

CONSTRUCTION OF VARIETIES OF LOW CODIMENSION WITH APPLICATIONS TO MODULI SPACES OF VARIETIES OF GENERAL TYPE

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Dedicated to our dear friend Miguel González on his 60th birthday

ABSTRACT. We develop a new way of systematically constructing infinitely many families of smooth subvarieties X of any given dimension m , $m \geq 3$, and any given codimension in \mathbb{P}^N , embedded by complete subcanonical linear series, and, in particular, in the range of Hartshorne's conjecture. We accomplish this by showing the existence of everywhere non-reduced schemes called ropes, embedded in \mathbb{P}^N , and by smoothing them. In the range $3 \leq m < N/2$, we construct smooth subvarieties, embedded by complete subcanonical linear series, that are not complete intersections. We also go beyond a question of Enriques on constructing simple canonical surfaces in projective spaces, and construct simple canonical varieties in all dimensions. The canonical map of infinitely many of these simple canonical varieties is finite birational but not an embedding. Finally, we show the existence of components of moduli spaces of varieties of general type (in all dimensions m , $m \geq 3$) that are analogues of the moduli space of curves of genus $g > 2$ with respect to the behavior of the canonical map and its deformations. In many cases, the general elements of these components are canonically embedded and their codimension is in the range of Hartshorne's conjecture.

1. INTRODUCTION

In this article, we study the deformations of morphisms φ to projective spaces that factor through a finite cover π of degree n of a complete intersection subvariety. As a result of this, we find a new, systematic method to construct infinitely many smooth subvarieties of *any* codimension, embedded in any projective space \mathbb{P}^N by complete linear series, including infinitely many smooth, non-complete intersection subvarieties and infinitely many simple canonical varieties in any dimension m , $m \geq 3$. We also find, for varieties of general type of any dimension $m \geq 3$, irreducible moduli components with a closed locus where the degree of the canonical map jumps up. This includes, for any $m \geq 3$, components whose general points correspond to varieties whose canonical map is either birational or an embedding. As we explain in more detail below, the subvarieties we construct and, in particular, the smooth canonical varieties corresponding to general points of the above mentioned moduli components, degenerate to certain everywhere non-reduced schemes of multiplicity n , called *ropes*, as it happens in the moduli of curves.

To carry out this program, we find out the conditions under which the deformation of φ can be deformed to an embedding or, more generally, to a morphism of lower degree, along a one-parameter family. In order for this to happen, we show first the existence of ropes of the right multiplicity and codimension embedded in \mathbb{P}^N , and prove that they can be smoothed. In particular, the smooth subvarieties constructed by our method are smoothings of these ropes. Therefore we produce one-parameter families whose general members are smooth subvarieties and whose special member is not even a local complete intersection, except if it is a rope of multiplicity 2. As already said, our study of the deformations of φ goes beyond the case of deformations to embeddings. Indeed, we prove general results (see Proposition 3.4 and Theorems 3.5, 3.7, 3.8) which show that the deformations of φ have very diverse behavior and the degree of the morphisms so obtained varies greatly (from degree 1 to degree $n/2$ and n). In particular, we obtain criteria for φ to be deformed to birational morphisms which are not necessarily embeddings.

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Although we extensively use our general results when π is a simple cyclic cover, a cover acted by $\mathbb{Z}_{n/2} \times \mathbb{Z}_2$ or a dihedral cover of a complete intersection subvariety, this article shows the way to follow when π is an arbitrary Galois cover or, even, certain finite covers of complete intersections, which are not necessarily Galois (see Setup 3.1). We detail now the several applications, given in the paper, of our method and techniques.

Construction of smooth, small codimension subvarieties. Among the smooth subvarieties we construct, there are infinitely many families of subvarieties of small codimension r , embedded by complete linear series in \mathbb{P}^N . They include subvarieties in the range $r < \frac{1}{3}N$, that is, in the range of Hartshorne's conjecture on complete intersections. The smooth codimension two subvarieties all turn out to complete intersections by Corollary 5.3 (see also Proposition 5.4), in higher codimensions this is not clear. Among them, those at or near the boundary of the range of Hartshorne's conjecture could be especially interesting. In the following two tables we give a very small sample of the smooth subvarieties we construct in the range $r = (1/3)N - 1$. In the tables, X' is a smooth projective s -subcanonical (see Definition 4.10), m -dimensional subvariety, obtained by deforming φ , where φ factors through either a simple cyclic or a dihedral cover of degree n of a complete intersection Y of multidegree \underline{d} (see Section 10 for notation).

m	n	N	s	\underline{d}	$\deg(X')$
9	4	12	5	(2, 4, 6)	192
11	5	15	12	(2, 4, 6, 8)	1920
15	7	21	32	(2, 4, 6, 8, 10, 12)	322560

TABLE 1. Deforming simple cyclic covers to obtain low codimension subvarieties

m	n	N	s	\underline{d}	$\deg(X')$
13	6	18	5	(2, 2, 4, 4, 6)	2304
17	8	24	15	(2, 2, 4, 4, 6, 6, 8)	147456
21	10	30	29	(2, 2, 4, 4, 6, 6, 8, 8, 10)	14745600

TABLE 2. Deforming simple dihedral covers to obtain low codimension subvarieties

The general deformation of the morphisms φ of Table 1 is an embedding. By Proposition 5.4, the image of this general deformation of φ is a complete intersection. However, we do not know whether the same is true for some special deformations (see Question 12.1). In the case of Table 2, we do not know whether the images of the deformations of φ are complete intersection subvarieties or not (see Question 12.2). For comprehensive results on the construction of small codimension, smooth subvarieties, see Sections 6 and 10 and Subsection 9.1.

Construction of smooth, non-complete intersection subvarieties. We also construct, in a systematic way, smooth subvarieties, embedded in \mathbb{P}^N by complete linear series, which are not complete intersections. In the table below, we give examples of non-complete intersection s -subcanonical threefolds (see Example 7.5 for further details):

m	n	N	s	\underline{d}	$\deg(\varphi'(X'))$
3	2	7	6	(3, 3, 3, 3)	162
3	2	7	7	(3, 3, 3, 4)	216
3	3	8	15	(4, 4, 4, 4, 4)	3072
3	3	8	16	(4, 4, 4, 4, 5)	3840

TABLE 3. Deforming simple cyclic covers of degree n to obtain non-complete intersections

More generally, for any $3 \leq m < \frac{N}{2}$, we construct non-complete intersection, m -dimensional smooth subvarieties of \mathbb{P}^N of infinitely many different degrees (see Theorems 7.6 and 7.7). A standard way of

constructing smooth non-complete intersection subvarieties in this range is to realize them as degeneracy loci of vector bundle homomorphisms. Our method is quite different from that, since our smooth subvarieties come as general members of families which smooth ropes.

Construction of simple canonical varieties. In 1943 Enriques raised the question of the existence of simple canonical surfaces in projective spaces, i.e., surfaces of general type for which the canonical map is birational. In the present article we systematically construct simple canonical varieties of general type in all dimensions $m \geq 3$. More generally, we deform the morphisms φ to morphisms φ' , birational onto their image, from smooth, s -subcanonical varieties X' . When $s = 1$, X' are simple canonical varieties. In most of those cases, the morphisms φ' are embeddings and the varieties $\varphi'(X')$ are canonically embedded, smooth subvarieties; however, we produce infinitely many examples of smooth varieties equipped with finite birational canonical maps, in fact, morphisms, that are not embeddings (see Theorem 8.3 and Example 8.4).

Moduli of varieties of general type. Deep results on existence and boundedness of moduli spaces of varieties of general type have been accomplished in recent years (see [Kol13], [Kol23], [Kov15], [HMX18]). The moduli space of curves \mathcal{M}_g and its compactification has been a chief inspiration for them and considering one-parameter families of varieties of general type has also been crucial. For each $m \geq 3$ it is natural to ask if one can systematically construct nontrivial examples of moduli spaces of m -dimensional varieties of general type X that have components analogous to the moduli space of curves, with respect to the behavior of canonical maps and their deformations. To be precise, one would like to construct moduli components with locally closed loci that resemble the hyperelliptic locus of \mathcal{M}_g ($g > 2$) in the following sense. The hyperelliptic locus parametrizes curves whose canonical maps are finite morphisms φ of degree 2 onto its image such that a general deformation of φ is an embedding. More subtly, a general deformation of φ gives rise to a one-parameter family of subvarieties whose central member is a rope of multiplicity 2 (a canonical ribbon), while its general member is a smooth canonically embedded curve.

In Section 11, we produce moduli components of higher dimensional varieties of general type that capture the features, described above, of the moduli space of curves and its hyperelliptic locus. Indeed, we show how to systematically construct nontrivial examples of moduli spaces of varieties of general type (for instance, not being products) of dimension m , $m \geq 3$, with an irreducible component having a locally closed locus that parametrizes varieties whose canonical map is a finite morphism φ of degree n , $n \geq 2$, onto its image such that its general deformation is an embedding. Moreover, like in the case of curves, the image of the central fiber of a general first order deformation φ is an embedded rope of multiplicity n . Such a rope is the one-parameter degeneration of smooth, canonically embedded varieties that correspond to general points of the moduli component. We find these canonically embedded varieties for any codimension r and, in particular, for any r in the range of the Hartshorne's conjecture. For each $m \geq 3$, we also construct two distinct kinds of moduli components which differ from the moduli space of curves. Firstly, we exhibit moduli components (see Corollary 11.2) such that a general deformation of a canonical morphism φ corresponding to a point in the special locus is not an embedding but a birational morphism, and the image of the central fiber of a general first order deformation of φ is not an embedded rope, but a possibly locally non-Cohen-Macaulay, everywhere non-reduced scheme. Secondly, we also construct moduli components (see Corollary 11.3) where the degree of the canonical map φ drops from n to $n/2$, as φ deforms from a locally closed locus to the general stratum. For surfaces, the existence of moduli components with locally closed loci where the degree of the canonical map jumps up was previously known (see [Cat81], [Cat87], [CS02], [CPT00], [AK90], [GGP10], [GGP13] and [BGMR21]). The special loci seen in Section 11 are but a particular case of a more general phenomenon. In fact, our results imply (see Remarks 6.7, 7.8, 9.10, 10.7) the existence of infinitely many irreducible components possessing loci which correspond to the jumping up of the degree of subcanonical maps. These loci for higher dimensional varieties motivate interesting questions concerning the moduli space of curves itself,

namely, if the moduli of curves has analogous locally closed loci, in that case, in relation to the possible change of degree of morphisms induced by theta-characteristics or by other subcanonical divisors.

Hilbert scheme components with ropes in their boundary. As already mentioned, the construction and smoothing of ropes is key in our method to produce subvarieties in projective space (see Theorem 3.7). A priori, it is not clear why embedded ropes of codimensions in the range of Hartshorne’s conjecture and beyond should deform to smooth subvarieties. When these ropes have multiplicity $n \geq 3$, they are not even local complete intersections. We systematically construct, in any codimension (this includes the range of Hartshorne’s conjecture), non–complete intersection, embedded ropes, lying in the boundary of an irreducible component of the Hilbert scheme parametrizing smooth complete intersection subvarieties (see Proposition 5.4). We do this when φ satisfies Theorem 6.6 (a) and factors through a simple cyclic cover or, more generally, a suitable composition of simple cyclic covers. However, it is not clear that any one-parameter deformation of these ropes is a complete intersection, except if the codimension is $r \leq 2$, where we use the results in [KPR03] to prove Corollary 5.3 (indeed, the question of complete intersections is better known for codimension 2 subvarieties of projective space, see e.g. [Hor64], [KPR03], and has even been studied in more general varieties, see the works [Ott89], [AC00], [Mad00], [CM04], [KRR07a], [KRR07b] [Rav09], [RT19], to name just a few). Furthermore, when φ factors through other Galois covers, such as, for instance, dihedral covers, it is also unclear that the general deformation of such ropes is a complete intersection subvariety. The situation is further complicated by the existence of examples (see Remark 12.5) where we show that φ is unobstructed in its deformation space, but the rope that appears when we deform φ is obstructed, in some of the cases because it lies in the intersection of two components of the Hilbert scheme. In addition, in the range $r < N/2$, we also show the existence of ropes that can be deformed to smooth subvarieties but do not lie in the boundary of any irreducible component of the Hilbert scheme parameterizing smooth complete intersection subvarieties (see Theorems 7.6, 7.7).

Organization. We now provide the structure of this article. In Section 2, we recall the definition of ropes and several results, including the results on the deformation theory of finite morphisms developed by the first two authors and Miguel González. In Section 3 we study the deformations of finite covers of complete intersection subvarieties. In Section 4 we give the details of the construction of abelian and dihedral covers. Section 5 is devoted to finding the necessary and sufficient conditions to ensure that a general deformation of a finite cover is not a complete intersection. We apply general results of Sections 3, 4 and 5 to deform simple cyclic covers to embeddings of smooth subvarieties of any dimension (including small codimensional subvarieties), embeddings of non–complete intersections, and birational maps and, more generally, to non–embeddings, in Sections 6, 7, and 8 respectively. We study the deformations of $\mathbb{Z}_{n/2} \times \mathbb{Z}_2$ and simple dihedral covers of complete intersections in Sections 9 and 10. In Section 11, we apply the results of the previous sections to prove the existence of irreducible moduli components that behave like the moduli space of curves. Finally, we devote Section 12 to open questions.

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2. PRELIMINARIES

We work over the field of complex numbers. One of the central tools for deforming a finite morphism to a morphism of smaller degree is to construct a suitable multiple structure, called rope, on the image of the morphism.

Definition 2.1. Let Y be a reduced connected scheme and let \mathcal{E} be a vector bundle of rank $m - 1$ on Y . A *rope of multiplicity m on Y with conormal bundle \mathcal{E}* is a scheme Y' with $Y'_{\text{red}} = Y$ such that

- (1) $\mathcal{I}_{Y/Y'}^2 = 0$,
- (2) $\mathcal{I}_{Y/Y'} = \mathcal{E}$ as \mathcal{O}_Y modules.

If \mathcal{E} is a line bundle then Y' is called a *ribbon* on Y .

For basic definitions and terminologies in deformation theory, we refer to [Ser06]. For a vector bundle \mathcal{E} on Y , the space $H^1(T_Y \otimes \mathcal{E})$ parametrizes ropes \tilde{Y} with conormal bundle \mathcal{E} , and such a rope is split if and only if its class in $H^1(T_Y \otimes \mathcal{E})$ is zero. If $i \hookrightarrow \mathbb{P}^N$ is a closed embedding, it is shown in [Gon06, Proposition 2.1], that the space $H^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E})$ parametrizes the pairs (\tilde{Y}, \tilde{i}) where \tilde{Y} is a rope on Y with conormal bundle \mathcal{E} and $\tilde{i}: \tilde{Y} \rightarrow \mathbb{P}^N$ is a morphism that extends i .

Typically, we will work with the following situation: let $\varphi: X \rightarrow \mathbb{P}^N$ be a morphism from a smooth, projective variety X , which is finite onto a smooth variety $Y \hookrightarrow \mathbb{P}^N$. The space $H^0(\mathcal{N}_\varphi)$, where \mathcal{N}_φ is the normal sheaf of φ , parametrizes the infinitesimal deformations of φ (see, e.g., [Ser06, Section 3.4.2]). We will further set \mathcal{E} to be the trace zero module of the induced morphism $\pi: X \rightarrow Y$. In this situation, the relation between $H^0(\mathcal{N}_\varphi)$ and $H^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E})$ is given by the following proposition:

Proposition 2.2. ([Gon06, Proposition 3.7]) *Let X be a smooth variety and let $\varphi: X \rightarrow \mathbb{P}^N$ be a morphism that factors as $\varphi = i \circ \pi$, where π is a finite cover of a smooth variety Y and $i: Y \hookrightarrow \mathbb{P}^N$ is an embedding. Let \mathcal{E} be the trace zero module of π and let \mathcal{I} be the ideal sheaf of $i(Y)$. There exists a homomorphism*

$$H^0(\mathcal{N}_\varphi) \xrightarrow{\Psi} \text{Hom}(\pi^*(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_X)$$

that appears when taking cohomology on the commutative diagram [Gon06, (3.3.2)]. Since

$$\text{Hom}(\pi^*(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_X) = H^0(\mathcal{N}_{Y/\mathbb{P}^N}) \oplus H^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E}),$$

the homomorphism ψ has two components;

$$H^0(\mathcal{N}_\varphi) \xrightarrow{\Psi_1} H^0(\mathcal{N}_{Y/\mathbb{P}^N}) \text{ and } H^0(\mathcal{N}_\varphi) \xrightarrow{\Psi_2} H^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E}).$$

Throughout this article, we will use the following result from [GGP10]. Note the misprint in [GGP10], where the word "generically" in the last line of the statement is missing.

Theorem 2.3. ([GGP10, Theorem 1.4]) *Let X be a smooth projective variety and let $\varphi: X \rightarrow \mathbb{P}^N$ be a morphism that factors through an embedding $Y \hookrightarrow \mathbb{P}^N$ with Y smooth. Let $\pi: X \rightarrow Y$ be the induced morphism which we assume to be finite of degree $n \geq 2$. Let $\tilde{\varphi}: \tilde{X} \rightarrow \mathbb{P}_\Delta^N$ ($\Delta = \text{Spec}\left(\frac{\mathbb{C}[\epsilon]}{\epsilon^2}\right)$) be a locally trivial first order infinitesimal deformation of φ and let $v \in H^0(\mathcal{N}_\varphi)$ be the class of $\tilde{\varphi}$. If*

- (a) the homomorphism $\psi_2(v)$ has rank $k > \frac{n}{2} - 1$, and
- (b) there exists an algebraic formally semiuniversal deformation of φ and φ is unobstructed,

then there exists a flat family of morphisms, $\Phi: \mathcal{X} \rightarrow \mathbb{P}_T^N$ over T , where T is a smooth irreducible algebraic curve with a distinguished point 0 , such that

- (1) \mathcal{X}_t is a smooth irreducible projective variety,
- (2) the restriction of Φ to the first order infinitesimal neighborhood of 0 is $\tilde{\varphi}$, and
- (3) for $t \neq 0$, Φ_t is finite and generically one-to-one onto its image in \mathbb{P}_t^N .

The first and the second author, along with González, in fact gave a criterion under which a finite morphism deforms to an embedding.

Theorem 2.4. ([GGP13, Theorem 1.5]) *Under the assumption of Theorem 2.3, suppose moreover $\psi_2(v)$ is a surjective homomorphism. Then there exists a flat family of morphisms, $\Phi : \mathcal{X} \rightarrow \mathbb{P}_T^N$ over T , where T is a smooth irreducible algebraic curve with a distinguished point 0 , such that*

- (1) \mathcal{X}_t is a smooth irreducible projective variety,
- (2) the restriction of Φ to the first order infinitesimal neighborhood of 0 is $\tilde{\varphi}$, and
- (3) for $t \neq 0$, Φ_t is a closed immersion into \mathbb{P}_t^N .

The authors of [GGP13] in fact showed that under the assumptions above, $(\text{Im}\Phi)_0$ is an embedded rope \tilde{Y} corresponding to $\psi_2(v)$. It is clear from the above theorems that we need to produce homomorphisms of appropriate rank between the vector bundles $\mathcal{S}/\mathcal{S}^2$ and \mathcal{E} . In order to do that, we need a theorem of Bănică. We need the following definition in order to state the theorem.

Definition 2.5. Let $v : \mathcal{E} \rightarrow \mathcal{F}$ be a morphism between vector bundles of rank e and f respectively, on an irreducible complex projective variety X . For any positive integer $k \leq \min\{e, f\}$, we define the k -th *degeneracy locus* $D_k(v)$ as the subscheme cut out by the minors of order $k+1$ of the matrix locally representing v .

Now we state the result of Bănică. We will use this result to deform a finite morphism to a birational morphism.

Theorem 2.6. ([Băn91, §4.1]) *Let \mathcal{E} and \mathcal{F} be vector bundles on a projective variety X of rank e and f respectively. Assume $\mathcal{E}^* \otimes \mathcal{F}$ is globally generated. Then, for a general morphism $v : \mathcal{E} \rightarrow \mathcal{F}$, the subschemes $D_k(v)$ either are empty or have pure codimension $(e-k)(f-k)$ in X and the singular locus $\text{Sing}(D_k(v)) = D_{k-1}(v)$.*

The double covers of complete intersections are studied in [BG23] in more detail. In case of double covers, there are two possibilities, namely the general deformation is either birational, or it is finite of degree 2. However, for higher degree covers, the degree of a general deformation might drop, but it might not be birational onto its image. We will show that this case indeed appears. In order to prove that, the following proposition and Proposition 4.5 will be crucial.

The following result is the consequence of [Weh86, Proposition 1.10].

Proposition 2.7. *Let $\pi : X \rightarrow Y$ be a finite, flat morphism between smooth projective varieties with trace zero module \mathcal{E} and let $\psi : Y \rightarrow Z$ be a non-degenerate morphism to a smooth variety Z . Let $\varphi = \psi \circ \pi$ be the composed morphism. If $H^0(\mathcal{N}_\psi \otimes \mathcal{E}) = 0$ then the natural map between the functors $\mathbf{Def}(\pi|Z) \rightarrow \mathbf{Def}_\varphi$ is smooth.*

Proof. This is proven in [BGMR21, Proposition 2.14], we recall the argument here. Apply [Weh86, Proposition 1.10] to the following commutative diagram.

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ \varphi \searrow & & \swarrow \psi \\ & Z & \end{array}$$

The maps β_1 and β_2 of [Weh86, Proposition 1.10] become the following:

$$\beta_1 : H^0(\mathcal{N}_\psi) \rightarrow H^0(\pi^* \mathcal{N}_\psi), \text{ and } \beta_2 : H^1(\mathcal{N}_\psi) \rightarrow H^1(\pi^* \mathcal{N}_\psi).$$

Recall that $\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}$. Thus the assertion follows since the map β_2 is always injective and β_1 is surjective if $H^0(\mathcal{N}_\psi \otimes \mathcal{E}) = 0$, both by projection formula. \square

3. DEFORMATION THEORY OF FINITE COVERS OF COMPLETE INTERSECTIONS

In this section we develop the deformation theory of finite covers of complete intersection subvarieties of projective space. We will use our theory to construct in a systematic way other subvarieties, of

dimension $m \geq 3$, of projective spaces of any dimension. These new subvarieties cover all possibilities in terms of dimension and codimension and infinitely many of them are non–complete intersections.

We will require our covers to satisfy two additional conditions (see (2.1) and (2.2) of Setup 3.1 below), namely, the splitting of its trace zero module as a direct sum of line bundles and the vanishing of the first cohomology group of its normal sheaf. As Subsection 4.1 makes clear, abelian and simple dihedral covers of complete intersections satisfy these two conditions, so there are plenty of covers to which our theory applies. This will be our setup:

Setup 3.1. *Throughout the remaining of the article, unless otherwise stated, we will use the following setup.*

- (1) *Let $i : Y \hookrightarrow \mathbb{P}^N$ be a smooth, non–degenerate, complete intersection subvariety of multidegree $\underline{d} = (d_1, d_2, \dots, d_r)$, with $r \geq 1$ and $2 \leq d_1 \leq d_2 \leq \dots \leq d_r$. Let m be the dimension of Y and assume $m \geq 3$. Set*

$$\delta = \sum_{i=1}^r d_i \text{ and } d = \prod_{i=1}^r d_i.$$

- (2) *Let X be a smooth, irreducible variety, let $\pi : X \rightarrow Y$ be a finite morphism of degree n , $n \geq 2$ and let $\varphi = i \circ \pi$. Let $\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}$, where \mathcal{E} is the trace zero module of π . Let \mathcal{N}_π be the normal sheaf of π . Assume furthermore that*

- (2.1) \mathcal{E} splits as direct sum of line bundles;
(2.2) $H^1(\mathcal{N}_\pi) = 0$.

Finally, let $\mathcal{O}_Y(k) := i^* \mathcal{O}_{\mathbb{P}^N}(k)$ for any $k \in \mathbb{Z}$ and let

$$\mathcal{E} = \bigoplus_{i=1}^{n-1} \mathcal{O}_Y(-k_i), \text{ i.e., } \pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \left(\bigoplus_{i=1}^{n-1} \mathcal{O}_Y(-k_i) \right),$$

(see Remark 3.2 (1) below), where $k_1 \leq k_2 \leq \dots \leq k_{n-1}$ are, necessarily positive, integers.

Remark 3.2. We make a note of the following well known facts that we will use without explicitly stating.

- (1) Since Y is a complete intersection in \mathbb{P}^N , it follows from the Grothendieck–Lefschetz theorem (recall that $m \geq 3$) that any line bundle on Y is the restriction of a line bundle on \mathbb{P}^N . In particular, $\text{Pic}(Y) = \mathbb{Z}$.
(2) Since Y is a complete intersection of multidegree $\underline{d} = (d_1, \dots, d_r)$, it follows that

$$\mathcal{N}_{Y/\mathbb{P}^N} = \bigoplus_{i=1}^r \mathcal{O}_Y(d_i).$$

- (3) For $a \in \mathbb{Z}$, $H^0(\mathcal{O}_Y(a)) \neq 0 \iff a \geq 0 \iff \mathcal{O}_Y(a)$ is globally generated.
(4) For $a \in \mathbb{Z}$, $H^i(\mathcal{O}_Y(a)) = 0$, for all $1 \leq i \leq m-1$.
(5) Let \mathcal{I} be the ideal sheaf of $i(Y)$ inside \mathbb{P}^N and let $H_*^i(\mathcal{I}) = \bigoplus_{v \in \mathbb{Z}} H^i(\mathcal{I}(v))$. For all $1 \leq i \leq m$, $H_*^i(\mathcal{I}) = 0$.
(6) Because of (4), the variety X is regular.
(7) The morphism φ is induced by the complete linear series $|\pi^* \mathcal{O}_Y(1)|$ if and only if $k_1 \geq 2$.

We now state and prove our main results on deformations of finite covers of complete intersections. First we need the following lemma.

Lemma 3.3. *In the situation of Setup 3.1, φ has an algebraic formally semiuniversal deformation space and it is unobstructed. Moreover, the map (see Proposition 2.2)*

$$\psi_2 : H^0(\mathcal{N}_\varphi) \rightarrow H^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E})$$

is surjective, where \mathcal{E} is the trace zero module of π .

Proof. Notice that $H^2(\mathcal{O}_X) = H^2(\mathcal{O}_Y) \oplus H^2(\mathcal{E})$. Consequently, by [BGG20, Proposition 1.7], the morphism φ has an algebraic formally semiuniversal deformation space since by Remark 3.2, (4) $H^2(\mathcal{O}_X) = 0$. It follows from our assumption that $H^1(\mathcal{N}_\pi) = 0$. In particular, ψ_2 surjects due to the following exact sequence (see [Gon06, Lemma 3.3]) and projection formula:

$$(3.1) \quad 0 \rightarrow \mathcal{N}_\pi \rightarrow \mathcal{N}_\varphi \rightarrow \pi^* \mathcal{N}_{Y/\mathbb{P}^N} \rightarrow 0.$$

Moreover, by Remark 3.2, (4), we get that $H^1(\pi^* \mathcal{N}_{Y/\mathbb{P}^N}) = H^1(\mathcal{N}_{Y/\mathbb{P}^N}) \oplus H^1(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E}) = 0$. Then, from the short exact sequence (3.1) we get $H^1(\mathcal{N}_\varphi) = 0$. \square

The following proposition gives a criterion under which the degree of a finite morphism remains unchanged for any of its deformation.

Proposition 3.4. *Let Setup 3.1 hold except maybe hypothesis (2.2). If $d_r < k_1$ then any deformation of φ is finite of degree n onto its image, which is a smooth, complete intersection subvariety of \mathbb{P}^N of multidegree $\underline{d} = (d_1, d_2, \dots, d_r)$.*

Proof. By hypothesis, and Remark 3.2, (2), we obtain $H^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E}) = 0$. The conclusion follows from Proposition 2.7 and [Ser75]. \square

Our methods of deformations of the finite morphism to an embedding is intimately related to the existence of some special rope structure on Y . We construct ropes of suitable ranks on Y and we smooth them. This in turn deforms the finite morphism to an embedding or a birational map. One can explicitly construct smoothing family of split multiple structures, but our constructions are general.

Now we state our main theorems that give criteria to determine when the general deformation of the covering morphism is birational (see Theorem 3.5) and when is not only birational but also an embedding (see Theorem 3.7). Our results are a consequence of the existence of appropriate ropes.

Theorem 3.5. *In the situation of Setup 3.1, let $r > \lfloor \frac{n}{2} \rfloor - 1$ and $d_{r - \lfloor \frac{n}{2} \rfloor + 1} \geq k_{n-1}$. Then a general element of the algebraic formally semiuniversal deformation space of φ is finite and birational onto its image.*

Remark 3.6. Observe that $2r + 2 - n - N = r - (N - r) - (n - 2) < r$ as $n \geq 2$ and $N - r \geq 3$.

Proof of Theorem 3.5. Let $\mathcal{N}' = \mathcal{O}_Y(d_a) \oplus \mathcal{O}_Y(d_{a+1}) \oplus \dots \oplus \mathcal{O}_Y(d_r)$ where $a = r - \lfloor \frac{n}{2} \rfloor + 1$ and let \mathcal{E} be the trace zero module of π . Notice that $\text{rank}(\mathcal{N}') = \lfloor \frac{n}{2} \rfloor$. The condition $d_{r - \lfloor \frac{n}{2} \rfloor + 1} \geq k_{n-1}$ guarantees that $\mathcal{N}' \otimes \mathcal{E}$ is globally generated. It follows from Theorem 2.6 that, for a general homomorphism $\sigma : \mathcal{N}'^* \rightarrow \mathcal{E}$, $D_{\lfloor \frac{n}{2} \rfloor - 1}(\sigma)$ is a proper closed subvariety inside Y . Consequently, there exists a rank $\lfloor \frac{n}{2} \rfloor$ homomorphism in $\text{Hom}(\mathcal{N}'^*, \mathcal{E})$, which in turn implies the existence of a rank $\lfloor \frac{n}{2} \rfloor$ homomorphism in $\text{Hom}(\mathcal{N}_{Y/\mathbb{P}^N}^*, \mathcal{E})$. The assertion follows from Theorem 2.3 and Lemma 3.3, since birationality is an open condition. \square

Theorem 3.7. *In the situation of Setup 3.1, let either of the following conditions (a) or (b) hold;*

(a) $r \geq n - 1$ and there exists $1 \leq l_1 < l_2 < \dots < l_{n-1}$, such that $d_{l_j} = k_j$ for all $1 \leq j \leq n - 1$;

(b) $r > \frac{N+n-2}{2}$, and $d_{2r+2-n-N} \geq k_{n-1}$;

then there exist embedded ropes $\tilde{Y} \hookrightarrow \mathbb{P}^N$ on Y with conormal bundle \mathcal{E} , which are non-split if φ is induced by a complete linear series.

Moreover, for any embedded rope $\tilde{Y} \hookrightarrow \mathbb{P}^N$ on Y with conormal bundle \mathcal{E} , there exists a flat family $\Phi : \mathcal{X} \rightarrow \mathbb{P}_T^N$ over T , where T is a smooth irreducible algebraic curve with a distinguished point 0 , such that

(I) \mathcal{X}_t is a smooth irreducible projective variety,

(II) $\mathcal{X}_0 = X$, $\Phi_0 = \varphi$, and

(III) for $t \neq 0$, Φ_t is a closed immersion into \mathbb{P}_t^N and $(\text{Im } \Phi)_0$ is \tilde{Y} .

Consequently, a general element of the algebraic formally semiuniversal deformation space of φ is an embedding.

Proof. We aim to show that if (a) or (b) holds, then there exists a surjective homomorphism in $\text{Hom}(\mathcal{N}_{Y/\mathbb{P}^N}^*, \mathcal{E})$. Clearly that is the case if (a) holds. Now, assume (b) holds. As before, let $a = 2r + 2 - n - N$ and set $\mathcal{N}'' = \mathcal{O}_Y(d_a) \oplus \mathcal{O}_Y(d_{a+1}) \oplus \cdots \oplus \mathcal{O}_Y(d_r)$. Notice that \mathcal{N}'' is a vector bundle of rank $(N - r) + (n - 1)$ and the condition $d_{2r+2-n-N} \geq k_{n-1}$ guarantees that $\mathcal{N}'' \otimes \mathcal{E}$ is globally generated. It follows from Theorem 2.6 that, for a general homomorphism $\sigma' : \mathcal{N}''^* \rightarrow \mathcal{E}$, $D_{n-2}(\sigma')$ is empty. Indeed, otherwise it has expected codimension

$$((N - r) + (n - 1) - (n - 2))((n - 1) - (n - 2)) = N - r + 1 > N - r$$

which is impossible. Thus, σ' can be extended to a surjective homomorphism in $\text{Hom}(\mathcal{N}_{Y/\mathbb{P}^N}^*, \mathcal{E})$.

Now, any surjective homomorphism of $\text{Hom}(\mathcal{N}_{Y/\mathbb{P}^N}^*, \mathcal{E})$ corresponds to an embedded rope \tilde{Y} on Y , with conormal bundle \mathcal{E} . Assume φ is induced by the complete linear series $|\pi^* \mathcal{O}_Y(1)|$ (equivalently, $k_1 \geq 2$). Then any embedded rope $\tilde{Y} \hookrightarrow \mathbb{P}^N$ on Y with conormal bundle \mathcal{E} is non-split. Indeed, it follows from the long exact sequence associated to the restricted Euler sequence twisted by \mathcal{E} :

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{E}(1)^{\oplus N+1} \rightarrow T_{\mathbb{P}^N|_Y} \otimes \mathcal{E} \rightarrow 0$$

that $H^0(T_{\mathbb{P}^N|_Y} \otimes \mathcal{E}) = H^1(T_{\mathbb{P}^N|_Y} \otimes \mathcal{E}) = 0$. Consequently, by the long exact sequence associated to the following exact sequence,

$$0 \rightarrow T_Y \otimes \mathcal{E} \rightarrow T_{\mathbb{P}^N|_Y} \otimes \mathcal{E} \rightarrow \mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E} \rightarrow 0,$$

it follows that $H^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E}) \cong H^1(T_Y \otimes \mathcal{E})$. Thus, the assertion follows from the fact that the class of a surjective homomorphism in $H^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E})$ is nonzero.

The rest of the assertions of the statement of the theorem are consequences of Theorem 2.4, Lemma 3.3, [GGP13, Theorem 2.2], and the fact that being an embedding is an open condition. \square

Next theorem gives a criterion under which the degree of a general deformed morphism is one half of the initial degree.

Theorem 3.8. *In the situation of Setup 3.1, assume n is even, $n \geq 4$, and the following holds;*

- (1) $\pi = p_1 \circ \pi_1$ where $\pi_1 : X \rightarrow X_1$ is a abelian cover of degree $n/2$ with trace zero module

$$\mathcal{E}_2 = p_1^* \left(\mathcal{O}_Y(-k'_1) \oplus \cdots \oplus \mathcal{O}_Y(-k'_{\frac{n}{2}-1}) \right), \quad k'_1 \leq \cdots \leq k'_{\frac{n}{2}-1}$$

and branch divisor

$$E = \sum_i p_1^*(D_i)$$

where $p_1^* D_i$ are the irreducible components of E and D_i are divisors on Y and $p_1 : X_1 \rightarrow Y$ is a double cover with trace zero module $\mathcal{E}_2 = \mathcal{O}_Y(-l)$,

- (2) $k'_1 > \max\{2l, d_r\}$ and $d_r \geq l$.

Then, a general element φ' of the algebraic formally semiuniversal deformation space of φ is a morphism, which is finite and of degree $\frac{n}{2}$ onto its image. If, moreover, one of the following holds;

- (1') $d_s = l$ for some $1 \leq s \leq r$, or

- (2') $r > \frac{N}{2}$, and $d_{2r-N} \geq l$,

then, a general element φ' of the algebraic formally semiuniversal deformation space of φ is a finite morphism of degree $\frac{n}{2}$, onto a smooth variety. In particular, φ' is flat. Moreover in this case the algebraic semiuniversal deformation spaces of both φ and X are smooth and uniruled.

Proof. We have the following short exact sequence where $\varphi_1 = i \circ p_1$ (see [Gon06, Lemma 3.3]);

$$(3.2) \quad 0 \rightarrow \mathcal{N}_{p_1} \rightarrow \mathcal{N}_{\varphi_1} \rightarrow p_1^* \mathcal{N}_{Y/\mathbb{P}^N} \rightarrow 0.$$

Note that $h^0(\mathcal{N}_{p_1} \otimes \mathcal{E}_2) = \sum_{i=1}^{\frac{n}{2}-1} h^0(\mathcal{O}_B(2l - k'_i))$, where $B \in |\mathcal{O}_Y(2l)|$ is the branch divisor of p_1 . It is easy to see from the following exact sequence;

$$0 \rightarrow \mathcal{O}_Y(-k'_i) \rightarrow \mathcal{O}_Y(2l - k'_i) \rightarrow \mathcal{O}_B(2l - k'_i) \rightarrow 0,$$

that $h^0(\mathcal{N}_{p_1} \otimes \mathcal{E}_2) = \sum_{i=1}^{\frac{n}{2}-1} h^0(\mathcal{O}_Y(2l - k'_i))$. By assumption, $k'_i > 2l$, and consequently, $k'_i > 2l$, for all i . Thus, $h^0(\mathcal{N}_{p_1} \otimes \mathcal{E}_2) = 0$. Also, since $k'_j > d_i$, we obtain

$$h^0(p_1^*(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E}_2)) = \sum_{i=1}^r \sum_{j=1}^{\frac{n}{2}-1} h^0(\mathcal{O}_Y(d_i - k'_j)) + \sum_{i=1}^r \sum_{j=1}^{\frac{n}{2}-1} h^0(\mathcal{O}_Y(d_i - k'_j - l)) = 0.$$

Consequently, by tensoring the exact sequence (3.2) by \mathcal{E}_2 and taking the long exact sequence of cohomology, one finds that $h^0(\mathcal{N}_{\varphi_1} \otimes \mathcal{E}_2) = 0$. It follows from Proposition 2.7 that any deformation of φ is of degree greater than or equal to $\frac{n}{2}$, since it factors through a deformation of π_1 .

Since φ is unobstructed (Lemma 3.3), $h^1(\mathcal{N}_{p_1}) = 0$ and $h^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{O}_Y(-l)) \neq 0$, it follows that there exists a deformation $\Psi : \mathcal{X}_1 \rightarrow \mathbb{P}_T^N$ over a smooth pointed algebraic curve $(T, 0)$ for which $\Psi_0 = \varphi_1$ and Ψ_t is of degree 1 for $t \neq 0$. Now notice that $H^i(\mathcal{O}_{X_1}) = H^i(\mathcal{O}_Y) \oplus H^i(\mathcal{O}_Y(-l)) = 0$ for $i = 1, 2$ since Y is at least a threefold and the intermediate cohomology vanishes. Also we have $H^1(p_1^*D_i) = H^1(D_i) \oplus H^1(D_i \otimes \mathcal{O}_Y(-l)) = 0$ for the same reason. It follows from Proposition 4.5, that, possibly after shrinking T , there exists $\Pi : \mathcal{X} \rightarrow \mathcal{X}_1$ such that $\Pi_0 = \pi_1$. Thus, $\Phi = \Psi \circ \Pi : \mathcal{X} \rightarrow \mathbb{P}_T^N$ is a deformation of $\Phi_0 = \varphi$ such that Φ_t of degree $\frac{n}{2}$ for all $t \neq 0$. Thus, a general element of the algebraic formally semiuniversal deformation space of φ will have degree less than or equal to $\frac{n}{2}$. The conclusion follows since we have already showed that any deformation of φ is of degree greater than or equal to $\frac{n}{2}$.

The last assertion is clear from Theorem 3.7, since under the assumption, φ_1 deforms to an embedding, and being an embedding is an open condition.

The deformation space of φ is fibered over the deformation space of φ_1 by a product of projective spaces and is hence uniruled. Since φ_1 is unobstructed and $H^1(N_{\varphi_1}) \rightarrow H^1(T_{X_1})$ surjects, we have that the deformation space of X is fibered over the deformation space of X_1 by a product of projective spaces and is hence smooth and uniruled. That completes the proof. \square

Remark 3.9. Let Y' be the image of φ' . Note that, if, in addition to (1) and (2), hypothesis (1') and (2') in Theorem 3.8 are assumed, then, the length of the fiber of φ' , over any closed point of Y' , is $n/2$. If (1') and (2') are not assumed, then Y' might be singular and one can only guarantee that the length of the fiber of φ' , over any closed point of a dense open set of Y' , is $n/2$.

Remark 3.10. In the situation of Setup 3.1, assume that $k_1 \geq 2$. This is equivalent to φ being induced by the complete linear series $|\varphi^* \mathcal{O}_{\mathbb{P}^N}(1)|$. Then, for any deformation $\Phi : \mathcal{X} \rightarrow \mathbb{P}_T^N$ of φ over a smooth variety $(T, 0)$ (in particular, for those in Proposition 3.4 and, provided $k_1 \geq 2$, Theorems 3.5, 3.7 and 3.8), we may assume (possibly after shrinking T), that Φ_t is induced by a complete linear series. This is a consequence of semicontinuity and the fact that being a non-degenerate morphism (in the sense that the image is not contained in any hyperplane) is an open condition. In particular, if φ' corresponds to a general element of the algebraic formally semiuniversal deformation space of φ , then φ' is induced by a complete linear series. In particular, if $k_i \geq 2$ for all i , then the subvarieties $\Phi_t(\mathcal{X}_t)$ of Theorem 3.7 are embedded in \mathbb{P}^N by complete linear series.

We end this section with a Barth-type result for covers of complete intersections, with a flavor similar to [Laz80, Proposition 3.1] for covers of projective space. In what follows, for a variety W , we write $\rho(W)$ to denote its Picard rank.

Corollary 3.11. *In the situation of Setup 3.1, assume the hypothesis of Theorem 3.7 (a) is satisfied and $N \geq 2r + 2$. Then $\rho(X) = 1$.*

Proof. Let $(T, 0)$ be a smooth algebraic curve and let $\Phi : \mathcal{X} \rightarrow \mathbb{P}_T^N$ be a deformation for which Φ_t is an embedding for all $t \neq 0$. Since $H^2(\mathcal{O}_X) = 0$, it follows that $\rho(X) \leq \rho(\mathcal{X}_t)$ (see for e.g. [Man91, Proofs of Lemma 2 and Theorem 4]). By the theorem of Barth-Larsen (see [Lar73], see also [Laz04, Corollary 3.2.3]), we know that $\text{Pic}(\mathcal{X}_t) = \mathbb{Z}$. The conclusion follows from the projectivity of X . \square

4. ABELIAN AND DIHEDRAL COVERS

Theorems 3.5 and 3.7 work for finite covers of complete intersections of dimension $m \geq 3$ that satisfy conditions (2.1) and (2.2) of Setup 3.1. Theorem 3.8 also works for finite covers of complete intersections satisfying conditions (2.1) and (2.2) of Setup 3.1 and possessing an intermediate cover of certain kind. We see now that these conditions are satisfied by large classes of covers of complete intersection subvarieties, namely, abelian covers and simple dihedral covers. Now we give the details of the construction of both kinds of covers. In this section, Y is a smooth variety which is *not necessarily a complete intersection*.

4.1. Abelian covers. We recall the definition of an abelian cover.

Definition 4.1. Let Y be a variety and let G be a finite group. A *Galois cover of Y with Galois group G* is a finite flat morphism $\pi : X \rightarrow Y$ together with a faithful action of G on X that exhibits Y as a quotient of X via G . We call a Galois cover *abelian*, if G is abelian.

Remark 4.2. If $\pi : X \rightarrow Y$ is an abelian cover, then the vector bundle $\pi_* \mathcal{O}_X$ splits as a direct sum of line bundles on Y . More precisely (see e.g. [Par91, (1.1)])

$$(4.1) \quad \pi_* \mathcal{O}_X = \bigoplus_{\chi \in G^*} L_\chi^{-1},$$

where G^* is the character group of G , L_χ is a line bundle on Y , G acts on L_χ via the character χ and the invariant summand L_1 is isomorphic to \mathcal{O}_Y . In particular, an abelian cover of a complete intersection subvariety satisfies (2.1) of Setup 3.1.

In order to check abelian covers satisfy (2.2) of Setup 3.1 we need to know the structure of $\pi_* \mathcal{N}_\pi$. For this we need to introduce some notation (see [Par91] for further details). Let D be the branch divisor of π . Let \mathcal{C} be the set of cyclic subgroups of G and for all $H \in \mathcal{C}$, denote by S_H the set of generators of the group of characters H^* . Then, we may write

$$D = \sum_{H \in \mathcal{C}} \sum_{\psi \in S_H} D_{H,\psi}$$

where $D_{H,\psi}$ is the sum of all the components of D that have inertia group H and character ψ . For every $\chi \in G^*$, and $H \in \mathcal{C}$, and for every $\psi \in S_H$, one may write $\chi|_H = \psi^{i_\chi}$, $i_\chi \in \{0, \dots, m_H - 1\}$, where m_H is the order of H . For every character $\chi \in G^*$, let

$$S_\chi = \{(H, \psi) : \chi|_H \neq \psi^{m_H - 1}\}.$$

Proposition 4.3. ([Par91, Corollary 4.1]) *Let $\pi : X \rightarrow Y$ be an abelian cover with Galois group G , with X and Y smooth. Assume the branch divisor D of π is normal crossing. Then,*

$$(\pi_* \mathcal{N}_\pi)^\chi \cong \bigoplus_{(H,\psi) \in S_\chi} \mathcal{O}_{D_{H,\psi}}(D_{H,\psi}) \otimes L_\chi^{-1}.$$

Corollary 4.4. *Let $\pi : X \rightarrow Y$ be as in Setup 3.1. If π is abelian, then (2.2) of Setup 3.1 is satisfied, i.e., $H^1(\mathcal{N}_\pi) = 0$.*

Proof. It follows from Proposition 4.3 and Remark 3.2, (4). □

We end this subsection by proving a result on deformations of abelian covers:

Proposition 4.5. *Let $\pi : X \rightarrow Y$ be an abelian cover with X and Y smooth, projective varieties with $H^1(\mathcal{O}_Y) = H^2(\mathcal{O}_Y) = H^1(D_i) = 0$ for all irreducible components D_i of the branch divisor D . Suppose $p : \mathcal{Y} \rightarrow T$ be a deformation of Y (p is proper, flat and surjective) over a smooth algebraic curve T . Then there exists a deformation $\Pi : \mathcal{X} \rightarrow \mathcal{Y} \rightarrow T$ of π over T .*

Proof. By [Par91, Theorem 2.1], if G is the Galois group of π , then π is completely determined by the line bundles L_χ , the divisors $D_{H,\psi}$ above, corresponding to a choice of a cyclic subgroup $H \subseteq G$ and a generator $\psi \in H^*$, and linear equivalent relations of the form

$$(4.2) \quad L_\chi + L_{\chi'} = L_{\chi\chi'} + \sum_H \sum_\psi \epsilon_{\chi,\chi'}^{H,\psi} D_{H,\psi}.$$

Now note that, since $H^2(\mathcal{O}_Y) = 0$, by [Ser06, Propositions 2.2.5 (iv), 2.3.6], all the line bundles L_χ lift to $\mathcal{Y} \rightarrow T$. Also since $H^1(D_i) = 0$, by [BGM21, Remark 2.10], every $D_{H,\psi}$ lifts to a divisor on $\mathcal{Y} \rightarrow T$. Now, since (4.2) is satisfied in the central fibre \mathcal{Y} , and Y is regular, the same linear equivalence relation holds on \mathcal{Y}_t , for a general $t \in T$ and, once again, by [Par91, Theorem 2.1] we have a family of abelian covers $\mathcal{X} \rightarrow \mathcal{Y} \rightarrow T$ extending π . \square

4.2. Simple dihedral covers. We will see now that conditions (2.1) and (2.2) of our Setup 3.1 also hold for simple dihedral covers. Dihedral covers were studied by Catanese and Perroni in [CP17]. We denote the dihedral group of order $2n'$ by $D_{n'}$ and we recall the following theorem.

Theorem 4.6. ([CP17, Theorem 6.1]). *Let Y be a smooth variety and let $n' \geq 3$ be an integer. Let $L \rightarrow Y$ be a line bundle with $s_1 \in H^0(L^{\otimes n'})$ and $s_2 \in H^0(L^{\otimes 2})$ such that*

- (1) *The zero locus of $s_1^2 - s_2^{n'}$ is smooth in the open set $s_2 \neq 0$, and*
- (2) *The divisors $\{s_1 = 0\}$ and $s_2 = 0\}$ intersect transversally.*

Define $X \subset L \oplus L$ to be the variety defined by $(u, v) \in L \oplus L$ that satisfies the following equations:

$$uv = s_2, \quad u^{n'} - 2s_1 + v^{n'} = 0.$$

Then the restriction to X of the fiber bundle projection $L \oplus L \rightarrow Y$ is a Galois cover with group $D_{n'}$ and branch divisor $\{s_2^{n'} - s_1^2 = 0\}$. Furthermore, if $\{s_1 = 0\} \cap \{s_2 = 0\} \neq \emptyset$, then X is irreducible.

It was shown in [CP17, Proposition 6.3] that the condition (2) in Theorem 4.6 is satisfied for a general choice of s_1 and s_2 . Furthermore, it is easy to see that the condition (2) of the theorem and the irreducibility of X can be simultaneously guaranteed if L is ample and globally generated.

Definition 4.7. A simple $D_{n'}$ cover of Y is the Galois cover $\pi : X \rightarrow Y$ with group $D_{n'}$ given as in Theorem 4.6 by the restriction to X of the fiber bundle projection $L \oplus L \rightarrow Y$.

Note that a simple $D_{n'}$ cover π has degree $n = 2n'$. Given a simple $D_{n'}$ cover π , the following formulas for the push-forward of the structure sheaf and for the canonical bundle hold ([CP17, §6.1]):

$$(4.3) \quad \pi_* \mathcal{O}_X = \bigoplus_{i=0}^{n'-1} [\mathcal{O}_Y(-iL) \oplus \mathcal{O}_Y(-(n'-i)L)], \quad K_X = \pi^*(K_Y(n'L)).$$

In particular, simple dihedral covers satisfy (2.1) of Setup 3.1.

Theorem 4.8. *Let $\pi : X \rightarrow Y$ be a simple $D_{n'}$ cover corresponding to a line bundle L and sections $s_1 \in H^0(L^{\otimes n'})$ and $s_2 \in H^0(L^{\otimes 2})$. Then we have the following exact sequence of sheaves on Y*

$$0 \rightarrow \pi_* \pi^*(L^{\otimes 2}) \rightarrow [\mathcal{O}_Y(2L) \oplus \mathcal{O}_Y(n'L)] \otimes \pi_* \mathcal{O}_X \rightarrow \pi_*(N_\pi) \rightarrow 0$$

Proof. Let $V = L \oplus L$. By abuse of notation, we use V to denote the total space of the vector bundle as well. We have the following exact sequences (see [CP17])

$$0 \rightarrow \pi_* T_X \rightarrow \pi_*(T_V|_X) \rightarrow [\mathcal{O}_Y(2L) \oplus \mathcal{O}_Y(n'L)] \otimes \pi_*(\mathcal{O}_X) \rightarrow 0$$

$$0 \rightarrow \pi_*(T_X) \rightarrow \pi_* \pi^*(T_Y) \rightarrow \pi_*(N_\pi) \rightarrow 0$$

$$0 \rightarrow \pi_*(T_{V/Y}|_X) \rightarrow \pi_*(T_V|_X) \rightarrow \pi_* \pi^* T_Y \rightarrow 0$$

The exact sequences above induce a commutative diagram with exact rows and columns.

$$\begin{array}{ccccccc}
& & & 0 & & & \\
& & & \downarrow & & & \\
& & & \pi_*(T_{V/Y}|_X) & & & \\
& & & \downarrow & & & \\
0 & \longrightarrow & \pi_* T_X & \longrightarrow & \pi_*(T_{V|X}) & \longrightarrow & (\mathcal{O}_Y(2L) \oplus \mathcal{O}_Y(n'L)) \otimes \pi_*(\mathcal{O}_X) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \pi_* T_X & \longrightarrow & \pi_* \pi^*(T_Y) & \longrightarrow & \pi_*(N_\pi) \longrightarrow 0 \\
& & & & \downarrow & & \\
& & & & 0 & &
\end{array}$$

Applying the snake lemma and noting that $(T_{V/Y}|_X) = \pi^*(L^{\oplus 2})$, we have our exact sequence. \square

Corollary 4.9. *In the situation of Setup 3.1 (1), let π be a smooth, simple dihedral cover of Y . Then $H^1(\mathcal{N}_\pi) = 0$, i.e., (2.2) of Setup 3.1 is satisfied.*

Proof. The corollary follows from the short exact sequence of Theorem 4.8, projection formula, (4.3), and Remark 3.2 (4). \square

4.3. Subcanonical covers. If π is simple cyclic (see Remark 6.2), an iteration of simple cyclic covers as in Proposition 5.4 (i), or simple dihedral (see (4.3)), the canonical bundle of X is a pullback of a line bundle of \mathbb{P}^N . This property generalizes subcanonical subvarieties. We collect here all these related definitions:

Definition 4.10. Let \hat{X} be a smooth projective variety and let $\hat{\varphi} : \hat{X} \rightarrow \mathbb{P}^N$ be a morphism. Let

$$L = \hat{\varphi}^* \mathcal{O}_{\mathbb{P}^N}(1) \text{ and } s \in \mathbb{Z}.$$

- (1) The polarized variety (\hat{X}, L) is said to be *s-subcanonical* if $K_{\hat{X}} = \hat{\varphi}^* \mathcal{O}_{\mathbb{P}^N}(s)$.
- (2) The morphism $\hat{\varphi}$ is called *s-subcanonical* if (\hat{X}, L) is *s-subcanonical* and $\hat{\varphi}$ is induced by the complete linear series $|L|$.
- (3) If \hat{X} is embedded in \mathbb{P}^N , then \hat{X} is an *s-subcanonical subvariety* if $K_{\hat{X}} = \mathcal{O}_{\mathbb{P}^N}(s)$.

Remark 4.11. We note that, an *s-subcanonical polarized scheme* (\hat{X}, L) is

- (1) a Fano polarized scheme of index $-s$, in the sense of Fujita (see [Fuj83, Definition 1.5]), if $s < 0$;
- (2) a polarized scheme of Calabi–Yau, if $s = 0$;
- (3) a canonically polarized scheme of general type, if $s = 1$;
- (4) an *s-subcanonical polarized scheme of general type*, if $s > 0$.

Proposition 3.4 and Theorems 3.5, 3.7 and 3.8, when applied to a subcanonical morphism φ , produce subcanonical morphisms, and, when applicable, subcanonical subvarieties:

Proposition 4.12. *In the situation of Setup 3.1, assume φ is *s-subcanonical*. Let $\Phi : \mathcal{X} \rightarrow \mathbb{P}_T^N$ be a flat family of deformations of $\varphi : X \rightarrow \mathbb{P}^N$ over a smooth variety $(T, 0)$. After shrinking T if necessary, the morphism Φ_t is *s-subcanonical*. In particular, if $\varphi' : X' \rightarrow \mathbb{P}^N$ is a general element of the algebraic formally semiuniversal deformation space of φ , then φ' is *s-subcanonical*. In particular, if Φ_t is an embedding, then $\Phi_t(\mathcal{X}_t)$ is an *s-subcanonical subvariety*.*

Proof. The result follows from Remark 3.10, the regularity of X (see Remark 3.2 (6)) and the fact that $R^2 p_* \mathbb{Z}_2$ is locally constant, where $p : \mathcal{X} \rightarrow T$ is the deformation of X induced by Φ . \square

5. NECESSARY AND SUFFICIENT CONDITIONS FOR NON-COMPLETE INTERSECTIONS

Theorem 3.7 gives a systematic way to construct subvarieties of projective space, starting from complete intersection subvarieties. It is therefore natural to ask whether the subvarieties so obtained are complete intersections or not. The answer is that both things may occur. Next Propositions 5.1 and 5.4 and Corollary 5.3 give sufficient conditions for each situation to happen. In either of the two cases, we show that, in the boundary of the irreducible Hilbert component of a subvariety produced by using Theorem 3.7, lie points that correspond to ropes. Note that a rope of multiplicity bigger than 2 is not a complete intersection, neither locally and, therefore, nor globally.

The following proposition gives a sufficient condition for a subvariety obtained from a deformation of φ to be a non-complete intersection.

Proposition 5.1. *In the situation of Setup 3.1, assume φ is s -subcanonical, Assume $\Phi : \mathcal{X} \rightarrow \mathbb{P}_T^N$ is a flat family of deformations of $\varphi : X \rightarrow \mathbb{P}^N$ over a smooth variety $(T, 0)$, such that $\Phi_t : \mathcal{X}_t \rightarrow \mathbb{P}^N$ is an embedding for $t \neq 0$. Suppose there are no integers $d'_1, d'_2, \dots, d'_r, d'_i > 1$ such that*

$$(a) \sum_{i=1}^r d'_i = s + N + 1, \text{ and } (b) \prod_{i=1}^r d'_i = n \prod_{i=1}^r d_i.$$

Then, shrinking T if necessary, $\Phi_t(\mathcal{X}_t)$ is not a complete intersection if $t \neq 0$.

Proof. It follows from Proposition 4.12 that, after shrinking T if necessary, $\Phi_t(\mathcal{X}_t)$ is s -subcanonical. Suppose now that $\Phi_t(\mathcal{X}_t)$, $t \neq 0$, is a complete intersection of multidegree $\underline{d} = (d'_1, d'_2, \dots, d'_r)$. Since, $K'_X = \Phi_t^* \mathcal{O}_{\mathbb{P}^N}(\sum_{i=1}^r d'_i - N - 1)$, we get that $\sum_{i=1}^r d'_i - N - 1 = s$, consequently (a) should hold. To see that (b) should also hold, notice that $n \prod_{i=1}^r d_i$ is the top self-intersection of the pullback, by φ to X , of the hyperplane section of \mathbb{P}^N and $\prod_{i=1}^r d'_i$ is the top self-intersection of the pullback, by Φ_t to \mathcal{X}_t , of the hyperplane section of \mathbb{P}^N , so they are equal. The proof is now complete. \square

5.1. Smooth subcanonical varieties of codimension two. We now show that codimension two examples produced by Theorem 3.7 (a) are always complete intersections. We prove this fact by means of the following lemma and its corollary. In what follows, as always, for a subvariety $j : Z \hookrightarrow \mathbb{P}^M$, we will have $\mathcal{O}_Z(k) = j^* \mathcal{O}_{\mathbb{P}^M}(k)$, and further we introduce the notation $H_*^i(\mathcal{F}) := \bigoplus_{v \in \mathbb{Z}} H^i(\mathcal{F}(v))$.

Lemma 5.2. *In the situation of Setup 3.1, assume the hypothesis of Theorem 3.7 (a) or (b) is satisfied. Let*

$$\Phi : \mathcal{X} \rightarrow \mathbb{P}_T^N$$

be a family over a smooth irreducible curve $(T, 0)$ satisfying the conditions (I), (II), and (III) of Theorem 3.7. Then, possibly after shrinking T , $H_^i(\mathcal{I}_{\mathcal{X}_t}) = 0$ for all $2 \leq i \leq m$ and $t \neq 0$, where $\mathcal{I}_{\mathcal{X}_t}$ is the ideal sheaf of \mathcal{X}_t inside \mathbb{P}_t^N .*

Proof. We have the short exact sequence

$$0 \rightarrow \mathcal{I}_{\mathcal{X}_t} \rightarrow \mathcal{O}_{\mathbb{P}_t^N} \rightarrow \mathcal{O}_{\mathcal{X}_t} \rightarrow 0.$$

Observe that

$$H^i(\varphi^* \mathcal{O}_{\mathbb{P}^N}(k)) \cong H^i(\pi^* \mathcal{O}_Y(k)) \cong H^i(\mathcal{O}_Y(k)) \oplus H^i(\mathcal{E}(k)) = 0$$

for all $1 \leq i \leq m - 1$, $k \in \mathbb{Z}$ since \mathcal{E} is split. The conclusion follows by semicontinuity and the fact that $H_*^i(\mathcal{O}_{\mathbb{P}_t^N}) = 0$ for all $2 \leq i \leq m$. \square

Corollary 5.3. *In the situation of Setup 3.1, assume $r = 2$ and $m \geq 4$. Assume the hypothesis of Theorem 3.7 (a) is satisfied, and let $(T, 0)$ be a smooth irreducible curve satisfying the conditions (I), (II), and (III) of Theorem 3.7. Then (shrinking T if necessary), $\Phi_t : \mathcal{X}_t \hookrightarrow \mathbb{P}_t^N$ embeds \mathcal{X}_t as a complete intersection, for $t \neq 0$.*

Proof. It follows from the Barth-Larsen theorem (see [Lar73], see also [Laz04, Corollary 3.2.3]) that any line bundle on \mathcal{X}_t extends to \mathbb{P}^N . Notice that $\text{codim}(\mathcal{X}_t/\mathbb{P}_t^N) = 2$. The following exact sequence follows from [OSS80, Theorem 5.1.1] ;

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_t^N} \rightarrow \mathcal{E}'_t \rightarrow \mathcal{I}_{\mathcal{X}_t}(l) \rightarrow 0,$$

where \mathcal{E}'_t is the rank 2 bundle associated to \mathcal{X}_t , and $\det(\mathcal{N}_{\mathcal{X}_t/\mathbb{P}_t^N}) = \mathcal{O}_{\mathbb{P}_t^N}(l)|_{\mathcal{X}_t}$. We also know from [OSS80, Lemma 5.2.1] that \mathcal{E}'_t is split if and only if \mathcal{X}_t is a complete intersection. To this end, we apply [KPR03, Theorem 1] (see also [Hol89] and [HS85]). We obtain from Lemma 5.2 that $H_*^i(\mathcal{I}_{\mathcal{X}_t}(k)) = 0$ for $2 \leq i \leq N-2$. Thus, $H_*^i(\mathcal{E}'_t) = 0$ for all $2 \leq i \leq N-2$, since $H_*^i(\mathcal{O}_{\mathbb{P}_t^N}) = 0$ when i is in the same range. \square

Proposition 5.4 below, which was inspired by conversations with Nori, shows that some of the ropes that appear in Theorem 3.7 (only the ones satisfying (i) and (ii) below) correspond to points that lie in an irreducible component of the Hilbert scheme whose general points correspond to smooth complete intersections. However, the general arguments below cannot determine if the general members of every one-parameter smoothing of those ropes are complete intersections (see Question 12.1).

Proposition 5.4. *Let X, Y, φ, π, n and \mathcal{E} be as in Setup 3.1. Assume that*

- (i) $\pi = \pi_1 \circ \dots \circ \pi_l$ is the composition of simple cyclic covers π_1, \dots, π_l such that, for each $1 \leq l' \leq l$, $\pi_{l'}$ is branched along the pull back by $\pi_{l'} \circ \dots \circ \pi_1$ of a divisor of $|\mathcal{O}_Y(n_{l'} \kappa_{l'})|$ ($n = n_1 \cdots n_l$);
- (ii) the unordered multidegree of Y is

$$\underline{d}_{\text{unord}} = (\kappa_1, \dots, \kappa_l, \beta_1, \dots, \beta_{r-l}).$$

Furthermore, assume the hypotheses of Theorem 3.7 are satisfied. Then:

- (1) A general member of the algebraic formally semiuniversal deformation of φ is an embedding whose image is a complete intersection of unordered multidegree

$$\underline{d}'_{\text{unord}} = (n_1 \kappa_1, \dots, n_l \kappa_l, \beta_1, \dots, \beta_{r-l}).$$

- (2) If $\tilde{Y} \hookrightarrow \mathbb{P}^N$ is an embedded rope on Y with conormal bundle \mathcal{E} and Φ is a flat family of morphisms satisfying (I), (II) and (III) of Theorem 3.5, then \tilde{Y} and, for any $t \neq 0$, $\Phi_t(\mathcal{X}_t)$, correspond to points of an irreducible component of the Hilbert scheme whose general point corresponds to a smooth, complete intersection subvariety of unordered multidegree \underline{d}' .

Proof. We do in detail the proof when π is simple cyclic, the general case follows from iterating the arguments used to prove the simple cyclic case.

Thus, let π be a simple cyclic cover branched along a (smooth) divisor of $|\mathcal{O}_Y(nk)|$, for some $k \in \mathbb{Z}$, $k > 0$. Recall that Y is a complete intersection $H_1 \cap \dots \cap H_r$ of multidegree

$$\underline{d}_{\text{unord}} = (k, \beta_1, \dots, \beta_{r-1})$$

and let k be the degree of H_1 . Let $Y' = H'_1 \cap H_2 \cap \dots \cap H_r$ be a smooth complete intersection of unordered multidegree

$$\underline{d}'_{\text{unord}} = (nk, \beta_1, \dots, \beta_{r-1}),$$

where H'_1 has degree nk . By letting H'_1 degenerate to n times H_1 , we obtain a smooth algebraic curve S , with a distinguished point $0 \in S$ and a flat family \mathcal{Y} of subschemes of \mathbb{P}^N over S , whose general member of \mathcal{Y} is a smooth complete intersection of unordered multidegree \underline{d}' and whose member at 0 is a primitive multiple structure \hat{Y} of multiplicity n , supported on Y . In fact, H'_1 can be chosen in such a way that, after base change, normalization and, if necessary, a linear automorphism of Y , we obtain a flat family Ψ of morphisms to \mathbb{P}^N , over an algebraic curve T' with a distinguished point $0 \in T'$, such that $\Psi_0 = \varphi$ and Ψ_t

is an embedding whose image is a smooth, complete intersection subvariety of unordered multidegree \underline{d}' . Then there is a point in the base Z of the algebraic formally semiuniversal deformation of φ (see Lemma 3.3) which corresponds to an embedding whose image is a smooth, complete intersection variety. Since being a complete intersection is an open property (see [Ser75]), then there is a non-empty open set U of Z consisting of points that correspond to embeddings (see Theorem 3.7 (2)), whose images are smooth, complete intersection subvarieties. This proves (1).

Recall that Z is irreducible (see Lemma 3.3). Hence there is a rational map ρ from Z to the Hilbert scheme. Now $\text{Im}(\rho)$ is irreducible and is hence contained inside a unique irreducible component H of the Hilbert scheme. Any rope \tilde{Y} as in the statement of (2) corresponds to a point in the closure of $\text{Im}(\rho)$ and is hence contained in H . Also, for any $t \neq 0$, any $\Phi_t(\mathcal{X}_t)$ as in the statement of (2) corresponds to a point in $\text{Im}(\rho)$, which is therefore a point of H . By part (1), $\text{Im}(\rho)$ contains at least one subvariety which is a complete intersection subvariety of multidegree \underline{d}' . Since, by [Ser75], being a complete intersection is an open property in the Hilbert scheme, a general point of H corresponds to a complete intersection subvariety of multidegree \underline{d}' . This proves (2). \square

Remark 5.5. To prove Proposition 5.4 (1) we use, among other things, an elementary construction from which we obtain the flat family Ψ . This construction is a particular case of the one in [CL19] (see [CL19, Definition 2.3, Theorem 2.4 and Remark 2.5]). M. Nori also showed us a similar example.

Question 5.6. Let \mathcal{E} be as in Proposition 5.4. If $n > 2$, there is one irreducible component of the Hilbert scheme containing points corresponding to two different kind of multiple structures, namely, ropes embedded in \mathbb{P}^N , supported on Y with conormal bundle \mathcal{E} on the one hand, and complete intersection multiple structures obtained by intersecting multiple hypersurfaces and smooth hypersurfaces on the other hand. In the case in which π is simple cyclic, then the latter multiple structures are primitive (and are like \hat{Y}). Thus, a natural question to ask is how, in the Hilbert scheme, the loci parameterizing each kind of multiple structure are related.

6. DEFORMING FINITE MORPHISMS TO CONSTRUCT SMALL CODIMENSION SUBVARIETIES

In this section we use the results proven in the previous sections to produce, in a systematic way, subvarieties of projective space, of infinitely many degrees, for any given codimension. In particular, we produce infinitely many subvarieties of small codimension in \mathbb{P}^N . In order to do so, we deform morphisms to \mathbb{P}^N , finite onto their image, to embeddings. More precisely, in this section we study the deformations of simple cyclic covers of complete intersections (in Sections 9 and 10 we extend this study to more general abelian covers and to dihedral covers). We first describe our setup, that we will follow in this and in Sections 7 and 8.

Setup 6.1. *In the situation of Setup 3.1, let $k \in \mathbb{Z}_{>0}$ and assume π is simple cyclic, branched along a smooth divisor in $|\mathcal{O}_Y(nk)|$. Consequently,*

$$\mathcal{E} = \mathcal{O}_Y(-k) \oplus \cdots \oplus \mathcal{O}_Y(-(n-1)k),$$

and, therefore, $k_i = ik$ for all $i = 1, \dots, n-1$.

We now make two remarks. The first one recalls that simple cyclic covers are subcanonical and tells what kind of variety X is, depending on Y and the cover. The second one shows in particular that, when φ is the canonical morphism, the geometric genus of X is bounded in terms of the dimension of X and the degree of π .

Remark 6.2. Assume Setup 6.1 and let $L = \pi^* \mathcal{O}_Y(1)$. Then (X, L) is subcanonical. Indeed, by adjunction and the ramification formula (see e.g. [BHPV04, Lemma I.17.1]), $K_X = \pi^* \mathcal{O}_Y(\delta + (n-1)k - N - 1)$. In particular, (X, L) is s -subcanonical if and only if

$$(6.1) \quad N + 1 + s = \delta + (n-1)k.$$

From Remark 4.11 we get the following facts:

- (1) X is Fano variety if and only if $\delta + (n-1)k \leq N$ (then, Y is also Fano). In this case (X, L) is a Fano polarized variety of index $-s$ if and only if $N + 1 + s = \delta + (n-1)k$.
- (2) X is a Calabi–Yau variety if and only if $\delta + (n-1)k = N + 1$ (in this case, Y is Fano).
- (3) X is a variety of general type if and only if $\delta + (n-1)k \geq N + 2$. The morphism φ (respectively (X, L)) is canonical (respectively, the canonical polarization) if and only if $\delta + (n-1)k = N + 2$ and $k \geq 2$ (resp. $\delta + (n-1)k = N + 2$); in this case, Y is Fano (resp. Y is Fano, unless $n = 2$ and $k = 1$, in which case Y is Calabi–Yau).

Remark 6.3. In the situation of Setup 6.1, the following happens.

- (1) If (X, L) is s -subcanonical, then $m + 1 \leq N \leq 2m + s + 1 - (n-1)k \leq 2m + s - n + 2$.
- (2) If φ is s -subcanonical, then $m + 1 \leq N \leq 2m + s - 2n + 3$. In particular, if φ is a canonical morphism, then $p_g(X) \leq 2m - 2n + 4$.

Remark 6.4. Although in this sections we will use Theorem 3.7 (a) only to produce s -subcanonical morphisms, we note it can also be used to produce morphisms induced by non-complete linear series, in the way shown in Examples 7.1 and 8.1.

Thus, from now on in this section we will only look at the cases in which φ is s -subcanonical, in particular, it is induced by a complete linear series. Recall that in these cases $k_1 \geq 2$, i.e., $k \geq 2$ (see Setups 3.1 and 6.1 for notations). In this subsection we apply Theorem 3.7 (a) when π is simple cyclic. As we will see, Theorem 3.7 (a) produces in a systematic way, for any given dimension m , $m \geq 3$ and any given codimension r , subvarieties of infinitely many different degrees. In particular, it produces subvarieties of small given codimension r and infinitely many different degrees.

First we study what invariants m, n and N in Setup 6.1 and s in Definition 4.10 satisfy the hypothesis of Theorem 3.7 (a).

Proposition 6.5. *In the situation of Setup 6.1, if the hypothesis of Theorem 3.7 (a) holds and φ is s -subcanonical, then,*

$$(6.2) \quad m + n - 1 \leq N \leq 2(m + n - 1) + s - (n-1)(n+2) + 1 = 2m - n(n-1) + s + 1,$$

so, in particular,

$$(6.3) \quad s \geq n^2 - m - 2.$$

In particular,

$$(6.4) \quad m + 1 \leq N \leq 2m + s - 1,$$

and, if $n \geq 3$, then

$$(6.5) \quad m + 2 \leq N \leq 2m + s - 5.$$

Further, if $s = (n-1)(n+2) - (m+n)$, then $N = m + n - 1$ and $\underline{d} = (2, 4, \dots, 2(n-1))$.

Proof. Assume the hypothesis of Theorem 3.7 (a) holds. Then, obviously $r = N - m \geq n - 1$, which gives the lower bound of N in (6.2). Under the assumption, the *unordered* multidegree of Y is the following;

$$(6.6) \quad \underline{d}_{\text{unord}} = (k, 2k, \dots, (n-1)k, \beta_1, \beta_2, \dots, \beta_{N-m-n+1}),$$

where each $\beta_i \geq 2$. Thus, $\delta \geq \frac{k(n-1)n}{2} + 2(N - m - n + 1)$. Since $\delta + k(n-1) = N + s + 1$, we get the following;

$$(6.7) \quad N + s + 1 \geq \frac{k(n-1)n}{2} + 2(N - m - n + 1) + k(n-1).$$

An elementary computation completes the proof of (6.2), since $k \geq 2$. If we set $n = 2$ in (6.2) (respectively $n = 3$), then (6.2) becomes (6.4) (respectively (6.5)). Consequently, (6.4) and (6.5) are obvious consequences of (6.2). The last assertion is a consequence of (6.2), (6.6) and (6.7). \square

Now we show that, for all set of invariants satisfying (6.2), there exist morphisms φ to which Theorem 3.7 (a) can be applied, therefore producing infinitely many subvarieties in the range $3 \leq m \leq N - 1$:

Theorem 6.6. *Given any integers n, m, s and N such that $m \geq 3$ and $n \geq 2$, if (6.2) holds, then there exist smooth varieties X' of dimension m and s -subcanonical embeddings $\varphi' : X' \rightarrow \mathbb{P}^N$ such that*

- (a) *the morphisms φ' are deformations of morphisms φ , where φ, m and N are as in Setup 6.1 and φ satisfies the hypothesis of Theorem 3.7 (a);*
- (b) *the subvarieties $\varphi'(X')$ are one-parameter deformations, as described in Theorem 3.7, of multiplicity n rope subschemes.*

Moreover, for any given integers n, m, N and $s \geq 1$ satisfying $m \geq 3, n \geq 2$ and (6.2), there are infinitely many non-isomorphic subvarieties $\varphi'(X')$ as above.

Proof. Let $m, n, s \in \mathbb{Z}$, with $m \geq 3, n \geq 2$ satisfying (6.2). For any integer k such that

$$2 \leq k \leq \frac{2(2(m+n) + s - N - 1)}{(n-1)(n+2)},$$

(e.g., for $k = 2$), then there are integers $\beta_1, \beta_2, \dots, \beta_{N-m-n+1} \geq 2$ satisfying the equation

$$\sum \beta_i + \frac{kn(n-1)}{2} + k(n-1) = N + s + 1.$$

For any such choices of β_i 's, let Y be a complete intersection in \mathbb{P}^N of multidegree

$$\underline{d}_{\text{unord}} = (k, 2k, \dots, (n-1)k, \beta_1, \dots, \beta_{N-m-n+1}).$$

Let $\pi : X \rightarrow Y$ be a simple cyclic cover branched along a smooth member of $|\mathcal{O}_Y(kn)|$. Then the corresponding morphism φ is s -subcanonical and satisfies the hypothesis of Theorem 3.7 (a). Thus, a general deformation of φ is an embedding.

We now show that there are infinitely many such non-isomorphic subvarieties $\varphi'(X')$. We first show that deformations of X are unobstructed. Note that since $H^1(\mathcal{O}_X(1)) = H^2(\mathcal{O}_X) = 0$, by [GGP10, Lemma 2.1], X is unobstructed if and only if φ is unobstructed. On the other hand since $H^1(N_\varphi) = 0$, deformations of φ are unobstructed and hence X is unobstructed. Now we show that $h^1(T_X) > 0$. This can be seen either by direct computation or as follows: the morphism φ is induced by the complete linear series of the s -subcanonical morphism of X . Further since $H^1(\mathcal{O}_X) = 0$, any deformation φ' of φ is once again the s -subcanonical morphism of the deformed variety X' . If every deformation of X is trivial then any deformation φ' of φ would once again be $n : 1$ onto its image, which is a contradiction to the fact that a general deformation of φ is an embedding. This also shows that the s -subcanonical map of a general deformation of X is an embedding. Now since X has ample canonical bundle, the unique irreducible moduli component of canonical models of varieties of general type containing X is locally isomorphic to quotient of the space of deformations of X with its group of automorphisms. Since X has $h^0(T_X) = 0$ by Nakano vanishing theorem, the quotient is a finite map. Now the moduli space parameterizes isomorphism classes of canonical models and is positive dimensional. Further since a general element X' of the moduli has an s -subcanonical map which is an embedding and $h^0(\mathcal{O}_{X'}(s)) = h^0(\mathcal{O}_X(s))$ for a general X' , we have infinitely many non-isomorphic subvarieties $\varphi'(X')$ as mentioned in the Theorem. \square

Remark 6.7. (1) For m, N, n fixed with $m \geq 3, n \geq 2, N \geq m + n - 1$, there exist smooth varieties X' of dimension m as in Theorem 6.6 belonging to infinitely many different moduli spaces. This is because we can choose the d_i 's and/or k arbitrarily large (then s also grows), so K_X^m also grows arbitrarily. These moduli spaces possess reduced and irreducible components with a locally closed locus that parametrizes smooth varieties with an s -subcanonical morphism, of degree n and finite onto its image, whereas the general points of the components correspond to smooth varieties with an s -subcanonical morphism which is an embedding. In Section 11 we will describe more specifically this phenomenon in the case of the canonical map, i.e., if $s = 1$ (see Corollary 11.1).

- (2) For m, n fixed with $m \geq 3, n \geq 2$ or, simply, for m fixed with $m \geq 3$, if we let s grow arbitrarily, then N , and, hence, p_g , can be chosen arbitrarily large. Therefore there exist smooth varieties X' of dimension m as in Theorem 6.6 with $p_g(X')$ arbitrarily large.

Theorem 6.6 yields this corollary:

Corollary 6.8. *Given any integers m and N such that $3 \leq m \leq N - 1$, there exist smooth varieties X' of dimension m and embeddings $\varphi' : X' \rightarrow \mathbb{P}^N$, with $\varphi'(X')$ having infinitely many different degrees, such that*

- (a) *the morphisms φ' are deformations of morphisms φ , where φ, m and N are as in Setup 6.1 and φ satisfies the hypothesis of Theorem 3.7 (a);*
- (b) *the subvarieties $\varphi'(X')$ are one-parameter deformations, as described in Theorem 3.7, of rope subschemes.*

More precisely:

- (1) If

$$(6.8) \quad s \geq N - 2m + 1,$$

then there exist s -subcanonical embeddings φ' as above and, for any such $s \geq 1$, there are infinitely many non-isomorphic subvarieties $\varphi'(X')$ as above.

- (2) If

$$(6.9) \quad m \leq N - 2 \text{ and } s \geq N - 2m + 5,$$

then the above ropes can be chosen to be of multiplicity greater than 2, and hence not a complete intersection.

Proof. Under our assumption that $m \leq N - 1$, (6.4) is equivalent to (6.8). As we saw in the proof of Proposition 6.5, if we set $n = 2$ in (6.2), then we obtain (6.4), so setting $n = 2$ in Theorem 6.6, we obtain morphisms φ' as required in (1). Furthermore, for any s satisfying (6.8), let δ satisfy (6.1). Then, as s goes to infinity, we have a sequence of values of δ that gives rise to a sequence of s -subcanonical morphisms φ as in Theorem 6.6 such that L^m goes to infinity, for $L = \varphi^* \mathcal{O}_{\mathbb{P}^N}(1)$. Finally, (6.5) is equivalent to (6.9). If we set $n = 3$ in (6.2), then we obtain (6.5), so setting $n = 3$ in Theorem 6.6, we obtain morphisms φ' as required in (2), since ropes of multiplicity $n \geq 3$ are locally not a complete intersection. \square

To demonstrate the power of Theorem 6.6 and Corollary 6.8, we construct some explicit examples of small codimension subvarieties of Corollary 6.8 (2). We will consider only examples in which $n \geq 3$ so all the subvarieties in them are one-parameter deformations of locally non-complete intersections and, therefore, non-complete intersections, embedded ropes.

Example 6.9. For fixed $m \geq 3$ and $n \geq 3$, we look at the subvarieties $\varphi'(X')$ with smallest possible s (that is, according to Proposition 6.5, $s = n^2 - m - 2$), constructed in Theorem 6.6. By Proposition 6.5, these subvarieties have codimension $r = n - 1$ and $k = 2$. Due to Corollary 5.3, the subvarieties $\varphi'(X')$ corresponding to the rows shaded with light blue color in the table below are complete intersections. For the white rows, the general deformation of φ is an embedding whose image is a complete intersection, due to Proposition 5.4. A priori it is not known if the same is true for special deformations (see Question 12.1), although, they should also be complete intersections according to Hartshorne's conjecture. Further, these are the lowest codimension examples of subvarieties which are smoothings of non-complete intersections ropes. The number of different invariants of such varieties $\varphi'(X')$ is infinite, so we will just list a few of them in the table below, precisely those of codimension 2 and 3 and $-2 \leq s \leq 2$.

m	n	k	N	s	\underline{d}	$\deg(\varphi'(X'))$	$K_{X'}^m$	$p_g(X')$
9	3	2	11	-2	(2, 4)	24	$-24 \cdot 2^9$	0
16	4	2	19	-2	(2, 4, 6)	192	$192 \cdot 2^{16}$	0
8	3	2	10	-1	(2, 4)	24	24	0
15	4	2	18	-1	(2, 4, 6)	192	-192	0
7	3	2	9	0	(2, 4)	24	0	1
14	4	2	17	0	(2, 4, 6)	192	0	1
6	3	2	8	1	(2, 4)	24	24	9
13	4	2	16	1	(2, 4, 6)	192	192	17
5	3	2	7	2	(2, 4)	24	$24 \cdot 2^5$	37
12	4	2	15	2	(2, 4, 6)	192	$192 \cdot 2^{12}$	137

Example 6.10. Theorem 6.6 allows also the construction of smooth subvarieties near the boundary of, but inside, the range of Hartshorne's conjecture, like the very small sample of the smooth subvarieties in the range $r = (1/3)N - 1$, displayed in Table 1 of the introduction, which are obtained by deforming φ , which factors through a simple cyclic cover π of degree n of Y , when π is branched along a smooth divisor in $|\mathcal{O}_Y(2n)|$. By Proposition 5.4, the image of the general deformation of φ is a complete intersection, although we do not know whether the same is true for some special deformations of φ (see Question 12.1).

7. DEFORMING FINITE MORPHISMS TO CONSTRUCT SMOOTH NON-COMPLETE INTERSECTION SUBVARIETIES

In this section we construct smooth, non-complete intersection subvarieties of projective space, by applying Theorem 3.7 (b) when π is a simple cyclic cover. As we will see, Theorem 3.7 (b) produces in a systematic way, for any given dimension m , $m \geq 3$ and any given codimension $r \geq m + 1$, smooth subvarieties of infinitely many different degrees. Most importantly, for any m and r such that $r \geq m + 1$, Theorem 3.7 (b) produces smooth, non-complete intersection subvarieties, embedded by complete linear series. First we study what invariants m, n and N in Setup 6.1 and s in Definition 4.10 satisfy the hypothesis of Theorem 3.7 (b).

Although, as already remarked, our main interest are morphisms induced by complete linear series, we give first an example of how Theorem 3.7 (b) yields embeddings induced by non-complete linear series:

Example 7.1. Set $k = 1$, and assume $N \leq \frac{nm+s}{n-1}$. Let Y be a complete intersection of a hypersurface of degree $\alpha = nm + N(1 - n) + 2 + s$ and $N - m - 1$ hypersurfaces of degree n . By assumption, $\alpha \geq 2$. If one wants to write the multidegree of Y that is consistent with the convention of Setup 3.1, it will be

$$\underline{d} = \begin{cases} \overbrace{(n, \dots, n, \alpha)}^{N-m-1}; & \text{if } N \leq \frac{nm+2+s-n}{n-1}, \\ \underbrace{(\alpha, n, \dots, n)}_{N-m-1}; & \text{if } N > \frac{nm+2+s-n}{n-1}. \end{cases}$$

Recall that $\pi : X \rightarrow Y$ is a simple cyclic cover of Y branched along a smooth divisor in $|\mathcal{O}_Y(n)|$ and φ is the corresponding morphism (which is induced by an incomplete linear series).

If one of the following conditions hold;

$$(7.1) \quad 2m + n - 1 \leq N \leq \min \left\{ \frac{nm+2+s}{n-1} - 1, 2m + s + 2 - n \right\} \text{ or}$$

$$(7.2) \quad 2m + n \leq N \leq 2m + s + 2 - n,$$

then, since Theorem 3.7 (b) applies, a general deformation of φ is an embedding.

From now on, in this section we look only at morphisms induced by complete linear series.

Proposition 7.2. *In the situation of Setup 6.1, if the hypothesis of Theorem 3.7 (b) holds and φ is s -subcanonical, then,*

$$(7.3) \quad 2m + n - 1 \leq N \leq 2(2m + n - 1) - 2(n - 1)(m + n) + s + 1,$$

so, in particular,

$$(7.4) \quad s \geq 2m(n - 2) + n(2n - 3).$$

In particular,

$$(7.5) \quad 2m + 1 \leq N \leq 2m + s - 1.$$

and, if $n \geq 3$, then

$$(7.6) \quad 2m + 2 \leq N \leq s - 7.$$

Further, if $s = 2m(n - 2) + n(2n - 3)$, then $N = 2m + n - 1$, $k = 2$, and Y has multidegree

$$\underbrace{(2(n - 1), \dots, 2(n - 1))}_{m+n-1}.$$

Proof. We just prove (7.3), the other assertions will easily follow from it. Assume the hypothesis of Theorem 3.7 (b) holds. Then, $2r \geq N + n - 1$ and consequently $N \geq 2m + n - 1$ since $r = N - m$. Now, under the assumption, δ will be least if the *unordered* multidegree is the following:

$$\underline{d}_{\text{unord}} = \underbrace{(k(n - 1), \dots, k(n - 1))}_{m+n-1}, \underbrace{(2, \dots, 2)}_{N-2m-n+1},$$

consequently, $\delta \geq k(n - 1)(m + n - 1) + 2(N - 2m - n + 1)$. Since $\delta + k(n - 1) = N + s + 1$, we get,

$$N + s + 1 \geq k(n - 1)(m + n) + 2(N - 2m - n + 1).$$

Since $k \geq 2$, (7.3) follows. \square

Now we see that, for all set of invariants satisfying (7.3), there exist morphisms φ to which Theorem 3.7 (b) can be applied, therefore producing infinitely many subvarieties in the range $3 \leq m < \frac{N}{2}$.

Theorem 7.3. *Given any integers n, m, s and N such that $m \geq 3$ and $n \geq 2$, if (7.3) holds, then there exist smooth varieties X' of dimension m and s -subcanonical embeddings $\varphi' : X' \rightarrow \mathbb{P}^N$ such that*

- (a) *the morphisms φ' are deformations of morphisms φ , where φ, m and N are as in Setup 6.1 and φ satisfies the hypothesis of Theorem 3.7 (b);*
- (b) *the subvarieties $\varphi'(X')$ are one-parameter deformations, as described in Theorem 3.7, of multiplicity n rope subschemes.*

Moreover, for any given integers n, m, N and $s \geq 1$ satisfying $m \geq 3$, $n \geq 2$ and (7.3), there are infinitely many non-isomorphic subvarieties $\varphi'(X')$ as above.

Proof. Let m, n, s be integers satisfying $m \geq 3$, $n \geq 2$ and (7.3). For any integer k such that

$$2 \leq k \leq \frac{2(2m + n) + s - N - 1}{(n - 1)(m + n)},$$

(e.g., $k = 2$), there exist integers $\beta_1, \beta_2, \dots, \beta_{N-2m-n+1} \geq 2$, and $\alpha_1, \dots, \alpha_{m+n-1} \geq k(n - 1)$ satisfying the equation

$$\sum \beta_i + \sum \alpha_j + k(n - 1) = N + s + 1.$$

For any such choices of α_i 's and β_j 's, let Y be a complete intersection in \mathbb{P}^N of multidegree

$$\underline{d}_{\text{unord}} = (\alpha_1, \dots, \alpha_{m+n-1}, \beta_1, \dots, \beta_{N-2m-n+1}).$$

Let $\pi : X \rightarrow Y$ be a simple cyclic cover branched along a smooth member of $|\mathcal{O}_Y(kn)|$. Then π is an s -subcanonical cover of degree n satisfying the hypothesis of Theorem 3.7 (b). Thus, a general deformation of φ is an embedding. \square

We omit the proof of the following which is analogous to the proof of Corollary 6.8.

Corollary 7.4. *Given any integers m and N such that $3 \leq m < \frac{N}{2}$ there exist smooth varieties X' of dimension m and embeddings $\varphi' : X' \rightarrow \mathbb{P}^N$, with $\varphi'(X')$ having infinitely many different degrees, such that*

- (a) *the morphisms φ' are deformations of morphisms φ , where φ , m and N are as in Setup 6.1 and φ satisfies the hypothesis of Theorem 3.7 (b);*
- (b) *the subvarieties $\varphi'(X')$ are one-parameter deformations, as described in Theorem 3.7, of rope subschemes.*

More precisely:

- (1) If

$$(7.7) \quad s \geq N - 2m + 1,$$

then there exist s -subcanonical embeddings φ' as above and, for any such $s \geq 1$, there are infinitely many non-isomorphic subvarieties $\varphi'(X')$ as above.

- (2) If

$$(7.8) \quad m \leq \frac{N}{2} - 1 \text{ and } s \geq N + 7,$$

then the above rope subschemes can be chosen not to be complete intersections.

Next thing we want to know is which ones among the subvarieties $\varphi'(X')$ of Theorem 7.3 are non-complete intersections. As we will see in Theorem 7.6, for each codimension in the range of Theorem 7.3, there will be infinitely many of them. Now, in the next example we start displaying invariants of the lowest degree, non-complete intersection threefolds obtained by deforming double, triple and quadruple simple cyclic covers.

Example 7.5. There exist s -subcanonical, non-complete intersection, smooth threefolds $\varphi'(X')$ in \mathbb{P}^N , embedded by complete linear series, where φ' is a deformation of φ , and φ , n and k are as in Setup 6.1, with the following invariants.

n	k	N	s	\underline{d}	$\deg(\varphi'(X'))$
2	2	7	6	(3, 3, 3, 3)	162
2	2	7	7	(3, 3, 3, 4)	216
3	2	8	15	(4, 4, 4, 4, 4)	3072
3	2	8	16	(4, 4, 4, 4, 5)	3840
4	2	9	32	(6, 6, 6, 6, 6, 6)	186624
4	2	9	33	(6, 6, 6, 6, 6, 7)	217728

The subvarieties $\varphi'(X')$ are not complete intersections because of Proposition 5.1.

In the next two theorems, for any pair (m, N) such that $3 \leq m < N/2$, we construct non-complete intersection, m -dimensional smooth subvarieties of \mathbb{P}^N of infinitely many different degrees by deforming suitable simple cyclic covers of suitable complete intersection subvarieties. In particular, both theorems provide subvarieties of dimension m in \mathbb{P}^N such that the ratio m/N goes to $1/2$ from below as m approaches infinity. The subvarieties constructed in Theorem 7.6 and the subvarieties constructed in Theorem 7.7 are different since, for instance, their degrees are different.

Theorem 7.6. *Let m and N be any integers such that $3 \leq m < \frac{N}{2}$ and let p be any odd prime integer.*

- (1) *There exist smooth, non-complete intersection subvarieties X'_m , of dimension m and degree $2p^{N-m}$, embedded in \mathbb{P}^N by a complete linear series.*

- (2) For any m, N, p as above, there are subvarieties X' as in (1) which are s -subcanonical, where $s = (p-1)N - pm + k - 1$, for any integer k such that $2 \leq k < p$.

Proof. Let Y be a complete intersection of multidegree

$$\underline{d} = \underbrace{(p, \dots, p)}_{N-m}$$

and let $i : Y \hookrightarrow \mathbb{P}^N$ be its embedding into \mathbb{P}^N . Let k be an integer such that $2 \leq k < p$ and let $\pi : X \rightarrow Y$ be a double cover of Y branched along a smooth member of $|\mathcal{O}_Y(2k)|$. Then the hypothesis of Theorem 3.7 (b) are satisfied, so $i \circ \pi$ can be deformed to an embedding. Let X' the embedded variety. The subvariety X' has degree $2p^{N-m}$ and

$$\omega_X = \mathcal{O}_X(p(N-m) + k - N - 1).$$

If X' were a complete intersection, of multidegree $(d'_1, d'_2, \dots, d'_{N-m})$, then Proposition 5.1 would imply

$$\sum d'_i = p(N-m) + k, \text{ and } \prod d'_i = 2p^{N-m}$$

The second equality implies either $d'_1 = 2, d'_2 = \dots = d'_{N-m-1} = p, d'_{N-m} = p^2$ or $d'_1 = \dots = d'_{N-m-1} = p, d'_{N-m} = 2p$. In the first case, $\sum d'_i = p(N-m) + k$ would be equivalent to $p(p-2) = k-2$, so either $k=2$ and $p=2$ or $k > 2$ and p divides $k-2$ and both contradict our hypothesis. In the second case, $\sum d'_i = p(N-m) + k$ would be equivalent to $p = k$, which again contradicts our hypothesis. Then X' is not a complete intersection. The claim about complete linear series follows from Remark 3.10. \square

Theorem 7.7. Let m and N be any integers such that $3 \leq m < \frac{N-1}{2}$.

- (1) There exist smooth, non-complete intersection subvarieties X'_m , of dimension m and degree $(N-2m+1)(2(N-2m))^{N-m}$, embedded in \mathbb{P}^N by a complete linear series.
- (2) A subvariety X' as in (1) is s -subcanonical, where $s = 2(N-m+1)(N-2m) - N - 1$.

Proof. Fix an integer $m \geq 3$. Let $n = N - 2m + 1$. Note that $n \geq 3$. Let Y_m be a complete intersection of multidegree

$$\underline{d} = \underbrace{(2(n-1), \dots, 2(n-1))}_{m+n-1}$$

inside \mathbb{P}^N , let $\pi_m : X_m \rightarrow Y_m$ be an n simple cyclic cover branched along a smooth member of $|\mathcal{O}_Y(2n)|$ and let $i_m : Y_m \hookrightarrow \mathbb{P}^N$ be the embedding of Y_m in \mathbb{P}^N . We know by Theorem 3.7 (b) that $i_m \circ \pi_m$ deforms to an embedding. Let the embedded variety be X'_m . Assume X'_m is a complete intersection of multidegree $(d'_1, d'_2, \dots, d'_{m+n-1})$. Then, by Proposition 5.1, we know that

$$\sum d'_i = 2(n-1)(m+n), \text{ and } \prod d'_i = n((2(n-1))^{m+n-1}).$$

By arithmetic mean–geometric mean inequality, we know that

$$\begin{aligned} \left(\frac{\sum d'_i}{m+n-1} \right)^{m+n-1} &\geq \prod d'_i \\ \implies \left(\frac{m+n}{m+n-1} \right)^{m+n-1} &= \left(1 + \frac{1}{m+n-1} \right)^{m+n-1} \geq n, \end{aligned}$$

which is a contradiction since $(1 + \frac{1}{m+n-1})^{m+n-1}$ is an increasing function of $m+n$ and the limit at infinity is e , but $n \geq 3$. The claim about complete linear series follows from Remark 3.10. \square

Remark 7.8. Arguing as in Remark 6.7 we conclude that:

- (1) Theorem 7.3 implies the existence, for fixed m, N, n with $m \geq 3, n \geq 2, N \geq 2m + n - 1$, of infinitely many different moduli spaces with the properties of the ones of Remark 6.7.
- (2) For m, n fixed with $m \geq 3, n \geq 2$ or, simply, for m fixed with $m \geq 3$, there exist smooth varieties of dimension m as in Theorem 7.3 with p_g arbitrarily large.

- (3) Theorem 7.6 implies, for fixed m, N with $3 \leq m < N/2$, the existence of non-complete intersection subvarieties of dimension m in \mathbb{P}^N belonging to infinitely many moduli spaces with the properties of the ones of Remark 6.7.
- (4) For m fixed with $m \geq 3$ there exist smooth non-complete intersection subvarieties of dimension m as in Theorem 7.6 and Theorem 7.7 with p_g arbitrarily large.

8. DEFORMING FINITE MORPHISMS TO NON-EMBEDDINGS

8.1. Varieties with birational morphisms. Now we study the cases for which Theorem 3.5 is applicable to simple cyclic covers π . Recall that, in this case φ can be deformed to a birational morphism φ' , which a priori, is not an embedding. We will show that these birational morphisms $\varphi' : X' \rightarrow \mathbb{P}^N$ exist for any N and for varieties X' of any dimension m , $3 \leq m \leq N-1$. Some of these morphisms φ' are strictly birational. Although we will mostly be interested in producing birational subcanonical morphisms, we start by showing an example of birational morphisms induced by a non-complete linear series:

Example 8.1. Set $k = 1$, and assume $N \leq \frac{nm+s}{n-1}$. Assume the notation of Example 7.1 regarding X, Y and π . Furthermore, assume that one of the following conditions hold;

$$(8.1) \quad \left\lfloor \frac{n}{2} \right\rfloor + m \leq N \leq \min \left\{ \frac{nm+2+s}{n-1} - 1, 2m+s+2-n \right\}, \text{ or,}$$

$$(8.2) \quad \left\lfloor \frac{n}{2} \right\rfloor + m + 1 \leq N \leq 2m+s+2-n.$$

Then the general deformation of φ is a morphism induced by a non-complete linear series, finite and birational onto its image. Indeed, (8.1) guarantees that $\alpha \geq n-1$ and (8.2) guarantees that $N-m-1 \geq \lfloor n/2 \rfloor$ so that Theorem 3.5 applies.

In the remaining of this subsection, we focus in producing birational morphisms induced by complete linear series. We start with a result that proves some numerical inequalities.

Proposition 8.2. *In the situation of Setup 6.1, assume φ is s -subcanonical. If the hypothesis of Theorem 3.5 holds, then,*

$$(8.3) \quad m + \lfloor n/2 \rfloor \leq N \leq 2(m + \lfloor n/2 \rfloor) - 2(n-1)(\lfloor n/2 \rfloor + 1) + s + 1 = 2m - 2((n+2)\lfloor n/2 \rfloor - n) + s + 3$$

and, consequently,

$$s \geq (2n+5)\lfloor n/2 \rfloor - 2n - m - 3.$$

In particular,

$$(8.4) \quad m + 1 \leq N \leq 2m + s - 1.$$

Proof. Assume the hypothesis of Theorem 3.5 holds. Then $r = N - m \geq \lfloor n/2 \rfloor$ and that gives the lower bound.

Now, under the assumption, δ will be least if the *unordered* multidegree is the following;

$$\underline{d}_{\text{unord}} = \underbrace{(k(n-1), \dots, k(n-1))}_{\lfloor n/2 \rfloor}, \beta_1, \dots, \beta_{N-m-\lfloor n/2 \rfloor},$$

where $\beta_i \geq 2$ for all i . Since $\delta + k(n-1) = N + s + 1$, we obtain;

$$N + s + 1 \geq \lfloor n/2 \rfloor k(n-1) + 2(N - m - \lfloor n/2 \rfloor) + k(n-1).$$

An elementary computation completes the proof since $k \geq 2$. □

Now we see that, for all set of invariants satisfying (8.3) and (8.4), there exist morphisms φ to which Theorem 3.5 can be applied, therefore producing infinitely many birational morphisms in the ranges (8.3) and (8.4).

Theorem 8.3. *Given any integers n, m, s and N such that $m \geq 3$ and $n \geq 2$ and (8.3) holds, then there exist smooth varieties X' of dimension m and s -subcanonical birational morphisms $\varphi' : X' \rightarrow \mathbb{P}^N$ which are deformations of morphisms φ , where φ, n, m and N are as in Setup 6.1 and φ satisfies the hypothesis of Theorem 3.5. In particular, there are X' and φ' as above for m, s and N satisfying (8.4).*

Proof. Assume $m \geq 3, n \geq 2$ and (8.3) holds. For any integer k such that

$$2 \leq k \leq \frac{2(m + \lfloor n/2 \rfloor) + s + 1 - N}{(\lfloor n/2 \rfloor + 1)(n - 1)},$$

(e.g., $k = 2$), there are integers $\beta_1, \beta_2, \dots, \beta_{N-m-\lfloor n/2 \rfloor} \geq 2$, and $\alpha_1, \dots, \alpha_{\lfloor n/2 \rfloor} \geq k(n-1)$ satisfying the following equation;

$$\sum \beta_i + \sum \alpha_i + k(n-1) = N + s + 1.$$

For any such choices of α_i 's, and β_j 's, let Y be a complete intersection in \mathbb{P}^N of multidegree

$$\underline{d}_{\text{unord}} = (\alpha_1, \dots, \alpha_{\lfloor n/2 \rfloor}, \beta_1, \dots, \beta_{N-m-\lfloor n/2 \rfloor}).$$

Let $\pi : X \rightarrow Y$ be a simple cyclic cover branched along a smooth member of $|\mathcal{O}_Y(kn)|$. Then φ is an s -subcanonical cover of degree n satisfying the hypothesis of Theorem 3.5. Thus, a general deformation of φ is birational onto its image. \square

Example 8.4. The following table describes the invariants of the first few varieties X' and birational morphisms φ' of Theorem 8.3, for $k = 2$ and $n = 3, 4$. Proposition 5.1 and Corollary 5.3 imply that, for light blue rows φ' is not an embedding while, for the white rows, φ' would not be an embedding if Hartshorne's conjecture is true.

m	n	k	N	s	\underline{d}	$K_{X'}^m$	$p_g(X')$
17	4	2	19	-1	(6, 7)	-168	0
17	4	2	20	-1	(2, 6, 6)	-288	0
16	4	2	18	0	(6, 7)	0	1
16	4	2	19	0	(2, 6, 6)	0	1
15	4	2	17	1	(6, 7)	168	18
15	4	2	18	1	(2, 6, 6)	288	19

Remark 8.5. Arguing as in Remark 6.7 we conclude that:

- (1) Theorem 8.3 implies the existence, for fixed m, N, n with $m \geq 3, n \geq 2, N \geq m + \lfloor n/2 \rfloor$, of infinitely many different moduli spaces having reduced and irreducible components with a locally closed locus that parametrizes smooth varieties with an s -subcanonical morphism, finite and of degree n onto its image, whereas the general points of the components correspond to smooth varieties with an s -subcanonical morphism which is a finite birational morphism onto its image. In Section 11 we will describe more specifically this phenomenon in the case of the canonical map, i.e., if $s = 1$ (see Corollary 11.2).
- (2) For m, n fixed with $m \geq 3, n \geq 2$ or, simply, for m fixed with $m \geq 3$, there exist smooth varieties of dimension m as in Theorem 8.3 with p_g arbitrarily large.

8.2. Degree n subcanonical morphisms whose deformations are of degree n . We now study the cases for which Proposition 3.4 is applicable to simple cyclic covers π . In these cases the degree of any deformation of φ in these cases remains unchanged.

Proposition 8.6. *In the situation of Setup 6.1, assume φ is s -subcanonical and the hypothesis of Proposition 3.4 is satisfied. Then,*

(1)

$$(8.5) \quad \max \left\{ \frac{s + 1 + m(k-1) - k(n-1)}{k-2}, m+1 \right\} \leq N \leq 2m + s + 1 - k(n-1)$$

and

$$3 \leq k \leq \frac{m+s}{n-1}.$$

(2) If $k = 3$, then $N = 2m + s + 1 - 3(n - 1)$ and $\underline{d} = (2, \dots, 2)$.

Proof. Since $\delta + k(n - 1) = N + s + 1$ and $\delta \geq 2(N - m)$, the upper bound follows.

Since $\underline{d} = (d_1, \dots, d_r)$ and $d_i \leq k - 1$, it follows that $\delta \leq (N - m)(k - 1)$. Consequently,

$$N + s + 1 \leq (N - m)(k - 1) + k(n - 1).$$

An easy computation completes the proof of (1), and (2) is a consequence of (1). \square

Remark 8.7. Let

$$\kappa = \left\lfloor \frac{m+s}{n-1} \right\rfloor.$$

The function on k , $k \geq 3$,

$$\frac{k(m-n+1) + s - m + 1}{k-2}$$

is strictly decreasing. Then, with the hypothesis of Proposition 8.6, (8.5) implies

$$(8.6) \quad \max \left\{ \frac{\kappa(m-n+1) + s - m + 1}{\kappa - 2}, m + 1 \right\} \leq N \leq 2m + s - 3n + 4;$$

in particular

$$s \geq 3n - m - 3.$$

We show now that, for invariants satisfying (8.5) and (8.6), there are morphisms φ satisfying the hypothesis of Proposition 3.4. Thus, the degree of these morphisms φ remains constant under deformation.

Proposition 8.8. *Let k, m, n, s, N be integers satisfying $k \geq 3$, $m \geq 3$ and $n \geq 2$ such that (8.5) holds or let m, n, s, N be integers satisfying $m \geq 3$ and $n \geq 2$ such that (8.6) holds. Then there are s -subcanonical morphisms φ , where φ, n, m and N are as in Setup 6.1, all whose deformations have degree n onto their image.*

Proof. Let m, n, s, N integers satisfying $m \geq 3$ and $n \geq 2$ such that (8.6) holds. Then we can choose an integer k , with $3 \leq k \leq \kappa$, such that k, m, n, s, N satisfy (8.5). Then, there are integers $\beta_r \geq \dots \geq \beta_1 \geq 2$ such that $\beta_i \leq k - 1$ and $\sum \beta_i + k(n - 1) = N + s + 1$. Let Y be of multidegree $\underline{d} = (\beta_1, \dots, \beta_r)$ inside \mathbb{P}^N . Let $\pi : X \rightarrow Y$ be a simple cyclic n cover branched along a smooth divisor in $|\mathcal{O}_Y(nk)|$. Then φ is s -subcanonical and satisfies the hypothesis of Proposition 3.4, consequently any deformation of φ has degree n . \square

9. DEFORMATIONS OF FINITE MORPHISMS: THE CASE OF $\mathbb{Z}_{n/2} \times \mathbb{Z}_2$

In this section, we shift the focus to the study of deformations of morphisms which factor through iterations of simple cyclic covers. More precisely, we will deform $\mathbb{Z}_{n/2} \times \mathbb{Z}_2$ covers to produce smooth subvarieties embedded by complete linear series. Even though we will only explicitly exhibit the invariants of subvarieties of small codimension, using Theorems 3.5 and 3.7 as we did in Sections 6, 7 and 8, one can easily produce

- (1) for any m, N such that $3 \leq m \leq N - 1$, m -dimensional smooth subvarieties in \mathbb{P}^N of infinitely many different degrees (see Theorem 9.4);
- (2) smooth subvarieties in \mathbb{P}^N of the dimension m in the range $3 \leq m < N/2$ which are not complete intersections;
- (3) smooth varieties equipped with birational subcanonical morphisms which are not embeddings;
- (4) smooth varieties equipped with birational subcanonical morphisms, which would not be complete intersections if they were embeddings.

Furthermore, applying Theorem 3.8 we will show the existence of $\mathbb{Z}_{n/2} \times \mathbb{Z}_2$ covers whose degree under deformation drops to half. We remark that we have not applied Theorem 3.8 in the previous sections devoted to simple cyclic covers. Although in this section we only deal with $\mathbb{Z}_{n/2} \times \mathbb{Z}_2$ covers, our arguments show how to proceed for more general iterations of simple cyclic covers and also for simple cyclic covers of even degree.

We describe now our setup for this section.

Setup 9.1. Let X, Y, m, n and π be as in Setup 3.1 (1). Let $p_1 : X_1 \rightarrow Y$ be a double cover branched along a smooth divisor D_2 in $|\mathcal{O}_Y(2l)|$ for some $l \in \mathbb{Z}_{>0}$. Let n be an even integer, $n \geq 4$, and let $p_2 : X_2 \rightarrow Y$ be a simple cyclic cover of degree $n/2$, branched along a smooth divisor D_1 in $|\mathcal{O}_Y(\frac{n}{2}k)|$ for some $k \in \mathbb{Z}_{>0}$. Assume D_1 and D_2 intersect transversally. Assume $X := X_1 \times_Y X_2$ and D_1 and D_2 intersect transversally (therefore, X is smooth) and assume $\pi : X \rightarrow Y$ is the natural morphism from the fiber product to Y .

Remark 9.2. In the situation of Setup 9.1, $K_X = \pi^* \mathcal{O}_Y(-N - 1 + \delta + l + k(\frac{n}{2} - 1))$. Consequently, (X, L) is s -subcanonical if and only if $\delta + l + (\frac{n}{2} - 1)k = N + s + 1$. From Remark 4.11, we get the following facts;

- (1) X is a Fano variety if and only if $\delta + l + (\frac{n}{2} - 1)k \leq N$ (in this case, Y is also Fano). (X, L) is a Fano polarized variety of index $-s$ if and only if $N + 1 + s = \delta + l + (\frac{n}{2} - 1)k$.
- (2) X is a Calabi–Yau variety if and only if $N + 1 = \delta + l + (\frac{n}{2} - 1)k$ (in this case, Y is Fano).
- (3) X is a variety of general type if and only if $\delta + l + (\frac{n}{2} - 1)k \geq N + 2$. The morphism φ (respectively (X, L)) is:
 - (a) Canonical if and only if $\delta + l + (n - 1)k = N + 2$ and $k, l \geq 2$ (resp. $\delta + l + (\frac{n}{2} - 1)k = N + 2$); in this case Y is Fano.
 - (b) Subcanonical if and only if $\delta + l + (\frac{n}{2} - 1)k \geq N + 3$ and $k, l \geq 2$ (resp. $\delta + l + (\frac{n}{2} - 1)k \geq N + 3$).

9.1. Subvarieties with small codimension. We study the cases in which Theorem 3.7 (a) applies. The following proposition, whose proof we omit, is the analogue of Proposition 6.5.

Proposition 9.3. In the situation of Setup 9.1, if φ is s -subcanonical and the hypothesis of Theorem 3.7 (a) holds, then,

$$(9.1) \quad m + n - 1 \leq N \leq 2(m + n - 1) - n(n/2 + 1) + s + 1 = 2m - n(n/2 - 1) + s - 1$$

and, therefore,

$$s \geq \frac{n^2}{2} - m.$$

Further, if $s = n^2/2 - m$, then $N = m + n - 1$ and the unordered multidegree of Y is

$$\underline{d}_{\text{unord}} = (2, 4, \dots, n - 2, 2, 4, \dots, n).$$

Theorem 9.4. Given any integers n, m, s and N such that $m \geq 3$, n even, and $n \geq 4$, if (9.1) holds, then there exist smooth varieties X' of dimension m and s -subcanonical embeddings $\varphi' : X' \rightarrow \mathbb{P}^N$ such that

- (a) the morphisms φ' are deformations of morphisms φ , where φ, m and N are as in Setup 6.1 and φ satisfies the hypothesis of Theorem 3.7 (a);
- (b) the subvarieties $\varphi'(X')$ are one-parameter deformations, as described in Theorem 3.7, of multiplicity n rope subschemes.

Moreover, for any given integers n, m, N and $s \geq 1$ satisfying $m \geq 3$, n even, $n \geq 4$ and (9.1), there are infinitely many non-isomorphic subvarieties $\varphi'(X')$ as above.

Proof. Let $m, n, s \in \mathbb{Z}$, with $m \geq 3$, n even, $n \geq 4$ satisfying (9.1). For any integer $k, l \geq 2$ such that

$$k(n^2/4 - 1) + l(n/2) \leq 2(m + n) + s - N - 1,$$

then there are integers $\beta_1, \beta_2, \dots, \beta_{N-m-n+1} \geq 2$ satisfying the equation

$$\sum \beta_i + k \left(\frac{n^2}{4} - 1 \right) + \frac{n}{2} l = N + s + 1.$$

For any such choices of β_i 's, let Y be a complete intersection in \mathbb{P}^N of multidegree

$$\underline{d}_{\text{unord}} = (k, 2k, \dots, (n/2 - 1)k, l, l + k, l + 2k, \dots, l + k(n/2 - 1), \beta_1, \dots, \beta_{N-m-n+1}).$$

Let $\pi : X \rightarrow Y$. Let $\varphi : X \rightarrow Y$ be the natural morphism from the fiber product $X := X_1 \times_Y X_2$ of a double cover $p_1 : X_1 \rightarrow Y$ branched along a smooth member D_2 of $|\mathcal{O}_Y(2l)|$, and a simple cyclic cover $p_2 : X_2 \rightarrow Y$ of degree $n/2$, branched along a smooth member D_1 of $|\mathcal{O}_Y((nk/2)|$ such that D_1 and D_2 intersect transversally. Then X is smooth and φ is an s -subcanonical cover of degree n satisfying the hypothesis of Theorem 3.7 (a). Thus, a general deformation of φ is an embedding. \square

Corollary 9.5. *Given any integers m and N such that $3 \leq m \leq N - 3$, there exist smooth varieties X' of dimension m and embeddings $\varphi' : X' \rightarrow \mathbb{P}^N$, with $\varphi'(X')$ having infinitely many different degrees, such that*

- (a) *the morphisms φ' are deformations of morphisms φ , where φ, m and N are as in Setup 9.1, φ is a $\mathbb{Z}_{n/2} \times \mathbb{Z}_2$ cover with even $n \geq 4$ and it satisfies the hypothesis of Theorem 3.7 (a);*
- (b) *the subvarieties $\varphi'(X')$ are one-parameter deformations, as described in Theorem 3.7, of rope subschemes.*

More precisely, in the above cases we have $s \geq N - 2m + 5$ and the above rope subschemes can be chosen to be non-complete intersections.

9.2. Varieties with degree $n/2$ subcanonical morphisms. Finally, we study the cases for which Theorem 3.8 applies. This is a new case that did not appear in the previous section.

Proposition 9.6. *In the situation of Setup 9.1, let*

$$\kappa = \left\lfloor \frac{2(m+s-1)}{n-2} \right\rfloor \text{ and } \kappa' = \left\lfloor \frac{2(m+s-2)}{n-2} \right\rfloor,$$

let φ be s -subcanonical and assume hypothesis (1) and (2) of Theorem 3.8 are satisfied.

- (1) *If $l = 1$, then*

$$(9.2) \quad \max \left\{ m + 1, \frac{(m - n/2 + 1)\kappa + s - m}{\kappa - 2} \right\} \leq N \leq 2m - 3n/2 + s + 3;$$

in particular,

$$(9.3) \quad s \geq 3n/2 - m - 2.$$

- (2) *If hypothesis (1') of Theorem 3.8 is satisfied, then*

$$(9.4) \quad \max \left\{ m + 1, \frac{(m - n/2 + 2)\kappa' + s - m + 1}{\kappa' - 2} \right\} \leq N \leq 2m - 5n/2 + s + 4;$$

in particular,

$$(9.5) \quad s \geq 5n/2 - m - 3.$$

- (3) *If $l \geq 2$, then*

$$(9.6) \quad \max \left\{ m + 1, \frac{(m - n/2 + 1/2)\kappa' + s - m + 3/2}{\kappa' - 2} \right\} \leq N \leq 2m - 5n/2 + s + 4;$$

in particular,

$$(9.7) \quad s \geq 5n/2 - m - 3.$$

Proof. We prove (1) first, so we assume $l = 1$. By Remark 9.2 and Theorem 3.8 (2),

$$(9.8) \quad \max\left\{m+1, \frac{(m-n/2+1)k+s-m}{k-2}\right\} \leq N \leq 2m+s-(n/2-1)k.$$

The function

$$\frac{(m-n/2+1)k+s-m}{k-2}$$

on $k, k \geq 3$, is strictly decreasing. Since (9.8) implies

$$k \leq \frac{2(m+s-1)}{n-2},$$

the minimum value of

$$\frac{(m-n/2+1)k+s-m}{k-2}$$

is attained at $k = \kappa$. This proves (9.2) and (9.2) implies (9.3).

Now we prove (2). Recall that hypothesis (1') of Theorem 3.8 is satisfied and, in particular, $l \geq 2$. By Remark 9.2,

$$(9.9) \quad \max\left\{m+1, \frac{(m-n/2+2)k-2l+s-m}{k-2}\right\} \leq N \leq 2m+s-(n/2-1)k-2l+3,$$

and, by hypothesis (2) of Theorem 3.8,

$$(9.10) \quad \max\left\{m+1, \frac{(m-n/2+1)k+s-m+1}{k-2}\right\} \leq N \leq 2m+s-(n/2-1)k-1.$$

The function

$$\frac{(m-n/2+1)k+s-m+1}{k-2}$$

on $k, k \geq 3$, is strictly decreasing. Since (9.10) implies

$$k \leq \frac{2(m+s-2)}{n-2},$$

the minimum value of

$$\frac{(m-n/2+1)k+s-m+1}{k-2}$$

is attained at $k = \kappa'$. This proves (9.4) and (9.4) implies (9.5).

Finally we prove (3), so we now assume $l \geq 2$. Then

$$\max\left\{m+1, \frac{(m-n/2+1)k-l+s-m+1}{k-2}\right\} \leq N \leq 2m+s-(n/2-1)k-2l+3,$$

and, by hypothesis (2) of Theorem 3.8,

$$\max\left\{m+1, \frac{(m-n/2+1/2)k+s-m+3/2}{k-2}\right\} \leq N \leq 2m+s-(n/2-1)k-1.$$

The function

$$\frac{(m-n/2+1/2)k+s-m+3/2}{k-2}$$

on $k, k \geq 3$, is strictly decreasing and, arguing as in the proof of (2), we see that its minimum is attained at $k = \kappa'$. Then (9.6) holds and (9.6) implies (9.7). \square

Remark 9.7. (1) If

$$(9.11) \quad \kappa \geq \frac{2(m+s+2)}{n}$$

(this happens for example if $s \geq n(n/4+1) - m - 2$), then

$$\frac{(m-n/2+1)\kappa + s - m}{\kappa - 2} \leq m + 1,$$

so (9.2) becomes

$$(9.12) \quad m + 1 \leq N \leq 2m - 3n/2 + s + 3$$

in this case.

(2) If

$$(9.13) \quad \kappa' \geq \frac{2(m+s+3)}{n}$$

(this happens for example if $s \geq n(n/4+2) - m - 3/2$), then

$$\frac{(m-n/2+1)\kappa' + s - m + 1}{\kappa' - 2} \leq m + 1,$$

so (9.4) becomes

$$(9.14) \quad m + 1 \leq N \leq 2m - 5n/2 + s + 4$$

in this case.

Theorem 9.8. *Let m, n, s, N be integers, with $m \geq 3$ and $n \geq 4$, even.*

- (1) *If (9.2) holds, or if (9.11) and (9.12) hold, then there exist smooth varieties X of dimension m and s -subcanonical morphisms $\varphi : X \rightarrow \mathbb{P}^N$ such that a general deformation of φ is a morphism which is finite and of degree $n/2$ onto its image.*
- (2) *If (9.4) holds, or if (9.13) and (9.14) hold, then there exist smooth varieties X of dimension m and s -subcanonical morphisms $\varphi : X \rightarrow \mathbb{P}^N$ such that a general deformation of φ is a flat morphism which is finite and of degree $n/2$ onto its image, which is smooth.*

Proof. We first set $l = 1$. For any integers k, N that satisfy $k \geq 3$ and (9.8), there are integers

$$2 \leq \beta_1, \beta_2, \dots, \beta_{N-m} \leq k - 1$$

such that

$$\sum \beta_i + 1 + k(n/2 - 1) = N + s + 1.$$

Then, for any such choices of β_i 's, let Y be a smooth complete intersection of multidegree

$$\underline{d}_{\text{unord}} = (\beta_1, \dots, \beta_{N-m}).$$

Let $\varphi : X \rightarrow Y$ be the morphism from the fiber product $X := X_1 \times_Y X_2$ of a double cover $p_1 : X_1 \rightarrow Y$ branched along a smooth member D_2 of $|\mathcal{O}_Y(2)|$, and a simple cyclic cover $p_2 : X_2 \rightarrow Y$ of degree $n/2$, branched along a smooth member D_1 of $|\mathcal{O}_Y(nk/2)|$, such that D_1 and D_2 intersect transversally. Then X is smooth, φ is an s -subcanonical cover of degree $n/2$ satisfying the hypothesis (1) and (2) of Theorem 3.8. Thus, a general deformation of φ is a morphism which is finite and of degree $n/2$ onto its image. Therefore, if k, N are integers that satisfy $k \geq 3$ and (9.8), then there exist smooth varieties X of dimension m and s -subcanonical morphisms $\varphi : X \rightarrow \mathbb{P}^N$ such that a general deformation of φ is a morphism, which is finite and of degree $n/2$ onto its image. This implies (1).

Now let us prove (2). For any integers k, l, N that satisfy $l \geq 2$, $k \geq 2l + 1$ and (9.9), there exist integers $2 \leq \beta_1, \beta_2, \dots, \beta_{N-m-1} \leq k - 1$ such that

$$\sum \beta_i + 2l + k(n/2 - 1) = N + s + 1.$$

Then, for any such choices of β_i 's, let Y be a smooth complete intersection of multidegree

$$\underline{d}_{\text{unord}} = (l, \beta_1, \dots, \beta_{N-m-1}).$$

Let now $\varphi : X \rightarrow Y$ be a morphism constructed as in the previous paragraph, except that this time D_2 is a smooth member of $|\mathcal{O}_Y(2l)|$. Then X is smooth, φ is an s -subcanonical cover of degree $n/2$ satisfying the hypothesis (1), (2) and (1') of Theorem 3.8. Thus, a general deformation of φ is a flat morphism, which is finite and degree $n/2$ onto a smooth image. Therefore, if k, l, N are integers that satisfy $l \geq 2$, $k \geq 2l+1$ and (9.9), then there exist smooth varieties X of dimension m and s -subcanonical morphisms $\varphi : X \rightarrow \mathbb{P}^N$ such that a general deformation of φ is a flat morphism, which is finite and degree $n/2$ onto a smooth image. This implies (2). \square

Example 9.9. The following table describes the first few varieties with $l = 2$ and $k = 5$, illustrating Theorem 9.8.

m	n	k	l	N	s	\underline{d}	K_X^m	$p_g(X)$
9	4	5	2	11	-1	(2,2)	-16	0
14	6	5	2	16	-1	(2,2)	-24	0
8	4	5	2	10	0	(2,2)	0	1
13	6	5	2	15	0	(2,2)	0	1
7	4	5	2	9	1	(2,2)	16	10
12	6	5	2	14	1	(2,2)	24	15

Remark 9.10. Arguing as in Remark 6.7 we conclude that:

- (1) Theorem 9.8 implies, for fixed m, N, n with $m \geq 3, N \geq m+1, n \geq 4, n$ even, the existence of infinitely many different moduli spaces having reduced and irreducible components with a locally closed locus that parametrizes smooth varieties with an s -subcanonical morphism, finite and of degree n onto a smooth image, whereas the general points of the components correspond to smooth varieties with an s -subcanonical morphism which is a morphism, which is finite and of degree $n/2$ onto its image, that can even be smooth. In Section 11 we will describe more specifically this phenomenon in the case of the canonical map, i.e., if $s = 1$ (see Corollary 11.3).
- (2) For m, n fixed with $m \geq 3, n \geq 4, n$ even or, simply, for m fixed with $m \geq 3$, there exist smooth varieties of dimension m as in Theorem 9.8 with p_g arbitrarily large.

10. DEFORMATIONS OF FINITE MORPHISMS: DIHEDRAL COVER CASE

In this section, we look at *non-abelian covers*, specifically, at simple dihedral covers. With the aid of the results of Section 3, we will deform simple dihedral covers to construct small codimensional subvarieties embedded by complete linear series inside projective space. As in the case of $\mathbb{Z}_{n/2} \times \mathbb{Z}_2$ covers, we will only explicitly exhibit small codimensional subvarieties. However, using the results of Section 3, one can produce m -dimensional smooth subvarieties in \mathbb{P}^N of infinitely many different degrees for any m, N such that $3 \leq m \leq N-1$; smooth, non-complete intersection, m -dimensional subvarieties in \mathbb{P}^N , in the range $3 \leq m < N/2$; smooth varieties equipped with birational subcanonical morphisms which are not embeddings; smooth varieties equipped with birational subcanonical morphisms, which would not be complete intersections if they were embeddings; and simple dihedral covers whose degree under deformation drops to half (by applying Theorem 3.8 to the factorization that the index 2, cyclic subgroup of $D_{n/2}$ induces on a $D_{n/2}$ cover).

Now we describe our setup for the simple dihedral covers that we will deform.

Setup 10.1. Let X, Y, π, n and m be as in Setup 3.1 (1). Let n be even, $n \geq 6$ and let $\pi : X \rightarrow Y$ be a smooth irreducible simple $D_{n/2}$ cover associated to sections $s_1 \in H^0(\mathcal{O}_Y(\frac{n}{2}k))$ and $s_2 \in H^0(\mathcal{O}_Y(2k))$ such that $s_1^2 -$

$s_2^{n/2}$ is smooth in the zero locus of $s_2 \neq 0$ and s_1 and s_2 intersect transversally (all this happens for instance for general choices of s_1 and s_2).

Remark 10.2. In the situation of Setup 10.1, $K_X = \pi^* \mathcal{O}_Y(-N-1 + \delta + \frac{n}{2}k)$. Consequently, (X, L) is s -subcanonical if and only if $\delta + \frac{n}{2}k = N + s + 1$. We also make a note of the following facts;

- (1) The variety X is a Fano variety if and only if $\delta + \frac{n}{2}k \leq N$ (in this case, Y is also Fano). If $\delta + \frac{n}{2}k \leq N$, then (X, L) is a Fano polarized variety of index $-s$ if and only if $N + 1 + s = \delta + \frac{n}{2}k$.
- (2) The variety X is a Calabi–Yau variety if and only if $N + 1 = \delta + \frac{n}{2}k$ (in this case, Y is Fano).
- (3) The variety X is a variety of general type if and only if $\delta + \frac{n}{2}k \geq N + 2$. Moreover The morphism φ (respectively (X, L)) is canonical if and only if $\delta + \frac{n}{2}k = N + 2$ and $k \geq 2$ (resp. $\delta + \frac{n}{2}k = N + 2$); in this case Y is Fano.

We study the cases in which Theorem 3.7 (a) applies, and as before, we omit the proof of the following

Proposition 10.3. *In the situation of Setup 10.1, assume φ is s -subcanonical. If the hypothesis of Theorem 3.7 (a) holds, then,*

$$(10.1) \quad m + n - 1 \leq N \leq 2(m + n - 1) - n(n/2 + 1) + s + 1 = 2m - n(n/2 - 1) + s - 1,$$

so, in particular,

$$s \geq n^2/2 - m.$$

Further, if $s = n^2/2 - m$, then $N = m + n - 1$ and the unordered multidegree of Y is

$$\underline{d}_{\text{unord}} = (2, 2, 4, 4, \dots, 2(n/2 - 1), 2(n/2 - 1), n).$$

In the next theorem we show the existence of smooth subvarieties $\varphi'(X')$ obtained by deforming s -subcanonical morphisms φ for which the inequalities of Proposition 10.3 hold.

Theorem 10.4. *Given any integers n, m, s and N such that $m \geq 3$, $n \geq 6$ even, and (10.1) holds, there exist smooth varieties X' of dimension m and s -subcanonical embeddings $\varphi' : X' \rightarrow \mathbb{P}^N$ such that*

- (a) *the morphisms φ' are deformations of morphisms φ , where φ, m and N are as in Setup 10.1 and φ satisfies the hypothesis of Theorem 3.7 (a);*
- (b) *the subvarieties $\varphi'(X')$ are one-parameter deformations, as described in Theorem 3.7, of multiplicity n rope subschemes.*

Moreover, for any given integers n, m, N and $s \geq 1$ satisfying $m \geq 3$, $n \geq 6$ even, satisfying (10.1), there are infinitely many non-isomorphic subvarieties $\varphi'(X')$ as above.

Proof. Under the assumption, there are integers $\beta_1, \beta_2, \dots, \beta_{N-m-n+1} \geq 2$ satisfying the following equation;

$$\sum \beta_i + n^2/2 = N + s + 1.$$

For any such choices of β_i 's, let Y be a complete intersection in \mathbb{P}^N of multidegree

$$\underline{d}_{\text{unord}} = (2, 2, 4, 4, \dots, 2(n/2 - 1), 2(n/2 - 1), n, \beta_1, \dots, \beta_{N-m-n+1}).$$

Let $\varphi : X \rightarrow Y$ be a smooth simple dihedral cover associated to the line bundle $L = \mathcal{O}_Y(2)$ satisfying the conditions of Theorem 4.6. Then φ is an s -subcanonical cover of degree n satisfying the hypothesis of Theorem 3.7 (a). Thus, a general deformation of φ is an embedding. \square

Example 10.5. Now we describe the invariants of the first few smooth, subvarieties $\varphi'(X')$ obtained when we deform s -subcanonical morphisms φ as in Setup 10.1, with $-1 \leq s \leq 1$, for which the hypothesis of Theorem 3.7 (a) holds (this includes Calabi–Yau and canonically embedded subvarieties). It is interesting to note that the subvarieties in rows 1, 2, 4, 5, 8 are near the boundary of, but inside, the range of Hartshorne's conjecture. Precisely, in row 7, the codimension r satisfies $r = (1/3)N - 2$ and in rows 1, 2, 4, 5, 8 the codimension r satisfy $(1/3)N - 2 < N \leq (1/3)N - 3$.

m	n	k	N	s	\underline{d}	$K_{X'}^m$	$p_g(X')$
20	6	2	26	-1	(2, 2, 2, 4, 4, 6)	-4608	0
19	6	2	24	-1	(2, 2, 4, 4, 6)	-2304	0
33	8	2	40	-1	(2, 2, 4, 4, 6, 6, 8)	-147456	0
19	6	2	25	0	(2, 2, 2, 4, 4, 6)	0	1
18	6	2	23	0	(2, 2, 4, 4, 6)	0	1
32	8	2	39	0	(2, 2, 4, 4, 6, 6, 8)	0	1
18	6	2	24	1	(2, 2, 2, 4, 4, 6)	4608	25
17	6	2	22	1	(2, 2, 4, 4, 6)	2304	23
31	8	2	38	1	(2, 2, 4, 4, 6, 6, 8)	147456	39

Example 10.6. Theorem 10.4 allows also the construction of other smooth subvarieties closer to the boundary of the range of Hartshorne’s conjecture, like the very small sample of the smooth subvarieties in the range $r = (1/3)N - 1$, displayed in Table 2 of the introduction, which are obtained by deforming φ , which factors through a simple $D_{n/2}$ cover of degree n , with $k = 2$. It is therefore interesting to know whether these subvarieties and the ones of Example 10.5 is a complete intersection, although we do not know whether the same is true for some special deformations of φ (see Question 12.2).

Remark 10.7. Arguing as in Remark 6.7 we conclude that:

- (1) Theorem 10.4 implies the existence, for fixed m, N, n with $m \geq 3, n \geq 6, n$ even, and $N \geq m + n - 1$, of infinitely many different moduli spaces having reduced and irreducible components with a locally closed locus that parametrizes smooth varieties with an s -subcanonical morphism, finite and of degree n onto its image, whereas the general points of the components correspond to smooth varieties with an s -subcanonical morphism which is an embedding onto its image. In Section 11 we will describe more specifically this phenomenon in the case of the canonical map, i.e., if $s = 1$ (see Corollary 11.5).
- (2) For m, n fixed with $m \geq 3, n \geq 6, n$ even or, simply, for m fixed with $m \geq 3$, there exist smooth varieties of dimension m as in Theorem 8.3 with p_g arbitrarily large.

11. MODULI OF VARIETIES OF GENERAL TYPE

In this section we construct components of the moduli space of varieties of general type that are analogues of moduli space of curves with respect to the canonical map and its deformations. More precisely, we show the existence of a locally closed locus in these moduli components, which is analogous to the hyperelliptic locus of curves of genus bigger than 2, where the degree of the canonical map jumps up. The deformation of the canonical maps gives raise to one-parameter families where the special member is a rope of multiplicity $m \geq 2$ and the general member is a smooth canonically embedded subvariety. An interesting point to note is that we find these canonically embedded varieties for any codimension r and, in particular, for any r in the range of the Hartshorne’s conjecture. The existence of the special loci proved in this section is but a particular case of a more general phenomenon. In fact, our results imply (see Remarks 6.7, 7.8, 9.10, 10.7) the existence of infinitely many irreducible components possessing loci which correspond to the jumping up of the degree of the subcanonical maps. As it happens in this section, the deformations of the canonical map in these cases give rise to analogous, interesting one-parameter families. Because of the significance of the canonical map, now we focus and give the details only of the loci related to this map. Finally, as explained in the introduction, in this section we also construct two distinct kinds of moduli components which differ from the moduli space of curves. These are made precise in Corollaries 11.2 and 11.3 below.

Corollary 11.1. *Let $m \geq 3$. For each integer n such that $2 \leq n \leq \sqrt{m+3}$, there is at least one reduced and irreducible component \mathcal{M}_n of the moduli space of varieties of general type of dimension m such that:*

- (i) \mathcal{M}_n has a locally closed locus that parametrizes varieties whose canonical map φ is a morphism, finite of degree n onto a smooth image; more precisely, φ factors through simple cyclic cover of degree n .
- (ii) The general points of \mathcal{M}_n correspond to varieties whose canonical map is an embedding.
- (iii) The canonical models of the varieties corresponding to general points of \mathcal{M}_n degenerate to a rope of multiplicity n , along a general one-parameter family that intersects the special locus described in (i).

For different values of n , the components \mathcal{M}_n are different. Letting m vary, the general points of the \mathcal{M}_n parametrize canonically embedded varieties of any codimension.

Proof. Let $m \geq 3$. Let $2 \leq n \leq \sqrt{m+3}$. There exists an integer N , satisfying (6.2) for $s = 1$. Then Theorem 6.6 implies the existence of \mathcal{M}_n satisfying (i) and (ii). Note that N is the geometric genus of the varieties parametrized by \mathcal{M}_n and that, for each n and $s = 1$, one can choose a different N satisfying (6.2). Note also that (6.2) can be rephrase in terms of r , so the last statement follows. Finally \mathcal{M}_n is reduced since the simple cyclic cover constructed in Theorem 6.6 is unobstructed by Lemma 3.3. \square

Arguing as in the proof of Corollary 11.1, Theorem 8.3 yields the following:

Corollary 11.2. *Let $m \geq 3$. For each integer n , $n \geq 2$ such that*

$$2(n-1)(\lfloor n/2 \rfloor + 1) - \lfloor n/2 \rfloor \leq m+2,$$

there is at least one reduced and irreducible component \mathcal{M}_n of the moduli space of varieties of general type of dimension m such that:

- (i) \mathcal{M}_n has a locally closed locus that parametrizes varieties whose canonical map φ is a morphism, finite of degree n onto a smooth image; more precisely, φ factors through simple cyclic cover of degree n .
- (ii) The general points of \mathcal{M}_n correspond to varieties whose canonical map is a finite birational morphism onto its image.

For different values of n , the components \mathcal{M}_n are different.

Theorem 9.8 yields the following:

Corollary 11.3. *Let $m \geq 6$. For each even integer n such that*

(1)

$$4 \leq n \leq (2m+8)/5;$$

(2) *and there exists an integer v satisfying*

$$2 \frac{m+4}{n} \leq v \leq 2 \frac{m-1}{n-2},$$

there is at least one reduced, irreducible, uniruled component \mathcal{M}_n of the moduli space of varieties of general type of dimension m such that:

- (i) \mathcal{M}_n has a locally closed locus that parametrizes varieties whose canonical map φ is a morphism, finite of degree n onto a smooth image; more precisely, φ factors through a $\mathbb{Z}_{n/2} \times \mathbb{Z}_2$ Galois cover.
- (ii) The general points of \mathcal{M}_n correspond to varieties whose canonical map is a flat morphism, finite and of degree $n/2$ onto a smooth image.

For different values of n , the components \mathcal{M}_n are different.

Proof. (1) and (2) of the statement imply, when $s = 1$, formula (9.13). If $s = 1$, (9.14) becomes

$$(11.1) \quad m+1 \leq N \leq 2m - 5n/2 + 5,$$

and (1) implies the existence of integers N satisfying (11.1). Then, arguing as in the proof of Corollary 11.1, Theorem 9.8 implies the result. By Theorem 3.8 the deformation space of X is smooth and uniruled. Now, let $X \hookrightarrow \mathbb{P}(H^0(lK_X)^*)$ be the closed embedding for some $l \gg 0$. Then $\text{Aut}(X) \hookrightarrow \text{PGL}(H^0(lK_X)^*)$ is an embedding. Thus, X has finite automorphism group since $H^0(T_X) = 0$ by Nakano vanishing theorem (as

$H^0(T_X)$ is the tangent space of $\text{Aut}(X)$ at identity¹. Hence the corresponding moduli component is also uniruled. \square

Remark 11.4. The inequalities

$$4 \leq n \leq 2(\sqrt{m+8}-2)$$

imply (1) and (2) of the statement of Corollary 11.3.

Arguing as in the proof of Corollary 11.1, Theorem 10.4 yields the following:

Corollary 11.5. *Let $m \geq 18$. For each even integer n such that $6 \leq n \leq \sqrt{2m}$, there is at least one reduced and irreducible component \mathcal{M}_n of the moduli space of varieties of general type of dimension m such that:*

- (i) \mathcal{M}_n has a locally closed locus that parametrizes varieties whose canonical map φ is a morphism, finite of degree n onto a smooth image; more precisely, φ factors through simple dihedral cover of degree n .
- (ii) The general points of \mathcal{M}_n correspond to varieties whose canonical map is an embedding.
- (iii) The canonical models of the varieties corresponding to general points of \mathcal{M}_n degenerate to a rope of multiplicity n , along a general one-parameter family that intersects the special locus described in (i).

For different values of n , the components \mathcal{M}_n are different. Letting m vary, the general points of the \mathcal{M}_n parametrize canonically embedded varieties of any codimension $r \geq 5$.

Remark 11.6. As hinted in the statements of Corollaries 11.1, 11.2, 11.3 and 11.5, once we fix m and n in any of these corollaries, there could be more than one reduced and irreducible component \mathcal{M}_n , even if we also fix p_g . This is because we can make different choices for the multidegree of Y and different choices for the ramification of π . For example, regarding Corollary 11.1, for $m = 6$, we can exhibit two components \mathcal{M}_2 and \mathcal{M}'_2 , the former constructed from a double cover π of a $(2, 4)$ -complete intersection Y , branched along a smooth member of $|\mathcal{O}_Y(4)|$, and the latter constructed from a double cover π' of a $(2, 6)$ -complete intersection Y' , branched along a smooth member of $|\mathcal{O}_{Y'}(2)|$. In both cases, $p_g = 9$; however, $K^6 = 16$ in the former case and $K^6 = 24$ in the latter case.

12. OPEN QUESTIONS AND FINAL REMARKS

We end this article by asking some questions regarding the subvarieties we have constructed.

Question 12.1. Recall that the subvarieties obtained in these article when applying Theorem 3.7 to cyclic covers (and, more generally, to suitable iterated cyclic covers) are one-parameter smoothing of ropes. Proposition 5.4 implies that general members of some of these smoothings are complete intersections, for example, when the hypotheses of Theorem 6.6 or Theorem 9.4 are satisfied. However, Proposition 5.4 *does not* imply that general members of every one-parameter smoothing of those ropes are complete intersections. Then, in the above cases, is every one-parameter smoothing a complete intersection or are there some one-parameter deformations yielding non-complete intersections subvarieties? This question is quite intriguing, since Theorem 6.6 and Theorem 9.4 produce m -dimensional, smooth subvarieties of \mathbb{P}^N of any codimension and, in particular, in the range $m \geq N/2 + 1$, where constructing non-complete intersection is quite difficult, and, more remarkably, fall also in the range of Hartshorne's conjecture.

Question 12.2. Are some of the smooth subvarieties $\varphi'(X')$ constructed in Theorem 10.4 by deforming dihedral covers, non-complete intersection subvarieties? Recall that these subvarieties are s -subcanonical (see (4.3) and Proposition 4.12). Although their degree and s are such that there exist smooth complete intersection, s -subcanonical subvarieties with the same degree in the same projective space, this doesn't necessarily implies that the subvarieties $\varphi'(X')$ are complete intersections. Since some of them have small

¹We thank the referee for suggesting this argument.

codimension, in the same range mentioned in Question 12.1 (see a few cases of this in Examples 10.5 and 10.6), the question is relevant. We note that the methods that work in the case of iterated cyclic covers to prove that a general deformation is a complete intersection, do not work for dihedral covers.

Question 12.3. Is the morphism φ' of Example 8.4 an embedding for the varieties appearing in the white rows of the table? More generally, using our methods, one can construct infinitely many smooth varieties with birational canonical morphisms which would be non-complete intersections if they were embeddings. The question is to determine if there are any special one-parameter family for these covers along which they deform to embeddings. Again, the question is relevant, for the images of many of these embeddings would be small codimension subvarieties.

Question 12.4. It is well known that one can construct m -dimensional, smooth subvarieties in \mathbb{P}^N , in the range $m < \frac{N}{2} + 1$ as degeneracy loci of vector bundle homomorphisms. The method we use in this paper to construct smooth subvarieties is completely different. Therefore, are the smooth, non-complete intersection subvarieties constructed by our methods degeneracy loci of vector bundle homomorphisms? Particularly, are the smooth, non-complete intersection subvarieties constructed in Theorems 7.6 and 7.7 degeneracy loci of vector bundle homomorphisms?

Next remark shows the subtleties of the situation we handle, as it introduces an example of an unobstructed morphism φ to which we can associate an embedded rope which is obstructed. In some of the cases, even if the deformation space of φ is smooth, the rope lies in at least two components of the Hilbert scheme:

Remark 12.5. With the notation of Setup 3.1, we give an example of the following situation:

- (1) a morphism $\varphi : X \rightarrow \mathbb{P}^N$ with Y smooth such that there exists a smooth curve T with a distinguished point 0 and a flat family of morphisms $\Phi : \mathcal{X} \rightarrow \mathbb{P}_T^N$ over T such that $\Phi_0 = \varphi$ and Φ_t is an embedding for all $t \neq 0$;
- (2) $(\text{Im}\Phi)_0$ is a rope \tilde{Y} in \mathbb{P}^N supported on Y with conormal bundle \mathcal{E} ;
- (3) the morphism φ is unobstructed and, by (1), a general element of the algebraic formally semi-universal deformation of φ is an embedding;
- (4) However, the rope \tilde{Y} does not correspond to a smooth point of its Hilbert scheme.

For such an example, consider the rational normal scroll $Y = \mathbb{F}_e \hookrightarrow \mathbb{P}^N$ with $e = 3$ or $e = 4$. By [Rei76], there exists a double $K3$ cover $\pi : X \rightarrow Y$, branched along a reasonably singular curve $C \in |-2K_Y|$ (see [Rei76, Theorem 2.2]). Then $H^1(\mathcal{N}_\pi) = H^1(\pi_*\mathcal{N}_\pi) = H^1(-2K_Y|_C) = H^1(2K_C) = 0$ by Proposition 4.3. Since $H^1(\mathcal{N}_{Y/\mathbb{P}^N}) = H^1(\mathcal{N}_{Y/\mathbb{P}^N} \otimes K_Y) = 0$, we have $H^1(\mathcal{N}_\varphi) = 0$ and φ is unobstructed. But \tilde{Y} is a singular point of its Hilbert scheme by [GP97, Theorem 4.1]. Moreover, if $N \geq 10$ and N is congruent with 1 modulo 4, then the Hilbert point of \tilde{Y} lies in two irreducible components of the Hilbert scheme (see [GP97, Theorem 4.3]).

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