

# Polychronakos-Frahm spin chain of $BC_N$ type and the Berry-Tabor conjecture

J. C. Barba, F. Finkel, A. González-López,<sup>\*</sup> and M. A. Rodríguez

*Departamento de Física Teórica II, Universidad Complutense, 28040 Madrid, Spain*

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We compute the partition function of the  $su(m)$  Polychronakos-Frahm spin chain of  $BC_N$  type by means of the freezing trick. We use this partition function to study several statistical properties of the spectrum, which turn out to be analogous to those of other spin chains of Haldane-Shastry type. In particular, we find that when the number of particles is sufficiently large the level density follows a Gaussian distribution with great accuracy. We also show that the distribution of (normalized) spacings between consecutive levels is of neither Poisson nor Wigner type but is qualitatively similar to that of the original Haldane-Shastry spin chain. This suggests that spin chains of Haldane-Shastry type are exceptional integrable models since they do not satisfy a well-known conjecture of Berry and Tabor, according to which the spacings distribution of a generic integrable system should be Poissonian. We derive a simple analytic expression for the cumulative spacings distribution of the  $BC_N$ -type Polychronakos-Frahm chain using only a few essential properties of its spectrum such as the Gaussian character of the level density and the fact that the energy levels are equally spaced. This expression is shown to be in excellent agreement with the numerical data.

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## I. INTRODUCTION

Solvable spin chains often provide a natural setting for testing or modeling interesting physical phenomena and mathematical results in such disparate fields as fractional statistics, random matrix theory, or orthogonal polynomials. Among these chains, those of Haldane-Shastry (HS) type occupy a distinguished position due to their remarkable integrability and solvability properties. The original chain of this type was independently introduced twenty years ago by Haldane<sup>1</sup> and Shastry<sup>2</sup> in an attempt to construct a model whose ground state coincided with Gutzwiller's variational wave function for the Hubbard model in the limit of large on-site interaction.<sup>3-5</sup> In the original HS chain, the spins are equally spaced on a circle and present pairwise interactions inversely proportional to their chord distance.

An essential feature of the spin chains of HS type is their close connection with the spin versions of the Calogero<sup>6</sup> and Sutherland<sup>7,8</sup> models, and their generalizations due to Olshanetsky and Perelomov.<sup>9</sup> This observation—already pointed out by Shastry in his original paper—was elegantly formulated by Polychronakos in Ref. 10. In the latter reference, the author showed that the original HS chain can be obtained from the spin Sutherland model<sup>11-13</sup> in the strong coupling limit, in which the dynamical and spin degrees of freedom decouple so that the particles “freeze” at the equilibrium positions of the scalar part of the potential. In this regime, the integrals of motion of the spin Sutherland model directly yield first integrals of the HS chain, thereby explaining its complete integrability. This procedure was also applied in Ref. 10 to construct a new integrable spin chain related to the original Calogero model. The spectrum of this chain was numerically studied by Frahm,<sup>14</sup> who found that the levels are grouped in highly degenerate multiplets. In a subsequent publication, Polychronakos computed the partition function of this chain [usually referred to in the literature as the Polychronakos-Frahm (PF) chain] by the “freezing trick” argument described above.<sup>15</sup> Although there is a considerable

amount of work on the spectrum and thermodynamics of the original HS chain (see, e.g., Refs. 16–18), a simple closed-form expression for its partition function was found only very recently.<sup>19</sup>

Both the HS and the PF chains are obtained from the Sutherland and Calogero models associated with the  $A_N$  root system in Olshanetsky and Perelomov's approach. The  $BC_N$  versions of both chains have also been studied in the literature. More precisely, the integrability of the PF chain of  $BC_N$  type was established by Yamamoto and Tsuchiya<sup>20</sup> using again the freezing trick. On the other hand, the partition function of the HS chain of  $BC_N$  type was computed in closed form in Ref. 21. The explicit knowledge of the partition function made it possible to study certain statistical properties of the spectrum of this chain. In particular, it was observed that for a large number of spins the level density is Gaussian. As a matter of fact, this property also holds for the original HS chain, as shown in Ref. 19. The analysis of the distribution of the spacing between consecutive levels of the original HS chain was also undertaken in the latter reference. Rather unexpectedly, it was found that this distribution is not of Poisson type, as should be the case for a “generic” integrable model according to a long-standing conjecture of Berry and Tabor.<sup>22</sup> This behavior has also been recently reported for a supersymmetric version of the HS chain.<sup>23</sup>

The aim of this paper is twofold. In the first place, we shall compute in closed form the partition function of the PF chain of  $BC_N$  type by means of the freezing trick. Using the partition function, we shall perform a numerical study of the density of levels and the distribution of the spacing between consecutive energies. We shall see that the level density is again Gaussian and that the spacings distribution is analogous to that of the original HS chain. In particular, our results show that the distribution of spacings is neither Poissonian nor of Wigner type (characteristic of chaotic systems). We shall next derive a simple analytic expression for the cumulative spacings distribution, which reproduces the numerical data with much greater accuracy than the empiric formula proposed in Ref. 19. In fact, we have strong numerical evi-

dence that the new expression can also be applied to the HS and PF chains of  $A_N$  type. In view of the Berry-Tabor conjecture, our results suggest that spin chains of HS type are exceptional among the class of integrable models.

## II. PARTITION FUNCTION OF THE PF CHAIN OF $BC_N$ TYPE

The Hamiltonian of the (antiferromagnetic)  $su(m)$  PF chain of  $BC_N$  type is defined by

$$\mathcal{H}^\epsilon = \sum_{i \neq j} \left[ \frac{1 + S_{ij}}{(\xi_i - \xi_j)^2} + \frac{1 + \tilde{S}_{ij}}{(\xi_i + \xi_j)^2} \right] + \beta \sum_i \frac{1 - \epsilon S_i}{\xi_i^2}, \quad (1)$$

where the sums run from one to  $N$  (as always hereafter, unless otherwise stated),  $\beta > 0$ ,  $\epsilon = \pm 1$ ,  $S_{ij}$  is the operator that permutes the  $i$ th and  $j$ th spins,  $S_i$  is the operator reversing the  $i$ th spin, and  $\tilde{S}_{ij} = S_i S_j S_{ij}$ . Note that the spin operators  $S_{ij}$  and  $S_i$  can be expressed in terms of the fundamental  $su(m)$  spin generators  $J_k^\alpha$  at the site  $k$  [with the normalization  $\text{tr}(J_k^\alpha J_k^\gamma) = \frac{1}{2} \delta^{\alpha\gamma}$ ] as

$$S_{ij} = \frac{1}{m} + 2 \sum_{\alpha=1}^{m^2-1} J_i^\alpha J_j^\alpha, \quad S_i = \sqrt{2m} J_i^1.$$

The chain sites  $\xi_i$  are the coordinates of the unique minimum in  $C = \{ \mathbf{x} | 0 < x_1 < \dots < x_N \}$  of the potential

$$U(\mathbf{x}) = \sum_{i \neq j} \left[ \frac{1}{(x_{ij}^-)^2} + \frac{1}{(x_{ij}^+)^2} \right] + \sum_i \frac{\beta^2}{x_i^2} + \frac{r^2}{4}, \quad (2)$$

where  $x_{ij}^\pm = x_i \pm x_j$  and  $r^2 = \sum_i x_i^2$ . The existence of this minimum follows from the fact that  $U$  tends to  $+\infty$  on the boundary of  $C$  and as  $r \rightarrow \infty$ , and its uniqueness was established in Ref. 24 by expressing the potential  $U$  in terms of the logarithm of the ground state of the  $BC_N$  Calogero model

$$H^{\text{sc}} = - \sum_i \partial_{x_i}^2 + a(a-1) \sum_{i \neq j} \left[ \frac{1}{(x_{ij}^-)^2} + \frac{1}{(x_{ij}^+)^2} \right] + b(b-1) \sum_i \frac{1}{x_i^2} + \frac{a^2}{4} r^2, \quad (3)$$

with  $b = \beta a$  and  $a > 1/2$ . Moreover, it can be shown<sup>25</sup> that  $\xi_i = \sqrt{2} y_i$ , where  $y_i$  is the  $i$ th zero of the generalized Laguerre polynomial  $L_N^{\beta-1}$ . From this fact, one can infer<sup>26</sup> that for  $N \gg \beta$ , the density of sites (normalized to unity)  $\rho_N(x)$  is given by the circular law

$$\rho_N(x) = \frac{1}{2\pi N} \sqrt{8N - x^2}. \quad (4)$$

Note that in this limit the sites' density is independent of  $\beta$  and is qualitatively similar to that of the PF chain of  $A_N$  type.<sup>14</sup> Integrating the previous equation, we obtain the implicit asymptotic relation

$$4\pi k = \xi_k \sqrt{8N - \xi_k^2} + 8N \arcsin\left(\frac{\xi_k}{\sqrt{8N}}\right),$$

valid also for  $N \gg \beta$ .

The spin chain in Eq. (1) can be expressed in terms of the spin Calogero model of  $BC_N$  type

$$H^\epsilon = - \sum_i \partial_{x_i}^2 + a \sum_{i \neq j} \left[ \frac{a + S_{ij}}{(x_{ij}^-)^2} + \frac{a + \tilde{S}_{ij}}{(x_{ij}^+)^2} \right] + b \sum_i \frac{b - \epsilon S_i}{x_i^2} + \frac{a^2}{4} r^2, \quad (5)$$

and its scalar reduction [Eq. (3)] as

$$H^\epsilon = H^{\text{sc}} + a \tilde{\mathcal{H}}^\epsilon, \quad (6)$$

where  $\tilde{\mathcal{H}}^\epsilon$  is obtained from  $\mathcal{H}^\epsilon$  replacing the chain sites  $\xi$  by the particles' coordinates  $\mathbf{x}$ . In the limit  $a \rightarrow \infty$ , the particles in the spin dynamical model in Eq. (5) concentrate at the coordinates of the minimum of the potential  $U$ , that is, at the sites  $\xi_i$  of the chain in Eq. (1). Hence, the spin and dynamical degrees of freedom of the Hamiltonian in Eq. (5) decouple, so that by Eq. (6) its eigenvalues are approximately given by

$$E_{ij}^\epsilon \simeq E_i^{\text{sc}} + a \mathcal{E}_j^\epsilon, \quad a \gg 1, \quad (7)$$

where  $E_i^{\text{sc}}$  and  $\mathcal{E}_j^\epsilon$  are two arbitrary eigenvalues of  $H^{\text{sc}}$  and  $\mathcal{H}^\epsilon$ , respectively. The asymptotic relation in Eq. (7) immediately yields the following exact formula for the partition function  $\mathcal{Z}^\epsilon$  of the chain in Eq. (1):

$$\mathcal{Z}^\epsilon(T) = \lim_{a \rightarrow \infty} \frac{\mathcal{Z}^\epsilon(aT)}{\mathcal{Z}^{\text{sc}}(aT)}, \quad (8)$$

where  $\mathcal{Z}^\epsilon$  and  $\mathcal{Z}^{\text{sc}}$  are the partition functions of  $H^\epsilon$  and  $H^{\text{sc}}$ , respectively.

We shall next evaluate the partition function  $\mathcal{Z}^\epsilon$  of the chain in Eq. (1) by computing the partition functions  $\mathcal{Z}^\epsilon$  and  $\mathcal{Z}^{\text{sc}}$  in Eq. (8). In order to determine the spectra of the corresponding Hamiltonians  $H^\epsilon$  and  $H^{\text{sc}}$ , following Ref. 19 we introduce the auxiliary operator

$$H' = - \sum_i \partial_{x_i}^2 + \sum_{i \neq j} \left[ \frac{a}{(x_{ij}^-)^2} (a - K_{ij}) + \frac{a}{(x_{ij}^+)^2} (a - \tilde{K}_{ij}) \right] + \sum_i \frac{b}{x_i^2} (b - K_i) + \frac{a^2}{4} r^2, \quad (9)$$

where  $K_{ij}$  permutes the  $i$ th and  $j$ th coordinates,  $K_i$  reverses the sign of the  $i$ th coordinate, and  $\tilde{K}_{ij} \equiv K_i K_j K_{ij}$ . We then have the obvious relations

$$H^\epsilon = H' |_{K_{ij} \rightarrow -S_{ij}, K_i \rightarrow \epsilon S_i}, \quad H^{\text{sc}} = H' |_{K_{ij}, K_i \rightarrow 1}. \quad (10)$$

On the other hand, the spectrum of  $H'$  can be easily computed by noting that this operator can be written in terms of the rational Dunkl operators of  $BC_N$  type<sup>27</sup>

$$\mathcal{J}_i^- = \partial_{x_i} + a \sum_{j \neq i} \left[ \frac{1}{x_{ij}^-} (1 - K_{ij}) + \frac{1}{x_{ij}^+} (1 - \tilde{K}_{ij}) \right] + \frac{b}{x_i} (1 - K_i), \quad i = 1, \dots, N, \quad (11)$$

as follows:<sup>28</sup>

$$H' = \mu \left[ - \sum_i (\mathcal{J}_i^-)^2 + a \sum_i x_i \partial_{x_i} + E_0 \right] \mu^{-1}, \quad (12)$$

where

$$\mu(\mathbf{x}) = e^{-ar^2/4} \prod_i |x_i|^b \cdot \prod_{i < j} |x_i^2 - x_j^2|^a \quad (13)$$

is the ground state of the Hamiltonian [Eq. (3)] and

$$E_0 = Na \left[ b + a(N-1) + \frac{1}{2} \right]. \quad (14)$$

Since the Dunkl operators in Eq. (11) map any monomial  $\prod_i x_i^{n_i}$  into a polynomial of total degree  $n_1 + \dots + n_N - 1$ , by Eq. (12) the operator  $H'$  is represented by an upper triangular matrix in the (nonorthonormal) basis with elements

$$\phi_{\mathbf{n}} = \mu \prod_i x_i^{n_i}, \quad \mathbf{n} \equiv (n_1, \dots, n_N) \in (\mathbb{N} \cup \{0\})^N, \quad (15)$$

ordered according to the total degree  $|\mathbf{n}| \equiv n_1 + \dots + n_N$  of the monomial part. More precisely,

$$H' \phi_{\mathbf{n}} = E'_{\mathbf{n}} \phi_{\mathbf{n}} + \sum_{|\mathbf{m}| < |\mathbf{n}|} c_{\mathbf{m}\mathbf{n}} \phi_{\mathbf{m}}, \quad (16)$$

where the coefficients  $c_{\mathbf{n}}$  are real constants and

$$E'_{\mathbf{n}} = a|\mathbf{n}| + E_0. \quad (17)$$

We shall now construct a basis of the Hilbert space of the Hamiltonian  $H^\epsilon$ , in which this operator is also represented by an upper triangular matrix. To this end, let us denote by  $\Lambda^\epsilon$  the projector on states antisymmetric under simultaneous permutations of spatial and spin coordinates, and with parity  $\epsilon$  under sign reversals of coordinates and spins. If

$$|\mathbf{s}\rangle \equiv |s_1, \dots, s_N\rangle, \quad s_i = -M, -M+1, \dots, M \equiv \frac{m-1}{2},$$

denotes a state of the  $su(m)$  spin basis, the functions

$$\psi_{\mathbf{n},\mathbf{s}}(\mathbf{x}) = \Lambda^\epsilon [\phi_{\mathbf{n}}(\mathbf{x})|\mathbf{s}\rangle], \quad (18)$$

form a basis of the Hilbert space of the Hamiltonian  $H^\epsilon$ , provided that:

- (i)  $n_1 \geq \dots \geq n_N$ .
- (ii)  $s_i > s_j$  whenever  $n_i = n_j$  and  $i < j$ .
- (iii)  $s_i \geq 0$  for all  $i$ , and  $s_i > 0$  if  $(-1)^{n_i} = -\epsilon$ .

The first two conditions are a consequence of the antisymmetry of the states in Eq. (18) under particle permutations, while the last condition is due to the fact that these states must have parity  $\epsilon$  under sign reversals. Since  $K_{ij}\Lambda^\epsilon = -S_{ij}\Lambda^\epsilon$  and  $K_i\Lambda^\epsilon = \epsilon S_i\Lambda^\epsilon$ , it follows that  $H^\epsilon\Lambda^\epsilon = H'\Lambda^\epsilon$ . Using this identity and the fact that  $H'$  obviously commutes with  $\Lambda^\epsilon$ , from Eq. (16) we easily obtain

$$H^\epsilon \psi_{\mathbf{n},\mathbf{s}} = \Lambda^\epsilon [(H' \phi_{\mathbf{n}})|\mathbf{s}\rangle] = E'_{\mathbf{n}} \psi_{\mathbf{n},\mathbf{s}} + \sum_{|\mathbf{m}| < |\mathbf{n}|} c_{\mathbf{m}\mathbf{n}} \psi_{\mathbf{m},\mathbf{s}}.$$

Thus  $H^\epsilon$  is represented by an upper triangular matrix in the basis in Eq. (18), ordered according to the degree  $|\mathbf{n}|$ . The diagonal elements of this matrix are given by

$$E_{\mathbf{n},\mathbf{s}}^\epsilon = a|\mathbf{n}| + E_0, \quad (19)$$

where  $\mathbf{n}$  and  $\mathbf{s}$  satisfy conditions (i)–(iii) above. Note that, although the numerical value of  $E_{\mathbf{n},\mathbf{s}}^\epsilon$  is independent of  $\mathbf{s}$ , the degeneracy of each level clearly depends on the spin through the latter conditions.

Turning next to the scalar Hamiltonian  $H^{\text{sc}}$ , in view of Eq. (10), we now need to consider scalar functions of the form

$$\psi_{\mathbf{n}}(\mathbf{x}) = \Lambda_s \phi_{\mathbf{n}}(\mathbf{x}), \quad (20)$$

where  $\Lambda_s$  is the symmetrizer with respect to both permutations and sign reversals. These functions form a (nonorthonormal) basis of the Hilbert space of  $H^{\text{sc}}$ , provided that  $n_i = 2k_i$  are even integers and  $k_1 \geq \dots \geq k_N$ . Just as before, the matrix of the scalar Hamiltonian  $H^{\text{sc}}$  in the basis in Eq. (20) ordered by the degree is upper triangular with diagonal elements  $E_{\mathbf{n}}^{\text{sc}}$  also given by the right-hand side (RHS) of Eq. (19).

Let us next compute the partition functions  $Z^{\text{sc}}$  and  $Z^\epsilon$  of the models in Eqs. (3) and (5). To begin with, one can drop without loss of generality the common ground-state energy  $E_0$  in both models, since by Eq. (8) it does not contribute to the partition function  $Z^\epsilon$ . An unexpected consequence of this simple observation is the fact that the partition function of the chain in Eq. (1) does not depend on the parameter  $\beta$ . With the above convention, the partition function of the scalar Hamiltonian  $H^{\text{sc}}$  is given by

$$Z^{\text{sc}}(aT) = \sum_{k_1 \geq \dots \geq k_N \geq 0} q^{2|\mathbf{k}|},$$

where  $q = e^{-1/(k_B T)}$ . The previous sum can be evaluated by expressing it in terms of the differences  $p_i = k_i - k_{i+1}$ ,  $i = 1, \dots, N-1$ , with  $p_N \equiv k_N$ . Since  $k_j = \sum_{i=j}^N p_i$ , we easily obtain

$$\begin{aligned} Z^{\text{sc}}(aT) &= \sum_{p_1, \dots, p_N \geq 0} q^{2 \sum_{j=1}^N \sum_{i=j}^N p_i} = \sum_{p_1, \dots, p_N \geq 0} q^{2 \sum_{i=1}^N i p_i} \\ &= \prod_i \sum_{p_i \geq 0} (q^{2i})^{p_i} = \prod_i (1 - q^{2i})^{-1}. \end{aligned} \quad (21)$$

In order to compute the partition function of the spin Hamiltonian  $H^\epsilon$ , we shall first assume that  $m$  is even so that condition (iii) simplifies to

$$(iii') \quad s_i > 0 \text{ for all } i.$$

As neither the value of  $E_{\mathbf{n},\mathbf{s}}^\epsilon$  nor conditions (i), (ii), and (iii') depend on  $\epsilon$ , in this case the partition functions  $Z^\epsilon$  and  $Z^{\text{sc}}$  cannot depend on  $\epsilon$ . Hence, from now on we shall drop the superscript  $\epsilon$  when  $m$  is even, writing simply  $Z$  and  $\mathcal{Z}$ . By Eq. (19), after dropping  $E_0$  the partition function of the Hamiltonian in Eq. (5) can be written as

$$Z(aT) = \sum_{n_1 \geq \dots \geq n_N \geq 0} d_{\mathbf{n}} q^{|\mathbf{n}|}, \quad (22)$$

where the spin degeneracy factor  $d_{\mathbf{n}}$  is the number of spin states  $|\mathbf{s}\rangle$  satisfying conditions (ii) and (iii'). Writing

$$\mathbf{n} = (\overbrace{k_1, \dots, k_1}^{v_1}, \dots, \overbrace{k_r, \dots, k_r}^{v_r}), \quad k_1 > \dots > k_r \geq 0,$$

by conditions (ii) and (iii'), we have

$$d_{\mathbf{n}} = \prod_{i=1}^r \binom{m/2}{v_i} \equiv d(\mathbf{v}), \quad \mathbf{v} = (v_1, \dots, v_r). \quad (23)$$

Note that  $\sum_{i=1}^r v_i = N$ , so that the multi-index  $\mathbf{v}$  can be regarded as an element of the set  $\mathcal{P}_N$  of partitions of  $N$  (taking

order into account). With the previous notation, Eq. (22) becomes

$$\begin{aligned}
 Z(aT) &= \sum_{\mathbf{v} \in \mathcal{P}_N} d(\mathbf{v}) \sum_{k_1 > \dots > k_r \geq 0} q^{\sum_{i=1}^r v_i k_i} \\
 &= \sum_{\mathbf{v} \in \mathcal{P}_N} d(\mathbf{v}) \sum_{\substack{p_1, \dots, p_{r-1} > 0 \\ p_r \geq 0}} q^{\sum_{i=1}^r v_i \sum_{j=i}^r p_j} \\
 &= \sum_{\mathbf{v} \in \mathcal{P}_N} d(\mathbf{v}) \sum_{\substack{p_1, \dots, p_{r-1} > 0 \\ p_r \geq 0}} \prod_{j=1}^r q^{p_j \sum_{i=1}^j v_i} \\
 &= q^{-N} \sum_{\mathbf{v} \in \mathcal{P}_N} d(\mathbf{v}) \prod_{j=1}^r \frac{q^{N_j}}{1 - q^{N_j}}, \quad (24)
 \end{aligned}$$

where  $N_j = \sum_{i=1}^j v_i$ . From Eqs. (8), (21), and (24), we finally obtain the following explicit expression for the partition function of the  $\text{su}(m)$  PF chain of  $BC_N$  type in the case of even  $m$ :

$$Z(T) = q^{-N} \prod_i (1 - q^{2i}) \sum_{\mathbf{v} \in \mathcal{P}_N} d(\mathbf{v}) \prod_{j=1}^{\ell(\mathbf{v})} \frac{q^{N_j}}{1 - q^{N_j}}, \quad (25)$$

where  $\ell(\mathbf{v}) = r$  is the number of components of the multi-index  $\mathbf{v}$ . For instance, for spin 1/2 we have  $v_i = 1$  for all  $i$ , and, therefore,  $\ell(\mathbf{v}) = N$ ,  $d(\mathbf{v}) = 1$ , and  $N_j = j$ , so that the previous formula simplifies to

$$Z(T) = q^{N(N-1)/2} \prod_i (1 + q^i), \quad m = 2. \quad (26)$$

Thus, for spin 1/2 the spectrum is given by

$$\mathcal{E}_j = \frac{1}{2}N(N-1) + j, \quad j = 0, 1, \dots, \frac{1}{2}N(N+1), \quad (27)$$

and the degeneracy of the energy  $\mathcal{E}_j$  is the number  $Q_N(j)$  of partitions of the integer  $j$  into distinct parts no larger than  $N$  [with  $Q_N(0) \equiv 1$ ]. For  $j \leq N$  this number coincides with the number  $Q(j)$  of partitions of  $j$  into distinct parts, which has been extensively studied in the mathematical literature.<sup>29</sup> It is also interesting to observe that the partition function in Eq. (26) is closely related to Ramanujan's fifth order mock theta function<sup>30</sup>

$$\psi_1(q) = \sum_{N=0}^{\infty} q^N Z_N(q),$$

where  $Z_N(q)$  denotes the RHS of Eq. (26).

Equation (26) also shows that for spin 1/2, the chain in Eq. (1) is equivalent to a system of  $N$  species of noninteracting fermions [with vacuum energy  $\mathcal{E}_0 = N(N-1)/2$ ], where the  $i$ th species of fermion has energy  $E_i = i$ . A similar result was obtained in Ref. 31 for the supersymmetric  $\text{su}(1|1)$  (ferromagnetic) HS chain, although in the latter case the energy of the  $i$ th fermion is  $E_i = i(N-i)$ .

Let us consider now the case of odd  $m$ . In this case, it is convenient to slightly modify condition (i) above by first grouping the components of  $\mathbf{n}$  with the same parity and then ordering separately the even and odd components. In other words, we shall write  $\mathbf{n} = (\mathbf{n}_e, \mathbf{n}_o)$ , where

$$\begin{aligned}
 \mathbf{n}_e &= (\overbrace{2k_1, \dots, 2k_1}^{\nu_1}, \dots, \overbrace{2k_s, \dots, 2k_s}^{\nu_s}), \\
 \mathbf{n}_o &= (\overbrace{2k_{s+1} + 1, \dots, 2k_{s+1} + 1}^{\nu_{s+1}}, \dots, \overbrace{2k_r + 1, \dots, 2k_r + 1}^{\nu_r}),
 \end{aligned}$$

and

$$k_1 > \dots > k_s \geq 0, \quad k_{s+1} > \dots > k_r \geq 0.$$

By conditions (ii) and (iii), the spin degeneracy factor is now

$$d_{\mathbf{n}}^{\epsilon} = \prod_{i=1}^s \binom{\frac{m+\epsilon}{2}}{\nu_i} \cdot \prod_{i=s+1}^r \binom{\frac{m-\epsilon}{2}}{\nu_i} \equiv d_s^{\epsilon}(\mathbf{v}). \quad (28)$$

Calling

$$\tilde{N}_j = \sum_{i=s+1}^j \nu_i, \quad j = s+1, \dots, r,$$

and proceeding as before, we obtain

$$\begin{aligned}
 Z^{\epsilon}(aT) &= \sum_{\mathbf{v} \in \mathcal{P}_N} \sum_{s=0}^r d_s^{\epsilon}(\mathbf{v}) \sum_{\substack{k_1 > \dots > k_s \geq 0 \\ k_{s+1} > \dots > k_r \geq 0}} q^{\sum_{i=1}^s 2\nu_i k_i} q^{\sum_{i=s+1}^r \nu_i (2k_i + 1)} \\
 &= \sum_{\mathbf{v} \in \mathcal{P}_N} \sum_{s=0}^r d_s^{\epsilon}(\mathbf{v}) q^{\tilde{N}_r} \left[ \sum_{k_1 > \dots > k_s \geq 0} q^{\sum_{i=1}^s 2\nu_i k_i} \right] \\
 &\quad \times \left[ \sum_{k_{s+1} > \dots > k_r \geq 0} q^{\sum_{i=s+1}^r 2\nu_i k_i} \right] \\
 &= \sum_{\mathbf{v} \in \mathcal{P}_N} \sum_{s=0}^{\ell(\mathbf{v})} d_s^{\epsilon}(\mathbf{v}) q^{-(N+N_s)} \prod_{j=1}^s \frac{q^{2N_j}}{1 - q^{2N_j}} \cdot \prod_{j=s+1}^{\ell(\mathbf{v})} \frac{q^{2\tilde{N}_j}}{1 - q^{2\tilde{N}_j}}. \quad (29)
 \end{aligned}$$

Substituting the previous expression and Eq. (21) into Eq. (8), we immediately deduce the following explicit formula for the partition functions of the  $\text{su}(m)$  PF chain of  $BC_N$  type for odd  $m$ :

$$\begin{aligned}
 Z^{\epsilon}(T) &= \prod_i (1 - q^{2i}) \sum_{\mathbf{v} \in \mathcal{P}_N} \sum_{s=0}^{\ell(\mathbf{v})} d_s^{\epsilon}(\mathbf{v}) q^{-(N+N_s)} \\
 &\quad \times \prod_{j=1}^s \frac{q^{2N_j}}{1 - q^{2N_j}} \cdot \prod_{j=s+1}^{\ell(\mathbf{v})} \frac{q^{2\tilde{N}_j}}{1 - q^{2\tilde{N}_j}}. \quad (30)
 \end{aligned}$$

Although we have chosen, for definiteness, to study the antiferromagnetic chain in Eq. (1), a similar analysis can be performed for its ferromagnetic counterpart,

$$\mathcal{H}_F^{\epsilon} = \sum_{i \neq j} \left[ \frac{1 - S_{ij}}{(\xi_i - \xi_j)^2} + \frac{1 - \tilde{S}_{ij}}{(\xi_i + \xi_j)^2} \right] + \beta \sum_i \frac{1 - \epsilon S_i}{\xi_i^2}. \quad (31)$$

Since now



$$H_F^\epsilon = H' |_{K_{ij} \rightarrow S_{ij}, K_i \rightarrow \epsilon S_i}, \quad (32)$$

we must replace the operator  $\Lambda^\epsilon$  in Eq. (18) by the projector on states *symmetric* under simultaneous permutations of the particles' spatial and spin coordinates, and with parity  $\epsilon$  under sign reversal of coordinates and spin. Hence, condition (ii) above on the basis states  $\psi_{\mathbf{n},s}$  should now read

$$(ii') \quad s_i \geq s_j \text{ whenever } n_i = n_j \text{ and } i < j.$$

As a result, the degeneracy factors  $d(\nu)$  and  $d_s^\epsilon(\nu)$  in Eqs. (23) and (28) should be replaced by their “bosonic” versions

$$d_F(\nu) = \prod_{i=1}^r \binom{\frac{m}{2} + \nu_i - 1}{\nu_i},$$

$$d_{F,s}^\epsilon(\nu) = \prod_{i=1}^s \binom{\frac{m+\epsilon}{2} + \nu_i - 1}{\nu_i} \cdot \prod_{i=s+1}^r \binom{\frac{m-\epsilon}{2} + \nu_i - 1}{\nu_i}.$$

Therefore the partition function of the ferromagnetic  $\text{su}(m)$  PF chain of  $BC_N$  type [Eq. (31)] is still given by Eq. (25) (for even  $m$ ) or Eq. (30) (for odd  $m$ ), but with  $d(\nu)$  and  $d_s^\epsilon(\nu)$  replaced respectively by  $d_F(\nu)$  and  $d_{F,s}^\epsilon(\nu)$ .

On the other hand, the chains in Eqs. (1) and (31) are obviously related by

$$\mathcal{H}_F^\epsilon + \mathcal{H}^{-\epsilon} = 2 \left\{ \sum_{i \neq j} [(\xi_i - \xi_j)^{-2} + (\xi_i + \xi_j)^{-2}] + \beta \sum_i \xi_i^{-2} \right\}.$$

The RHS of this equation clearly coincides with the largest eigenvalue  $\mathcal{E}_{\max}$  of the antiferromagnetic chains  $\mathcal{H}^\epsilon$ , whose corresponding eigenvectors are the spin states symmetric under permutations and with parity  $\epsilon$  under spin reversal. This eigenvalue is most easily computed for the spin 1/2 chains, since in this case the spectrum is explicitly given in Eq. (27). We thus obtain

$$\mathcal{E}_{\max} = \frac{1}{2}N(N-1) + \frac{1}{2}N(N+1) = N^2, \quad (33)$$

so that  $\mathcal{H}_F^\epsilon = N^2 - \mathcal{H}^{-\epsilon}$ . Hence, the partition functions  $\mathcal{Z}^\epsilon$  and  $\mathcal{Z}_F^\epsilon$  of  $\mathcal{H}^\epsilon$  and  $\mathcal{H}_F^\epsilon$  satisfy the remarkable identity

$$\mathcal{Z}_F^\epsilon(q) = q^{N^2} \mathcal{Z}^{-\epsilon}(q^{-1}).$$

This is a manifestation of the boson-fermion duality discussed in detail in Ref. 32 for the  $\text{su}(m|n)$  supersymmetric HS spin chain since the ferromagnetic (respectively antiferromagnetic) chain can be regarded as purely bosonic (respectively fermionic). For instance, using the latter identity and Eq. (26), we easily obtain the following expression for the partition function of the ferromagnetic spin 1/2 chains:

$$\mathcal{Z}_F(T) = \prod_i (1 + q^i), \quad m = 2. \quad (34)$$

(Note that, as in the antiferromagnetic case,  $\mathcal{Z}_F^\epsilon$  is actually independent of  $\epsilon$  for even  $m$ .) This is, again, the partition function of a system of  $N$  species of free fermions of energy  $E_i = i$ , but now the vacuum energy vanishes.

Equation (25) for the partition function of the antiferromagnetic chains with even  $m$  can be easily simplified to

$$\mathcal{Z}(T) = \prod_i (1 + q^i) \sum_{\nu \in \mathcal{P}_N} d(\nu) q^{\sum_{j=1}^{\ell(\nu)} N_j} \prod_{j=1}^{N-\ell(\nu)} (1 - q^{N'_j}),$$

where the positive integers  $N'_j$  are defined by

$$\{N'_1, \dots, N'_{N-\ell(\nu)}\} = \{1, \dots, N-1\} - \{N_1, \dots, N_{\ell(\nu)-1}\}.$$

The sum in the RHS is easily recognized as the partition function  $\mathcal{Z}^{(A)}(T; m/2)$  of the  $\text{su}(m/2)$  (antiferromagnetic) PF chain of  $A_N$  type.<sup>31</sup> We thus obtain the remarkable factorization

$$\mathcal{Z}(T; m) = \mathcal{Z}_F(T; 2) \cdot \mathcal{Z}^{(A)}(T; m/2), \quad m \in 2\mathbb{N}, \quad (35)$$

where the second argument in  $\mathcal{Z}$  and  $\mathcal{Z}_F$  denotes the number of internal degrees of freedom. Replacing  $d(\nu)$  by  $d_F(\nu)$  in Eq. (25), we obtain a similar factorization for the partition function of the ferromagnetic chains,

$$\mathcal{Z}_F(T; m) = \mathcal{Z}_F(T; 2) \cdot \mathcal{Z}_F^{(A)}(T; m/2), \quad m \in 2\mathbb{N}. \quad (36)$$

Thus, for even  $m$ , the PF chains of  $BC_N$  type [Eqs. (1) and (31)] can be described by an effective model of two simpler noninteracting chains. This remarkable property, which to the best of our knowledge is unique among the class of chains of Haldane–Shastry type, certainly deserves further investigation. Note that the situation for odd  $m$  is quite different since in this case it is not even clear how to simplify the denominators appearing in Eq. (30).

### III. SPACINGS DISTRIBUTION AND THE BERRY-TABOR CONJECTURE

For fixed values of the number of particles  $N$  and the internal degrees of freedom  $m$ , it is straightforward to obtain the spectrum of the chain in Eq. (1) by expanding in powers of  $q$  the expressions in Eq. (25) or (30) for its partition function. In this way, we have been able to compute the spectrum of the latter chain for relatively large values of  $N$  (for instance, up to  $N=22$  for  $m=3$ ). Our calculations conclusively show that the spectrum consists of a set of consecutive integers. For even  $m$ , this observation follows immediately from the expressions in Eqs. (34) and (35), and the fact that the energies of the PF chain of  $A_N$  type are also consecutive integers. For odd  $m$  we have been unable to deduce this property from Eq. (30) for the partition function although we have verified it numerically for many different values of  $N$  and  $m$ .

Our computations also evidence that for  $N \geq 10$ , the level density (normalized to unity) can be approximated with great accuracy by a normal distribution,

$$g(\mathcal{E}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\mathcal{E}-\mu)^2}{2\sigma^2}}, \quad (37)$$

where  $\mu$  and  $\sigma$  are the mean and the variance of the energy, respectively. For instance, in Fig. 1, we compare the cumulative level density

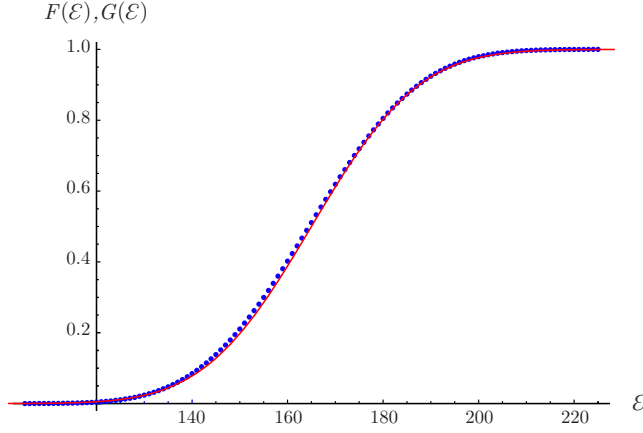


FIG. 1. (Color online) Cumulative distribution functions  $F(\mathcal{E})$  (at its discontinuity points) and  $G(\mathcal{E})$  (continuous red line) for  $N=15$  and  $m=2$ .

$$F(\mathcal{E}) = m^{-N} \sum_{i: \mathcal{E}_i \leq \mathcal{E}} d_i,$$

where  $\mathcal{E}_i$  is the  $i$ th energy and  $d_i$  its degeneracy, with the cumulative Gaussian density

$$G(\mathcal{E}) = \int_{-\infty}^{\mathcal{E}} g(\mathcal{E}') d\mathcal{E}' = \frac{1}{2} \left[ 1 + \operatorname{erf} \left( \frac{\mathcal{E} - \mu}{\sqrt{2}\sigma} \right) \right] \quad (38)$$

for  $N=15$  and  $m=2$ . Note, in this respect, that the approximately Gaussian character of the level density has already been verified for other chains of HS type such as the original Haldane–Shastry chain,<sup>19</sup> its  $\operatorname{su}(m|n)$  supersymmetric version,<sup>23</sup> and the HS spin chain of  $BC_N$  type.<sup>21</sup>

The mean energy  $\mu$  and its standard deviation  $\sigma^2$ , which by the previous discussion characterize the approximate level density of the chain in Eq. (1) for large  $N$ , can be computed in closed form. Indeed, in Appendix A we show that

$$\mu = \frac{1}{2} \left( 1 + \frac{1}{m} \right) N^2 - \frac{N}{2m} (1 + \epsilon p), \quad (39)$$

$$\sigma^2 = \frac{N}{36} (4N^2 + 6N - 1) \left( 1 - \frac{1}{m^2} \right) + \frac{N(1-p)}{4m^2}, \quad (40)$$

where  $p \in \{0, 1\}$  is the parity of  $m$ . Thus, when  $N$  tends to infinity,  $\mu$  and  $\sigma^2$  respectively diverge as  $N^2$  and  $N^3$ , as for the original Polychronakos–Frahm chain.<sup>33</sup> By contrast, it is known that  $\mu \sim N^3$  and  $\sigma^2 \sim N^5$  for the trigonometric HS chains of both  $A_N$  (Ref. 19) and  $BC_N$  (Ref. 21) types. It is also interesting to observe that the standard deviation of the energy is independent of  $\epsilon$  even for odd  $m$ , when the spectrum does depend on  $\epsilon$  according to the previous section’s results on the partition function.

We have next studied the probability density  $p(s)$  of the spacing  $s$  between consecutive (unfolded) levels of the chain in Eq. (1). For many important integrable systems, it is known that  $p(s)$  is Poissonian,<sup>34,35</sup> in agreement with a well-known conjecture of Berry and Tabor.<sup>22</sup> On the other hand, it has been recently shown that for the HS chain of  $A_N$  type<sup>19</sup> (and its supersymmetric extension<sup>23</sup>) the cumulative density

$P(s) = \int_0^s p(x) dx$  is well approximated by an empiric law of the form

$$\tilde{P}(s) = \left( \frac{s}{s_{\max}} \right)^\alpha \left[ 1 - \gamma \left( 1 - \frac{s}{s_{\max}} \right)^\beta \right], \quad (41)$$

where  $s_{\max}$  is the largest normalized spacing,  $\alpha$  and  $\beta$  are adjustable parameters in the interval  $(0, 1)$ , and  $\gamma$  is fixed by requiring that the average spacing be equal to 1. Thus, the cumulative density of spacings for the HS chains of  $A_N$  type follows neither Poisson’s nor Wigner’s law

$$P(s) = 1 - e^{-\pi s^2/4},$$

characteristic of a chaotic system. Our aim is to ascertain whether the cumulative density of spacings for the PF chain of  $BC_N$  type in Eq. (1) resembles that of the  $A_N$ -type HS chain, or is rather Poissonian, as expected for a generic integrable model.

In order to compare the spacings distributions of spectra with different level densities, it is necessary to transform the “raw” spectrum by applying what is known as the *unfolding* mapping.<sup>36</sup> This mapping is defined by decomposing the cumulative level density  $F(\mathcal{E})$  as the sum of a fluctuating part  $F_{fl}(\mathcal{E})$  and a continuous part  $\eta(\mathcal{E})$ , which is then used to transform each energy  $\mathcal{E}_i$ ,  $i=1, \dots, n$ , into an unfolded energy  $\eta_i = \eta(\mathcal{E}_i)$ . In this way one obtains a uniformly distributed spectrum  $\{\eta_i\}_{i=1}^n$ , regardless of the initial level density. One finally considers the normalized spacings  $s_i = (\eta_{i+1} - \eta_i)/\Delta$ , where  $\Delta = (\eta_n - \eta_1)/(n-1)$  is the mean spacing of the unfolded energies, so that  $\{s_i\}_{i=1}^{n-1}$  has unit mean.

By the above discussion, in our case we can take the unfolding mapping  $\eta(\mathcal{E})$  as the cumulative Gaussian distribution [Eq. (38)] with parameters  $\mu$  and  $\sigma$ , respectively given by Eqs. (39) and (40). Just as for the level density, in order to compare the discrete distribution function  $p(s)$  with a continuous distribution, it is more convenient to work with the cumulative spacings distribution  $P(s)$ . Our computations show that for a wide range of values of  $N$ ,  $m$ , and  $\epsilon = \pm 1$ , the distribution  $P(s)$  is well approximated by the empiric law in Eq. (41) with suitable values of  $\alpha$  and  $\beta$ . For instance, for  $N=20$  and  $m=2$ , the largest spacing is  $s_{\max}=3.13$ , and the least-squares fit parameters  $\alpha$  and  $\beta$  are respectively 0.29 and 0.24, with a mean-square error of  $6.0 \times 10^{-4}$  (see Fig. 2). Thus the PF spin chain of  $BC_N$  type behaves in this respect as the HS chain of  $A_N$  type, and unlike most known integrable systems. In fact, we have also studied the spacings’ distribution of the original ( $A_N$ -type) PF chain, obtaining completely similar results.<sup>33</sup> These findings (and also those of Ref. 23) suggest that a spacings distribution qualitatively similar to the empiric law in Eq. (41) is characteristic of all spin chains of HS type.

Our next objective is to explain this characteristic behavior of the cumulative spacings distribution  $P(s)$  of the chain in Eq. (1) using only a few essential properties of its spectrum. We shall find a simple analytic expression without any adjustable parameters approximating  $P(s)$  even better than the empiric law in Eq. (41). Moreover, we have strong nu-

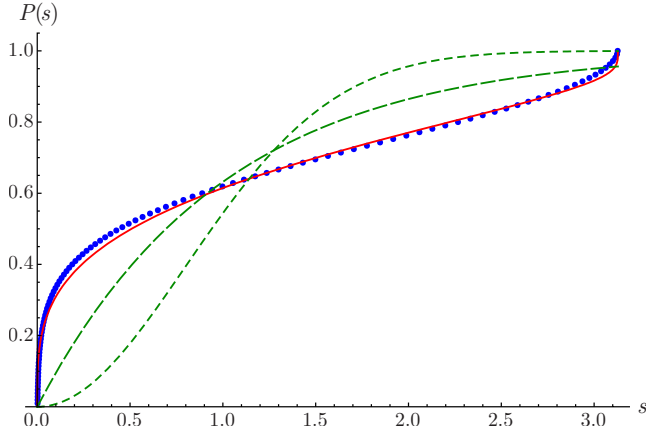


FIG. 2. (Color online) Cumulative spacings distribution  $P(s)$  and its approximation  $\tilde{P}(s)$  (continuous red line) for  $N=20$  and  $m=2$ . For convenience, we have also represented Poisson's (green, long dashes) and Wigner's (green, short dashes) cumulative distributions.

merical evidence that the new expression also provides a very accurate approximation to the cumulative spacings distribution of the original HS and PF chains.

Consider, to begin with, any spectrum  $\mathcal{E}_1 \equiv \mathcal{E}_{\min} < \dots < \mathcal{E}_n \equiv \mathcal{E}_{\max}$  obeying the following conditions:

- (i) The energies are equispaced, i.e.,  $\mathcal{E}_{i+1} - \mathcal{E}_i = \delta$  for  $i = 1, \dots, n-1$ .
- (ii) The level density (normalized to unity) is approximately Gaussian [cf. Eq. (37)].
- (iii)  $\mathcal{E}_n - \mu, \mu - \mathcal{E}_1 \gg \sigma$ .
- (iv)  $\mathcal{E}_1$  and  $\mathcal{E}_n$  are approximately symmetric with respect to  $\mu$ , namely  $|\mathcal{E}_1 + \mathcal{E}_n - 2\mu| \ll \mathcal{E}_n - \mathcal{E}_1$ .

As discussed above, the spectrum of the chain in Eq. (1) satisfies the first condition with  $\delta=1$ , while condition (ii) holds for sufficiently large  $N$ . As to the third condition, from Eqs. (33), (39), (40), (B1), and (B2), it follows that both  $(\mathcal{E}_n - \mu)/\sigma$  and  $(\mu - \mathcal{E}_1)/\sigma$  grow as  $N^{1/2}$  when  $N \rightarrow \infty$ . The last condition is also satisfied for large  $N$ , since by the equations just quoted  $|\mathcal{E}_1 + \mathcal{E}_n - 2\mu| = O(N)$  while  $\mathcal{E}_n - \mathcal{E}_1 = O(N^2)$ .

From conditions (i) and (ii), it follows that

$$\eta_{i+1} - \eta_i = G(\mathcal{E}_{i+1}) - G(\mathcal{E}_i) \simeq G'(\mathcal{E}_i) \delta = g(\mathcal{E}_i) \delta.$$

On the other hand, by condition (iii), we have

$$\eta_n = G(\mathcal{E}_n) \simeq 1, \quad \eta_1 = G(\mathcal{E}_1) \simeq 0,$$

so that  $\Delta \simeq 1/(n-1)$ . Thus

$$s_i = \frac{\eta_{i+1} - \eta_i}{\Delta} \simeq Wg(\mathcal{E}_i) = \frac{W}{\sqrt{2\pi}\sigma} e^{-\frac{(\mathcal{E}_i - \mu)^2}{2\sigma^2}}, \quad (42)$$

where

$$W \equiv (n-1)\delta = \mathcal{E}_n - \mathcal{E}_1 \quad (43)$$

on account of the first condition. The cumulative probability density  $P(s)$  is, by definition, the quotient of the number of normalized spacings  $s_i \leq s$  by the total number of spacings, that is,

$$P(s) = \frac{\#(s_i \leq s)}{n-1}.$$

By Eq. (42),

$$\#(s_i \leq s) = \#(\mathcal{E}_1 \leq \mathcal{E}_i \leq \mathcal{E}_n) + \#(\mathcal{E}_+ \leq \mathcal{E}_i < \mathcal{E}_n), \quad (44)$$

where

$$\mathcal{E}_{\pm} = \mu \pm \sqrt{2}\sigma \sqrt{\log\left(\frac{s_{\max}}{s}\right)} \quad (45)$$

are the roots of the equation  $s = Wg(\mathcal{E})$  expressed in terms of the maximum normalized spacing

$$s_{\max} = \frac{W}{\sqrt{2\pi}\sigma}. \quad (46)$$

Using the first condition to estimate the RHS of Eq. (44), we easily obtain

$$P(s) \simeq \frac{1}{W} [\max(\mathcal{E}_n - \mathcal{E}_1, 0) + \max(\mathcal{E}_n - \mathcal{E}_+, 0)]. \quad (47)$$

In fact, we can replace the latter approximation to  $P(s)$  by the simpler one

$$P(s) \simeq \frac{1}{W} (\mathcal{E}_n - \mathcal{E}_1 + \mathcal{E}_n - \mathcal{E}_+), \quad (48)$$

since the error involved is bounded by  $|\mathcal{E}_1 + \mathcal{E}_n - 2\mu|/W$ , which is vanishingly small by condition (iv). Substituting the explicit expression in Eq. (45) for  $\mathcal{E}_{\pm}$  into Eq. (48) and, using Eqs. (43) and (46), we finally obtain

$$P(s) \simeq 1 - \frac{2}{\sqrt{\pi}s_{\max}} \sqrt{\log\left(\frac{s_{\max}}{s}\right)}. \quad (49)$$

The RHS of this remarkable expression depends only on the quantity  $s_{\max}$ , which for the PF chain of  $BC_N$  type is completely determined as a function of  $N$  and  $m$  by Eqs. (33), (43), (46), (B1), and (B2). In particular, for large  $N$ , we have the asymptotic expression

$$s_{\max} = \frac{3}{\sqrt{2\pi}} \sqrt{\frac{m-1}{m+1}} N^{1/2} + O(N^{-1/2}).$$

Our numerical computations indicate that Eq. (49) is in excellent agreement with the data for a broad range of values of  $N$ ,  $m$ , and  $\epsilon = \pm 1$ , providing much greater accuracy than the empiric formula (41). For instance, for  $N=20$  and  $m=2$ , we have  $s_{\max} = 6\sqrt{35}/(41\pi) \simeq 3.12765$ , which differs from the numerically computed maximum spacing by  $4.5 \times 10^{-4}$ . In Fig. 3, we compare the corresponding cumulative spacings distribution with its approximation in Eq. (49) using the above value of  $s_{\max}$ . The mean-square error is in this case  $2.2 \times 10^{-5}$ , smaller than that of the empiric law in Eq. (41) by more than an order of magnitude.

A natural question in view of these results is to what extent the approximation in Eq. (49) is applicable to other

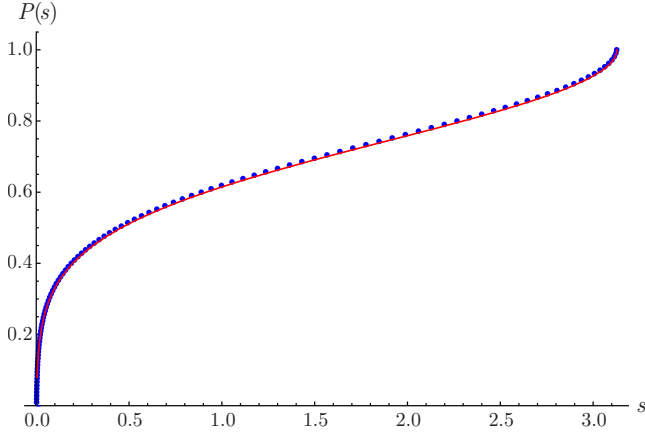


FIG. 3. (Color online) Cumulative spacings distribution  $P(s)$  and its analytic approximation (49) (continuous red line) for  $N=20$  and  $m=2$ .

spin chains of HS type. For the PF chain of  $A_N$  type, one can check that the spectrum satisfies conditions (i)–(iv) of this section and, in fact, we have verified that Eq. (49) holds with remarkable accuracy in this case.<sup>33</sup> The situation is less clear for the original HS chain, whose spectrum is certainly not equispaced.<sup>19</sup> Nevertheless, our computations show that the formula (49) still fits the numerical data much better than our previous approximation [Eq. (41)], a fact clearly deserving further study.

#### ACKNOWLEDGMENTS

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#### APPENDIX A: COMPUTATION OF $\mu$ AND $\sigma^2$

In this appendix we shall compute in closed form the mean energy  $\mu$  of the spin chain in Eq. (1) and its standard deviation  $\sigma^2$  as functions of the number of particles  $N$  and internal degrees of freedom  $m$ .

In the first place, using the formulas for the traces of the operators  $S_{ij}$ ,  $\tilde{S}_{ij}$ , and  $S_i$  in Ref. 21, we easily obtain

$$\mu = \frac{\text{tr } \mathcal{H}^\epsilon}{m^N} = \left(1 + \frac{1}{m}\right) \sum_{i \neq j} (h_{ij} + \tilde{h}_{ij}) + \left(1 - \frac{\epsilon p}{m}\right) \sum_i h_i, \quad (\text{A1})$$

where  $p$  is the parity of  $m$  and

$$h_{ij} = (\xi_i - \xi_j)^{-2}, \quad \tilde{h}_{ij} = (\xi_i + \xi_j)^{-2}, \quad h_i = \beta \xi_i^{-2}.$$

The sums appearing in Eq. (A1) can be expressed in terms of sums involving the zeros  $y_i = \xi_i^2/2$ ,  $1 \leq i \leq N$ , of the Laguerre polynomial  $L_N^{\beta-1}$  as follows:

$$\sum_{i \neq j} (h_{ij} + \tilde{h}_{ij}) = \sum_{i \neq j} \frac{2y_i}{(y_i - y_j)^2}, \quad \sum_i h_i = \frac{\beta}{2} \sum_i \frac{1}{y_i}. \quad (\text{A2})$$

The latter sums can be easily computed using the following identities satisfied by the zeros  $y_i$ , which can be found in Ref. 37:

$$\sum_{j \neq i} \frac{2y_i}{y_i - y_j} = y_i - \beta, \quad (\text{A3})$$

$$\sum_{j \neq i} \frac{12y_i^2}{(y_i - y_j)^2} = -y_i^2 + 2(2N + \beta)y_i - \beta(\beta + 4). \quad (\text{A4})$$

Indeed, from the first of these identities, we easily obtain

$$\sum_i y_i = N(N + \beta - 1), \quad \sum_i \frac{1}{y_i} = \frac{N}{\beta}, \quad (\text{A5})$$

so that by Eq. (A4),

$$\begin{aligned} \sum_{i \neq j} \frac{2y_i}{(y_i - y_j)^2} &= \frac{1}{6} \left( -\sum_i y_i + 2N(2N + \beta) - \beta(\beta + 4) \sum_i \frac{1}{y_i} \right) \\ &= \frac{1}{2} N(N - 1). \end{aligned}$$

Combining the last two equations with Eqs. (A1) and (A2), we immediately arrive at Eq. (39) for the level density  $\mu$ .

Turning now to the standard deviation of the energy  $\sigma^2$ , from Eqs. (66)–(68) in Ref. 21, we have

$$\begin{aligned} \sigma^2 &= \frac{\text{tr}(\mathcal{H}^\epsilon)^2}{m^N} - \mu^2 = \left(1 - \frac{1}{m^2}\right) \left( 2 \sum_{i \neq j} (h_{ij}^2 + \tilde{h}_{ij}^2) + \sum_i h_i^2 \right) \\ &\quad + \frac{4}{m^2} (1 - p) \left( \frac{1}{4} \sum_i h_i^2 - \sum_{i \neq j} h_{ij} \tilde{h}_{ij} \right). \end{aligned} \quad (\text{A6})$$

All of the sums appearing in the latter expression can be readily evaluated. Indeed, we have

$$\sum_i h_i^2 = \frac{\beta^2}{4} \sum_i \frac{1}{y_i^2} = \frac{N(N + \beta)}{4(\beta + 1)}, \quad (\text{A7})$$

$$\sum_{i \neq j} h_{ij} \tilde{h}_{ij} = \frac{1}{4} \sum_{i \neq j} \frac{1}{(y_i - y_j)^2} = \frac{N(N - 1)}{16(\beta + 1)}, \quad (\text{A8})$$

where we have used Eqs. 15 and 17 from Ref. 37. On the other hand,

$$\begin{aligned} \sum_{i \neq j} (h_{ij}^2 + \tilde{h}_{ij}^2) &= \frac{1}{2} \sum_{i \neq j} \frac{y_i^2 + y_j^2 + 6y_i y_j}{(y_i - y_j)^4} = 2 \sum_{i \neq j} \frac{y_i^2 + y_j^2}{(y_i - y_j)^4} \\ &\quad - \frac{3}{2} \sum_{i \neq j} \frac{1}{(y_i - y_j)^2} = 4 \sum_{i \neq j} \frac{y_i^2}{(y_i - y_j)^4} - \frac{3N(N - 1)}{8(\beta + 1)}, \end{aligned} \quad (\text{A9})$$

while from Theorem 5.1 of Ref. 38 it follows that



$$\begin{aligned}
720 \sum_{i \neq j} \frac{y_i^2}{(y_i - y_j)^4} &= \sum_i y_i^2 - 4(2N + \beta) \sum_i y_i \\
&\quad + 2N(8N^2 + 2(4N - 1)\beta + 3\beta^2) \\
&\quad - 4(2N + \beta)(\beta^2 - 2\beta - 18) \sum_i \frac{1}{y_i} \\
&\quad + \beta(\beta^3 - 4\beta^2 - 104\beta - 144) \sum_i \frac{1}{y_i^2}.
\end{aligned} \tag{A10}$$

All the sums in the right-hand side of the latter expression have already been evaluated in Eqs. (A5) and (A7) except the first one. In order to compute this sum, we multiply Eq. (A3) by  $y_i$  and sum over  $i$ , obtaining

$$\begin{aligned}
\sum_{i \neq j} \frac{2y_i^2}{y_i - y_j} &= \sum_{i \neq j} \frac{y_i^2 - y_j^2}{y_i - y_j} \\
&= \sum_{i \neq j} (y_i + y_j) = 2(N - 1) \sum_i y_i = \sum_i y_i^2 - \beta \sum_i y_i,
\end{aligned}$$

and hence, by Eq. (A5),

$$\sum_i y_i^2 = N(N + \beta - 1)(2N + \beta - 2). \tag{A11}$$

Substituting the value of the latter sum into Eq. (A10) and using Eqs. (A5) and (A7), we obtain

$$\sum_{i \neq j} \frac{y_i^2}{(y_i - y_j)^4} = \frac{N(N - 1)((2N + 5)\beta + 2N + 14)}{144(\beta + 1)}. \tag{A12}$$

Equation (40) now follows by inserting Eqs. (A7)–(A9) and (A12) into Eq. (A6).

## APPENDIX B: COMPUTATION OF THE MINIMUM ENERGY

In this appendix we shall obtain an explicit expression for the minimum energy  $\mathcal{E}_{\min}^\epsilon$  of the spin chain in Eq. (1). Our starting point is Eq. (7), which implies that

$$\mathcal{E}_{\min}^\epsilon = \lim_{a \rightarrow \infty} \frac{1}{a} (E_{\min}^\epsilon - E_{\min}^{\text{sc}}),$$

where  $E_{\min}^{\text{sc}}$  and  $E_{\min}^\epsilon$  are the minimum energies of the scalar and spin dynamical models in Eqs. (3) and (5), respectively. By the discussion in Sec. II [cf. Eq. (19)],  $\mathcal{E}_{\min}^\epsilon$  is the minimum value of  $|\mathbf{n}|$ , where  $\mathbf{n}$  is any multi-index compatible with conditions (i)–(iii) in the latter section. From these conditions, it follows that the multi-index  $\mathbf{n}$  minimizing  $|\mathbf{n}|$  is given by

$$\mathbf{n} = (\overbrace{k, \dots, k}^l, \overbrace{k-1, \dots, k-1}^{l_{k-1}}, \dots, \overbrace{0, \dots, 0}^{l_0}),$$

where  $0 \leq l < l_k$  and  $l_i$  is given by

$$l_i = \begin{cases} \frac{m}{2}, & m \text{ even} \\ \frac{m + \epsilon}{2}, & m \text{ odd and } i \text{ even} \\ \frac{m - \epsilon}{2}, & m \text{ and } i \text{ odd.} \end{cases}$$

In view of the above expression, it is convenient to treat separately the cases of even and odd  $m$ . For even  $m$ , we have  $l_i = \frac{m}{2}$  for all  $i$ , so that

$$k = \lfloor 2N/m \rfloor, \quad l = N \bmod \frac{m}{2},$$

and

$$\begin{aligned}
\mathcal{E}_{\min}^\epsilon &= \frac{m}{2} \sum_{i=1}^{k-1} i + lk = \frac{m}{4} k(k-1) + lk \\
&= \frac{N^2}{m} - \frac{N}{2} + \frac{l(m-2l)}{2m} \quad (m \text{ even}). \tag{B1}
\end{aligned}$$

Suppose now that  $m$  is odd. If  $k=2j$  is an even number, then  $l_0=l_2=\dots=l_{2j-2}=\frac{m+\epsilon}{2}$ ,  $l_1=l_3=\dots=l_{2j-1}=\frac{m-\epsilon}{2}$ , and, thus,  $N = jm + l$ , so that

$$j = \lfloor N/m \rfloor, \quad l = (N \bmod m) < l_{2j} = \frac{m + \epsilon}{2}.$$

The minimum energy in this case is thus given by

$$\begin{aligned}
\mathcal{E}_{\min}^\epsilon &= \frac{m + \epsilon}{2} \sum_{i=0}^{j-1} 2i + \frac{m - \epsilon}{2} \sum_{i=0}^{j-1} (2i + 1) + 2jl \\
&= \frac{N^2}{m} - \frac{N(m + \epsilon)}{2m} + \frac{l(m + \epsilon - 2l)}{2m}.
\end{aligned}$$

On the other hand, if  $k=2j+1$  is odd, then  $l_0=l_2=\dots=l_{2j}=\frac{m+\epsilon}{2}$ ,  $l_1=l_3=\dots=l_{2j-1}=\frac{m-\epsilon}{2}$ , and, thus,  $N = jm + l + \frac{m+\epsilon}{2}$ , with  $0 \leq l < l_{2j+1} = \frac{m-\epsilon}{2}$ . Calling  $l' = l + \frac{m+\epsilon}{2}$ , we have

$$j = \lfloor N/m \rfloor, \quad l' = (N \bmod m) \geq \frac{m + \epsilon}{2},$$

and

$$\begin{aligned}
\mathcal{E}_{\min}^\epsilon &= \frac{m + \epsilon}{2} \sum_{i=0}^j 2i + \frac{m - \epsilon}{2} \sum_{i=0}^{j-1} (2i + 1) + (2j + 1) \left( l' - \frac{m + \epsilon}{2} \right) \\
&= \frac{N^2}{m} - \frac{N(m + \epsilon)}{2m} + \frac{(l' - m)(m + \epsilon - 2l')}{2m}.
\end{aligned}$$

Hence, we can express the minimum energy for odd  $m$  in a unified way as

$$\begin{aligned}
\mathcal{E}_{\min}^\epsilon &= \frac{N^2}{m} - \frac{N(m + \epsilon)}{2m} + \frac{1}{2m} [l - m\theta(2l - m - \epsilon)] \\
&\quad \times (m + \epsilon - 2l) \quad (m \text{ odd}), \tag{B2}
\end{aligned}$$

where  $l = N \bmod m$  and  $\theta(x) = 1$  for  $x \geq 0$  and 0 otherwise.

\*Corresponding author. artemio@fis.ucm.es

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