THE PRODUCT OF TWO UNIFORM REALCOMPACTIFICATIONS

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Abstract. For a uniform space (X, μ), $s_\mu X$ denotes the well-known Samuel compactification of X. We say that a realcompactification αX of X is a *uniform realcompactification* whenever it is a topological subspace of $s_\mu X$, i.e., $X \subset \alpha X \subset s_\mu X$. In this paper we are mainly concerned with the following equation:

$$\alpha(X \times Y) = \alpha X \times \alpha Y$$

for three specific uniform realcompactifications: the Samuel realcompactification $H(U_\mu(X))$, the G_δ-realcompactification $K(X)$, and the countable-modification realcompactification $e_\mu X$. For each of these, we provide necessary and sufficient conditions for the corresponding equality to hold. Moreover, we give a characterization of those uniform realcompactifications of (X, μ) that are, in addition, G_δ-closed in $s_\mu X$.

1. Introduction

For a uniform space (X, μ), the most natural and suitable compactification of X is the classical Samuel compactification, denoted by $s_\mu X$. Recall that $s_\mu X$ is the completion of the uniform space $(X, f\mu)$ where $f\mu$ denotes the uniformity on X induced by all the finite covers from μ. Moreover, note that this uniformity $f\mu$ is equivalent to the weak uniformity $w_{U_\mu^*(X)}$ induced by the family $U_\mu^*(X)$ of all the real-valued bounded uniformly continuous functions on X.

Recently, some authors have considered different realcompactifications for a uniform space. For example, Garrido and Meroño in [12], [13], Chekeev in [4], Vroegrijk in [28], [29]. We also refer to the survey by Hušek [21] where various notions of realcompactness for uniform spaces are collected.

By a *uniform realcompactification* of (X, μ) we simply mean a realcompactification of X that is a topological subspace of its Samuel compactification $s_\mu X$. The uniform realcompactifications of (X, μ) that we consider in this paper are:

- The *Samuel realcompactification* $H(U_\mu(X))$, which is the completion of the uniform space $(X, w_{U_\mu(X)})$ where $w_{U_\mu(X)}$ denotes the weak uniformity induced by $U_\mu(X)$, the set of all the real-valued uniformly continuous functions on X.
- The *G_δ-realcompactification* $K(X)$, defined as the G_δ-closure of X in $s_\mu X$.
- The *countable-modification realcompactification* $e_\mu X$, which is the completion of the uniform space $(X, e\mu)$ where $e\mu$ denotes the uniformity on X induced by all the countable covers from μ.

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For uniform spaces (X, μ) and (Y, ν), we denote by $(X \times Y, \mu \times \nu)$ their usual uniform product space. If αX represents any of the three above realcompactifications of (X, μ), our main goal in this paper is to find sufficient and necessary conditions under which the following equality

$$\alpha(X \times Y) = \alpha X \times \alpha Y$$

holds, where this equality is understood as an equivalence between two realcompactifications of the same topological space $X \times Y$; that is, between $\alpha(X \times Y)$ and $\alpha X \times \alpha Y$. (In Section 2 we recall the notion of equivalent realcompactifications).

This problem has several topological precedents. The most notable is the famous Glicksberg Theorem for the Stone-Čech compactification βX, which states that for (infinite) Tychonoff spaces X and Y, $\beta(X \times Y) = \beta X \times \beta Y$ if and only if $X \times Y$ is pseudocompact ([17]). A similar problem has been considered for the well-known Hewitt realcompactification υX, with the aim of characterizing those Tychonoff spaces X and Y for which $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$. Unlike the Stone-Čech case, this problem is still certainly not solved, but many partial results are known. See, for instance, the papers by Hušek [19], Ohta [23], and Render [27] for additional references.

There exists also some precedents concerning the Samuel compactification $s_\mu X$, studied by Čech in [3] and by Woods in [30]. More precisely, they proved that, in the frame of uniform spaces and in the frame of metric spaces, respectively, $s_{\mu \times \nu}(X \times Y) = s_\mu X \times s_\nu Y$ if and only if at least one of the spaces is totally bounded, in other words, when the Samuel compactification of one of the factors coincides with its completion.

Note that βX can be also viewed as the Samuel compactification of the uniform space X endowed with the so-called *fine uniformity* $\mathtt{u}$, so that $\beta X = s_{\mathtt{u}} X$. Recall that the fine uniformity is characterized by the universal property that every continuous function from a fine space X to a uniform space Y is uniformly continuous. However the result mentioned above does not extend to the topological case of the Stone-Čech compactification because the product of fine uniformities is not, in general, fine [5].

More recently, we proved that a similar result to Woods' theorem holds for the so-called *Lipschitz realcompactification* $H(Lip_d(X))$ in the context of metric spaces ([14]). Namely, the equality between the corresponding realcompactifications of $X \times Y$ is true if and only if the Lipschitz realcompactification of one of the factors is just its completion.

The aforementioned results on the Samuel compactification and on the Lipschitz realcompactifications motivate our study of the equality $\alpha(X \times Y) = \alpha X \times \alpha Y$, for other uniform realcompactifications. First we prove that, for some of our uniform realcompactifications, the corresponding expected results give only "sufficient conditions". Specifically, we obtain that for the Samuel realcompactification $H(U_\mu(X))$ and for the countable-modification realcompactification $e_\mu X$, if the completion of one of the factors coincides with the given realcompactification then $\alpha(X \times Y) = \alpha X \times \alpha Y$ (Theorem 4 and Theorem 25, respectively). Unfortunately, these sufficient conditions are not necessary, even in the realm of metric spaces, as shown by Examples 5 and 26. Nevertheless, we provide some "necessary conditions" for the equality (Theorems 9, 10 and 28).

For the specific case of the G_δ-realcompactification $K(X)$, we obtain a result giving necessary and sufficient conditions for the equality, involving not the usual completion of the uniform space, but the known as the weak completion (i.e., the G_δ-closure of the space in its completion). Thus, Theorem 17 says that $K(X \times Y) = K(X) \times K(Y)$ if and only if the weak completion of at least one the factors coincides with its G_δ-realcompactification. As an interesting consequence, we can derive a result by Ohta [23] concerning the equation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$, since, when X is a metric space, we have $K(X) = \upsilon X$.

The structure of the paper is as follows. We begin with a preliminary section (Section 2) where we recall general facts concerning realcompactifications for Tychonoff spaces, mainly taken from [9]. Sections 3, 4 and 5 are devoted to study, respectively, our three uniform realcompactifications. In each section, we present key properties leading to our main results. Notably, we provide Katetov-Shirota type characterizations of those uniform spaces that topologically coincide with the corresponding realcompactification in terms of completeness properties together with some fact related with measurable cardinals (Propositions 1, 13 and 21). In fact, these Katetov-Shirota results give us the necessary conditions in the cases where they are needed. We conclude the paper by noting that all of these realcompactifications have a common property, namely, they are not only topological subspaces of $s_\mu X$, but they are in fact G_δ-closed in it. Accordingly, we include a final section where we characterize this special class of uniform realcompactifications.

For general terminology, see [8] and [15].

2. Some preliminaries on realcompactifications

In this section we present some basic facts about realcompactifications that we have taken from [9]. Recall that a realcompactification of a Tychonoff space X is a realcompact space αX in which X is densely embedded. There are several methods to construct realcompactifications for a space X. One of them consists of starting with a family $\mathcal{L}$ of real-valued continuous functions on X, that we suppose having the algebraic structure of a unital vector lattice. Next, we take $H(\mathcal{L})$ the set of all the real unital vector lattice homomorphisms on $\mathcal{L}$, endowed with the topology inherited as a subspace of the product space $\mathbb{R}^{\mathcal{L}}$, where the real line $\mathbb{R}$ has the usual topology. It is easy to check that $H(\mathcal{L})$ is closed in $\mathbb{R}^{\mathcal{L}}$, and therefore it is a realcompact space. Analogously, we can consider $\mathcal{L}^*$ the unital vector sublattice formed by the bounded functions in $\mathcal{L}$. Now the space $H(\mathcal{L}^*)$ is in fact compact, and it is easy to see that $H(\mathcal{L})$ can be thought of as a topological subspace of $H(\mathcal{L}^*)$. Hence, we can write $H(\mathcal{L}) \subset H(\mathcal{L}^*)$.

If, in addition, the family $\mathcal{L}$ separates points and closed sets of X, i.e., when for every closed subset F of X and $x \in X \setminus F$ there exists some $f \in \mathcal{L}$ such that $f(x) \notin \overline{f(F)}$, then we can embed the topological space X (in a dense way) into $H(\mathcal{L})$. Indeed, this is achieved if we consider the evaluation map

$$e : X \to H(\mathcal{L}) \subset \mathbb{R}^{\mathcal{L}}$$

$$x \rightsquigarrow e(x) = (f(x))_{f \in \mathcal{L}}.$$

And this means, in particular, that $H(\mathcal{L})$ is a realcompactification of X. Analogously, $H(\mathcal{L}^*)$ is a compactification of X, and we have

$$X \subset H(\mathcal{L}) \subset H(\mathcal{L}^*).$$

We can easily check that every realcompactification of X can be obtained in this way for a suitable family $\mathcal{L}$. The advantage of working with this type of realcompactifications $H(\mathcal{L})$ is that every function in $\mathcal{L}$ (respectively in $\mathcal{L}^*$) admits a unique continuous extension to $H(\mathcal{L})$ (resp. to $H(\mathcal{L}^*)$). In fact, $H(\mathcal{L})$ (resp. $H(\mathcal{L}^*)$) is characterized (up to equivalence) as the smallest realcompactification (resp. compactification) of X with this property. Note that we are here considering the usual order in the set of all the realcompactifications and compactifications on X. Recall that for two realcompactifications of X, $\alpha_1 X$ and $\alpha_2 X$, we say that $\alpha_1 X \leq \alpha_2 X$ whenever there is a continuous mapping $h : \alpha_2 X \to \alpha_1 X$ leaving X pointwise fixed. And we say that $\alpha_1 X$ and $\alpha_2 X$ are equivalent whenever $\alpha_1 X \leq \alpha_2 X$ and $\alpha_2 X \leq \alpha_1 X$, and this implies the existence of a homeomorphism between $\alpha_1 X$ and $\alpha_2 X$ leaving X pointwise fixed.

Another important property in connection with the extension of continuous functions is the following: "each $f \in \mathcal{L}$ can be extended to a unique continuous function $f^* : H(\mathcal{L}^*) \to \mathbb{R} \cup \{\infty\}$, where $\mathbb{R} \cup \{\infty\}$ denotes the one point compactification of the real line". In particular, this allows us to describe the space $H(\mathcal{L})$ as follows,

$$H(\mathcal{L}) = \big\{\xi \in H(\mathcal{L}^*) :\ f^*(\xi) \neq \infty \ \text{ for all } \ f \in \mathcal{L}\big\}.$$

Examples of such realcompactifications are the following. For a Tychonoff space X, if we consider $C(X)$, the set of all the real-valued continuous functions on X, then $H(C(X)) = \upsilon X$ and $H(C^*(X)) = \beta X$, i.e., they are the well-known Hewit-Nachbin realcompactification and Stone-Čech compactification of X, respectively. For a uniform space (X, μ), we denote by $U_\mu(X)$ the set of all uniformly continuous real functions on X, and by $U^*_\mu(X)$ the set of all bounded functions from $U_\mu(X)$. Then $H(U_\mu(X))$ is the so-called Samuel realcompactification of X, and $H(U^*_\mu(X))$ is the Samuel compactification $s_\mu X$ of X. For a metric space (X, d), if $Lip_d(X)$ denotes the set of all the real-valued Lipschitz functions on X then $H(Lip_d(X))$ is known as the Lipschitz realcompactification of X, whereas $H(Lip^*_d(X)) = H(U^*_\mu(X)) = s_\mu X$. We refer to [12] and [13] where these Samuel and Lipschitz realcompactifications are introduced and studied.

3. The product of two Samuel realcompactifications

We start this section recalling some properties of the Samuel realcompactification $H(U_\mu(X))$ of a uniform space (X, μ), taken from [12] and [13]. Thus, we know that:

- $X \subset H(U_\mu(X)) \subset s_\mu X$.
- $H(U_\mu(X)) = \big\{\xi \in s_\mu X :\ f^*(\xi) \neq \infty \ \text{ for all } \ f \in U_\mu(X)\big\}$.
- $H(U_\mu(X)) = H(U_{\widetilde{\mu}}(\widetilde{X}))$, where $(\widetilde{X}, \widetilde{\mu})$ is the completion of the space (X, μ).
- The completion of the uniform space X endowed with the weak uniformity $w_{U_\mu(X)}$ given by $U_\mu(X)$, is precisely $H(U_\mu(X))$ with the uniformity inherited as a uniform subspace of the product $\mathbb{R}^{U_\mu(X)}$.

We say that X is *Samuel realcompact* when $X = H(U_\mu(X))$, where this equality is topological not necessarily uniform. The following theorem proved in [13] (Theorem 12 and Remark 13) is a Katetov-Shirota type result since it gives a characterization of those spaces that coincide with one of their realcompactifications in terms of a certain completeness property together with some considerations involving measurable cardinals.

Proposition 1. ([13]) *The uniform space (X, μ) is Samuel realcompact if and only if (X, μ) is Bourbaki-complete and every uniform partition of X has a non-measurable cardinal.*

Recall that the uniform property of Bourbaki-completeness was introduced and studied in [10] for metric spaces, and in [11] for general uniform spaces. We will use some aspects of this property later.

As we have already announced, we are interested in determining the equivalence between the two realcompactifications, of the product $X \times Y$, $H(U_{\mu\times\nu}(X \times Y))$ and $H(U_\mu(X)) \times H(U_\nu(Y))$. At this respect, we have a first result saying that one of the inequalities is always true.

Proposition 2. *Let (X, μ) and (Y, ν) be uniform spaces. Then,*

$$H(U_{\mu\times\nu}(X \times Y)) \geq H(U_\mu(X)) \times H(U_\nu(Y)).$$

Proof. In order to see this inequality we need to prove that the inclusion map,

$$i : X \times Y \to H(U_\mu(X)) \times H(U_\nu(Y))$$

can be continuously extended to $H(U_{\mu\times\nu}(X \times Y))$. For that, it is enough to note that i is in fact uniformly continuous when $X \times Y$ is endowed with the weak uniformity $w_{U_{\mu\times\nu}(X\times Y)}$. Therefore, by [8, Theorem 8.3.10], it can be extended to a (uniformly) continuous map to its completion $H(U_{\mu\times\nu}(X \times Y))$. And this completes the proof. □

The next example shows that the non equivalence between the two realcompactifications of $X \times Y$ could occur.

Example 3. Let X and Y be two Banach spaces of infinite dimension. Since the Samuel realcompactification of every Banach space coincides with its Lipschitz realcompactification (see [12]), the equivalence between the above considered realcompactifications will be true whenever the equivalence between the corresponding Lipschitz realcompactifications is true. But according to Corollary 12 in [14] this happens if and only if one of the spaces has finite dimension.

Next, we are going to give a sufficient condition for the desired equivalence. Note that this condition says, in particular, that at least one of the uniform spaces has the Samuel realcompactification as simple as possible, that is, it is just its completion. Namely, $H(U_\mu(X)) = H(U_{\tilde{\mu}}(\widetilde{X})) = \widetilde{X}$.

Theorem 4. *Let (X, μ) be a uniform space such that its completion $(\widetilde{X}, \widetilde{\mu})$ is Samuel realcompact, and let (Y, ν) be a uniform space. Then $H(U_{\mu\times\nu}(X \times Y)) = H(U_\mu(X)) \times H(U_\nu(Y))$.*

Proof. Firstly note that, as we have said before, the Samuel realcompactification of a uniform space is equivalent to the Samuel realcompactification of its completion, and this means that the above equality can be written as,

$$H(U_{\tilde{\mu}\times\tilde{\nu}}(\widetilde{X}\times\widetilde{Y})) = H(U_{\tilde{\mu}}(\widetilde{X}))\times H(U_{\tilde{\nu}}(\widetilde{Y})).$$

Then, we can suppose without loss of generality that both spaces are complete. Now, if we assume that the complete space (X,μ) is Samuel realcompact, we have that $H(U_\mu(X)) = X$. From all the above, we have only to see that

$$X \times H(U_\nu(Y)) \geq H(U_{\mu\times\nu}(X\times Y))$$

or, equivalently, that the inclusion map

$$i : X \times Y \to H(U_{\mu\times\nu}(X\times Y)) \subset \mathbb{R}^{U_{\mu\times\nu}(X\times Y)}$$

can be continuously extended to $X \times H(U_\nu(Y))$. To this end, we are going to extend every coordinate function $\pi_f \circ i$, for every $f \in U_{\mu\times\nu}(X\times Y)$. Note that $\pi_f \circ i = f$.

Indeed, let $f \in U_{\mu\times\nu}(X\times Y)$ and, for $x \in X$ fixed, define $f_x(y) = f(x,y)$, for every $y \in Y$. Then $f_x \in U_\nu(Y)$, and hence f_x can be extended to a continuous function f_x^* on $H(U_\nu(Y))$. Next, we can define $\widetilde{f}(x,\xi) = f_x^*(\xi)$, for every $(x,\xi) \in X \times H(U_\nu(Y))$. In order to check that $\widetilde{f}$ is continuous on $X \times H(U_\nu(Y))$, it is enough to prove that $\widetilde{f}$ is continuous on $(X\times Y)\cup\{(x_0,\xi_0)\}$, for every $x_0 \in X$ and $\xi_0 \in H(U_\nu(Y))\backslash Y$ (see 6H in [15]).

Let $x_0 \in X$, $\xi_0 \in H(U_\nu(Y))\backslash Y$, and $\varepsilon > 0$. Since $f_{x_0}^*$ is continuous on $H(U_\nu(Y))$, there exists an open set $V^{\xi_0} \subset H(U_\nu(Y))$ such that $\xi_0 \in V^{\xi_0}$ and

$$f_{x_0}^*(V^{\xi_0}) \subset B_{d_u}(f_{x_0}^*(\xi_0), \varepsilon/2)$$

where $B_{d_u}(f_{x_0}^*(\xi_0), \varepsilon/2)$ denotes the open ball centered in $f_{x_0}^*(\xi_0)$ and radius $\varepsilon/2$ with the usual metric d_u in $\mathbb{R}$.

On the other hand, as f is uniformly continuous, there exists a uniform cover $\mathcal{U}\times\mathcal{W} \in \mu\times\nu$ such that $|f(x,y) - f(u,w)| < \varepsilon/2$ whenever $(x,y),(u,w) \in U\times W$, for every $U\times W \in \mu\times\nu$.

Now, take $U \in \mathcal{U}$ such that $x_0 \in U$, then for every $(x,y) \neq (x_0,\xi_0)$ belonging to $(U\times V^{\xi_0})\cap((X\times Y)\cup\{(x_0,\xi_0)\})$ we have,

$$|\widetilde{f}(x,y) - \widetilde{f}(x_0,\xi_0)| = |f(x,y) - f_{x_0}^*(\xi_0)| \leq$$
$$\leq |f(x,y) - f(x_0,y)| + |f(x_0,y) - f_{x_0}^*(\xi_0)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and $\widetilde{f}$ is continuous at (x_0,ξ_0), as we claimed. □

The next example shows that the condition appearing in the above result, namely, the completion of one of the factors being Samuel realcompact, is not necessary even in the realm of metric spaces. This fact exhibits an important difference between this realcompactification and the Samuel compactification of a uniform space, and even with the Lipschitz realcompactification of a metric space. Recall that, in both of these cases, the corresponding condition is also necessary (see Theorem 39B.9 in [3] and Theorem 9 in [14]).

Example 5. Let $X = J(\omega_0)$ be the hedgehog space of countable spines with its natural metric d (see for instance [8]), and let Y be a uniformly discrete metric space, endowed with the $0-1$ metric ρ, such that the cardinality of Y is a measurable cardinal. Note that we can consider the metric $d+\rho$ on the uniform product $X\times Y$. It is well known that both metric spaces X and Y are complete, but neither of them is Samuel realcompact. Indeed, X is not Bourbaki-complete ([10]) and $H(U_\rho(Y)) = \upsilon Y \neq Y$.

Nevertheless, we are going to see that the equality $H(U_{d+\rho}(X\times Y)) = H(U_d(X))\times H(U_\rho(Y))$ holds. Recall that we only have to prove one of the inequalities, and for this what we need to do is just to continuously extend every $f \in U_{d+\rho}(X \times Y)$ to a real-valued function defined on $H(U_d(X)) \times H(U_\rho(Y))$.

Firstly, for every $y \in Y$ take f_y^* the continuous extension to $H(U_d(X))$ of the function $f_y(\cdot) = f(\cdot, y) \in U_d(X)$. Next, we can consider the map $\phi : Y \to C(H(U_d(X)))$, defined by $y \in Y \rightsquigarrow f_y^* \in C(H(U_d(X)))$. Since Y is discrete, ϕ will be continuous regardless of the topology we place on the target space. Thus, if we consider on $C(H(U_d(X)))$ the topology of the uniform convergence, then this space is metrizable and, moreover, it has a non-measurable cardinal since X has a non-measurable cardinal. Then, ϕ is a continuous function from Y to a realcompact space, that can be continuously extended to υY (Theorem 8.7 in [15]). Let $\phi^\upsilon : \upsilon Y \to C(H(U_d(X)))$ be such an extension.

We finish taking $\tilde{f} : H(U_d(X)) \times \upsilon Y \to \mathbb{R}$, by $\tilde{f}(\xi, \eta) = \phi^\upsilon(\eta)(\xi)$, for every $(\xi, \eta) \in H(U_d(X)) \times \upsilon Y$. The continuity of this function at every (ξ_0, η_0) can be seen as follows. Let $\varepsilon > 0$. From the continuity of ϕ^υ at the point η_0, there exists an open set V in υY containing η_0 such that

$$\sup_{\xi \in H(U_d(X))} |\phi^\upsilon(\eta)(\xi) - \phi^\upsilon(\eta_0)(\xi)| < \varepsilon/2, \text{ for every } \eta \in V.$$

On the other hand, since $\phi^\upsilon(\eta_0)$ is continuous at ξ_0, there exists an open set $U \subset H(U_d(X))$ containing ξ_0 such that

$$|\phi^\upsilon(\eta_0)(\xi) - \phi^\upsilon(\eta_0)(\xi_0)| < \varepsilon/2, \text{ for every } \xi \in U.$$

Summarizing, whenever $(\xi, \eta) \in U \times V$, we have

$$|\tilde{f}(\xi, \eta) - \tilde{f}(\xi_0, \eta_0)| = |\phi^\upsilon(\eta)(\xi) - \phi^\upsilon(\eta_0)(\xi_0)| \leq$$
$$\leq |\phi^\upsilon(\eta)(\xi) - \phi^\upsilon(\eta_0)(\xi)| + |\phi^\upsilon(\eta_0)(\xi) - \phi^\upsilon(\eta_0)(\xi_0)| \leq \varepsilon.$$

Remark 6. In the above example, the key idea (deeply inspired by Theorem 2.8 of [6]) is to use the realcompactness of the space $C(H(U_\mu(X)))$ when it is endowed with the uniform convergence topology. According to a result contained in [20] (p. 168), we can assert that the realcompactness of $C(H(U_\mu(X)))$, with the uniform convergence topology, holds if and only if the space $H(U_\mu(X))$ has a non-measurable cardinal. On the other hand, note that $H(U_\mu(X))$ has a non-measurable cardinal if and only if X has a non-measurable cardinal.

Thus, taking into account the above remark and following the same arguments as in Example 5, the next result can be derived at once.

Theorem 7. *Let (X, μ) be a uniform space with a non-measurable cardinal. Then, for every uniformly discrete space (Y, ν), we have that $H(U_{\mu\times\nu}(X \times Y)) = H(U_\mu(X)) \times H(U_\nu(Y))$.*

Nevertheless, combining Theorem 4 and Theorem 7, we can obtain many examples of uniform spaces X and Y whose respective completions are not Samuel realcompact for which the equality holds, and such that none of them is discrete.

Example 8. Let $X = J(\omega_0)$ and $Y = D \times \beta D$, being D any uniformly discrete space and βD its Stone-Čech compactification. Or more generally, take (X, μ) a uniform space with a non-measurable cardinal, and, $Y = D \times Z$ where D is uniformly discrete, and (Z, ν) is a uniform space whose completion is Samuel realcompact. Then, considering the suitable uniformity on each space, the equivalence between the Samuel realcompactifications holds. Indeed,

$$H(U(X \times Y)) = H(U(X \times (D \times Z))) = H(U((X \times D) \times Z)) =$$
$$= H(U(X \times D)) \times H(U(Z)) = (H(U(X)) \times H(U(D))) \times H(U(Z)) =$$
$$= H(U(X)) \times (H(U(D)) \times H(U(Z))) = H(U(X)) \times H(U(D \times Z)) =$$
$$= H(U(X)) \times H(U(Y)).$$

Let us remember that for a uniform space to be Samuel realcompact two properties are necessary (Proposition 1). Namely, to be "Bourbaki-complete" and "having no uniform partitions of a measurable cardinal". Next, we are going to see how both of these ingredients provide, independently, necessary conditions for the desired equivalence.

We start with the second property, related to the non-existence of uniform partitions of a measurable cardinal, but previously we need to state a simple lemma.

Lemma 1. *Let (A, μ_A) be a uniform subspace of (X, μ). Then, $H(U_{\mu_A}(A))$ is a (topological) subspace of $H(U_\mu(X))$.*

Proof. According to Theorem 2.9 in [30], we know that $s_{\mu_A}A = \mathrm{cl}_{s_\mu X}(A)$, where $\mathrm{cl}_{s_\mu X}$ denotes the closure in $s_\mu X$. As $H(U_{\mu_A}(A)) \subset s_{\mu_A}A$, we have that both spaces $H(U_{\mu_A}(A))$ and $H(U_\mu(X))$ are contained in the same space $s_\mu X$. In order to see that $H(U_{\mu_A}(A)) \subset H(U_\mu(X))$, suppose that there is $\xi \in s_\mu X$ such that $\xi \in H(U_{\mu_A}(A)) \setminus H(U_\mu(X))$. Then, there is a function $f \in U^*_\mu(X)$, such that $f^*(\xi) = \infty$. Taking the function $f_{|A} \in U^*_{\mu_A}(A)$, we have that $(f_{|A})^*(\xi) = f^*(\xi) = \infty$, and this is a contradiction. □

Theorem 9. *Let (X, μ) and (Y, ν) be uniform spaces such that $H(U_{\mu\times\nu}(X\times Y)) = H(U_\mu(X))\times H(U_\nu(Y))$. Then, at least one of the spaces, (X, μ) or (Y, ν), has no uniform partition of a measurable cardinal.*

Proof. Suppose on the contrary that both (X, μ) and (Y, ν) have a uniform partition of a measurable cardinal. This means that both spaces have a uniformly discrete subspace, $(D, \mu_{|_D})$ and $(D', \nu_{|_{D'}})$ respectively, of a measurable cardinal. In particular, the product space

$$(D \times D', \mu_{|_D} \times \nu_{|_{D'}}) = (D \times D', \mu \times \nu_{|_{D\times D'}})$$

is a uniformly discrete subspace of $(X \times Y, \mu \times \nu)$.

By Theorem 3 in [19], we know that $\upsilon(D \times D') \neq \upsilon D \times \upsilon D'$. Since $\upsilon(D \times D') \geq \upsilon D \times \upsilon D'$ is always true, there must exist a real-valued continuous function f on $D \times D'$ that cannot be continuously extended to $\upsilon D \times \upsilon D'$. On the other hand, as f is also uniformly continuous on the uniformly discrete $(D \times D', \mu \times \nu_{|_{D\times D'}})$, we can easily extend it to a uniformly continuous function $F : (X \times Y, \mu \times \nu) \to \mathbb{R}$ simply by giving the value $f((x_\lambda, y_{\lambda'}))$ on the whole member of the product of the corresponding partitions $P_\lambda \times Q_{\lambda'}$ which contains the point $(x_\lambda, y_{\lambda'})$. That is, $F((x, y)) = f((x_\lambda, y_{\lambda'}))$ whenever $(x, y) \in P_\lambda \times Q_{\lambda'}$.

Now, we can continuously extend F to the Samuel realcompactification $H(U_{\mu\times\nu}(X\times Y)) = H(U_\mu(X))\times H(U_\nu(Y))$. But this implies, in particular, that $f = F_{|_{D\times D'}}$ could be continuously extended to $\upsilon D\times\upsilon D'$ which is a contradiction. Indeed, recall that the Samuel realcompactification of a uniformly discrete space coincides with its Hewitt realcompactification, and applying Lemma 1 we have that

$$\upsilon D\times\upsilon D' = H(U_{\mu_{|D}}(D))\times H(U_{\nu_{|D'}}(D'))\subset H(U_\mu(X))\times H(U_\nu(Y)).$$

□

It is interesting to point out here that, as a byproduct of a result of the next section, we will see (Theorem 20) that the sufficient condition in the above result can be replaced by the stronger condition: "at least one of the spaces, (X,μ) or (Y,ν), has no uniformly discrete subspace of a measurable cardinal".

Now, we are going to see how the second ingredient of Samuel realcompactness, i.e., the Bourbaki-completeness, contributes to give another necessary condition for our purposes. Indeed, it is known that any Bourbaki-complete uniform space is complete and it satisfies that every Bourbaki-bounded subset is compact. Note that in the realm of metric spaces the converse is also true. Nevertheless, Example 1.3.10 in [22] shows that in the realm of the general uniform spaces the converse is not necessarily true. Then we can only affirm that if the completion of a uniform space is Bourbaki-complete then every Bourbaki-bounded subset of X must be totally bounded. And this is the necessary condition that we were looking for, as the next result shows. In order to establish it, we only need to take into account that a Bourbaki-bounded subset B in a uniform space X can be characterized by the fact that every real-valued uniform continuous function on X is bounded on B. This implies, in particular, that for a Bourbaki-bounded subset B, we have that $\mathrm{cl}_{H(U_\mu(X))}B = \mathrm{cl}_{s_\mu X}B$.

Theorem 10. *Let (X,μ) and (Y,ν) be uniform spaces with $H(U_{\mu\times\nu}(X\times Y)) = H(U_\mu(X))\times H(U_\nu(Y))$. Then, for at least one of the spaces, the Bourbaki-bounded subsets coincide with the totally bounded subsets.*

Proof. Suppose on the contrary that there exist Bourbaki-bounded subsets B of (X,μ) and B' of (Y,ν) which are not totally bounded. According to the above comments and Theorem 2.9 in [30], we have

$$\mathrm{cl}_{H(U_\mu(X))}B = \mathrm{cl}_{s_\mu X}B = s_{\mu_{|B}}B.$$

The same is true for B' in Y and for the Bourbaki-bounded subset $B\times B'$ in $X\times Y$. Then

$$s_{\mu\times\nu_{|B\times B'}}B\times B' = \mathrm{cl}_{H(U_{\mu\times\nu}(X\times Y))}(B\times B') = \mathrm{cl}_{H(U_\mu(X))\times H(U_\nu(Y))}(B\times B') =$$

$$= \mathrm{cl}_{H(U_\mu(X))}B\times\mathrm{cl}_{H(U_\nu(Y))}B' = s_{\mu_{|B}}B\times s_{\nu_{|B'}}B'.$$

But this fact contradicts Theorem 39B.9 in [3], because neither B nor B' are totally bounded.

□

We finish this section by stating the following natural question.

Question 11. Do there exist uniform spaces (X,μ) and (Y,ν) for which

$$H(U_{\mu\times\nu}(X\times Y))\neq H(U_\mu(X))\times H(U_\nu(Y))$$

and such that (X, μ) is Bourbaki-complete having a measurable uniform partition and (Y, ν) is not Bourbaki-complete without measurable uniform partitions?

4. The product of two uniform G_δ-realcompactifications

In this section we are considering another realcompactification for a uniform space (X, μ) that we will denote by $K(X)$. The name comes from the so-called class $\mathcal{K}_1$ of those uniform spaces that are G_δ-closed in its Samuel compactification, introduced by Curzer and Hager in [7]. Also Chekeev in [4] considers this realcompactification that he denotes by $\upsilon_u X$.

Thus, for a uniform space (X, μ), we define $K(X)$ as the G_δ-closure of X in its Samuel compactification $s_\mu X$, and we write $K(X) = G_\delta\text{-cl}_{s_\mu X}(X)$. Hence,

$$X \subset K(X) \subset s_\mu X.$$

Note that $K(X)$ is in fact a realcompactification of X, since every G_δ-closed subspace of a realcompact space is also realcompact (see [15]). The realcompactification $K(X)$ of the uniform space X will be called the *G_δ-realcompactification* of X.

The next result gives a useful characterization of the points $\xi \in s_\mu X$ belonging to $K(X)$.

Proposition 12. *A point $\xi \in s_\mu X$ belongs to $K(X)$ if and only if for every $f \in U_\mu^*(X)$ such that $f(x) > 0$, for all $x \in X$, we have that $f^*(\xi) > 0$, where f^* is the continuous extension to $s_\mu X$ of the function f.*

Proof. Let $\xi \in K(X)$ and let $f \in U_\mu^*(X)$ be such that, for all $x \in X$, $f(x) > 0$. Then, it is clear that $f^*(\xi) \geq 0$. If $f^*(\xi) = 0$, then the zero-set $Z(f^*)$ is a G_δ-set of $s_\mu X$ containing ξ, but it does not meet X, which is a contradiction.

Conversely, suppose $\xi \notin K(X)$, then there exists a G_δ-set G in $s_\mu X$ such that $\xi \in G$ and $G \cap X = \emptyset$. Since G is a G_δ-set in $s_\mu X$ containing ξ, there exists a zero-set $Z(h)$, for some real continuous function $h \geq 0$ on $s_\mu X$, such that $\xi \in Z(h) \subset G$. Then, the hypotheses fail for $f = h_{|X} \in U_\mu^*(X)$. □

Moreover, $K(X)$ can be described as being of the form $H(\mathcal{L})$ for some unital vector lattice $\mathcal{L}$ of real continuous functions on X. Indeed, combining different results contained in [18] (Theorem 2.1, Theorem 3.1 and Theorem 4.2) we can assert that $K(X) = H(\mathcal{L})$ where

$$\mathcal{L} = \{f/g : \ f, g \in U_\mu^*(X), g(x) > 0, \text{ for every } \ x \in X\}.$$

Contrary to what happens with other realcompactifications of a uniform space X, $K(X)$ can be different from $K(\widetilde{X})$, being $\widetilde{X}$ its completion. For instance, if we consider the usual metric on $\mathbb{R}$, then $K(\mathbb{Q}) = \mathbb{Q}$ whereas $K(\mathbb{R}) = \mathbb{R}$, since, as we will see later, in the realm of metric spaces $K(X) = \upsilon X$.

Nevertheless, according to the equality $s_\mu X = s_{\widetilde{\mu}} \widetilde{X}$, we have that both $\widetilde{X}$ and $K(X)$ are contained in $s_\mu X$, and then we can consider the subspace $\gamma(X) = K(X) \cap \widetilde{X}$. Note that $\gamma(X)$, known as the *weak completion* of (X, μ) (see [25]), is in fact the G_δ-closure of X in $\widetilde{X}$. Thus, when X is a metric space then $\gamma(X) = X$, since every point in the metric space $\widetilde{X}$ is a G_δ-set. Now, from $X \subset \gamma(X) \subset \widetilde{X}$ we have that the Samuel compactification of the uniform space $(\gamma(X), \widetilde{\mu}_{|\gamma(X)})$ is again $s_\mu X$, and from $X \subset \gamma X \subset K(X)$ it follows that $K(\gamma(X)) = K(X)$.

Next, we are going to recall two results taken from [7] that will be useful along this section. The first one can be seen as a result of type Katetov-Shirota's Theorem for this realcompactification.

Proposition 13. ([7]) *Let (X, μ) be a uniform space. Then, $K(X) = X$ if and only if $\gamma(X) = X$ (i.e., X is weakly complete) and every uniform cover of X has a refinement of a non-measurable cardinal.*

According to the above result, there are many uniform spaces X such that $K(X) = X$. For instance, any metric space with a non-measurable cardinal. An example of a uniform space such that $K(X) \neq X$ is the following. Take $X = \mathbb{R}$ (or any uncountable set) with the uniformity inherited by the unique uniformity of the one point compactification X^∞ of the discrete space X. It is easy to check that X^∞ is also the completion of X and that $K(X) = K(\gamma(X)) = \gamma(X) = X^\infty \neq X$.

Now, taking into account that the property that every uniform cover of X has a refinement of a non-measurable cardinal is equivalent to saying that X has no uniformly discrete subspace of a measurable cardinal (Theorem 2.3 in [16]), the following result is immediate.

Proposition 14. ([7]) *Let (X, μ) be a uniform space. Then $K(X) = \gamma(X)$ if and only if X has no uniformly discrete subspace of a measurable cardinal.*

Proof. The proof follows at once from Proposition 13 and noting that $K(\gamma(X)) = K(X)$. □

On the other hand, according to a known result by Blair and Hager (Proposition 4.4 [1]) stating that under G_δ-density assumption, z-embedding and C-embedding are equivalent properties, and taking into account that υX is the unique (up to equivalence) realcompactification of X where it is C-embedded, we can derive the following result.

Proposition 15. *For a uniform space (X, μ), the following conditions are equivalent:*

(a) $K(X) = \upsilon X$.
(b) *X is z-embedded in $K(X)$.*

There are many uniform spaces fulfilling condition (b) above. For instance, every Lindelöf space, since it is well known that these spaces are z-embedded in every (Tychonoff) superspace (see [1]). Also, every metric space X satisfies (b), since it is clear that it is z-embedded in $s_\mu X$ and hence in $K(X)$. On the other hand, there exist spaces where $K(X) \neq \upsilon X$. For instance, take X the uniform space appearing as an example after Proposition 13.

Now, in order to compare $K(X)$ with the Samuel realcompactification of X, we can say the following. As $H(U_\mu(X))$ is G_δ-closed in $H(U^*_\mu(X)) = s_\mu X$ (every $H(\mathcal{L})$ is G_δ-closed in $H(\mathcal{L}^*)$, see [9]), then we have that

$$X \subset K(X) \subset H(U_\mu(X)) \subset s_\mu X.$$

Spaces where $K(X) = H(U_\mu(X))$ are, for instance, any Banach space X of finite dimension, where $K(X) = X = H(U_\mu(X))$. On the other hand, for (realcompact) Banach spaces X of infinite dimension we have that $K(X) = X \neq H(U_\mu(X)$.

Now we return to our main objective, that is, when the equality $K(X\times Y)=K(X)\times K(Y)$ holds. In order to obtain our main result in this case we previously need to establish some useful lemmas.

Lemma 2. *Let (A,μ_A) be a uniform subspace of (X,μ). Then, $K(A)$ is the G_δ-closure of A in $K(X)$.*

Proof. From the identity $s_{\mu_A}A=\mathrm{cl}_{s_\mu X}(A)\subset s_\mu X$, it follows that $K(A)=G_\delta\text{-cl}_{s_{\mu_A}}(A)=G_\delta\text{-cl}_{s_\mu X}(A)\cap s_{\mu_A}A=G_\delta\text{-cl}_{s_\mu X}(A)\cap \mathrm{cl}_{s_\mu X}(A)=G_\delta\text{-cl}_{s_\mu X}(A)$. Thus, we obtain that $K(A)=G_\delta\text{-cl}_{s_\mu X}(A)\subset G_\delta\text{-cl}_{s_\mu X}(X)=K(X)$. Hence, $K(A)=G_\delta\text{-cl}_{s_\mu X}(A)\cap K(X)=G_\delta\text{-cl}_{K(X)}(A)$, as we wanted. □

Lemma 3. *Let (A,μ_A) and (B,ν_B) be uniform subspaces of (X,μ) and (Y,ν), respectively. If $K(X\times Y)=K(X)\times K(Y)$, then $K(A\times B)=K(A)\times K(B)$.*

Proof. This follows at once from the above Lemma. Indeed, first note that $K(A)\times K(B)$ is the G_δ-closure of $A\times B$ in $K(X)\times K(Y)$. Now, as $K(X\times Y)=K(X)\times K(Y)$, then $K(A)\times K(B)$ must be the G_δ-closure of $A\times B$ in $K(X\times Y)$, which is precisely $K(A\times B)$. □

Now we are going to see that, as in the case of the Samuel realcompactification, one of the inequalities is always true.

Proposition 16. *Let (X,μ) and (Y,ν) be uniform spaces. Then $K(X\times Y)\geq K(X)\times K(Y)$.*

Proof. We start with the inequality $s_{\mu\times\nu}(X\times Y)\geq s_\mu X\times s_\nu Y$, which means that the inclusion map $i: X\times Y\to s_\mu X\times s_\nu Y$ can be continuously extended to $s_{\mu\times\nu}(X\times Y)$. Denote by $i^*: s_{\mu\times\nu}(X\times Y)\to s_\mu X\times s_\nu Y$ such an extension. We finish showing that i^* maps $K(X\times Y)$ into $K(X)\times K(Y)$. Indeed, from the continuity of i^*, we have that

$$i^*(K(X\times Y))=i^*(G_\delta\text{-cl}_{s_{\mu\times\nu}(X\times Y)}(X\times Y))\subset G_\delta\text{-cl}_{s_\mu X\times s_\nu Y}(i^*(X\times Y))=$$
$$=G_\delta\text{-cl}_{s_\mu X}(X)\times G_\delta\text{-cl}_{s_\nu Y}(Y)=K(X)\times K(Y).$$

□

We are now ready to establish our main result in this section. Note that, for this realcompactification, we obtain necessary and sufficient conditions for the corresponding equality.

Theorem 17. *For uniform spaces (X,μ) and (Y,ν), the following conditions are equivalent:*

(a) $K(X\times Y)=K(X)\times K(Y)$.
(b) $K(X)=\gamma(X)$ *or* $K(Y)=\gamma(Y)$.
(c) *At least one of the spaces, (X,μ) or (Y,ν), has no uniformly discrete subspace of a measurable cardinal.*

Proof. The equivalence between (b) and (c) follows at once from Proposition 14.

(a) implies (c). Suppose (c) is not true, then there exist two uniformly discrete subspaces with a measurable cardinal D and D' in X and Y, respectively. Now applying Lemma 3, we have that $K(D\times D')=K(D)\times K(D')$. As these subspaces are in particular metric spaces then we deduce that $\upsilon(D\times D')=\upsilon(D)\times\upsilon(D')$, but this contradicts Theorem 3 in [19].

(b) implies (a). Suppose that $K(X)=\gamma(X)$. Note that it is only necessary to prove condition (a) whenever $K(X)=X$. Indeed, if (a) is fulfilled in this case and we have $K(X)=$

$\gamma(X)$, then the equality will be true for the uniform spaces $(\gamma(X), \widetilde{\mu}_{|\gamma(X)})$ and $(\gamma(Y), \widetilde{\nu}_{|Y})$, since $K(X) = K(\gamma(X)) = \gamma(X)$. Hence, we can write that

$$K(\gamma(X) \times \gamma(Y)) = K(\gamma(X)) \times K(\gamma(Y)) = K(X) \times K(Y).$$

On the other hand, it is easy to check that it is always true that $\gamma(X \times Y) = \gamma(X) \times \gamma(Y)$, and then

$$K(X \times Y) = K(\gamma(X \times Y)) = K(\gamma(X) \times \gamma(Y)) = K(X) \times K(Y)$$

as we wanted.

Thus, we are going to prove condition (a) when $K(X) = X$. Recall that we only need to check the inequality $X \times K(Y) \geq K(X \times Y)$, or, equivalently, that the inclusion map $i : X \times Y \to K(X \times Y)$ can be continuously extended to $X \times K(Y)$. Indeed, first consider $i : X \times Y \to K(X \times Y) \subset s_{\mu\times\nu}(X \times Y)$. So, if we proceed step by step as in the proof of Theorem 4, then we can extend i as a continuous function

$$i^* : X \times K(Y) \to s_{\mu\times\nu}(X \times Y) = H(U^*_{\mu\times\nu}(X \times Y)) \subset \mathbb{R}^{U^*_{\mu\times\nu}(X\times Y)}.$$

We finish the proof if we show that in fact $i^*(X \times K(Y)) \subset K(X \times Y)$. For that, let $(x, \xi) \in X \times K(Y)$, then in order to see that $i^*(x, \xi) \in K(X \times Y)$ we will apply Proposition 12. Let $f \in U^*_{\mu\times\nu}(X \times Y)$ be such that $f > 0$ on $X \times Y$, therefore we have that $f(i^*(x, \xi)) > 0$ since $f(i^*(x, \xi)) = \widetilde{f}(x, \xi) = f^*_x(\xi)$, but $f^*_x(\xi) > 0$ because $f^*_x \in U^*(s_\nu Y)$, $f^*_x > 0$ on Y and $\xi \in K(Y)$. □

From the last theorem, we are going to obtain a result essentially proved by Ohta in [23]. Here, we shall use the fine uniformity $\mathtt{u}$ defined on every uniform space (see Section 1). Recall that, for every Tychonoff space X, υX is in fact the G_δ-closure of X in βX. Now since $\beta X = s_{\mathtt{u}} X$, we have that $\upsilon X = K_{\mathtt{u}}(X)$, where $K_{\mathtt{u}}(X)$ denotes the G_δ-realcompactification of the uniform space $(X, \mathtt{u})$.

Theorem 18. ([23]) *Let (X, d) and (Y, ρ) be metric spaces. The following statements are equivalent:*

(a) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$.
(b) *At least one of the factors has a non-measurable cardinal.*

Proof. First of all note that, as we have said before, for any metric space X we have that $K(X) = \upsilon X$. Then the first condition is equivalent to say that $K(X \times Y) = K(X) \times K(Y)$. Then, by Theorem 17, it is clear that (b) implies (a).

For the converse, and using the fine uniformity, we have the following:

$$K_{\mathtt{u}}(X) \times K_{\mathtt{u}}(Y) = \upsilon X \times \upsilon Y = \upsilon(X \times Y) = K_{\mathtt{u}}(X \times Y) \geq$$

$$\geq K_{\mathtt{u}\times\mathtt{u}}(X \times Y) \geq K_{d\times\rho}(X \times Y) = K(X) \times K(Y) = \upsilon X \times \upsilon Y.$$

Therefore, we have that $K_{\mathtt{u}}(X)\times K_{\mathtt{u}}(Y) = K_{\mathtt{u}\times\mathtt{u}}(X\times Y)$, and again by Theorem 17 it follows that every uniformly discrete subspace of $(X, \mathtt{u})$ or of $(Y, \mathtt{u})$ has a non-measurable cardinal. Suppose $(X, \mathtt{u})$ has this property, then any uniform covering of $(X, \mathtt{u})$ has a refinement of a non-measurable cardinal (Theorem 2.3 [16]). On the other hand, since X is metrizable, it is paracompact and normal and this means in particular that every open cover belongs to $\mathtt{u}$. Then every open cover of X has a refinement with a non-measurable cardinal. Hence, every closed discrete subspace of X has a non-measurable cardinal. Finally, applying the classical Katetov-Shirota theorem to the metrizable space X it follows that X is realcompact and hence it has a non-measurable cardinal. □

Another consequence of Theorem 17 is the last result of this section giving a more general condition than the one obtained in Theorem 9, as we have already announced. Note that it can be obtained as an easy corollary of the next fact.

Proposition 19. *For uniform spaces (X,μ), (Y,ν), the following implication is satisfied:*

$$H(U_{\mu\times\nu}(X\times Y)) = H(U_\mu(X)) \times H(U_\nu(Y)) \Rightarrow K(X\times Y) = K(X)\times K(Y).$$

Proof. It follows at once since in particular $K(X)$ is also the G_δ-closure of X in $H(U_\mu(X))$. □

Theorem 20. *Let (X,μ) and (Y,ν) be uniform spaces. If $H(U_{\mu\times\nu}(X\times Y)) = H(U_\mu(X)) \times H(U_\nu(Y))$ then, at least one of the spaces, (X,μ) or (Y,ν), has no uniformly discrete subspace of a measurable cardinal.*

5. The product of two countable-modification realcompactifications

For a uniform space (X,μ), we denote by $e\mu$ the uniformity on X induced by all the countable covers in μ. This uniform space $(X, e\mu)$ is known as the countable-modification of (X,μ). Spaces being complete with this uniformity were studied by Reynolds and Rice in [26], and more recently by Hušek in [21]. Then, if we consider the completion of the uniform space $(X, e\mu)$ that we denote by $e_\mu X = (\widetilde{X}, \widetilde{e\mu})$, we are going to see that this space is another uniform realcompactification of X. We call $e_\mu X$ the *countable-modification realcompactification* of X.

In order to check that $e_\mu X$ is a uniform realcompactification of X, first note that if $f : (X,\mu) \to (Y,\nu)$ is a uniformly continuous function then $f : (X, e\mu) \to (Y, e\nu)$ is also uniformly continuous. This means, in particular, that $U_\mu(X) = U_{e\mu}(X)$ and $U^*_\mu(X) = U^*_{e\mu}(X)$. And hence, we have that

$$s_\mu X = s_{e\mu} X = s_{\widetilde{e\mu}} \widetilde{X}$$

$$H(U_\mu(X)) = H(U_{e\mu}(X)) = H(U_{\widetilde{e\mu}}(\widetilde{X})).$$

Thus, from the above, we have that $s_\mu X$ is also the Samuel compactification of the complete space $e_\mu X$. Since every uniform cover of $e_\mu X$ has a countable uniform cover refinement, we can assert that its G_δ-realcompactification $K(e_\mu X) = e_\mu X$ (see Proposition 13).

Summarizing, we can affirm that $e_\mu X$ is a uniform realcompactification of X which is in fact G_δ-closed in $s_\mu X$, and also that

$$X \subset K(X) \subset e_\mu X \subset H(U_\mu(X)) \subset s_\mu X.$$

All infinitely dimensional separable Banach spaces are examples of spaces showing that $e_\mu X$ can be different from $H(U_\mu(X))$, since for them $e_\mu X = X \neq H(U_\mu(X))$ (note that $e\mu = \mu$). Nevertheless, in the class of uniformly 0-dimensional spaces the equality $e_\mu X = H(U_\mu(X))$ holds, since these spaces have a base of partitions for their uniformity, and then $e\mu$ coincides with the weak uniformity given by the real-valued uniformly continuous functions.

On the other hand, $X = e_\mu X$ means that X is complete with the uniformity $e\mu$ and then X must be complete with μ. The converse is not true in general. Indeed, an example of this was given by Pelant in [24] (see also [21]). Namely, he proved that the Banach space $(\ell_\infty(\omega_1), \|\cdot\|_\infty)$ is not complete with the uniformity $e\mu_{\|\cdot\|_\infty}$, i.e., $X \neq e_\mu X$. Note that this space is also an example of a uniform space where $K(X) \neq e_\mu X$.

The next result, proved by Reynolds and Rice in [26], gives a Katetov-Shirota type result for this realcompactification $e_\mu X$. Recall that the point-finite modification of (X, μ) is the space X endowed with the uniformity generated by all the point-finite covers from μ. A cover of a set is *point-finite* if every point of the set lies in at most finitely many elements of the cover.

Proposition 21. ([26]) *Let (X, μ) be a uniform space. Then, $e_\mu X = X$ if and only if X is point-finite complete and every uniform cover of X has a refinement of a non-measurable cardinal.*

An interesting result is the following.

Proposition 22. *For any uniform space (X, μ), we have that $e_\mu X = e_{\widetilde{\mu}}\widetilde{X}$.*

Proof. The inclusion map $i : (X, e\mu) \to (\widetilde{X}, e\widetilde{\mu})$ is a uniform embedding (recall that every uniform cover on X can be isomorphically extended to a uniform cover on $\widetilde{\mu}$ (see [26], p. 370)). Therefore, both the functions i and i^{-1} can be extended to continuous functions on the respective completions (Theorem 8.3.10 in [8]). Then it follows from the density of X that the continuous extension of i over $e_\mu X$ is a homeomorphism of $e_\mu X$ onto $e_{\widetilde{\mu}}\widetilde{X}$. Clearly, for every $x \in X$, $i(x) = x$. This completes the proof. □

Now, according to the last Section and knowing that $K(e_\mu X) = e_\mu X$, we can also describe this realcompactification of X in the form $e_\mu X = H(\mathcal{L})$ just taking

$$\mathcal{L} = \{f/g : f, g \in U_\mu^*(X), g^*(\xi) > 0, \text{ for every } \xi \in e_\mu X\}$$

where g^* is the continuous extension to $s_\mu X$ of the function g.

It is clear that the last description is not good enough as it is not given in an internal way, i.e., only in terms of X, since we use points in $e_\mu X$. On the other hand, as every realcompact space Y coincides with $H(C(Y))$, then we have that $e_\mu X = H(C(e_\mu X))$. Taking into account that the real-valued continuous functions on the completion of a uniform space are precisely the extensions of the so-called Cauchy-continuous functions on the space (see [2]), it follows that $e_\mu X = H(CC(X, e\mu))$, where $CC(X, e\mu)$, denotes the family of the real-valued Cauchy-continuous functions on the uniform space $(X, e\mu)$. Recall that a function $f : (X, \mu) \to (Y, \nu)$ is *Cauchy-continuous* if it maps μ-Cauchy filters (nets) of X to ν-Cauchy filters (nets) of Y.

Next, we are going to analyze our main objective, that is, what happens with the product of two of these realcompactifications. Namely, what about the following equality:

$$e_{\mu\times\nu}(X \times Y) = e_\mu X \times e_\nu Y.$$

The next example proves that the equality is not always true.

Example 23. Let (X,μ) and (Y,ν) be two uniformly discrete spaces with a measurable cardinal. Since they are uniformly 0-dimensional therefore $e_{\mu\times\nu}(X\times Y)=\upsilon(X\times Y)$, $e_\mu X=\upsilon X$ and $e_\nu Y=\upsilon Y$. Then $e_{\mu\times\nu}(X\times Y)\neq e_\mu X\times e_\nu Y$, according to Theorem 3 in [19].

As usual, one of the inequalities is easily true, as the next result shows.

Proposition 24. *For uniform spaces (X,μ) and (Y,ν), we have:*

$$e_{\mu\times\nu}(X\times Y)\geq e_\mu X\times e_\nu Y.$$

Proof. The identity map $id:(X\times Y,e(\mu\times\nu))\to(X\times Y,e\mu\times e\nu)$ is uniformly continuous as $e\mu\times e\nu\subset e(\mu\times\nu)$. By [8, Theorem 8.3.10], it can be extended to a continuous map

$$\widetilde{id}:e_{\mu\times\nu}(X\times Y)\to e_\mu X\times e_\nu Y$$

and this completes the proof. □

Next, in order to get the reverse inequality, we are going to proceed as in Example 5. In fact, we could use this technique in both Theorem 4, for the Samuel realcompactification, and Theorem 17, for the G_δ-realcompactification, but we think that the proofs given there are perhaps more natural and simpler.

Thus, let $f\in U_{\mu\times\nu}(X\times Y)$ be a uniformly continuous function and, as usual, take for every $y\in Y$, $f_y:X\to\mathbb{R}$ defined by $f_y(x)=f(x,y)$, $x\in X$. Then, $f_y\in U_\mu(X)=U_{e\mu}X$, and therefore it can be extended to its completion. Denote $f_y^*:e_\mu X\to\mathbb{R}$ the corresponding (uniformly) continuous extension. Now, we can consider the map $\varphi_f:Y\to C(e_\mu X)$ (in fact we can take $U_{\widetilde{e\mu}}(e_\mu X)$ as the target set) defined by $\varphi_f(y)=f_y^*$. Thus, the following result follows.

Lemma 4. *The map $\varphi_f:Y\to C(e_\mu X)$ is uniformly continuous when Y is endowed with the uniformity ν, and $C(e_\mu X)$ with the uniformity of the uniform convergence.*

Proof. Let $\varepsilon>0$. Since, for every $y\in Y$, the function $f_y^*:e_\mu X\to\mathbb{R}$ is uniformly continuous on $e_\mu X$ with its uniformity $\widetilde{e\mu}$ therefore, for this $\varepsilon>0$, there is a cover $\mathcal{V}^y\in\widetilde{e\mu}$, such that

$$|f_y^*(\xi_1)-f_y^*(\xi_2)|<\tfrac{\varepsilon}{3}$$

whenever $\xi_1,\xi_2\in V\in\mathcal{V}^y$.

In addition, since f is uniformly continuous on $X\times Y$, given $\varepsilon>0$, we can fix $\mathcal{U}\in\mu$ and $\mathcal{W}\in\nu$ such that, if $x,u\in U\in\mathcal{U}$ and $y,w\in W\in\mathcal{W}$, then

$$|f(x,y)-f(u,w)|<\tfrac{\varepsilon}{3}.$$

Thus, for this $\varepsilon>0$ we are going to see that the above cover $\mathcal{W}\in\nu$ works for the uniform continuity of φ_f. Indeed, we are going to check that when $y,w\in W\in\mathcal{W}$,

$$\sup_{\xi\in e_\mu X}|\varphi_f(y)(\xi)-\varphi_f(w)(\xi)|<\varepsilon.$$

To see that, first note that for each $\xi\in e_\mu X$, we can select $V_y\in\mathcal{V}^y$ and $V_w\in\mathcal{V}^w$ such that $\xi\in V_y\cap V_w$. And, we can also choose some point $x(\xi)\in V_y\cap V_w\cap X$. Then, with all these ingredients we have the following:

$$\sup_{\xi\in e_\mu X}|\varphi_f(y)(\xi)-\varphi_f(w)(\xi)| = \sup_{\xi\in e_\mu X}|f^*_y(\xi)-f^*_w(\xi)| \leq$$
$$\leq \sup_{\xi\in e_\mu X}|f^*_y(\xi)-f^*_y(x(\xi))| + \sup_{\xi\in e_\mu X}|f^*_y(x(\xi))-f^*_w(x(\xi))| +$$
$$+ \sup_{\xi\in e_\mu X}|f^*_w(x(\xi))-f^*_w(\xi)| \leq \sup_{\xi\in e_\mu X}|f^*_y(\xi)-f^*_y(x(\xi))| +$$
$$+ \sup_{\xi\in e_\mu X}|f(x(\xi),y)-f(x(\xi),w)| + \sup_{\xi\in e_\mu X}|f^*_w(x(\xi))-f^*_w(\xi)| < \varepsilon.$$

Observe that $x(\xi)\in U$ for some $U\in\mathcal{U}$. This completes the proof. □

Lemma 5. *The space $e_{\mu\times\nu}(X\times Y)$ contains a uniform copy of $e_\mu X\times\{y\}$, for every $y\in Y$.*

Proof. Fix $y\in Y$. Observe that every cover in $e(\mu\times\nu)$ can be restricted to a uniform cover on $(X\times\{y\}, e\mu\times e\nu|_{\{y\}})$ and that every uniform cover of $(X\times\{y\}, e\mu\times e\nu|_{\{y\}})$ can be extended to a uniform cover on $(X\times Y, e\mu\times e\nu)$ which belongs to $e(\mu\times\nu)$ because $e\mu\times e\nu\subset e(\mu\times\nu)$.

Then, the identity map

$$id : (X\times\{y\}, e\mu\times e\nu|_{\{y\}}) \to (X\times Y, e(\mu\times\nu))$$

is a uniform homeomorphism on $X\times\{y\}$ onto itself. Now, by [8, Corollary 8.3.11], the result follows. □

Theorem 25. *Let (X,μ) and (Y,ν) be uniform spaces such that $\widetilde{X}=e_\mu X$ or $\widetilde{Y}=e_\mu Y$. Then*

$$e_{\mu\times\nu}(X\times Y) = e_\mu X\times e_\nu Y.$$

Proof. First, note that, by Proposition 22, we may suppose that both uniform spaces (X,μ) and (Y,μ) are complete because

$$e_\mu X\times e_\nu Y = e_{\widetilde{\mu}}\widetilde{X}\times e_{\widetilde{\nu}}\widetilde{Y}$$
$$e_{\mu\times\nu}(X\times Y) = e_{\widetilde{\mu\times\nu}}\widetilde{X\times Y} = e_{\widetilde{\mu}\times\widetilde{\nu}}(\widetilde{X}\times\widetilde{Y}).$$

Moreover, since $e_{\mu\times\nu}(X\times Y)\geq e_\mu X\times e_\nu Y$ is always satisfied, we just need to prove the reverse inequality. For that we will use that $e_{\mu\times\nu}(X\times Y)$ can be described as $H(\mathcal{L})$ where $\mathcal{L}$ is the family defined below Proposition 22. That means, in particular, that we need only to (continuously) extend to $e_\mu X\times e_\nu Y$ the functions belonging to $\mathcal{L}$. Then, let $f/g\in\mathcal{L}$, where $f,g\in U^*_{\mu\times\nu}(X\times Y)$ and the extension of g to $s_{\mu\times\nu}(X\times Y)$ does not vanish on $e_{\mu\times\nu}(X\times Y)$.

Thus, for any $f\in U^*_{\mu\times\nu}(X\times Y)$ consider the map $\varphi_f:(Y,\nu)\to C(e_\mu X)$ given in Lemma 4. If we suppose that $Y=e_\nu Y$, it is clear that $\varphi_f: e_\nu Y\to C(e_\mu X)$ is continuous.

Next, define the extension of f as

$$\widetilde{f} : e_\mu X\times e_\nu Y\to\mathbb{R}$$

by $\widetilde{f}(\xi,\eta)=\varphi_f(\eta)(\xi)$, for every $(\xi,\eta)\in e_\mu X\times e_\nu Y$. We will prove the continuity of $\widetilde{f}$ at (ξ_0,η_0). Indeed, let $\varepsilon>0$, since φ_f is continuous at η_0 there is an open set $V\subset e_\nu Y$ containing η_0 such that

$$\sup_{\xi\in e_\mu X}|\varphi_f(\eta)(\xi)-\varphi_f(\eta_0)(\xi)| < \varepsilon/2, \text{ for every } \eta\in V.$$

On the other hand, since $\varphi_f(\eta_0)$ is continuous at ξ_0, there exists an open set $U \subset e_\mu X$ containing ξ_0 such that

$$|\varphi_f(\eta_0)(\xi) - \varphi_f(\eta_0)(\xi_0)| < \varepsilon/2, \text{ for every } \xi \in U.$$

Summarizing, whenever $(\xi, \eta) \in U \times V$ we have

$$|\widetilde{f}(\xi,\eta) - \widetilde{f}(\xi_0,\eta_0)| = |\varphi_f(\eta)(\xi) - \varphi_f(\eta_0)(\xi_0)| \leq$$

$$\leq |\varphi_f(\eta)(\xi) - \varphi_f(\eta_0)(\xi)| + |\varphi_f(\eta_0)(\xi) - \varphi_f(\eta_0)(\xi_0)| < \varepsilon.$$

Now, if in addition a function $g \in U^*_{\mu\times\nu}(X \times Y)$ also satisfies $g^*(\zeta) \neq 0$, for every $\zeta \in e_{\mu\times\nu}(X \times Y)$ where g^* denotes the continuous extension of g to $s_{\mu\times\nu}(X \times Y)$, then by Lemma 5, $\varphi_g(y) = g^*_y \in C(e_\mu X)$ is a non-vanishing function on $e_\mu X$, for every $y \in Y$. Hence, it is clear that $\widetilde{g}(\xi, \eta) = \varphi_g(\eta)(\xi) \neq 0$, for every $(\xi, \eta) \in e_\mu X \times e_\nu Y$ and $1/g$ can be extended to a continuous function to $e_\mu X \times e_\nu Y$, as we wanted. □

The next example shows that the conditions given in Theorem 25 are not necessary in order to get the corresponding equivalence, even in the realm of metric spaces.

Example 26. Let $X = (\ell_\infty(\omega_1), \|\cdot\|_\infty)$ and $Y = (D, \nu)$ any uniformly discrete space with a measurable cardinal. As we have said before, $e_{\mu_{\|\cdot\|_\infty}} X \neq X = \widetilde{X}$. On the other hand, since Y is uniformly 0-dimensional and uniformly discrete, we have $e_\nu Y = H(U_\nu(Y)) = \upsilon Y \neq Y = \widetilde{Y}$. In order to see the equality $e_{\mu\times\nu}(X \times Y) = e_\mu X \times e_\nu Y$ we will proceed as in the proof of the last theorem, but doing some little changes. Indeed, let $f \in U^*_{\mu\times\nu}(X \times Y)$ and consider the continuous map $\varphi_f : (Y, \nu) \to C(e_\mu X)$ given in Lemma 4. Now, since X and also $e_\mu X$ has a non-measurable cardinal, we know that $C(e_\mu X)$ is realcompact with the topology of the uniform convergence (see Remark 6). And that means in particular that the function φ_f can be continuously extended to υY. Let $\varphi^\upsilon_f : \upsilon Y \to C(e_\mu X)$ the corresponding continuous extension.

Next, since $\upsilon Y = e_\nu Y$, we can extend f as

$$\widetilde{f} : e_\mu X \times e_\nu Y \to \mathbb{R}$$

by $\widetilde{f}(\xi, \eta) = \varphi^\upsilon_f(\eta)(\xi)$, for every $(\xi, \eta) \in e_\mu X \times e_\nu Y$. The continuity of $\widetilde{f}$ follows exactly as in the above proof.

On the other hand, if a function $g \in U^*_{\mu\times\nu}(X \times Y)$ has the additional property that its extension g^* is not vanishing on $e_{\mu\times\nu}(X \times Y)$, we need to check that $\widetilde{g}$ is not vanishing on $e_\mu X \times e_\nu Y$. Note that, if we repeat the same arguments as in the last result, we get that $\widetilde{g}(\xi, y) \neq 0$, for every $(\xi, y) \in e_\mu X \times Y$. Then, for every $\xi \in e_\mu X$, we have that the restriction of $\widetilde{g}$ to the subspace $\{\xi\} \times \upsilon Y$ is not vanishing on $\{\xi\} \times Y$. By realcompactness, we can derive that $\widetilde{g}$ is not vanishing on $\{\xi\} \times \upsilon Y$, and that completes the proof.

We finish this section giving a necessary condition for the equality of the product of these countable-modification realcompactifications. We will obtain it using the following easy result analogous to Proposition 19.

Proposition 27. *For uniform spaces (X,μ) and (Y,ν), the following implication is satisfied:*

$$e_{\mu\times\nu}(X\times Y)=e_\mu X\times e_\nu Y\Rightarrow K(X\times Y)=K(X)\times K(Y).$$

Proof. It follows at once since in particular $K(X)$ is also the G_δ-closure of X in $e_\mu X$. □

According to the last result together with Theorem 17 we can easily derive the announced necessary condition.

Theorem 28. *Let (X,μ) and (Y,ν) be uniform spaces. If $e_{\mu\times\nu}(X\times Y)=e_\mu X\times e_\nu Y$ then, at least one of the spaces, (X,μ) or (Y,ν), has no uniformly discrete subspace of a measurable cardinal.*

Note that the reverse implication in Proposition 27 is true if and only if the converse of Theorem 28 is true. Unfortunately, we do not know any example in order to clarify this point.

Moreover, according to Proposition 19 and Proposition 27, we wonder if the implication

$$H(U_{\mu\times\nu}(X\times Y))=H(U_\mu(X))\times H(U_\nu(Y))\Rightarrow e_{\mu\times\nu}(X\times Y)=e_\mu X\times e_\nu Y$$

holds. We think that an affirmative answer must occur, since the equality "$H(U_{\mu\times\nu}(X\times Y))=H(U_\mu(X))\times H(U_\nu(Y))$" is somehow a very strong condition into this framework.

6. Uniform realcompactifications G_δ-closed in the Samuel compactification

Along the paper we have seen that the three uniform realcompactifications of (X,μ) analyzed here have a common property. Namely, they are not only topological subspaces of $s_\mu X$, but they are in fact G_δ-closed in it. Then, we wonder which are these uniform realcompactifications. In this section we are going to see that a characterization of these spaces can be obtained by means of some families of real-valued continuous functions.

Firstly, note if αX is a uniform realcompactification of X being G_δ-closed in $s_\mu X$, then $K(X)\subset\alpha X$, since $K(X)$ is the G_δ-closure of X in $s_\mu X$. According to this, examples of uniform realcompactifications αX which are not G_δ-closed in $s_\mu X$ will be any realcompact space αX such that $X\subset\alpha X\subsetneq K(X)$. For instance, this happens when X is itself realcompact with $X\neq K(X)$, as in the example after Proposition 13.

On the other hand, we have seen (Section 4) that there exists a unital vector lattice of continuous functions describing the realcompactification $K(X)$. Namely, $K(X)=H(\mathcal{L}_K)$ where

$$\mathcal{L}_K=\{f/g:\ f,g\in U_\mu^*(X),\, g(x)>0,\ \text{for every}\ \ x\in X\}.$$

Next, by means of this family $\mathcal{L}_K$ we are going to state the announced characterization.

Theorem 29. *Let (X,μ) a uniform space. A uniform realcompactification αX of X is G_δ-closed in $s_\mu X$ if and only if there exists a unital vector lattice $\mathcal{L}$ such that $U_\mu^*(X)\subset\mathcal{L}\subset\mathcal{L}_K$ and $\alpha X=H(\mathcal{L})$.*

Proof. Suppose αX is G_δ-closed in $s_\mu X$, then as we have said before,

$$K(X)\subset\alpha X\subset s_\mu X.$$

Take $\mathcal{L} = \{f/g : f, g \in U_\mu^*(X), g^*(\xi) > 0, \text{ for every } \xi \in \alpha X\}$ where g^* is the continuous extension to $s_\mu X$ of the function g. It is easy to check that $\mathcal{L}$ is a unital vector lattice such that $U_\mu^*(X) \subset \mathcal{L} \subset \mathcal{L}_K$. Now, we are going to see that $\alpha X = H(\mathcal{L})$.

Firstly, note that $H(\mathcal{L})$ is a topological subspace of $s_\mu X$. Indeed, if we consider the restriction map:

$$r : H(\mathcal{L}) \to s_\mu X = H(U_\mu^*(X))$$

given by $\varphi \rightsquigarrow \varphi_{|_{U_\mu^*(X)}}$, for every φ real unital vector homomorphism on $\mathcal{L}$ (see Section 2), then r is in fact a topological embedding. The proof of that is not difficult, but rather cumbersome. The continuity of r is clear, since $U_\mu^*(X) \subset \mathcal{L}$. On the other hand, to see that r is injective we need to take into account that $\mathcal{L}$ is also an algebra and that every homomorphism $\varphi \in H(\mathcal{L})$ is also an algebra homomorphism (see Lemma 2.3 in [9]). Hence, r is injective because $\varphi(f/g) = \varphi(f)/\varphi(g)$, for every $f/g \in \mathcal{L}$. Indeed, if φ and φ' coincide on $U_\mu^*(X)$, then they coincide on $\mathcal{L}$. Finally, to check that r is an open map on its image $r(H(\mathcal{L}))$, let G be the open set in $H(\mathcal{L}) \subset \mathbb{R}^{\mathcal{L}}$ of the form:

$$G = \{\varphi \in H(\mathcal{L}) : \alpha_i < \varphi(f_i/g_i) < \beta_i,\ f_i/g_i \in \mathcal{L}, \alpha_i < \beta_i, i = 1, ..., n\}$$

where $f_i, g_i \in U_\mu^*(X)$, $g_i(x) > 0$, for every $x \in X$, $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, ..., n$, $n \in \mathbb{N}$. Then, we have that:

$$G = \{\varphi \in H(\mathcal{L}) : \alpha_i \cdot \varphi(g_i) < \varphi(f_i) < \beta_i \cdot \varphi(g_i),\ f_i/g_i \in \mathcal{L}, \alpha_i < \beta_i, i = 1, ..., n\} =$$

$$= \{\varphi \in H(\mathcal{L}) : 0 < \varphi(f_i - \alpha_i \cdot g_i),\ f_i/g_i \in \mathcal{L}, \alpha_i < \beta_i, i = 1, ..., n\} \cap$$

$$\cap \{\varphi \in H(\mathcal{L}) : \varphi(f_i - \beta_i \cdot g_i) < 0,\ f_i/g_i \in \mathcal{L}, \alpha_i < \beta_i, i = 1, ..., n\}.$$

Thus, G can be written as the intersection of two open sets G_1 and G_2 in $H(\mathcal{L})$ defined by functions in $U_\mu^*(X)$, namely, $f_i - \alpha_i \cdot g_i$ and $f_i - \beta_i \cdot g_i$, $i = 1, ..., n$, respectively. Note that $r(G) = r(G_1) \cap r(G_2)$, because r is injective. We finish since it is clear that both $r(G_1)$ and $r(G_2)$ are open in the image $r(H(\mathcal{L}))$.

Next, to see that $H(C(\alpha X))$ is also a topological subspace of $H(\mathcal{L})$, it is enough to consider the corresponding restriction map $\varphi \rightsquigarrow \varphi_{|_{\mathcal{L}}}$, for every φ real unital vector homomorphism on $C(\alpha X)$ (note that every function in $\mathcal{L}$ is in $C(\alpha X)$). Now, it is only necessary to see that this restriction map is injective since both the spaces are contained as topological subspaces of $s_\mu X$. Now, since αX is realcompact therefore $\alpha X = H(C(\alpha X))$, and this implies that every homomorphism $\varphi \in H(C(\alpha X))$ is given by evaluation at some point $\xi \in \alpha X$, i.e., $\varphi(f) = f(\xi)$ for every $f \in C(H(\alpha X))$. But, for different $\xi \neq \xi'$, $\xi, \xi' \in \alpha X$, as they are also points in $s_\mu X$, there exists $f \in U_\mu^*(X)$, such that $f^*(\xi) \neq f^*(\xi')$, and this means that different homomorphism on $C(H(\alpha X))$ must be different when restricted to $\mathcal{L}$.

Summarizing, we have that

$$\alpha X = H(C(\alpha X)) \subset H(\mathcal{L}) \subset s_\mu X.$$

Finally, in order to see that $\alpha X = H(\mathcal{L})$, suppose that there exists $\xi \in H(\mathcal{L}) \setminus \alpha X$. Since αX is G_δ-closed in $s_\mu X$, there exists a zero set $Z(g)$, $g \in U_\mu^*(X)$, containing ξ such that $Z(g) \cap \alpha X = \emptyset$. But it is a contradiction since the function $1/g$ belongs to $\mathcal{L}$ and it cannot be continuously extended to $H(\mathcal{L})$.

Conversely, suppose that αX is a uniform realcompactification of X, such that $\alpha X = H(\mathcal{L})$, for some unital vector lattice $\mathcal{L}$. We are going to see that αX is the biggest subspace of $s_\mu X$ containing X where every function in $\mathcal{L}$ continuously extends to it. That is, if Y is a subspace of $s_\mu X$ having these properties, then $Y \subset H(\mathcal{L})$. Recall that, as we have seen in Section 2, the map

$$e : X \to H(\mathcal{L}) \subset \mathbb{R}^{\mathcal{L}}$$
$$x \rightsquigarrow e(x) = (f(x))_{f\in\mathcal{L}}$$

is a topological embedding. Now, if every function in $\mathcal{L}$ can be continuously extended to Y, then we can also extend the map e to Y. Let $\widetilde{e} : Y \to H(\mathcal{L}) \subset s_\mu X$ such an extension. Finally note that $\widetilde{e}$ and the inclusion map from Y to $s_\mu X$ coincides on the dense subspace X, and that means in particular that $Y \subset H(\mathcal{L})$, as we wanted.

We finish by showing that αX is G_δ-closed in $s_\mu X$. Suppose that there is a point $\xi \in G_\delta\text{-cl}_{s_\mu X}(\alpha X) \setminus \alpha X$. According to the above, there exists a function in $\mathcal{L}$ that cannot be continuously extended to ξ. Since $\mathcal{L} \subset \mathcal{L}_K$, there exists a function of the form f/g, where $f, g \in U^*_\mu(X)$ that cannot be extended to ξ. Then, it follows that $g^*(\xi) = 0$, i.e., ξ belongs to the G_δ-set $Z(g^*)$. On the other hand, since f/g must extend to $H(\mathcal{L}) = \alpha X$, we have that $Z(g^*) \cap \alpha X = \emptyset$. And this is a contradiction. □

Remark 30. Recall that every realcompact space Y between X and βX is always G_δ-closed in βX. Indeed, if $X \subset Y \subset \beta X$ then $\beta Y = \beta X$. Hence, $G_\delta\text{-cl}_{\beta X}(Y) = G_\delta\text{-cl}_{\beta Y}(Y) = \upsilon Y = Y$. Clearly the same result does not work for realcompact spaces between X and $s_\mu X$, since in particular these spaces must contain $K(X)$, and this is not always true as the example appearing at the beginning of this section proves. Then the following question arises.

Question 31. Is every realcompact space Y with $K(X) \subset Y \subset s_\mu X$, G_δ-closed in $s_\mu X$?

According to Theorem 29, this question can be reformulated as follows: "Does there exist, for each realcompact space Y where $K(X) \subset Y \subset s_\mu X$, a unital vector lattice $\mathcal{L}$, with $U^*_\mu(X) \subset \mathcal{L} \subset \mathcal{L}_K$ such that $Y = H(\mathcal{L})$?

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