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A topology for the sets of shape morphisms[☆]

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Abstract

We introduce a topology on the set of shape morphisms between arbitrary topological spaces X , Y , $Sh(X, Y)$. These spaces allow us to extend, in a natural way, some classical concepts to the realm of topological spaces. Several applications are given to obtain relations between shape theory and \mathbb{N} -compactness and shape-theoretic properties of the spaces of quasicomponents. © 1999 Elsevier Science B.V. All rights reserved.

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This paper is dedicated to B.J. Ball, in memoriam

Introduction

This paper follows the line initiated in [17], where it is given a complete ultrametric on the sets $Sh(X, Y)$, of shape morphisms between compact metric spaces. The general aspects of the introduction in [17] are also valid for this article.

In this article we introduce a topology on the sets $Sh(X, Y)$, where X and Y are arbitrary topological spaces, in such a way that it extends topologically the construction given in [17]. The reader can find in [16] and [18] applications of these techniques.

From the fact that the composition $\Omega : Sh(X, Y) \times Sh(Y, Z) \rightarrow Sh(X, Z)$, $\Omega(\alpha, \beta) = \beta \circ \alpha$, is continuous it is possible to derive several consequences, for example:

- We construct new shape invariants.
- We define new categories from old (in particular we extend the internal shape theory to arbitrary spaces).

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Some relations are obtained between shape theory and \mathbb{N} -compactness (see [8] and its references), in particular, a new characterization of \mathbb{N} -compactness is given for spaces of nonmeasurable cardinality.

When $X = \{*\}$ (a one point space), the space $Sh(X, Y)$ is intimately related to the space of quasicomponents QY . This space, and also the corresponding space of components $\square Y$, has been extensively studied by Ball in the series of papers [1–3]. In particular, Ball established several shape-theoretic properties of the spaces of quasicomponents. The last part of our paper is dedicated to show some connections between our results and Ball's work on quasicomponents (see also [21] and [6] for previous results of the authors motivated by Ball's theorems).

Information about shape theory can be found in [5,7,13]. In this case we recommend specially [13] for notation and definitions used here.

1. The basic construction

Let X, Y be topological spaces. Assume $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ to be an inverse system in pro-HPol and let $\mathbf{q}: Y \rightarrow \mathbf{Y}$ be an HPol-expansion of Y . For every $\mu \in M$ and $F \in Sh(X, Y)$ take $V_\mu^F = \{G \in Sh(X, Y) \text{ such that } q_\mu \circ F = q_\mu \circ G \text{ as homotopy classes to } Y_\mu\}$.

Let us prove the following

Proposition 1. *The family $\{V_\mu^F: F \in Sh(X, Y), \mu \in M\}$ is a base for a topology T_q in $Sh(X, Y)$. Moreover, the topology so obtained depends only on X and Y , in the sense that if $\mathbf{q}': Y \rightarrow \mathbf{Y}' = (Y_{\nu}, q_{\nu\nu'}, N)$ is another HPol-expansion of Y , then the identity map*

$$(Sh(X, Y), T_q) \rightarrow (Sh(X, Y), T_{q'})$$

is a homeomorphism.

Proof. In order to see the first assertion, note that $F \in V_\mu^F$ for every $\mu \in M$ and every $F \in Sh(X, Y)$. On the other hand, given $F \in Sh(X, Y)$ and $\mu_1, \mu_2 \in M$ one has that $F \in V_\mu^F \subset V_{\mu_1}^F \cap V_{\mu_2}^F$ for all $\mu \geq \mu_1, \mu_2$.

Now take another HPol-expansion of Y $\mathbf{q}': Y \rightarrow \mathbf{Y}' = (Y_\nu, q_{\nu\nu'}, N)$. There is a unique isomorphism $\mathbf{i}: \mathbf{Y} \rightarrow \mathbf{Y}'$, given by (i_ν, ϕ) , such that $\mathbf{i} \circ \mathbf{q} = \mathbf{q}'$.

We can represent each shape morphism F as the approaching morphism $\mathbf{h}: X \rightarrow \mathbf{Y}$ or $\mathbf{h}': X \rightarrow \mathbf{Y}'$ such that $\mathbf{h} = \mathbf{q} \circ F$ and $\mathbf{h}' = \mathbf{i} \circ \mathbf{h} = \mathbf{q}' \circ F$ in pro-Sh.

Take $\nu \in N$ and $F \in Sh(X, Y)$. Consider $\phi(\nu) \in M$. For each $G \in V_{\phi(\nu)}^F$ we have $q_{\phi(\nu)} \circ F = q_{\phi(\nu)} \circ G$. Then, $i_\nu \circ q_{\phi(\nu)} \circ F = i_\nu \circ q_{\phi(\nu)} \circ G$ and $q'_\nu \circ F = q'_\nu \circ G$. Consequently, $V_{\phi(\nu)}^F \subset V_\nu^F$ and $T_{q'} \subset T_q$. A similar argument shows that $T_q \subset T_{q'}$. \square

We shall also denote, if there is no confusion, by $Sh(X, Y)$ the topological space so obtained.

The following corollary is immediate.

Corollary 1. *Let $Y \in \text{Ob}(\text{HPol})$. Then, $\text{Sh}(X, Y)$ is discrete for every topological space X .*

Consider X, Y to be compacta embedded in the Hilbert cube Q and take the HPol-expansion of Y generated by the $1/n$ -neighbourhood system of Y in Q . Using Proposition 1 and the fact that $d(F, G) < \varepsilon$ if and only if $S(i_{Y, B(Y, \varepsilon)}) \circ F = S(i_{Y, B(Y, \varepsilon)}) \circ G$, see [17], it is easy to check the next proposition.

Proposition 2. *If X, Y are compact metric spaces, the topology defined on $\text{Sh}(X, Y)$ is the same as that induced by the ultrametric d constructed in [17].*

The most useful result in this paper is the following theorem.

Theorem 1. *The map $\Omega : \text{Sh}(X, Y) \times \text{Sh}(Y, Z) \rightarrow \text{Sh}(X, Z)$, given by the composition, $\Omega(F, G) = G \circ F$ is continuous for arbitrary topological spaces X, Y, Z .*

Proof. Let us consider fixed HPol-expansions $p : X \rightarrow \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $q : Y \rightarrow \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ and $r : Z \rightarrow \mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ of X, Y and Z , respectively.

Let $F_0 \in \text{Sh}(X, Y)$ and $G_0 \in \text{Sh}(Y, Z)$. Then we have morphisms $f_0 : \mathbf{X} \rightarrow \mathbf{Y}$ and $g_0 : \mathbf{Y} \rightarrow \mathbf{Z}$ in pro-HPol such that the diagram

$$\begin{array}{ccccc} X & \xrightarrow{F_0} & Y & \xrightarrow{G_0} & Z \\ p \downarrow & & q \downarrow & & r \downarrow \\ \mathbf{X} & \xrightarrow{f_0} & \mathbf{Y} & \xrightarrow{g_0} & \mathbf{Z} \end{array}$$

commutes in pro-Sh.

Let $(f_{0\mu}, \psi)$ and $(g_{0\nu}, \phi)$ be representatives of f_0 and g_0 .

Take $\nu \in N$ and $V_\nu^{G_0 \circ F_0}$. We will show that $\Omega(V_\nu^{F_0} \times V_\nu^{G_0}) \subset V_\nu^{G_0 \circ F_0}$.

Indeed, for any $F \in V_\nu^{F_0}$ and $G \in V_\nu^{G_0}$, $r_\nu \circ G \circ F = g_{0\nu} \circ q_{\phi(\nu)} \circ F = g_{0\nu} \circ q_{\phi(\nu)} \circ F = r_\nu \circ G \circ F = r_\nu \circ G \circ F$. \square

It is now easy, by fixing one of the morphisms in the composition, to construct families of functors from the shape to the topological category and consequently to give many shape invariants.

Corollary 2. *Let X, Y be topological spaces and let $F : X \rightarrow Y$ be a shape morphism. Let Z be a topological space and consider $F^* : \text{Sh}(Y, Z) \rightarrow \text{Sh}(X, Z)$ and $F_* : \text{Sh}(Z, X) \rightarrow \text{Sh}(Z, Y)$ to be defined by $F^*(H) = H \circ F$ and $F_*(G) = F \circ G$, respectively. Then:*

- F_* and F^* are continuous, $(G \circ F)_* = G_* \circ F_*$, $(G \circ F)^* = F^* \circ G^*$ and Id_*, Id^* are the corresponding identity maps. Consequently,*
- Assume $\text{Sh}(X) \geq \text{Sh}(Y)$. Then $\text{Sh}(Y, Z)$ is homeomorphic to a retract of $\text{Sh}(X, Z)$ and $\text{Sh}(Z, Y)$ is homeomorphic to a retract of $\text{Sh}(Z, X)$ for every topological space Z .*

(b') Assume $Sh(X) = Sh(Y)$. Then $Sh(X, Z)$ is homeomorphic to $Sh(Y, Z)$ and $Sh(Z, X)$ is homeomorphic to $Sh(Z, Y)$ for every topological space Z .

Another consequence of Theorem 1 is established as follows

Proposition 3. Let Γ be a subcategory of the shape category. Assume $X, Y \in Ob(\Gamma)$. Denote by $\Gamma(X, Y)$ the set of all morphisms in Γ between X and Y . Consider $\overline{\Gamma}$ to be the family formed by all objects of Γ and let $\overline{\Gamma(X, Y)} \subset Sh(X, Y)$, the closure of $\Gamma(X, Y)$ in $Sh(X, Y)$. Then, $\overline{\Gamma}$ is a subcategory of the shape category. We will call $\overline{\Gamma}$ the closure of the category Γ .

Proof. It is enough to prove that for any $X, Y, Z \in Ob(\Gamma)$, $F \in \overline{\Gamma(X, Y)}$ and $G \in \overline{\Gamma(Y, Z)}$, the composition $G \circ F \in \overline{\Gamma(X, Z)}$. From the continuity of Ω , we have that $\Omega(\overline{\Gamma(X, Y)} \times \overline{\Gamma(Y, Z)}) \subset \Omega(\overline{\Gamma(X, Y)} \times \Gamma(Y, Z)) \subset \overline{\Gamma(X, Z)}$. \square

Now we consider the weak homotopy category (in this paper, this means the subcategory of the shape category whose morphisms are only those generated by maps) for arbitrary topological spaces. It is the category Γ_{wh} whose objects are topological spaces and whose morphisms are shape morphisms generated by maps. Using the last proposition, we obtain a new category $\overline{\Gamma_{wh}}$ which we will call the *internal shape category*. The morphisms in this category are called internal shape morphisms. They are shape morphisms in the closure of shape morphisms induced by maps. We shall use the notation $ISh(X, Y) \equiv \overline{\Gamma_{wh}}(X, Y)$.

The internal shape category, when restricted to metrizable spaces, is just that defined in [14].

We say that two topological spaces are *internally shape equivalent*, if they are isomorphic as objects of $\overline{\Gamma_{wh}}$. It follows from the definition that two internally shape equivalent spaces are shape equivalent. Next proposition points out the existence of many internal shape invariants.

Proposition 4. Let X, Y be two topological spaces. Suppose that $F: X \rightarrow Y$ is an internal shape equivalence with inverse G . Let Z be another topological space. Then, by composition, F induces a homeomorphism $F^*: Sh(Y, Z) \rightarrow Sh(X, Z)$ in such a way that $ISh(Y, Z)$ is mapped homeomorphically onto $ISh(X, Z)$. The same can be said about $Sh(Z, X)$, $Sh(Z, Y)$, $ISh(Z, X)$ and $ISh(Z, Y)$.

Example. Using the last proposition, it is very easy to see that the 1-sphere S^1 and the Warsaw circle Y are not internally shape equivalent, although they have the same shape, because taking $Z = S^1$, $ISh(S^1, S^1)$ is countable while $ISh(S^1, Y)$ is a one-point space. In fact, there is a map $f: Y \rightarrow S^1$ which is a shape equivalence such that the inverse is not even an internal shape morphism.

If one wants to extend now the concept of internal movability, see [4] for a definition, we can choose some of the results obtained in [17] to get an easy formulation of the desired extension.

Definition 1. Let X be a topological space. X is said to be internally movable if for every neighbourhood V of the identity shape morphism on X there are an ANR (for metric spaces) P , and maps $f : X \rightarrow P$, $g : P \rightarrow X$ such that $S(g \circ f) \in V$.

Remark.

- (a) Here, the restriction is that the morphism from $S(g) : P \rightarrow X$ is generated by a map. It could be thought that it would be more convenient to assume $S(g)$ to be internal. But it is easy to prove, using density arguments, that we obtain the same concept if in the last definition we change $S(g)$ by an internal shape morphism.
- (b) It is not difficult to check that if we restrict ourselves in the above definition to the metrizable case, we obtain the notion of internally movable space introduced in [14].
- (c) It seems that, in our context, the best extension of Borsuk's concept of movability to topological spaces is the following: a topological space X is movable if and only if for every neighbourhood V of the identity shape morphism of X there are an ANR (for metrizable spaces), P , and shape morphisms $F : X \rightarrow P$, $G : P \rightarrow X$ such that $G \circ F \in V$. Obviously, this is a uniform movability reformulation.

It is not hard to prove the following

Proposition 5. *If Y is an internally movable space and X is any space, then every shape morphism $F : X \rightarrow Y$ is internal. Consequently the shape and internal shape classification are the same in the class of internally movable spaces. Moreover, internal movability is an hereditary internal shape invariant but it is not a shape invariant. Finally, the class of internally movable spaces contains all the spaces of trivial shape and the class of approximate polyhedra defined in [12].*

In order to study the topological structure of the spaces of shape morphisms, next result is useful.

Theorem 2. *Let Y be a topological space and let $q : Y \rightarrow \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be an HPol-expansion of Y . Take*

$$\mathbf{Sh}(X, Y) = (Sh(X, Y_\mu), (q_{\mu\mu'})_*, M)$$

and consider the morphism

$$q_* : Sh(X, Y) \rightarrow \mathbf{Sh}(X, Y)$$

induced by q . Then, q_ is an inverse limit of $\mathbf{Sh}(X, Y)$ in \mathbf{Top} .*

Proof. Let Z be a topological space and let $g : Z \rightarrow \mathbf{Sh}(X, Y)$ be a morphism in $\mathbf{pro-Top}$. It suffices to see that there is a unique continuous map $g : Z \rightarrow Sh(X, Y)$ such that $q_* \circ g = g$ in $\mathbf{pro-Top}$. For any $z \in Z$ and $\mu \in M$ one has $g_\mu(z) \in Sh(X, Y_\mu)$. Since $g_\mu = (q_{\mu\mu'})_* \circ g_{\mu'}$, if we fix $z \in Z$, $g_\mu(z) = ((q_{\mu\mu'})_* \circ g_{\mu'})(z) = (q_{\mu\mu'} \circ g_{\mu'})(z)$. Consequently, for any $z \in Z$, $\{g_\mu(z), \mu \in M\}$ produces a shape morphism $g(z) \in Sh(X, Y)$ such that $q_\mu \circ g(z) = g_\mu(z)$. Hence, we have defined a map $g : Z \rightarrow Sh(X, Y)$. It is clear that $q_* \circ g = g$.

Let $F \in Sh(X, Y)$ and take $\mu \in M$. Because, $Sh(X, Y_\mu)$ is discrete,

$$g^{-1}(V_\mu^F) = \{z \in Z: q_\mu \circ g(z) = q_\mu \circ F\} = g_\mu^{-1}(q_\mu \circ F)$$

is an open and closed subset of Z . It follows that g is continuous.

The unicity of g is obvious. \square

Corollary 3. *Let X, Y be two topological spaces. Then $Sh(X, Y)$ is a Tychonov space having a base of open-closed (clopen) sets (i.e., having null small inductive dimension).*

More can be said about the spaces $Sh(X, Y)$ but we need first a few words about certain “gap” which arises in set theory.

By a $\{0, 1\}$ -valued measure on a set X we mean a countable additive function defined on the family of all subsets of X , and assuming only the values 0 or 1.

A cardinal m is said to be measurable if a set X of cardinal m admits a $\{0, 1\}$ -valued measure μ such that $\mu(X) = 1$ and $\mu(\{x\}) = 0$ for every $x \in X$.

A discrete space is \mathbb{N} -compact if and only if its cardinal is nonmeasurable (see [11, p. 163]).

The class of nonmeasurable cardinals is a closed class containing \aleph_0 .

The question whether every cardinal number is nonmeasurable is known as the problem of measurability of cardinal numbers. The assumption that all cardinal numbers are nonmeasurable is consistent with the axioms of set theory; on the other hand it is not known whether the assumption of the existence of measurable cardinals is also consistent with the axioms of set theory. See [9,11] for further information and references about measurability of cardinal numbers.

Corollary 4. *Let X, Y be topological spaces such that Y has nonmeasurable cardinal. Then, $Sh(X, Y)$ is a \mathbb{N} -compact space.*

Proof. Take the Čech system $(Y_\mu, q_{\mu\mu'}, M)$ associated to Y (see [7] or [13]). It is clear that the cardinal of the set of homotopy classes of maps from X into Y_μ is nonmeasurable for every $\mu \in M$; M has also nonmeasurable cardinal. Now, since

$$Sh(X, Y) = \varprojlim Sh(X, Y_\mu)$$

and $Sh(X, Y_\mu)$ is discrete and nonmeasurable, hence, \mathbb{N} -compact, we have that $Sh(X, Y)$ is also \mathbb{N} -compact. \square

Remark. Note that if we assume the nonexistence of measurable cardinals, which is consistent with the usual axioms of set theory, we have that all spaces of shape morphisms are \mathbb{N} -compact.

Let X be a topological space, we denote by QX , see [2,19], the space whose points are the quasicomponents of X and a base for its topology is the family of all sets $A \subset QX$ whose union in X are clopen sets of X . It is clear that QX is a Tychonov space having a base of clopen sets.

Recall that QX admits an \mathbb{N} -compactification, $v(QX)$, with the property that every continuous map $f: QX \rightarrow Y$, where Y is an \mathbb{N} -compact space has a continuous extension $\tilde{f}: v(QX) \rightarrow Y$ and $QX \hookrightarrow v(QX)$ as a dense set. Finally, we can identify QX with the space of functions from a point to QX with the compact-open topology. On the other hand, it is not difficult to prove that two maps from a one point space to an space X generate the same shape morphism if and only if their images are in the same quasicomponent of X .

In [17] we proved that if Z is a metric compactum of trivial shape and X is a compact metric space, then $Sh(Z, X)$ is homeomorphic to the space of components of X , $\square X$. One can ask if $Sh(Z, X)$ also has a good topological representation in the arbitrary case, supposing of course that Z is a space of trivial shape.

We can now prove the following

Theorem 3. *Let X be a topological space. Assume Z to be a topological space of trivial shape. Then:*

- (a) $QX \subset Sh(Z, X) \subset v(QX)$ where \subset are topological inclusions as dense subsets. Moreover, if the cardinal of X is nonmeasurable, then $Sh(Z, X) = v(QX)$.
- (b) If QX is a paracompact space with null covering dimension, then $Sh(Z, X) = QX$ (topologically).

Proof. It is enough to consider $Sh(*, X)$, where $*$ is a one point space.

Given $f \in Sh(*, X)$, for every discrete covering α of X , denote by X_α the nerve of α and $p_\alpha: X \rightarrow X_\alpha$ the natural projection. The map inducing $p_\alpha \circ f$ determines a clopen subset f_α of X . $U(f) = \{f_\alpha: \alpha \text{ is a discrete covering of } X\}$ is a clopen ultrafilter of QX with the countable intersection property. Then $U(f)$ is a point of $v(QX)$ see [15,19].

Using analogous arguments as in [19], the reader can check that the map $U: Sh(*, X) \rightarrow v(QX)$ just constructed, is a homeomorphism onto its image.

On the other hand, it is easy to see that QX can be topologically identified with the subspace of $Sh(*, X)$ induced by maps. \square

The next consequence allows us to obtain again all results in [15,19].

Corollary 5. *Let X and Y be topological spaces having the same shape. Then, $v(QX)$ and $v(QY)$ are homeomorphic. Moreover, if QX and QY are paracompacta having null covering dimension then, QX and QY are homeomorphic.*

Proof. We have a homeomorphism $f: Sh(*, X) \rightarrow Sh(*, Y)$. Using last theorem we can extend f to a homeomorphism $\tilde{f}: v(QX) \rightarrow v(QY)$. The remaining part follows from (b) of above theorem. \square

Next result is an interesting connection between shape theory and \mathbb{N} -compactness.

Theorem 4. *Let X be a topological space of nonmeasurable cardinal. Then, the conditions*

- (a) X is \mathbb{N} -compact

(b) X is homeomorphic to a space of shape morphisms, $Sh(Z, Y)$, between topological spaces Y, Z of nonmeasurable cardinal are equivalent.

Proof. (a) \Rightarrow (b) is a consequence of (a) of Theorem 3 taking $Z = *$ and $Y = X$.

(b) \Rightarrow (a) follows from Corollary 4. \square

Remark. Note that if we assume the nonexistence of measurable cardinals, then Theorem 3 is a full characterization of \mathbb{N} -compactness.

As another consequence we obtain a new construction of the \mathbb{N} -compactification of a Tychonov zero-dimensional space with nonmeasurable cardinal.

Corollary 6. *If X is a Tychonov space with null small inductive dimension and nonmeasurable cardinal, then $Sh(*, X)$ is the \mathbb{N} -compactification of X (denoted by $\nu(X)$), considering X as the subspace of $Sh(*, X)$ generated by maps.*

2. Relations with Ball's theorems

Suppose that X and Y are compact metric spaces and that $F: X \rightarrow Y$ is a shape morphism. In this case $Sh(*, X)$ and $Sh(*, Y)$ are the corresponding spaces of components $\square X$ and $\square Y$ (see [17, Proposition 2.5]). The map $F_*: Sh(*, X) \rightarrow Sh(*, Y)$ defined in Corollary 2 is just the map $\Lambda_F: \square X \rightarrow \square Y$ defined by Borsuk in Theorem 5.2 of [5, p. 214]. In particular, if F is a shape isomorphism then F_* is a homeomorphism satisfying the conditions in Theorem 2.2 [1], due to Ball. In that paper, Ball asked

Question 2. If X and Y are metrizable spaces with $Sh(X) = Sh(Y)$ and such that the correspondent decompositions into components are upper semicontinuous, must there be a homeomorphism $\Phi: \square X \rightarrow \square Y$ such that $Sh(X_0) = Sh(\Phi(X_0))$ for each component X_0 of X ?

In this direction we have

Proposition 6. *Let X and Y be paracompact Hausdorff spaces with upper semicontinuous decompositions into components and such that*

- (a) X and Y are locally compact, or
- (b) $\square X$ and $\square Y$ satisfy one of the following conditions:
 - (b₁) they have null covering dimension,
 - (b₂) they have null small inductive dimension and each point is a G_δ -set,
 - (b₃) they are \mathbb{N} -compact spaces.

If, in addition, $Sh(X) = Sh(Y)$ then there exists a homeomorphism $\Phi: \square X \rightarrow \square Y$ such that

$$Sh(X_0) = Sh(\Phi(X_0)) \quad \text{for every } X_0 \in \square X.$$

Sketch of the proof. First of all, under condition (a) or (b) we have that $QX = \square X$. If $F: X \rightarrow Y$ is a shape isomorphism then it follows from Corollary 2 that $F_*: Sh(*, X) \rightarrow Sh(*, Y)$ is a homeomorphism. From Theorem 3 (see also Corollary 6) we can consider $\square X \subset Sh(*, X)$, $\square Y \subset Sh(*, Y)$ topologically embedded. Conditions (a) or (b) imply that $F_*|_{\square X}$ is a homeomorphism from $\square X$ onto $\square Y$. Take $\Phi = F_*|_{\square X}$. The way to construct shape isomorphisms from X_0 to $\Phi(X_0)$ is based on Lemma 1 of [6]. In fact, for every $X_0 \in \square X$ there is a shape isomorphism $F_{X_0}: X_0 \rightarrow \Phi(X_0)$ which makes commutative the following diagram in the shape category:

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \uparrow i & & \uparrow j \\ X_0 & \xrightarrow{F_{X_0}} & \Phi(X_0) \end{array},$$

where i and j are inclusions.

We would like to point out that a similar argument gives a different proof of Ball's Theorem 3.2 in [3].

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