\\ \boldsymbol{\nu}'-\boldsymbol{\nu}\in(2\mathbb{Z})^N,[\boldsymbol{\nu}']\preceq[\boldsymbol{\nu}]}}\gamma_{\boldsymbol{\nu},k}^{\boldsymbol{\nu}'}\varphi_{\boldsymbol{\nu}'} \tag{56}$$

for some complex constants $\gamma_{\boldsymbol{\nu},k}^{\boldsymbol{\nu}'}$. Indeed, if $\boldsymbol{\nu}$ is as above, there is an element W belonging to the Weyl group of BC_N type $\mathfrak{W}^{(\mathrm{B})}$ such that $\varphi_{\boldsymbol{\nu}}=W\varphi_{[\boldsymbol{\nu}]}$. Proceeding as in Ref. [52], it is easy to show that

$$[J_k,W]=\sum_{j=1}^{2^N N!}c_{jk}W_j,\qquad c_{jk}\in\mathbb{R},$$

where $\mathfrak{W}^{(\mathrm{B})}\equiv\{W_j:j=1,\dots,2^N N!\}$. We thus have

$$J_k\varphi_{\boldsymbol{\nu}}=W(J_k\varphi_{[\boldsymbol{\nu}]})+\sum_{j=1}^{2^N N!}c_{jk}W_j\varphi_{[\boldsymbol{\nu}]},$$

and Eq. (56) follows immediately from (49) and the fact that the partial ordering $\prec$ and the parity of the components are invariant under the action of $\mathfrak{W}^{(\mathrm{B})}$.

From the previous results it is relatively straightforward to compute the spectrum of H'. More precisely, we shall next show that the action of H' on each Schauder subbasis $\{\varphi_{\mathbf{n}}^{(\delta)}\}_{\mathbf{n}\in\mathbb{Z}^N}$, ordered as explained above, is upper triangular:

$$H'\varphi_{\mathbf{n}}^{(\delta)}=E_{\mathbf{n}}^{(\delta)}\varphi_{\mathbf{n}}^{(\delta)}+\sum_{\boldsymbol{\nu}'\prec\boldsymbol{\nu}}c_{\mathbf{n}'\mathbf{n}}^{(\delta)}\varphi_{\mathbf{n}'}^{(\delta)},\qquad \nu_k\equiv 2n_k+\delta,\ \nu_k'\equiv 2n_k'+\delta, \tag{57}$$

where $c_{\mathbf{n}'\mathbf{n}}^{(\delta)}\in\mathbb{C}$ and

$$E_{\mathbf{n}}^{(\delta)}=\sum_k\bigl([\boldsymbol{\nu}]_k+2a(N-k)\bigr)^2. \tag{58}$$

Indeed, suppose first that the multiindex $\boldsymbol{\nu}$ in Eq. (57) is nonnegative and nonincreasing. Applying J_k to both sides of Eq. (49) we obtain

$$J_k^2\varphi_{\boldsymbol{\nu}}=\lambda_{\boldsymbol{\nu},k}^2\varphi_{\boldsymbol{\nu}}+\sum_{\substack{\boldsymbol{\nu}'-\boldsymbol{\nu}\in(2\mathbb{Z})^N\\ \boldsymbol{\nu}'\prec\boldsymbol{\nu}}}\lambda_{\boldsymbol{\nu},k}c_{\boldsymbol{\nu},k}^{\boldsymbol{\nu}'}\varphi_{\boldsymbol{\nu}'}+\sum_{\substack{\boldsymbol{\nu}'-\boldsymbol{\nu}\in(2\mathbb{Z})^N\\ \boldsymbol{\nu}'\prec\boldsymbol{\nu}}}c_{\boldsymbol{\nu},k}^{\boldsymbol{\nu}'}J_k\varphi_{\boldsymbol{\nu}'}.$$

By Eq. (56), the last sum is a linear combination of basis functions $\varphi_{\boldsymbol{\nu}''}$ with $\boldsymbol{\nu}''\prec\boldsymbol{\nu}$ and $\boldsymbol{\nu}''-\boldsymbol{\nu}\in(2\mathbb{Z})^N$. Therefore we can write

$$J_k^2\varphi_{\boldsymbol{\nu}}=\lambda_{\boldsymbol{\nu},k}^2\varphi_{\boldsymbol{\nu}}+\sum_{\substack{\boldsymbol{\nu}'-\boldsymbol{\nu}\in(2\mathbb{Z})^N\\ \boldsymbol{\nu}'\prec\boldsymbol{\nu}}}b_{\boldsymbol{\nu},k}^{\boldsymbol{\nu}'}\varphi_{\boldsymbol{\nu}'},$$

with $b^{\boldsymbol{\nu}'}_{\boldsymbol{\nu},k} \in \mathbb{C}$. Summing over k and using the identity (48) we obtain

$$H'\varphi_{\boldsymbol{\nu}} = \Bigl(\sum_k \lambda^2_{\boldsymbol{\nu},k}\Bigr)\varphi_{\boldsymbol{\nu}} + \sum_{\substack{\boldsymbol{\nu}'-\boldsymbol{\nu}\in(2\mathbb{Z})^N\\ \boldsymbol{\nu}'\prec\boldsymbol{\nu}}} \Bigl(\sum_k b^{\boldsymbol{\nu}'}_{\boldsymbol{\nu},k}\Bigr)\varphi_{\boldsymbol{\nu}'}. \tag{59}$$

Suppose, next, that $\boldsymbol{\nu} \notin [\mathbb{Z}]^N$, and let $W \in \mathfrak{W}^{(\mathrm{B})}$ be such that $\varphi_{\boldsymbol{\nu}} = W\varphi_{[\boldsymbol{\nu}]}$. Since H' is obtained from its BC_N counterpart in Ref. [42] by setting $b = b' = 0$, and the latter operator commutes with all the elements of $\mathfrak{W}^{(\mathrm{B})}$, it follows that $[H', W] = 0$. By Eq. (59) applied to $\varphi_{[\boldsymbol{\nu}]}$ we have

$$H'\varphi_{\boldsymbol{\nu}} = W\cdot H'\varphi_{[\boldsymbol{\nu}]} = \Bigl(\sum_k \lambda^2_{[\boldsymbol{\nu}],k}\Bigr)\varphi_{\boldsymbol{\nu}} + \sum_{\substack{\boldsymbol{\nu}'-[\boldsymbol{\nu}]\in(2\mathbb{Z})^N\\ \boldsymbol{\nu}'\prec[\boldsymbol{\nu}]}} \Bigl(\sum_k b^{\boldsymbol{\nu}'}_{[\boldsymbol{\nu}],k}\Bigr)W\varphi_{\boldsymbol{\nu}'},$$

which establishes (57) with

$$E^{(\delta)}_{\mathbf{n}} = \sum_k \lambda^2_{[\boldsymbol{\nu}],k}. \tag{60}$$

All that remains to be proven is Eq. (58) for the eigenvalue $E^{(\delta)}_{\mathbf{n}}$. To this end, let $\mathbf{p} = [\boldsymbol{\nu}]$ and suppose that $p_{k-1} > p_k = \cdots = p_{k+r} > p_{k+r+1} \geqslant 0$, so that $\ell(p_{k+j}) = k$ and $\#(p_{k+j}) = r+1$ for $j = 0,\dots,r$. Since

$$\lambda_{\mathbf{p},k+j} = -p_{k+j} + 2a(k+r-j-N) = -p_{k+r-j} + 2a(k+r-j-N), \quad j = 0,\dots,r,$$

we have

$$\sum_{j=k}^{k+r} \lambda^2_{\mathbf{p},j} = \sum_{j=k}^{k+r} \bigl(p_j + 2a(N-j)\bigr)^2. \tag{61}$$

If, on the other hand, $p_{k-1} > p_k = \cdots = p_N = 0$, the analog of Eq. (61) follows directly from (50). This completes the proof of Eq. (58).

3.4. Triangularization of H and H_{sc}

Using the results of the previous subsection, it is a straightforward matter to triangularize H and H_{sc}. Indeed, by the results in Section 3.1, this problem is equivalent to the triangularization of the extensions $\tilde{H}$ and $\tilde{H}_{\mathrm{sc}}$ acting on their respective Hilbert spaces $\mathfrak{H} \equiv \Lambda(L^2(C)\otimes\Sigma)$ and $\mathfrak{H}_{\mathrm{sc}} \equiv \Lambda_{\mathrm{sc}}(L^2(C))$

Let us start with the operator $\tilde{H}$. By Eqs. (35) and (45), its Hilbert space can be decomposed as the direct sum

$$\mathfrak{H} = \bigoplus_{\substack{\epsilon=\pm\\ \delta=0,1}} \Lambda^{\epsilon}\bigl(\mathfrak{H}^{(\delta)}\otimes\Sigma\bigr). \tag{62}$$

Let $f(\mathbf{x})$ be in the domain of H', and let $|s\rangle \in \Sigma$ denote an arbitrary spin state. Since $\tilde{H}$ coincides with $H'\otimes\mathbb{1}$ on $\mathfrak{H}$, and the latter operator commutes with Λ (indeed, it commutes with all the elements of $\mathfrak{W}^{(\mathrm{B})}$, and hence of $\mathfrak{W}$), we have

$$\tilde{H}\bigl[\Lambda^{\epsilon}\bigl(f(\mathbf{x})|s\rangle\bigr)\bigr] = \Lambda^{\epsilon}\bigl[\bigl(H'f(\mathbf{x})\bigr)|s\rangle\bigr]. \tag{63}$$

As we saw in the previous subsection, H' preserves the subspaces $\mathfrak{H}^{(\delta)}$, which by the latter equation implies that each of the four subspaces $\Lambda^\epsilon(\mathfrak{H}^{(\delta)} \otimes \Sigma)$ is invariant under $\tilde{H}$. We shall next verify that $\tilde{H}$ acts triangularly on a (non-orthogonal) basis of $\Lambda^\epsilon(\mathfrak{H}^{(\delta)} \otimes \Sigma)$ of the form

$$\psi^{\delta,\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x}) = \Lambda^\epsilon\big(\varphi^{(\delta)}_{\mathbf{n}}(\mathbf{x})|\mathbf{s}\rangle\big), \tag{64}$$

ordered in such a way that $\psi^{\delta,\epsilon}_{\mathbf{n},\mathbf{s}}$ precedes $\psi^{\delta,\epsilon}_{\mathbf{n}',\mathbf{s}'}$ whenever $\boldsymbol{\nu} \prec \boldsymbol{\nu}'$ (with $\boldsymbol{\nu}$ defined in (46), and similarly $\boldsymbol{\nu}'$). The spin functions (64) are obviously a complete set (since the functions (38) are a basis of $L^2(C)$), but their linear independence is not assured unless we impose suitable restrictions on the quantum numbers $\mathbf{n}$ and $\mathbf{s}$. More precisely, the states (64) are a (non-orthogonal) basis of the Hilbert space $\Lambda^\epsilon(\mathfrak{H}^{(\delta)} \otimes \Sigma)$ provided that the quantum numbers $\mathbf{n} \in \mathbb{Z}^N$ and $\mathbf{s}$ satisfy the following conditions:

$$\text{(i) } n_1 \geqslant \cdots \geqslant n_N \geqslant 0, \tag{65a}$$

$$\text{(ii) } s_i > s_j \text{ whenever } n_i = n_j \text{ and } i < j, \tag{65b}$$

$$\text{(iii) If } \delta = n_i = 0 \text{ then } s_i \geqslant 0 \text{ for } \epsilon = 1, \text{ while } s_i > 0 \text{ for } \epsilon = -1. \tag{65c}$$

Indeed, since

$$\Lambda^\epsilon(K_i S_i) = \epsilon \Lambda^\epsilon, \qquad \Lambda^\epsilon(K_{ij} S_{ij}) = -\Lambda^\epsilon,$$

acting with suitable operators $K_i S_i$ and $K_{ij} S_{ij}$ on a spin function $\varphi^{(\delta)}_{\mathbf{n}}(\mathbf{x})|\mathbf{s}\rangle$ with arbitrary $\mathbf{n} \in \mathbb{Z}^N$ and $\mathbf{s}$ one can easily show that the corresponding state $\psi^{\delta,\epsilon}_{\mathbf{n},\mathbf{s}}$ is either zero or proportional to a state (64) satisfying the above conditions. (Note, in this respect, that a state (64) with $\delta = n_i = s_i = 0$ is symmetric under $(x_i, s_i) \to (-x_i, -s_i)$, and must therefore vanish identically if $\epsilon = -1$.) This shows that the states (64) with $\mathbf{n} \in \mathbb{Z}^N$ and $\mathbf{s}$ satisfying the above conditions are complete. Their linear independence is easily checked.

Remark 6. Conditions (i)–(ii) above are identical to the corresponding ones for the spin Calogero model of D_N type studied in Ref. [44]. As to the third one, the key difference is that in the present case the action of a coordinate sign reversing operator K_i on a state $\varphi^{(\delta)}_{\mathbf{n}}(\mathbf{x})|\mathbf{s}\rangle$ no longer produces a state with the same quantum number $\mathbf{n}$ (up to a constant factor) unless $\delta = n_i = 0$.

Remark 7. Since the functions $\varphi^{(0)}_{\mathbf{n}}$ with $\mathbf{n} \in \mathbb{Z}^N$ form a basis of $L^2(C^{(B)}) \subset L^2(C)$ (cf. Ref. [42]), it follows that each subspace $\Lambda^\epsilon(\mathfrak{H}^{(0)} \otimes \Sigma)$ properly contains the Hilbert space $\Lambda^\epsilon(L^2(C^{(\mathrm{B})}) \otimes \Sigma)$ of the Sutherland spin model of BC_N type (1) with chirality ϵ. Note, however, that the other sector $\Lambda^+(\mathfrak{H}^{(1)} \otimes \Sigma) \oplus \Lambda^-(\mathfrak{H}^{(1)} \otimes \Sigma)$ of $\mathfrak{H}$ has no counterpart in the BC_N model. Thus, in contrast with the rational case [44], the Hilbert space of the D_N model is larger than the direct sum of the Hilbert spaces of its BC_N counterparts with both chiralities.

Let us now examine the action of the operator $\tilde{H}$ on the basis of $\Lambda^\epsilon(\mathfrak{H}^{(\delta)} \otimes \Sigma)$ given by Eqs. (64)–(65). It is easy to show that

$$\tilde{H}\psi^{\delta,\epsilon}_{\mathbf{n},\mathbf{s}} = E^{\delta,\epsilon}_{\mathbf{n},\mathbf{s}}\psi^{\delta,\epsilon}_{\mathbf{n},\mathbf{s}} + \sum_{\substack{\mathbf{n}',\mathbf{s}' \\ \boldsymbol{\nu}' \prec \boldsymbol{\nu}}} c^{\delta\epsilon}_{\mathbf{n}'\mathbf{s}',\mathbf{n}\mathbf{s}}\psi^{\delta,\epsilon}_{\mathbf{n}',\mathbf{s}'}, \tag{66}$$

where $c^{\delta\epsilon}_{\mathbf{n}'\mathbf{s}',\mathbf{n}\mathbf{s}} \in \mathbb{C}$, the quantum numbers $(\mathbf{n}', \mathbf{s}')$ satisfy conditions (65) and

$$E_{\mathbf{n},\mathbf{s}}^{\delta,\epsilon} = \sum_k \bigl(2n_k + \delta + 2a(N-k)\bigr)^2. \tag{67}$$

Indeed, from Eqs. (57)–(58) and the identity (63) one immediately obtains

$$\tilde{H}\psi_{\mathbf{n},\mathbf{s}}^{\delta,\epsilon} = E_{\mathbf{n},\mathbf{s}}^{\delta,\epsilon}\psi_{\mathbf{n},\mathbf{s}}^{\delta,\epsilon} + \sum_{\boldsymbol{\nu}'\prec\boldsymbol{\nu}} c_{\mathbf{n}',\mathbf{n}}^{(\delta)}\psi_{\mathbf{n}',\mathbf{s}}^{\delta,\epsilon}. \tag{68}$$

Although the quantum numbers $(\mathbf{n}', \mathbf{s})$ appearing in the RHS of this equation do not necessarily satisfy conditions (65), there is an element $W \in \mathfrak{W}^{(\mathrm{B})}$ such that $(W\mathbf{n}', W\mathbf{s}) \equiv (\mathbf{n}'', \mathbf{s}'')$ do satisfy these conditions. Since the corresponding state $\psi_{\mathbf{n}'',\mathbf{s}''}^{\delta,\epsilon}$ differs from $\psi_{\mathbf{n}',\mathbf{s}}^{\delta,\epsilon}$ by at most an overall sign, and $[\boldsymbol{\nu}''] = [\boldsymbol{\nu}'] \prec [\boldsymbol{\nu}]$ implies that $\boldsymbol{\nu}'' \prec \boldsymbol{\nu}$, it is clear that we can rewrite (68) in the form (66).

From Eq. (66) it follows that the operator $\tilde{H}$ acts triangularly on the (non-orthogonal) basis of $\Lambda^\epsilon(\mathfrak{H}^{(\delta)} \otimes \Sigma)$ in Eqs. (64)–(65), ordered in such a way that $\psi_{\mathbf{n},\mathbf{s}}^{\delta,\epsilon}$ precedes $\psi_{\mathbf{n}',\mathbf{s}'}^{\delta,\epsilon}$ whenever $\boldsymbol{\nu} \prec \boldsymbol{\nu}'$. Moreover, the eigenvalues of the restriction of $\tilde{H}$ to $\Lambda^\epsilon(\mathfrak{H}^{(\delta)} \otimes \Sigma)$ are given by Eq. (67), with $\mathbf{n} \in \mathbb{Z}^N$ and $\mathbf{s}$ satisfying conditions (65).

Remark 8. Since the numerical value of the eigenvalue (67) does not depend on $\mathbf{s}$ or ϵ, for any multiindex $\mathbf{n} \in [\mathbb{Z}^N]$ the corresponding eigenvalue $E_{\mathbf{n},\mathbf{s}}^{\delta,\epsilon}$ has an associated degeneracy

$$d_{\mathbf{n}}^{\delta} = d_{\mathbf{n}}^{\delta,+} + d_{\mathbf{n}}^{\delta,-}, \tag{69}$$

where $d_{\mathbf{n}}^{\delta,\epsilon}$ is the number of basic spin states $|\mathbf{s}\rangle$ satisfying conditions (65) for given ϵ and δ. These spin degeneracy factors will be computed below, when we discuss the partition function of this model.

The spectrum of the scalar Hamiltonian $\tilde{H}_{\mathrm{sc}}$ can be computed in a similar way by exploiting the fact that $\tilde{H}_{\mathrm{sc}}$ coincides with H' in its Hilbert space $\mathfrak{H}_{\mathrm{sc}}$, which by Eqs. (36) and (45) is given by

$$\mathfrak{H}_{\mathrm{sc}} = \bigoplus_{\substack{\epsilon=\pm\\ \delta=0,1}} \Lambda_{\mathrm{sc}}^\epsilon\bigl(\mathfrak{H}^{(\delta)}\bigr). \tag{70}$$

Due to the identity

$$\tilde{H}_{\mathrm{sc}}\bigl(\Lambda_{\mathrm{sc}}^\epsilon f(\mathbf{x})\bigr) = \Lambda_{\mathrm{sc}}^\epsilon\bigl(H' f(\mathbf{x})\bigr),$$

it is immediate to show that each of the four subspaces $\Lambda_{\mathrm{sc}}^\epsilon(\mathfrak{H}^{(\delta)})$ is invariant under $\tilde{H}_{\mathrm{sc}}$. Just as in Section 3.4, it can be verified that the functions

$$\psi_{\mathbf{n}}^{\delta,\epsilon}(\mathbf{x}) = \Lambda_{\mathrm{sc}}^\epsilon\bigl(\varphi_{\mathbf{n}}^{(\delta)}(\mathbf{x})\bigr), \tag{71}$$

where $\mathbf{n} \in \mathbb{Z}^N$ and

$$n_1 \geqslant \cdots \geqslant n_N \geqslant \frac{1}{2}(1-\epsilon)(1-\delta), \tag{72}$$

are a Schauder basis of $\Lambda_{\mathrm{sc}}^\epsilon(\mathfrak{H}^{(\delta)})$. (The last inequality is due to the fact that if $\delta = n_N = 0$ the function $\psi_{\mathbf{n}}^{0,-}$ is symmetric under $x_N \to -x_N$, and therefore vanishes identically.) Proceeding as above, it is straightforward to show that if we order the basis (71)–(72) so that $\psi_{\mathbf{n}}^{\delta,\epsilon}$ precedes $\psi_{\mathbf{n}'}^{\delta,\epsilon}$ whenever $\boldsymbol{\nu} \prec \boldsymbol{\nu}'$, the operator $\tilde{H}_{\mathrm{sc}}$ acts triangularly on it, with eigenvalues $E_{\mathbf{n}}^{\delta,\epsilon}$ given by the RHS of Eq. (67). Of course, due to the absence of internal degrees of freedom, in this case the degeneracy factors $d_{\mathbf{n}}^{\delta,\epsilon}$ are equal to one for all quantum numbers $\mathbf{n}$, $\epsilon = \pm 1$, and $\delta = 0, 1$.

Remark 9. It is well-known [22] that the eigenfunctions of the scalar BC_N Sutherland model are of the form

$$\rho(\mathbf{x})\prod_i |\sin x_i|^b |\cos x_i|^{b'} \cdot P_k(\mathbf{y}), \tag{73}$$

where $P_k(\mathbf{y})$ is a symmetric polynomial in the variables $y_i = \sin^2 x_i$ $(i = 1, \ldots, N)$. The polynomials P_k, which can be regarded as multivariate generalizations of the classical Jacobi polynomials, are orthogonal in the hypercube $[0, 1]^N$ with respect to the weight function

$$w^{(\mathrm{B})}(\mathbf{y}) = \prod_{i<j} |y_i - y_j|^{2a} \cdot \prod_i y_i^{b-\frac{1}{2}} (1 - y_i)^{b'-\frac{1}{2}} \tag{74}$$

(cf. Eq. (2.17) of Ref. [22]). In our case, from the identities

$$\Lambda_{\mathrm{sc}}^{+} \mathrm{e}^{\mathrm{i}\sum_k \nu_k x_k} = \Lambda_{\mathrm{sc}} \prod_k \cos(\nu_k x_k), \qquad \Lambda_{\mathrm{sc}}^{-} \mathrm{e}^{\mathrm{i}\sum_k \nu_k x_k} = \mathrm{i}^N \Lambda_{\mathrm{sc}} \prod_k \sin(\nu_k x_k),$$

it is straightforward to show that the (orthonormalized) eigenfunctions of $\tilde{H}_{\mathrm{sc}}$ in each of the invariant subspaces $\Lambda_{\mathrm{sc}}^{\epsilon}(\mathfrak{H}^{(\delta)})$ are of the form

$$\rho(\mathbf{x})\prod_i |\sin 2x_i|^{\frac{1-\epsilon}{2}} |\cos x_i|^{\delta\epsilon} \cdot P_k^{\delta,\epsilon}(\mathbf{y}), \tag{75}$$

where $P_k^{\delta,\epsilon}(\mathbf{y})$ is a polynomial in the variables $y_i = \sin^2 x_i$ symmetric under permutations. From the discussion in Section 3.1 it follows that the restrictions of the functions (75) to the open set A are a complete set of eigenfunctions of the scalar Sutherland model of D_N type H_{sc}. They are also orthogonal in the latter set, on account of their symmetry under coordinate permutations and sign changes. This is easily seen to imply that the polynomials $P_k^{\delta,\epsilon}$ are orthogonal in the hypercube $[0, 1]^N$ with respect to the weight

$$w^{\delta,\epsilon}(\mathbf{y}) = \prod_{i<j} |y_i - y_j|^{2a} \cdot \prod_i y_i^{-\frac{\epsilon}{2}} (1 - y_i)^{\epsilon(\delta-\frac{1}{2})}. \tag{76}$$

In view of Eqs. (73)–(74) and (75)–(76), it is clear that the three orthogonal polynomial families $\{P_k^{\delta,\epsilon}\colon\ k \in \mathbb{N}\}$ with $(\delta, \epsilon) = (0, -1), (1, \pm 1)$ are not limiting cases of the multivariate Jacobi polynomials studied by Baker and Forrester [22]. The analysis of the properties of these new orthogonal polynomials, and their relations with their BC_N counterparts, could lead to interesting new developments in the field of multivariate orthogonal polynomials.

4. Partition function of the spin chain

The purpose of this section is to evaluate in closed form the partition function of the Haldane–Shastry spin chain of D_N type (13). Our starting point is the freezing trick relation (17), which can be equivalently written as

$$\mathcal{E}_j = \lim_{a\to\infty} \frac{E_{ij} - E_i^{\mathrm{sc}}}{4a}. \tag{77}$$

This formula expresses each eigenvalue $\mathcal{E}_j$ of the chain (13) in terms of a certain eigenvalue E_{ij} of the spin Sutherland model of D_N type (8) and a corresponding eigenvalue E_i^{sc} of the scalar model (16). In practice, the fact that the eigenvalues E_{ij} and $E_i^{\rm sc}$ are obviously not independent makes it impossible to use Eq. (77) to completely determine the spectrum of the chain (13) in terms of the spectra of the Hamiltonians H and $H_{\rm sc}$ computed in the previous section (cf. Eq. (67)). The key idea behind the freezing trick method introduced by Polychronakos [36] is to use Eq. (77), or rather the equivalent relation (17), to directly compute the chain's partition function. Indeed, the latter equation immediately yields the identity

$$\mathcal{Z}(T)=\lim_{a\to\infty}\frac{Z(4aT)}{Z_{\rm sc}(4aT)}, \tag{78}$$

expressing the chain's partition function $\mathcal{Z}$ in terms of the partition functions Z and $Z_{\rm sc}$ of the Hamiltonians H and $H_{\rm sc}$.

Remark 10. Eq. (77) can be used to obtain nontrivial qualitative information on the spectrum of the chain (13). For instance, from the fact that the numerical values of the energies of both Hamiltonians H and $H_{\rm sc}$ are given by the RHS of Eq. (67) and Eq. (77) it easily follows that all the energies of the spin chain (13) are integers.

In the rest of this section, we shall compute the large a limits of $Z(4aT)$ and $Z_{\rm sc}(4aT)$ using Eq. (67) for the spectrum of H and $H_{\rm sc}$, thereby obtaining an exact expression for $\mathcal{Z}$ via Eq. (78). Before doing so, it is convenient to subtract from the spectra of H and $H_{\rm sc}$ the constant term

$$E_0=4a^2\sum_k(N-k)^2=\frac{2}{3}a^2N(N-1)(2N-1),$$

which is of course irrelevant for the purposes of computing $\mathcal{Z}$. The rationale behind this normalization is the fact that, by Eq. (67), the eigenvalues of H and $H_{\rm sc}$ become $O(a)$ for $a\to\infty$, so that the limits of $Z(4aT)$ and $Z_{\rm sc}(4aT)$ exist separately.

Let us start with the partition function of Hamiltonian H of the D_N-type spin Sutherland model (8). With the normalization of the energies explained above, the spectrum of this model satisfies

$$E^{\delta,\epsilon}_{\mathbf{n},\mathbf{s}}=4a\sum_k(2n_k+\delta)(N-k)+O(1), \tag{79}$$

and hence its partition function is given by

$$\lim_{a\to\infty}Z(4aT)=\sum_{\substack{n_1\geqslant\cdots\geqslant n_N\geqslant 0\\ \epsilon=\pm,\,\delta=0,1}} d^{\delta,\epsilon}_{\mathbf{n}}\,q^{\sum_i(2n_i+\delta)(N-i)},\qquad q\equiv \mathrm{e}^{-1/(k_{\rm B}T)}. \tag{80}$$

As mentioned in Remark 8, the degeneracy factor $d^{\delta,\epsilon}_{\mathbf{n}}$ is equal to the number of spin states $|\mathbf{s}\rangle$ satisfying conditions (65) for given $\epsilon=\pm1$ and $\delta=0,1$. Writing the quantum number $\mathbf{n}$ in the form

$$\mathbf{n}=\big(\overbrace{p_1,\ldots,p_1}^{k_1},\ldots,\overbrace{p_r,\ldots,p_r}^{k_r}\big),\qquad p_1>\cdots>p_r\geqslant 0, \tag{81}$$

and using conditions (65b) and (65c) we have

$$d^{\delta,\epsilon}_{\mathbf{n}}=\begin{cases}\binom{m_\epsilon}{k_r}\prod_{i=1}^{r-1}\binom{m}{k_i}, & \delta=p_r=0;\\ \prod_{i=1}^{r}\binom{m}{k_i}, & \text{otherwise},\end{cases} \tag{82}$$

with

$$m_\epsilon = \frac{1}{2}\big(m + \epsilon\pi(m)\big), \qquad \pi(m) \equiv (m \bmod 2). \tag{83}$$

Let us now define

$$Z^{(\delta)}(T) \equiv \sum_{\substack{n_1 \geqslant \cdots \geqslant n_N \geqslant 0 \\ \epsilon=\pm}} d_{\mathbf{n}}^{\delta,\epsilon} q^{\sum_i (2n_i+\delta)(N-i)},$$

so that

$$\lim_{a\to\infty} Z(4aT) = Z^{(0)}(T) + Z^{(1)}(T). \tag{84}$$

The function $Z^{(0)}(T)$ can be easily expressed in terms of the partition functions $Z_\pm^{(\mathrm{B})}$ of two spin Sutherland models of BC_N type (1) with opposite chiralities $\epsilon = \pm$. Indeed, when $\delta = 0$ Eq. (82) coincides with Eq. (51) in Ref. [42] for the degeneracy factor of the BC_N-type spin Sutherland model with chirality ϵ. Likewise, Eq. (67) with $\delta = 0$ is obtained from the analogous formula for the energies of the BC_N Hamiltonian (1) in Eq. (24) of the latter reference by setting $\beta = \beta' = 0$, and the same is true for Eq. (79). We thus have

$$Z^{(0)}(T) = \lim_{a\to\infty} \big(Z_+^{(\mathrm{B})}(4aT) + Z_-^{(\mathrm{B})}(4aT)\big)\big|_{\beta=\beta'=0}. \tag{85}$$

Using Eq. (53) from Ref. [42] we obtain the explicit expression

$$\begin{aligned} Z^{(0)}(T) = \sum_{(k_1,\dots,k_r)\in\mathcal{P}_N} \Bigg\{ \Bigg[\binom{m_+}{k_r} + \binom{m_-}{k_r} \\ + 2\binom{m}{k_r}\frac{q^{K_r}}{1-q^{K_r}}\Bigg] \prod_{i=1}^{r-1} \Bigg[\binom{m}{k_i}\frac{q^{K_i}}{1-q^{K_i}}\Bigg]\Bigg\}, \end{aligned} \tag{86}$$

where $\mathcal{P}_N$ is the set of partitions of the integer N (taking order into account), and

$$K_i = \bar{k}_i(2N - 1 - \bar{k}_i), \quad \bar{k}_i \equiv \sum_{j=1}^{i} k_i. \tag{87}$$

Note that, since $k_1 + \cdots + k_r = N$, the integers $\bar{k}_i$ are in the range $1, \dots, N$.

On the other hand, from Eq. (82) it easily follows that

$$Z^{(1)}(T) = 2q^{\frac{1}{2}N(N-1)} \sum_{n_1 \geqslant \cdots \geqslant n_N \geqslant 0} \prod_{i=1}^{r} \binom{m}{k_i} \cdot q^{\sum_j 2n_j(N-j)}.$$

Proceeding as in Ref. [42] we easily obtain

$$Z^{(1)}(T) = 2q^{\frac{1}{2}N(N-1)} \sum_{(k_1,\dots,k_r)\in\mathcal{P}_N} \big(1 - q^{K_r}\big)^{-1} \prod_{i=1}^{r} \binom{m}{k_i} \cdot \prod_{i=1}^{r-1} \frac{q^{K_i}}{1-q^{K_i}}. \tag{88}$$

The partition function $Z_{\mathrm{sc}}(4aT)$ of the scalar Hamiltonian (16) is also easily evaluated in the limit $a \to \infty$, since in this limit its spectrum (with the normalization discussed above) is still given by the RHS of Eq. (79). Using Eq. (72), and taking into account that in this case $d_{\mathbf{n}}^{\delta,\epsilon} = 1$, we have

$$
\begin{aligned}
\lim_{a\to\infty} Z_{\mathrm{sc}}(4aT) &= 2\sum_{n_1\geqslant\cdots\geqslant n_N\geqslant 0} q^{\sum_i (2n_i+1)(N-i)} + \sum_{n_1\geqslant\cdots\geqslant n_N\geqslant 0} q^{\sum_i 2n_i(N-i)} \\
&\quad + \sum_{n_1\geqslant\cdots\geqslant n_N> 0} q^{\sum_i 2n_i(N-i)} \\
&= \bigl(2q^{\frac12 N(N-1)}+1\bigr)\sum_{n_1\geqslant\cdots\geqslant n_N\geqslant 0} q^{\sum_i 2n_i(N-i)} \\
&\quad + \sum_{n_1\geqslant\cdots\geqslant n_N\geqslant 0} q^{\sum_i 2(n_i+1)(N-i)} \\
&= \bigl(1+q^{\frac12 N(N-1)}\bigr)^2\sum_{n_1\geqslant\cdots\geqslant n_N\geqslant 0} q^{\sum_i 2n_i(N-i)}.
\end{aligned}
$$

The last sum is easily recognized as the $a\to\infty$ limit of the partition function $Z_{\mathrm{sc}}^{(\mathrm{B})}(4aT)$ of the scalar Sutherland Hamiltonian of BC_N type

$$
H_{\mathrm{sc}}^{(\mathrm{B})} \equiv H^{(\mathrm{B})}\big|_{S_{ij}\to 1,\, S_i\to 1}
$$

with $\beta=\beta'=0$. Using Eq. (49) in Ref. [42] we thus obtain

$$
\begin{aligned}
\lim_{a\to\infty} Z_{\mathrm{sc}}(4aT) &= \bigl(1+q^{\frac12 N(N-1)}\bigr)^2 \lim_{a\to\infty} Z_{\mathrm{sc}}^{(\mathrm{B})}(4aT)\big|_{\beta=\beta'=0} \\
&= \bigl(1+q^{\frac12 N(N-1)}\bigr)^2 \prod_i \bigl(1-q^{i(2N-1-i)}\bigr)^{-1}.
\end{aligned} \tag{89}
$$

The partition function of the Haldane–Shastry spin chain of D_N type (13) is easily computed by inserting Eqs. (84), (86), (88) and (89) into the freezing trick identity (78). In order to simplify the resulting expression, we define $N-r$ integers $\bar{k}_1'<\cdots<\bar{k}_{N-r}'$ in the range $1,\dots,N-1$ by

$$
\bigl\{\bar{k}_1',\dots,\bar{k}_{N-r}'\bigr\} = \{1,\dots,N-1\} - \{\bar{k}_1,\dots,\bar{k}_{r-1}\},
$$

and set

$$
K_i' = \bar{k}_i'\bigl(2N-1-\bar{k}_i'\bigr). \tag{90}
$$

Using this notation, the partition function of the chain (13) can be written as

$$
\begin{aligned}
\mathcal{Z}(T) &= \bigl(1+q^{\frac12 N(N-1)}\bigr)^{-2} \sum_{(k_1,\dots,k_r)\in\mathcal{P}_N} \prod_{i=1}^{r-1}\binom{m}{k_i}\cdot q^{\sum_{i=1}^{r-1}K_i}\biggl\{2\binom{m}{k_r}\bigl(q^{K_r}+q^{\frac12 N(N-1)}\bigr) \\
&\quad + \biggl[\binom{m_+}{k_r}+\binom{m_-}{k_r}\biggr]\bigl(1-q^{K_r}\bigr)\biggr\}\prod_{i=1}^{N-r}\bigl(1-q^{K_i'}\bigr).
\end{aligned} \tag{91}
$$

Taking into account that $\bar{k}_r=\sum_{i=1}^{r}k_i=N$, so that $K_r=N(N-1)$ by Eq. (87), we finally obtain the more compact expression

$$\mathcal{Z}(T) = \left(1+q^{\frac{1}{2}N(N-1)}\right)^{-1} \sum_{(k_1,\dots,k_r)\in\mathcal{P}_N} \prod_{i=1}^{r-1}\binom{m}{k_i}\cdot\left\{2\binom{m}{k_r}q^{\frac{1}{2}N(N-1)}\right.$$
$$\left.+\left[\binom{m_+}{k_r}+\binom{m_-}{k_r}\right]\left(1-q^{\frac{1}{2}N(N-1)}\right)\right\}q^{\sum_{i=1}^{r-1}K_i}\prod_{i=1}^{N-r}\left(1-q^{K_i'}\right). \tag{92}$$

Remark 11. From [42, Eq. (53)] and Eq. (91) we easily obtain the identity

$$\mathcal{Z}(T) = \left(1+q^{\frac{1}{2}N(N-1)}\right)^{-2}\left[\left(\mathcal{Z}_+^{(\mathrm{B})}(T)+\mathcal{Z}_-^{(\mathrm{B})}(T)\right)\Big|_{\beta=\beta'=0} + 2q^{\frac{1}{2}N(N-1)}\mathcal{Q}_N(T)\right],$$

where

$$\mathcal{Q}_l(T) = \sum_{(k_1,\dots,k_r)\in\mathcal{P}_l}\prod_{i=1}^{r}\binom{m}{k_i}\cdot q^{\sum_{i=1}^{r-1}K_i}\prod_{i=1}^{l-r}\left(1-q^{K_i'}\right) \tag{93}$$

and the integers K_i, K_i' are defined by Eqs. (87) and (90) for all l. Thus, unlike what happens in the rational case (cf. [44, Eq. (46)]), it does not seem possible to express in a simple way the partition function $\mathcal{Z}(T)$ exclusively in terms of its BC_N counterparts $\mathcal{Z}_\pm^{(\mathrm{B})}$. Note also that the function $\mathcal{Q}_N(T)$ has the same structure as the partition function of the ordinary (A_{N-1}-type) Haldane–Shastry chain [37], the only difference being the "dispersion relation" defining the quantities K_i and K_i' in terms of $\bar{k}_i$ and $\bar{k}_i'$.

As mentioned in Remark 10, the eigenvalues of the spin chain (13) are integers, and they are nonnegative on account of the nonnegative character of the operators $1+S_{ij}$ and $1+\tilde{S}_{ij}$. Thus, the partition function $\mathcal{Z}(T)$ should be a polynomial in q, a fact which is not apparent from Eq. (92). In order to ascertain this fact, consider first a partition $(k_1,\dots,k_r)\in\mathcal{P}_N$ with $k_r=1$. In this case the term in curly brackets in Eq. (92) reduces to $m(1+q^{N(N-1)/2})$, and $\bar{k}_{r-1}=N-k_r=N-1$ implies that

$$\left\{\bar{k}_1',\dots,\bar{k}_{N-r}'\right\} = \{1,\dots,N-2\} - \{\bar{k}_1,\dots,\bar{k}_{r-2}\}.$$

Hence the contribution to $\mathcal{Z}(T)$ of the partitions with $k_r=1$ is given by $mq^{N(N-1)}\mathcal{Q}_{N-1}(T)$. Consider next a partition $(k_1,\dots,k_r)\in\mathcal{P}_N$ such that $k_r\equiv l>1$. In this case $(k_1,\dots,k_{r-1})$ is a partition of $N-l$, and $\bar{k}_{r-1}=N-l$ implies that

$$\bar{k}_{N-j-r+1}' = N-j,\quad j=1,\dots,l-1, \tag{94}$$

and hence

$$K_{N-j-r+1}' = (N-j)(N+j-1),\quad j=1,\dots,l-1.$$

Note, in particular, that $K_{N-r}'=N(N-1)$, so that

$$\left(1+q^{\frac{1}{2}N(N-1)}\right)^{-1}\left(1-q^{K_{N-r}'}\right) = 1-q^{\frac{1}{2}N(N-1)}.$$

Taking into account that, by Eq. (94),

$$\left\{\bar{k}_1',\dots,\bar{k}_{N-l-r+1}'\right\} = \{1,\dots,N-l-1\} - \{\bar{k}_1,\dots,\bar{k}_{r-2}\},$$

it is immediate to verify that the contribution to $\mathcal{Z}(T)$ of the partitions with $k_r=l\geqslant 2$ is given by

$$\begin{aligned}&\big(1-q^{\frac{1}{2}N(N-1)}\big)q^{(N-l)(N+l-1)}\prod_{i=1}^{l-2}\big(1-q^{(N-i-1)(N+i)}\big)\\&\quad\times\left\{2\binom{m}{l}q^{\frac{1}{2}N(N-1)}+\left[\binom{m_+}{l}+\binom{m_-}{l}\right]\big(1-q^{\frac{1}{2}N(N-1)}\big)\right\}\mathcal{Q}_{N-l}(T),\end{aligned}$$

with $\mathcal{Q}_0\equiv 1$. Thus the partition function (92) can be expressed as[3]

$$\begin{aligned}\mathcal{Z}(T)&=mq^{N(N-1)}\mathcal{Q}_{N-1}(T)\\&\quad+\big(1-q^{\frac{1}{2}N(N-1)}\big)\sum_{l=2}^{\min(m,N)}q^{(N-l)(N+l-1)}\prod_{i=1}^{l-2}\big(1-q^{(N-i-1)(N+i)}\big)\\&\quad\times\left\{2\binom{m}{l}q^{\frac{1}{2}N(N-1)}+\left[\binom{m_+}{l}+\binom{m_-}{l}\right]\big(1-q^{\frac{1}{2}N(N-1)}\big)\right\}\mathcal{Q}_{N-l}(T),\end{aligned}\tag{95}$$

where the RHS is clearly a polynomial in q on account of Eq. (93). This remarkable formula is one of the main results in the paper.

Remark 12. From Eqs. (92) or (95) it is apparent that the partition function of the D_N chain (13) has a much more complex structure than its BC_N counterpart with $\beta=\beta'=0$, cf. [42, Eq. (53)]. In particular, while for the BC_N chain one can find [53] a description of the spectrum in terms of a suitable generalization of Haldane's *motifs* [54], it is not clear how to implement such a description for the present chain. Note that, for HS chains of A_N type, the existence of such a description is the key ingredient in the proof of the Gaussian character of their level density when the number of sites tends to infinity [55], which is of importance in the context of quantum chaos.

5. Concluding remarks

As mentioned in the introduction, reductions of the BC_N Calogero and Sutherland models obtained by setting suitable coupling constants to zero have been largely ignored in the extensive literature devoted to these models. This is probably due to the fact that these reductions were mostly regarded as trivial limits of the above models. In a previous paper [44], we showed that this is not case by studying the D_N reduction of the (spin) BC_N Calogero model. Moreover, the spin chain of Haldane–Shastry type associated with this reduction was also seen to differ from its BC_N counterpart even more markedly, essentially due to the nontrivial nature of Polychronakos's "freezing trick". The aim of the present paper is to perform a comprehensive study of the D_N reduction of the BC_N spin Sutherland model and its associated spin chain.

A significant part of the paper is devoted to the exact computation of the spectrum of the dynamical spin model (8) and its scalar version (16). We have first provided a rigorous proof of the equivalence of these models to their extended versions $\tilde{H}$ and $\tilde{H}_{\mathrm{sc}}$ defined on the Weyl-invariant configuration space C. The latter set, which turns out to be the N-dimensional generalization (27) of a rhombic dodecahedron, is more complicated in nature than its BC_N counterpart (a hypercube). The motivation for constructing the extended operators $\tilde{H}$ and $\tilde{H}_{\mathrm{sc}}$ is the fact that on their natural domains they essentially coincide with the restriction of a simpler auxiliary operator H', which can be expressed as a sum of squares of a suitable set of Dunkl operators of D_N type. In

3 Note that the terms with $l>m$ in the previous equation vanish identically due to the binomial coefficients. This is in fact a consequence of conditions (65b) and (65c), cf. Eq. (82).

particular, from the spectrum of H' it is not difficult to deduce those of $\tilde{H}$ and $\tilde{H}_{\mathrm{sc}}$, and hence of H and H_{sc}. In order to compute the spectrum of H', we have constructed an appropriate (non-orthogonal) basis of the Hilbert space $L^2(C)$ where this operator acts. This is indeed the key difference with the rational case, for which this step is trivial due to the fact that the (extended) configuration spaces of both the D_N and BC_N models is $\mathbb{R}^N$. Using a method similar to that of Ref. [52], we have shown that H' acts triangularly on the above basis when ordered appropriately. Finally, we have shown how to construct from the latter basis a (non-orthogonal) basis of the Hilbert spaces of $\tilde{H}$ and $\tilde{H}_{\mathrm{sc}}$ on which the action of the latter operators is also upper triangular. In this way we have computed in closed form the spectra of the spin Sutherland model of D_N type (8) and its scalar version (16).

The second main result in the paper is the exact computation of the partition function of the Haldane–Shastry spin chain of D_N type (13) obtained from the spin dynamical model (8) by means of Polychronakos's freezing trick. The latter chain, as is apparent from Eq. (25), cannot be obtained from its BC_N counterpart by taking the limit $(\beta, \beta') \to 0$ due to the presence of an "impurity" term at both endpoints. Our starting point is the fundamental relation (78), expressing the chain's partition function as the large coupling constant limit of the quotient between the partition functions of the corresponding spin dynamical model H and its scalar version H_{sc}. Using the above mentioned results for the spectra of these models, we have been able to evaluate this limit, thereby obtaining Eq. (92) for the chain's partition function. In contrast with the rational case (cf. Remark 11), this partition function is not expressed in a simple way in terms of its BC_N counterparts, since it also involves the partition function of the original (type A) HS chain with a slightly different dispersion relation. We have further simplified Eq. (92) for the partition function, showing how to write it explicitly as a polynomial in $q \equiv \mathrm{e}^{-1/(k_{\mathrm{B}}T)}$ (see Eq. (95)), as should be the case for a finite system. In fact, this simplified formula turns out to be quite efficient for the numerical computation of the chain's spectrum, making it possible to perform a statistical analysis of the spectrum when the number of particles becomes very large. It would be worthwhile to carry out such a study, and compare its results with the corresponding ones for other spin chains of HS type [37,43,44,56–59].

The results of this paper suggest a number of further developments that we shall now discuss. In the first place, we have shown that the D_N reduction of the standard Sutherland model of BC_N type gives rise to an interesting new solvable model that had been previously overlooked. In fact, there are several additional reductions that could be considered, like e.g. those associated with the B_N and C_N root systems, or even more general ones, like the $b = 0$ reduction of the Sutherland model (1). It could also be of interest to consider similar reductions of the comparatively less studied hyperbolic Sutherland model of BC_N type [52]. As we have also shown in this work, these reductions can potentially yield unexpected results in other fields, as for instance the remarkable tiling of $\mathbb{R}^N$ with the N-dimensional generalization of the rhombic dodecahedron uncovered in Section 3 (see Remark 4).

The work presented in this paper has direct implications in the field of multivariate orthogonal polynomials. More precisely (see Remark 9), the eigenfunctions of the D_N reduction of the scalar Sutherland model yield new families of multivariate orthogonal polynomials that cannot be obtained as straightforward limits of the generalized Jacobi polynomials associated with the BC_N Sutherland model [22]. It is to be expected that the additional reductions mentioned above could lead to similar new families of orthogonal polynomials.

An important aspect of spin chains of Haldane–Shastry type that has not been dealt with in this paper is their integrability, which for the original HS chain of A_N type was established by constructing a transfer matrix satisfying the Yang–Baxter equation [27]. This matrix was also used in the latter reference to derive the full Yangian symmetry algebra of this model, which is ultimately responsible for the highly degenerate character of its spectrum. Moreover, the representation theory of the Yangian is closely related to Haldane's elegant description of the spectrum in terms of motifs [54]. It is natural to inquire whether a similar construction is possible for the D_N chain of HS type studied in this paper. In fact, our numerical calculations show that the spectrum of the D_N chain is also highly degenerate, which points to the existence of a large underlying symmetry algebra. The characterization of this algebra, and its precise connection with the Yang–Baxter equation, is yet another open problem motivated by the present work.

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