

On the interpolation of the measure of non-compactness of bilinear operators with weak assumptions on the boundedness of the operator

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Abstract

We complete the range of the parameters in the interpolation formula established by Mastyo and Silva for the measure of non-compactness of a bilinear operator interpolated by the real method.

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Dedicated to Professor Hans Triebel on the occasion of his 85th birthday.

1. Introduction

One of the questions considered by Calderón [4] in his seminal paper on the complex method was the interpolation of compact bilinear operators. The counterpart for the real method of Lions and Peetre [18] has been done recently, starting with the papers by Fernandez and Silva [11] and Fernández-Cabrera and Martínez [12, 13]. A motivation for the research has been the fact that compact bilinear operators arise rather naturally in harmonic analysis. Namely, commutators of bilinear Calderón-Zygmund operators and multiplication by functions in the subspace CMO of BMO are compact (see the papers by Bényi and Torres [1], Cobos, Fernández-Cabrera and Martínez [6] and Torres, Xue and Yan [26]). Other results on interpolation of compact bilinear operators can be

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found in the papers by Mastyo and Silva [22] and Cobos, Fernández-Cabrera and Martínez [7, 8].

Quantitative versions in terms of the measure of non-compactness of some of these qualitative results have been established by Mastyo and Silva [21] and Besoy and Cobos [3]. Both papers refer to the real method but they work with different assumptions on the operator. Mastyo and Silva assume that T is defined from $(A_0 \cap A_1) \times (B_0 \cap B_1)$ into $E_0 \cap E_1$ with

$$\|T(a, b)\|_{E_j} \leq M_j \|a\|_{A_j} \|b\|_{B_j}, \quad a \in A_0 \cap A_1, \quad b \in B_0 \cap B_1, \quad j = 0, 1,$$

and establish the following log-convexity inequality for the measure of non-compactness

$$(1.1) \quad \beta(T : \bar{A}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow \bar{E}_{\theta,r}) \\ \leq C \beta(T : A_0^\circ \times B_0^\circ \longrightarrow E_0^\circ)^{1-\theta} \beta(T : A_1^\circ \times B_1^\circ \longrightarrow E_1^\circ)^\theta,$$

where $1 \leq p, q < \infty$, $1 < r < \infty$ and $1/p + 1/q = 1 + 1/r$ (see [21, Theorem 3.2]). Besoy and Cobos ask a stronger assumption on T . Namely, they suppose that T is bounded from $(A_0 + A_1) \times (B_0 + B_1)$ into $E_0 + E_1$ and that the restrictions $T : A_j \times B_j \longrightarrow E_j$ are also bounded for $j = 0, 1$. In this case, they show that the log-convexity inequality holds for operators acting among couples of quasi-Banach spaces and for the whole range of the parameters p, q, r (see [3, Theorem 3.5]). More precisely, if (E_0, E_1) is a couple of s -Banach spaces, then the conditions on the parameters are

$$(1.2) \quad 0 < p, q, r \leq \infty \quad \text{with} \quad 1/r = \begin{cases} 1/p + 1/q - 1/s & \text{if } p, q \geq s, \\ 1/\max(p, q) & \text{if } p < s \text{ or } q < s. \end{cases}$$

Approaches followed in [21] and [3] are completely different.

In applications, the weaker assumption on T is more handy. For this reason, it is important to extend inequality (1.1) to the remaining range of parameters. For couples of Banach spaces, this means the values of (1.2) with $s = 1$. Accordingly, we prove in this paper such a result.

Our techniques are a refinement of those used by Mastyo and Silva [21] based on duality results for bilinear operators introduced by Ramanujan and Schock [24] and the corresponding results established by Cobos, Fernández-Martínez and Martínez [9] and Fernández-Martínez [14] for the measure of non-compactness in the linear case (see also the paper by Edmunds and Teixeira [25]). Duality is the reason why we work with Banach couples.

We start by reviewing in Section 2 the basic results on the real interpolation method. We also establish there the variant of the bilinear interpolation theorem that we will need later. In Section 3 we study the connections between the measure of non-compactness of a bilinear operator and the measure of non-compactness of its adjoint operator, which is a linear operator. Finally, in Section 4, we establish the log-convexity inequality.

2. Preliminaries

By a *Banach couple* $\bar{A} = (A_0, A_1)$ we mean two Banach spaces A_0, A_1 which are continuously embedded in some Hausdorff topological vector space.

We endow $A_0 + A_1 = \Sigma(\bar{A})$ and $A_0 \cap A_1 = \Delta(\bar{A})$ with the norms $K(1, \cdot)$ and $J(1, \cdot)$, respectively, where for $t > 0$

$$K(t, a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}, a \in \Sigma(\bar{A}),$$

and

$$J(t, a) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}, a \in \Delta(\bar{A}),$$

are the Peetre's functionals.

Let $0 < \theta < 1$ and $0 < q \leq \infty$. The *real interpolation space* $(A_0, A_1)_{\theta, q} = \bar{A}_{\theta, q}$ is formed by all $a \in A_0 + A_1$ such that

$$\|a\|_{\bar{A}_{\theta, q}} = \left(\int_0^\infty [t^{-\theta} K(t, a)]^q \frac{dt}{t} \right)^{1/q} < \infty,$$

(the integral should be replaced by the supremum when $q = \infty$). The space $(A_0, A_1)_{\theta, q}$ is a Banach space if $1 \leq q \leq \infty$ and a quasi-Banach space if $0 < q < 1$ (see [2, 27]).

The real interpolation method has the interpolation property for linear operators: Let $\bar{B} = (B_0, B_1)$ be another Banach couple and let R be a linear operator from $A_0 + A_1$ into $B_0 + B_1$ whose restrictions $R : A_j \rightarrow B_j$ are bounded for $j = 0, 1$. Then $R : \bar{A}_{\theta, q} \rightarrow \bar{B}_{\theta, q}$ is bounded with

$$\|R\|_{\bar{A}_{\theta, q}, \bar{B}_{\theta, q}} \leq \|R\|_{A_0, B_0}^{1-\theta} \|R\|_{A_1, B_1}^\theta.$$

The space $(A_0, A_1)_{\theta, q}$ can also be introduced using series instead of integrals because the functional

$$\|a\|_{\theta, q} = \left(\sum_{m=-\infty}^{\infty} [2^{-\theta m} K(2^m, a)]^q \right)^{1/q},$$

(the sum should be replaced by the supremum if $q = \infty$) is an equivalent norm (respectively, quasi-norm) to $\|\cdot\|_{\bar{A}_{\theta, q}}$ if $1 \leq q \leq \infty$ (respectively, if $0 < q < 1$).

The real interpolation space can be equivalently described by means of the J -functional. Indeed, $(A_0, A_1)_{\theta, q}$ consists of all $a \in A_0 + A_1$ for which there is $(u_m) \subseteq A_0 \cap A_1$ such that

$$a = \sum_{m=-\infty}^{\infty} u_m \quad (\text{convergence in } A_0 + A_1)$$

and

$$\left(\sum_{m=-\infty}^{\infty} [2^{-\theta m} J(2^m, u_m)]^q \right)^{1/q} < \infty.$$

Moreover, the functional

$$\|a\|_{\theta, q}^J = \inf \left\{ \left(\sum_{m=-\infty}^{\infty} [2^{-\theta m} J(2^m, u_m)]^q \right)^{1/q} : a = \sum_{m=-\infty}^{\infty} u_m, (u_m) \subseteq A_0 \cap A_1 \right\}$$

is equivalent to $\|\cdot\|_{\bar{A}_{\theta, q}}$.

Subsequently, if A is a quasi-Banach space, we put A^* for its dual space. If $A_0 \cap A_1 \subseteq A$, we write A° for the closure of $A_0 \cap A_1$ in A . Since $A_0 \cap A_1 =$

$A_0^\circ \cap A_1^\circ$, it follows from the J -description of the real interpolation space that $(A_0, A_1)_{\theta, q} = (A_0^\circ, A_1^\circ)_{\theta, q}$. The J -description also gives that $A_0 \cap A_1$ is dense in $\bar{A}_{\theta, q}$ if $q < \infty$. Therefore, $\bar{A}_{\theta, q}^\circ = \bar{A}_{\theta, q}$ if $0 < q < \infty$.

We say that the Banach couple \bar{A} is *regular* if $A_0 \cap A_1$ is dense in A_0 and in A_1 . If this is the case, then $\bar{A}^* = (A_0^*, A_1^*)$ is also a Banach couple and the following duality formulae hold with equivalence of norms:

(2.1) If $1 \leq q < \infty$, $1/q + 1/q' = 1$ and $0 < \theta < 1$, then

$$(A_0, A_1)_{\theta, q}^* = (A_0^*, A_1^*)_{\theta, q'}.$$

(2.2) If $q = \infty$ and $0 < \theta < 1$, then

$$((A_0, A_1)_{\theta, \infty}^\circ)^* = (A_0^*, A_1^*)_{\theta, 1}.$$

(2.3) If $0 < q < 1$ and $0 < \theta < 1$, then

$$(A_0, A_1)_{\theta, q}^* = (A_0^*, A_1^*)_{\theta, \infty}.$$

See [2, Theorem 3.7.1 and Remark in p. 55] and [23, (0.5) in p. 124].

Let A, B, E be Banach spaces and let $T : A \times B \rightarrow E$ be a bilinear operator. We say that T is *bounded* if

$$\|T\|_{A \times B, E} = \sup \{ \|T(a, b)\|_E : \|a\|_A \leq 1, \|b\|_B \leq 1 \} < \infty.$$

We write $\mathcal{L}(A \times B, E)$ for the space of all bounded bilinear operators from $A \times B$ into E .

Let $\bar{E} = (E_0, E_1)$ be another Banach couple. By $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$ we mean that T is a bilinear operator defined on $(A_0 \cap A_1) \times (B_0 \cap B_1)$ with values in $E_0 \cap E_1$ such that there are constants $M_j > 0$ with

$$(2.4) \quad \|T(a, b)\|_{E_j} \leq M_j \|a\|_{A_j} \|b\|_{B_j}, \quad a \in \Delta(\bar{A}), \quad b \in \Delta(\bar{B}), \quad j = 0, 1.$$

Using (2.4), it is not hard to check that T may be uniquely extended to a bilinear operator $T : A_j^\circ \times B_j^\circ \rightarrow E_j^\circ$ with $\|T\|_{A_j^\circ \times B_j^\circ, E_j^\circ} \leq M_j$.

The behaviour of bounded linear operators under real interpolation was already considered by Lions and Peetre [18] in their foundational paper on the real method. Their result was extended by Karadzhov [16] to the full range for the parameters (see also the paper by König [17] for a proof). In our later considerations we are going to need the following variant of the bilinear interpolation theorem.

Theorem 2.1. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be Banach couples and let T be a bilinear operator from $(A_0 \cap A_1) \times (B_0 \cap B_1)$ into $E_0 \cap E_1$ such that there are positive constants M_j with*

$$\|T(a, b)\|_{E_j} \leq M_j \|a\|_{A_j} \|b\|_{B_j}, \quad a \in A_0 \cap A_1, \quad b \in B_j, \quad j = 0, 1.$$

Let $0 < \theta < 1$ and $0 < p, q, r \leq \infty$ be positive numbers satisfying

$$\frac{1}{r} = \begin{cases} \frac{1}{p} + \frac{1}{q} - 1 & \text{if } p, q \geq 1, \\ \frac{1}{\max(p, q)} & \text{if } p < 1 \text{ or } q < 1. \end{cases}$$

Then there is a constant $C > 0$ independent of T such that

$$\|T(a, b)\|_{\bar{E}_{\theta, r}} \leq C M_0^{1-\theta} M_1^\theta \|a\|_{\bar{A}_{\theta, p}} \|b\|_{\bar{B}_{\theta, q}}, \quad a \in \Delta(\bar{A}), \quad b \in \bar{B}_{\theta, q}.$$

Moreover, T may be uniquely extended to a bounded bilinear operator

$$T : \bar{A}_{\theta, p}^\circ \times \bar{B}_{\theta, q} \longrightarrow \bar{E}_{\theta, r}.$$

Proof. Take $n \in \mathbb{Z}$ such that $2^n \leq M_1/M_0 < 2^{n+1}$. Given any $a \in A_0 \cap A_1$, $u \in B_0 \cap B_1$ and $m, k \in \mathbb{Z}$, if $a = a_0 + a_1$ with $a_j \in A_j$, we obtain

$$\begin{aligned} K(2^m, T(a, u)) &\leq \|T(a_0, u)\|_{E_0} + 2^m \|T(a_1, u)\|_{E_1} \\ &\leq M_0 \|a_0\|_{A_0} \|u\|_{B_0} + 2^m M_1 \|a_1\|_{A_1} \|u\|_{B_1} \\ &\leq \max(M_0, 2^{-n} M_1) (\|a_0\|_{A_0} + 2^{m-k} \|a_1\|_{A_1}) J(2^{k+n}, u). \end{aligned}$$

Hence,

$$(2.5) \quad K(2^m, T(a, u)) \leq 2M_0 K(2^{m-k}, a) J(2^{k+n}, u).$$

Now take any $b \in \bar{B}_{\theta, q}$ and let $b = \sum_{k=-\infty}^{\infty} u_k$ be any J -representation of b . Since

$$\sum_{k=-\infty}^{\infty} \|u_k\|_{B_0+B_1} < \infty, \text{ we also have that } b = \sum_{k=-\infty}^{\infty} u_{k+n} \text{ in } B_0 + B_1.$$

Suppose that $p < 1$ and $p \leq q$, so $r = q$. We have

$$K(2^m, T(a, b)) \leq \sum_{k=-\infty}^{\infty} K(2^m, T(a, u_{k+n})) \leq \left(\sum_{k=-\infty}^{\infty} K(2^m, T(a, u_{k+n}))^p \right)^{1/p}.$$

Combining this inequality with (2.5) and using Young's inequality with parameters $p/q = 1 + p/q - 1$, we derive

$$\begin{aligned} \|T(a, b)\|_{\bar{E}_{\theta, q}} &\leq 2M_0 \left\| \left(\sum_{k=-\infty}^{\infty} 2^{-\theta(m-k)p} K(2^{m-k}, a)^p 2^{-\theta kp} J(2^{k+n}, u_{k+n})^p \right) \right\|_{\ell_{q/p}}^{1/p} \\ &\leq 2M_0 \left\| (2^{-\theta m} K(2^m, a))^p \right\|_{\ell_1}^{1/p} \left\| (2^{-\theta k} J(2^{k+n}, u_{k+n})) \right\|_{\ell_q} \\ &\leq 2M_0 2^{\theta n} \|a\|_{\theta, p} \left\| (2^{-\theta k} J(2^k, u_k)) \right\|_{\ell_q}. \end{aligned}$$

Taking the infimum over all J -representations of b , we derive that

$$\|T(a, b)\|_{\bar{E}_{\theta, q}} \leq C M_0^{1-\theta} M_1^\theta \|a\|_{\bar{A}_{\theta, p}} \|b\|_{\bar{B}_{\theta, q}}, \quad a \in \Delta(\bar{A}), \quad b \in \bar{B}_{\theta, q}.$$

Finally, since $A_0 \cap A_1$ is dense in $\bar{A}_{\theta, p}$, the operator T may be uniquely extended to a bounded bilinear operator $T : \bar{A}_{\theta, p} \times \bar{B}_{\theta, q} \longrightarrow \bar{E}_{\theta, q}$.

If $q < p \leq 1$, then $r = p$ and we can proceed as before but using now that

$$K(2^m, T(a, b)) \leq \left(\sum_{k=-\infty}^{\infty} K(2^m, T(a, u_{k+n}))^q \right)^{1/q}$$

and Young's inequality with parameters $q/p = q/p + 1 - 1$. The cases $q < 1$ and $q \leq p$, and $p < q \leq 1$ are similar. Finally, if $1 \leq p, q$ then $1/r = 1/p + 1/q - 1$ and we can proceed directly with Young's inequality. \square

Others results on interpolation of bilinear operators can be found, for example, in the papers by Janson [15], Mastysłó [19, 20] and Cobos, Fernández-Cabrera and Martínez [6].

3. Measure of non-compactness

Let A, B be Banach spaces. We write U_A for the closed unit ball of A and define U_B similarly. Given any bounded linear operator $R \in \mathcal{L}(A, B)$, the (ball) measure of non-compactness $\beta(R) = \beta(R : A \rightarrow B)$ is defined to be the infimum of the set of all $\sigma > 0$ for which there is a finite subset $\{b_1, \dots, b_s\} \subseteq B$ such that

$$R(U_A) \subseteq \bigcup_{k=1}^s \{b_k + \sigma U_B\},$$

(see [5, 10]).

Clearly, the operator R is compact if and only if $\beta(R) = 0$.

Let $R^* \in \mathcal{L}(B^*, A^*)$ be the adjoint operator of R . A well-known result of Schauder says that R is compact if and only if R^* is compact. If the operator R is not compact, then the following inequalities hold for the measure of non-compactness

$$(3.1) \quad \frac{1}{2} \beta(R : A \rightarrow B) \leq \beta(R^* : B^* \rightarrow A^*) \leq 2 \beta(R : A \rightarrow B),$$

(see [10, Corollary 2.10, p. 12]).

Let E be another Banach spaces and let $T : A \times B \rightarrow E$ be a bounded bilinear operator. The (ball) *measure of non-compactness* $\beta(T) = \beta(T : A \times B \rightarrow E)$ of T is the infimum of all $\sigma > 0$ for which there exists a finite subset $\{z_1, \dots, z_s\} \subseteq E$ such that

$$T(U_A, U_B) = \{T(a, b) : a \in U_A, b \in U_B\} \subseteq \bigcup_{k=1}^s \{z_k + \sigma U_E\}.$$

The operator T is said to be *compact* if for any bounded sets $V \subseteq A, W \subseteq B$ we have that the closure of $T(V, W)$ is compact in E . Again, the operator T is compact if and only if its measure of non-compactness is 0.

Following Ramanujan and Schock [24], the *adjoint operator* T^\times of T is the linear map

$$T^\times : E^* \rightarrow \mathcal{L}(A \times B, \mathbb{K})$$

defined by $(T^\times f)(a, b) = f[T(a, b)]$. Here \mathbb{K} is the scalar field.

It turns out that $\|T\|_{A \times B, E} = \|T^\times\|_{E^*, \mathcal{L}(A \times B, \mathbb{K})}$. Moreover, T is compact if and only if T^\times is compact (see [24, Theorem 2.6]).

Next we study inequalities of the type (3.1) in the bilinear setting. For operators acting among Banach couples, this question has been studied in [21, Lemma 3.1],

Theorem 3.1. *Let A, B, E be Banach spaces and let $T \in \mathcal{L}(A \times B, E)$. Then we have*

$$\beta(T^\times : E^* \rightarrow \mathcal{L}(A \times B, \mathbb{K})) \leq 4 \beta(T : A \times B \rightarrow E).$$

Proof. Given any $\varepsilon > 0$, we can find finite subsets $\{a_1, \dots, a_s\} \subseteq U_A$ and $\{b_1, \dots, b_s\} \subseteq U_B$ such that for any $(a, b) \in U_A \times U_B$ there exists $1 \leq k \leq s$ with

$$(3.2) \quad \|T(a, b) - T(a_k, b_k)\| \leq 2 \beta(T) + \varepsilon.$$

Let $S : E^* \rightarrow \mathbb{K}^s$ be the linear operator assigning to each $f \in E^*$ the s -tuple $Sf = (f[T(a_1, b_1)], \dots, f[T(a_s, b_s)])$. Since S has finite rank, S is compact.

Therefore, there exists a finite subset of functionals $\{f_1, \dots, f_m\} \subseteq U_{E^*}$ such that for any $f \in U_{E^*}$ there is $1 \leq n \leq m$ with $\|Sf - Sf_n\|_{\mathbb{K}^*} \leq \varepsilon$. Whence,

$$(3.3) \quad |f[T(a_k, b_k)] - f_n[T(a_k, b_k)]| \leq \varepsilon \text{ for any } 1 \leq k \leq s.$$

Then, given any $f \in U_{E^*}$ if we take n satisfying (3.3), and for any $(a, b) \in U_A \times U_B$ we choose k satisfying (3.2), we obtain

$$\begin{aligned} |(T^\times f - T^\times f_n)(a, b)| &= |f[T(a, b)] - f_n[T(a, b)]| \\ &\leq |f[T(a, b)] - f[T(a_k, b_k)]| + |f[T(a_k, b_k)] - f_n[T(a_k, b_k)]| \\ &\quad + |f_n[T(a_k, b_k)] - f_n[T(a, b)]| \\ &\leq \|T(a, b) - T(a_k, b_k)\|_E + \varepsilon + \|T(a_k, b_k) - T(a, b)\|_E \leq 4\beta(T) + 3\varepsilon. \end{aligned}$$

This yields that $\|T^\times f - T^\times f_n\|_{A \times B, \mathbb{K}} \leq 4\beta(T) + 3\varepsilon$. Consequently,

$$\beta(T^\times : E^* \longrightarrow \mathcal{L}(A \times B, \mathbb{K})) \leq 4\beta(T : A \times B \longrightarrow E).$$

□

Theorem 3.2. *Let A, B, E be Banach spaces and let $T \in \mathcal{L}(A \times B, E)$. Then we have*

$$\beta(T : A \times B \longrightarrow E) \leq 8\beta(T^\times : E^* \longrightarrow \mathcal{L}(A \times B, \mathbb{K})).$$

Proof. Put $W = \overline{T^\times(E^*)}$ which is a Banach space with the induced norm from $\mathcal{L}(A \times B, \mathbb{K})$. Let $\{R_1, \dots, R_s\} \subseteq \mathcal{L}(A \times B, \mathbb{K})$ and $\sigma > 0$ satisfy that

$$T^\times(U_{E^*}) \subseteq \bigcup_{k=1}^s \{R_k + \sigma U_{\mathcal{L}(A \times B, \mathbb{K})}\}.$$

We may assume that $W \cap \{R_k + \sigma U_{\mathcal{L}(A \times B, \mathbb{K})}\} \neq \emptyset$ for $k = 1, \dots, s$. Pick $S_k \in W \cap \{R_k + \sigma U_{\mathcal{L}(A \times B, \mathbb{K})}\}$. Then

$$T^\times(U_{E^*}) \subseteq \bigcup_{k=1}^s \{S_k + 2\sigma U_W\}.$$

This yields that

$$\beta(T^\times : E^* \longrightarrow W) \leq 2\beta(T^\times : E^* \longrightarrow \mathcal{L}(A \times B, \mathbb{K})).$$

Now consider the adjoint operator $(T^\times)^* : W^* \longrightarrow E^{**}$ and let $R : A \times B \longrightarrow W^*$ the operator defined by $R(a, b) = g_{(a,b)}$ where

$$g_{(a,b)}(T^\times f) = f[T(a, b)].$$

The operator R is bilinear and has norm less than or equal to 1 because

$$\begin{aligned} \|R(a, b)\|_{W^*} &= \sup\{|g_{(a,b)}(T^\times f)| : \|T^\times f\|_{A \times B, \mathbb{K}} \leq 1\} \\ &= \sup\{|f[T(a, b)]| : \|T^\times f\|_{A \times B, \mathbb{K}} \leq 1\} \\ &= \sup\{|(T^\times f)(a, b)| : \|T^\times f\|_{A \times B, \mathbb{K}} \leq 1\} \leq \|a\|_A \|b\|_B. \end{aligned}$$

The following diagram is useful

$$\begin{array}{ccccc}
A \times B & \xrightarrow{T} & E & \xrightarrow{\pi} & E^{**} \\
& \searrow R & & \nearrow (T^\times)^* & \\
& & W^* & &
\end{array}$$

where $\pi(w) = \widehat{w}$ is the natural embedding from E into its bidual E^{**} . Note that the diagram commutes because

$$\begin{aligned}
((T^\times)^* R(a, b))(f) &= R(a, b)(T^\times f) = g_{(a,b)}(T^\times f) \\
&= f[T(a, b)] = \widehat{T(a, b)}(f) = \pi(T(a, b))(f).
\end{aligned}$$

Consequently, using the diagram and (3.1), we obtain

$$\begin{aligned}
\beta(T : A \times B \longrightarrow E) &\leq 2 \beta(\pi T : A \times B \longrightarrow E^{**}) \\
&\leq 2 \|R\|_{A \times B, W^*} \beta((T^\times)^* : W^* \longrightarrow E^{**}) \\
&\leq 4 \beta(T^\times : E^* \longrightarrow W) \\
&\leq 8 \beta(T^\times : E^* \longrightarrow \mathcal{L}(A \times B, \mathbb{K})).
\end{aligned}$$

□

4. Interpolation of the measure of non-compactness

We start with an auxiliary result which complements an embedding of [7, p. 5] (see also [21, Theorem 2.1]).

Subsequently, for $0 < r \leq \infty$ we put

$$\tilde{r} = \begin{cases} \infty & \text{if } 0 < r \leq 1, \\ r', 1/r + 1/r' = 1 & \text{if } 1 < r \leq \infty. \end{cases}$$

Lemma 4.1. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be regular Banach couples. For $j = 0, 1$, put $X_j = \mathcal{L}(A_j \times B_j, \mathbb{K})$ and let $0 < \theta < 1$ and $0 < p, q, r \leq \infty$ such that*

$$\frac{1}{r} = \begin{cases} \frac{1}{p} + \frac{1}{q} - 1 & \text{if } p, q \geq 1, \\ \frac{1}{\max(p, q)} & \text{if } p < 1 \text{ or } q < 1. \end{cases}$$

Then the following continuous embedding holds

$$(X_0, X_1)_{\theta, \tilde{r}} \hookrightarrow \mathcal{L}(\bar{A}_{\theta, p}^\circ \times \bar{B}_{\theta, q}^\circ, \mathbb{K}).$$

Proof. The pair $\bar{X} = (X_0, X_1)$ is a Banach couple because $X_j \hookrightarrow \mathcal{L}(\Delta(\bar{A}) \times \Delta(\bar{B}), \mathbb{K})$, $j = 0, 1$. Let

$$\Phi : \Delta(\bar{A}) \times \mathcal{L}(\Delta(\bar{A}) \times \Delta(\bar{B}), \mathbb{K}) \longrightarrow \Delta(\bar{B})^*$$

be the bilinear operator defined by $\Phi(a, R)(b) = R(a, b)$. Since

$$|R(a, b)| \leq \|R\|_{\Delta(\bar{A}) \times \Delta(\bar{B}), \mathbb{K}} \|a\|_{\Delta(\bar{A})} \|b\|_{\Delta(\bar{B})},$$

the operator Φ has norm less than or equal to 1. Moreover, if $a \in \Delta(\bar{A})$ and $R \in X_j$, we get

$$|R(a, b)| \leq \|R\|_{X_j} \|a\|_{A_j} \|b\|_{B_j}.$$

So

$$\Phi : (\Delta(\bar{A}), \|\cdot\|_{A_j}) \times X_j \longrightarrow B_j^*, \quad j = 0, 1,$$

is also bounded, with norm less than or equal to 1.

By the duality formulae (2.1), (2.2), (2.3), we know that

$$((B_0, B_1)_{\theta, q}^\circ)^* = (B_0^*, B_1^*)_{\theta, \tilde{q}} = \overline{B^*}_{\theta, \tilde{q}}.$$

Moreover, we have

$$\frac{1}{\tilde{q}} = \begin{cases} \frac{1}{p} + \frac{1}{r} - 1 & \text{if } p \geq 1, \\ \frac{1}{r} & \text{if } p < 1. \end{cases}$$

Whence, applying Theorem 2.1, we get that Φ may be uniquely extended to a bounded bilinear operator

$$\Phi : \bar{A}_{\theta, p}^\circ \times \bar{X}_{\theta, \tilde{r}} \longrightarrow \overline{B^*}_{\theta, \tilde{q}} = ((B_0, B_1)_{\theta, q}^\circ)^*.$$

Consequently, there is a constant $C > 0$ such that for any $R \in \bar{X}_{\theta, \tilde{r}}$ and any $a \in \Delta(\bar{A})$, $b \in \Delta(\bar{B})$, we have

$$|R(a, b)| = |\Phi(a, R)(b)| \leq \|\Phi(a, R)\|_{\overline{B^*}_{\theta, \tilde{q}}} \|b\|_{\bar{B}_{\theta, q}} \leq C \|R\|_{\bar{X}_{\theta, \tilde{r}}} \|a\|_{\bar{A}_{\theta, p}} \|b\|_{\bar{B}_{\theta, q}}.$$

This yields that $R \in \mathcal{L}(\bar{A}_{\theta, p}^\circ \times \bar{B}_{\theta, q}^\circ, \mathbb{K})$ and that the embedding

$$\bar{X}_{\theta, \tilde{r}} \hookrightarrow \mathcal{L}(\bar{A}_{\theta, p}^\circ \times \bar{B}_{\theta, q}^\circ, \mathbb{K})$$

is bounded with norm less than or equal to C . \square

Now we are ready to establish the announced result on interpolation of the measure of non-compactness of bilinear operators.

Theorem 4.2. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be Banach couples and let $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$. Let $0 < \theta < 1$ and $0 < p, q, r \leq \infty$ such that*

$$\frac{1}{r} = \begin{cases} \frac{1}{p} + \frac{1}{q} - 1 & \text{if } p, q \geq 1, \\ \frac{1}{\max(p, q)} & \text{if } p < 1 \text{ or } q < 1. \end{cases}$$

Then

$$\begin{aligned} \beta(T : \bar{A}_{\theta, p}^\circ \times \bar{B}_{\theta, q}^\circ \longrightarrow \bar{E}_{\theta, r}^\circ) \\ \leq C \beta(T : A_0^\circ \times B_0^\circ \longrightarrow E_0^\circ)^{1-\theta} \beta(T : A_1^\circ \times B_1^\circ \longrightarrow E_1^\circ)^\theta, \end{aligned}$$

Here C is a constant independent of T .

Proof. According to the bilinear interpolation theorem (see, for example [13, Theorem 4.1]), we know that T may be uniquely extend to a bounded bilinear operator $T : \bar{A}_{\theta, p}^\circ \times \bar{B}_{\theta, q}^\circ \longrightarrow \bar{E}_{\theta, r}^\circ$. Hence, by Theorem 3.2, we have

$$(4.1) \quad \beta(T : \bar{A}_{\theta, p}^\circ \times \bar{B}_{\theta, q}^\circ \longrightarrow \bar{E}_{\theta, r}^\circ) \leq 8 \beta(T^\times : (\bar{E}_{\theta, r}^\circ)^* \longrightarrow \mathcal{L}(\bar{A}_{\theta, p}^\circ \times \bar{B}_{\theta, q}^\circ, \mathbb{K})).$$

Let $X_j = \mathcal{L}(A_j^\circ \times B_j^\circ, \mathbb{K})$, $j = 0, 1$. Since $T : A_j^\circ \times B_j^\circ \rightarrow E_j^\circ$, is bounded, the restrictions $T^\times : (E_j^\circ)^* \rightarrow X_j$ are also bounded and, by Theorem 3.1, we have

$$(4.2) \quad \beta(T^\times : (E_j^\circ)^* \rightarrow X_j) \leq 4\beta(T : A_j^\circ \times B_j^\circ \rightarrow E_j^\circ), \quad j = 0, 1.$$

Using the formula for the measure of non-compactness of a linear operator interpolated by the real method of Cobos, Fernández-Martínez and Martínez [9, Theorem 1.2] (Banach case) and Fernández-Martínez [14, Theorem 3.1] (quasi-Banach case) and (4.2) we derive that

$$(4.3) \quad \begin{aligned} \beta(T^\times : ((E_0^\circ)^*, (E_1^\circ)^*)_{\theta, \tilde{r}} \rightarrow (X_0, X_1)_{\theta, \tilde{r}}) \\ \leq C_1 \beta(T^\times : (E_0^\circ)^* \rightarrow X_0)^{1-\theta} \beta(T^\times : (E_1^\circ)^* \rightarrow X_1)^\theta \\ \leq 4C_1 \beta(T : A_0^\circ \times B_0^\circ \rightarrow E_0^\circ)^{1-\theta} \beta(T : A_1^\circ \times B_1^\circ \rightarrow E_1^\circ)^\theta. \end{aligned}$$

The duality formulae (2.1), (2.2), (2.3) imply that

$$((E_0^\circ)^*, (E_1^\circ)^*)_{\theta, \tilde{r}} = (E_0^\circ, E_1^\circ)_{\theta, r}^*.$$

Hence, it follows from Lemma 4.1 and (4.3) that

$$\begin{aligned} \beta(T^\times : (E_0^\circ, E_1^\circ)_{\theta, r}^* \rightarrow \mathcal{L}(\bar{A}_{\theta, p}^\circ \times \bar{B}_{\theta, q}^\circ, \mathbb{K})) \\ \leq C_2 \beta(T^\times : (E_0^\circ, E_1^\circ)_{\theta, r}^* \rightarrow (X_0, X_1)_{\theta, \tilde{r}}) \\ \leq C_3 \beta(T : A_0^\circ \times B_0^\circ \rightarrow E_0^\circ)^{1-\theta} \beta(T : A_1^\circ \times B_1^\circ \rightarrow E_1^\circ)^\theta. \end{aligned}$$

This estimate combined with (4.1) complete the proof. \square

As a direct consequence of Theorem 4.2 we derive the following compactness result:

Theorem 4.3. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be Banach couples and let $T \in \mathcal{B}(\bar{A} \times \bar{B}, \bar{E})$ such that $T : A_j^\circ \times B_j^\circ \rightarrow E_j^\circ$ compactly for $j = 0$ or $j = 1$. Let $0 < \theta < 1$ and $0 < p, q, r \leq \infty$ such that*

$$\frac{1}{r} = \begin{cases} \frac{1}{p} + \frac{1}{q} - 1 & \text{if } p, q \geq 1, \\ \frac{1}{\max(p, q)} & \text{if } p < 1 \text{ or } q < 1. \end{cases}$$

Then T may be uniquely extended to a compact bilinear operator from $(A_0, A_1)_{\theta, p}^\circ \times (B_0, B_1)_{\theta, q}^\circ$ to $(E_0, E_1)_{\theta, r}^\circ$.

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