Buy and hold golden strategies in financial markets with frictions and depth constraints

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Abstract. This paper deals with coherent risk measures and golden strategies, that is, financial portfolios (or financial strategies) with a negative risk and a non positive price. Golden strategies are important because they enable us to outperform every portfolio in a return/risk approach. In fact, every portfolio of securities is beaten by adding the golden strategy, i.e. the portfolio plus the golden strategy is better than the portfolio alone. Computationally tractable algorithms will be presented, and the general framework will be very realistic. Indeed, the study will incorporate all the classical frictions provoked by the order book of a financial market, and it will be both buy-and-hold and model-free. Numerical experiments involving derivative markets will be analyzed.

Key words. Coherent Risk, Market Depth and Frictions, Golden Strategy, Computational Tractability.

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1 Introduction

Coherent risk measures were introduced in the seminar paper of Artzner *et al.* (1999), and then they were used to revisit many financial problems (Stoyanov *et al.*, 2007, Kim, 2007, Dupacová and Kopa, 2014, Lejeune and Shen, 2016, etc.), actuarial problems (Hamada and Sherris, 2003, Goovaerts and Laeven, 2008, Lia *et al.*, 2018, Cheung *et al.*, 2019, etc.), risk management problems (Tapiero, 2004, Nakano, 2004, Johnston, 2009, etc.), or even problems beyond the economic fields (Wu *et al.*, 2018, Filippi *et al.*, 2020, Jiekang *et al.*, 2020, Cui *et al.*, 2021, etc.).

Several applications in portfolio choice have led to anti-intuitive findings. In particular, Balbás *et al.* (2010) pointed out that the absence of arbitrage does not guarantee the absence of sequences of investment strategies such that the couple ($expected\_return, coherent\_risk$) tends to $(+\infty, -\infty)$.[1] For instance, the existence of such sequences holds under the assumptions of very famous and prestigious theoretical pricing models such as Black-Scholes-Merton (BSM) or Heston. Later, Balbás *et al.* (2019) proved the existence of a European option whose sale, along with the investment of the option price in a riskless asset, led to a self-financing strategy with negative coherent risk. This option was called "golden option" by the authors, and the sequence of Balbás *et al.* (2010) may be constructed by repeating n times the self-financing strategy above, for every natural number n. The results of Balbás *et al.* (2019) were extended in Balbás *et al.* (2022) for more complex buy and hold portfolios and under the constraints imposed by the market order book, giving rise to the notion of "golden strategy".

Balbás *et al.* (2022) provided us with efficient algorithms/procedures related to linear programming and goal programming. Nevertheless, the representation theorem of a coherent risk measure (Artzner *et al.*, 1999) implied that the mathematical programming problems of Balbás *et al.* (2022) had to involve infinite-dimensional spaces, and accordingly, even an efficient algorithm could become slow in some practical applications (Anderson and Nash, 1987). Actually, Balbás *et al.* (2022) were looking for very general efficient strategies, and the golden strategies were only a particular case. This paper extends and simplifies the methods of Balbás *et al.* (2022) in order to find buy and hold golden strategies under the order book constraints. The approach may be model free and is very realistic because only real market quotes are involved and no dynamic assumptions have to be imposed. All the potential market imperfections/frictions are taken into account. Furthermore, the algorithms of Balbás *et al.* (2022) are significantly accelerated here because the unique focus is on golden strategies, and more general optimal portfolios do not matter. Although this paper and Balbás *et al.* (2022) deal with similar theoretical methods, the focus is on different problems, and the derived algorithms become very distinct. This is the major difference between Balbás *et al.* (2022) and the content of this paper.

The paper's outline is as follows. Section 2 is devoted to presenting a general (static, or buy and hold) framework only involving finitely many securities and their real market quotes. The golden strategies and their theoretical properties are studied in Section 3, where the most important results, that is, Theorem 1 and Corollary 3, characterize the presence or absence of golden strategies by means of the risk measure sub-gradient. The analysis is extended in Section 4, where the market depth is also incorporated, and consequently all the order book-related restrictions are considered. If the conditions of Theorem 1 hold, and therefore there are available golden strategies, then Theorem 7 generates a procedure to detect them in practice, and Algorithm I is a clear application of Theorem 7. In fact, Algorithm I synthesizes the theoretical properties of both theorems above and allows us to apply them in practical situations. Numerical experiments are presented in Sections 5 and 6, where the effectiveness

[1] There are other interesting papers showing the existence of ill-posed problems in portfolio choice. For instance, Jin and Zhou (2008) proved this existence in general continuous time behavioral portfolio selection models under a cumulative prospect theory.

and the tractability of Algorithm I is illustrated. The selected examples involve derivative markets, since the usefulness of derivative securities for traditional investors is still under discussion (Ahn *et al.*, 1999, Constantinides *et al.*, 2011, Bondarenko, 2014, etc.). Moreover, the use of derivatives may help to test the consistency of a system of prices in a model free approach (Haugh and Lo, 2001). The numerical results of Sections 5 and 6 seem to reveal that the incorporation of derivatives may frequently become interesting to many agents, and furthermore, consistent and arbitrage-free prices do not prevent the existence of golden strategies. Section 5 deals with the BSM model, and the numerical results seem to show the lack of relationships between the model parameters (drift and volatility) and the composition of the golden strategy. The restrictions imposed by the order book seem to be much more relevant. Section 6 involves real market data, and the numerical results reinforce the idea that Algorithm I is very efficient in searching for golden strategies. Obviously, empirical tests about the practical performance of golden strategies are beyond the scope of this theoretical paper, but Algorithm I will make it possible to implement these empirical analyses. There are former and closely related empirical studies. Balbás *et al.* (2016a) dealt with the theoretical results of Balbás *et al.* (2010) and tested the related sequences of strategies. Accordingly, a continuous time framework was adopted, and the results illustrated how the performance of such sequences was better than the performance of several very important international stock indices. This superiority was also pointed out when the selected risk measure was replaced by the usual Sharpe ratio, though to the best of our knowledge, a theoretical argument justifying this empirical finding was never proved. More recently, Balbás and Serna (2024) empirically tested the performance of the golden options of Balbás *et al.* (2019), and the results were better than those of Balbás *et al.* (2016a). The superiority was still maintained when replacing the coherent measure of risk with the Sharpe ratio. Evidently, a buy and hold framework prevents a lot of transaction costs with respect to a continuous time one. As said above, Theorem 7 and Algorithm I will make it possible to implement empirical studies beyond the ones of Balbás *et al.* (2016a) and Balbás and Serna (2024), and is a natural and very attractive challenge for future research.

Section 7 presents the main conclusions of the paper.

2 Preliminaries and notation

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ composed of the set of states of nature Ω, the $\sigma-$algebra $\mathcal{F}$ reflecting the information available at a future planning period T, and the probability measure $\mathbb{P}$. As usual, denote by L^2 the space of real valued random variables y on Ω such that $\mathbb{E}\left(y^2\right) < \infty$, endowed with the inner product $(x, y) \rightarrow \mathbb{E}(xy)$ and the norm $\|y\|_2 = \left(\mathbb{E}\left(y^2\right)\right)^{1/2}$, where $\mathbb{E}(.)$ represents the mathematical expectation. We will be dealing with a finite collection of $1+m$ available securities whose ask prices at T will be given by $\{S_0, S_1, ..., S_m\} \subset L^2$ and whose bid prices at T will be given by $\{s_0, s_1, ..., s_m\} \subset L^2$. Inequality $s_j \leq S_j$ will hold for $j = 1, 2, ..., m$, and $S_j - s_j \geq 0$ will be a random friction (or transaction cost) at T. The first security will be risk-free and friction-free, and therefore $\mathbb{P}(s_0 = 1) = \mathbb{P}(S_0 = 1) = 1$ will be fulfilled. Traders will also face frictions at $t = 0$, and therefore $\{q_0, q_1, ..., q_m\} \subset \mathbb{R}$ will be the collection of bid prices (prices to be received for a sale), whereas $\{Q_0, Q_1, ..., Q_m\} \subset \mathbb{R}$ will be the collection of ask prices (prices to be paid for a purchase). Obviously, $q_j \leq Q_j$ will hold for $j = 1, 2, ..., m$, and we will assume that $q_0 = Q_0 = 1$. In other words, as usual, the riskless asset is friction-free, and the real quotations have been normalized so as to make the riskless rate vanish, that is, q_j and Q_j are actually the real quotations multiplied by e^{rT}, r denoting the continuously compounded riskless rate.

Portfolios will be denoted by vectors $(x_0, x_1, ..., x_m, y_1, ..., y_m) \in \mathbb{R} \times \mathbb{R}_+^m \times \mathbb{R}_+^m$, where, as usual, $\mathbb{R}_+ = \{k \in \mathbb{R};\ k \geq 0\}$, and $x_j \geq 0$ (respectively, $y_j \geq 0$) indicates the number of purchased (sold) units of the $j - th$ security for $j = 1, 2, ..., m$. x_0 is not affected by any sign restriction and indicates the amount of money invested in the riskless asset. The investor is lending (borrowing) if $x_0 > 0$ $(x_0 < 0)$. In order to simplify the notation, $(x_0, x_1, ..., x_m, y_1, ..., y_m)$ may be also represented by $(x_0, x, y) \in \mathbb{R} \times \mathbb{R}_+^m \times \mathbb{R}_+^m$.

Fix a risk measure $\rho : L^2 \longrightarrow \mathbb{R}$, and let us assume that ρ satisfies a representation theorem in the line of Artzner *et al.* (1999) and Rockafellar *et al.* (2006). More precisely, consider the sub-gradient at zero of ρ

$$\Delta_\rho = \left\{ z \in L^2; -\mathbb{E}(uz) \leq \rho(u), \forall u \in L^2 \right\} \subset L^2 \tag{1}$$

composed of those linear and continuous $\mathbb{R}-$valued functions lower than ρ. Δ_ρ will be convex and weakly$-$compact (Zeidler, 1995), and ρ will be its envelope, in the sense that

$$\rho(u) = Max\ \{-\mathbb{E}(uz);\ z \in \Delta_\rho\} \tag{2}$$

will hold for every $u \in L^2$. Furthermore, we will also assume that

$$\{1\} \subset \Delta_\rho \subset \left\{ z \in L^2;\ \mathbb{E}(z) = 1 \right\} \tag{3}$$

and

$$\Delta_\rho \subset \left\{ z \in L^2;\ \mathbb{P}(z \geq 0) = 1 \right\}. \tag{4}$$

These properties are equivalent to the usual ones of continuity, sub-additivity, homogeneity, mean dominance, translation invariance and monotonicity. To sum up, we have:

Assumption I *$\rho : L^2 \longrightarrow \mathbb{R}$ is continuous, sub-additive ($\rho(u_1 + u_2) \leq \rho(u_1) + \rho(u_2)$ if $u_1, u_2 \in L^2$), homogeneous ($\rho(\alpha u) = \alpha\rho(u)$ if $u \in L^2$ and $\alpha \geq 0$), mean dominating ($\rho(u) \geq -\mathbb{E}(u)$ if $u \in L^2$), translation invariant ($\rho(u + k) = \rho(u) - k$ if $u \in L^2$ and $k \in \mathbb{R}$) and decreasing ($\rho(u_1) \leq \rho(u_2)$ if $u_1, u_2 \in L^2$ and $\mathbb{P}(u_1 - u_2 \geq 0) = 1$).*[2] □

[2] Assumption I is in some sense redundant. For instance, if ρ is sub-additive, homogeneous and decreasing then ρ is continuous. Similarly, instead of "mean domination" one can impose "law invariance", in which case the "mean domination" fulfilment may be proved. Nevertheless, let us present Assumption I as it is because all the mentioned properties will be used.

There are many important risk measures satisfying Assumption I. For instance, the conditional value at risk (Rockafellar *et al.*, 2006), several distortion risk measures (Hamada and Sherris, 2003), the robust risk measures of Balbás *et al.* (2016*b*) and the expectile risk measure (Delbaen, 2013, and Tadese and Drapeau, 2020, among others).[3], [4] Recall that for $0 < \beta < 1$ and the conditional value at risk with the confidence level $1 - \beta$ ($\rho = CV@R_{1-\beta}$) one has (Rockafellar *et al.*, 2006)

$$\Delta_\rho = \left\{ z \in L^2;\ 0 \leq z \leq 1/\beta,\ \mathbb{E}(z) = 1 \right\}. \tag{5}$$

Similarly, for $0 < \beta < 1/2$ and the expectile with parameter β ($\rho = \mathcal{E}_\beta$) one has (Delbaen, 2013)

$$\Delta_\rho = \left\{ z \in L^2;\ \xi \leq z \leq \xi \frac{1-\beta}{\beta},\ \xi \in \mathbb{R},\ \mathbb{E}(z) = 1 \right\}. \tag{6}$$

3 Golden strategies

Let us deal with the optimization problem

$$\left\{ \begin{array}{l} Min\ \rho \left(x_0 + \sum_{j=1}^m (x_j s_j - y_j S_j) \right) \\ x_0 + \sum_{j=1}^m (Q_j x_j - q_j y_j) \leq 0 \\ x_j, y_j \geq 0,\ j = 1, 2, ..., m. \end{array} \right. \tag{7}$$

(7) minimizes the risk of an investment strategy whose price is non positive. (7) is obviously feasible because $(x_0, x, y) = (0, 0, 0)$ satisfies the restrictions. Let us analyze whether (7) is bounded and solvable.

Theorem 1 *(7) is bounded if and only if there exists $z \in \Delta_\rho$ such that*

$$\left\{ \begin{array}{ll} \mathbb{E}(s_j z) \leq Q_j, & j = 1, 2, ..., m \\ \mathbb{E}(S_j z) \geq q_j, & j = 1, 2, ..., m. \end{array} \right. \tag{8}$$

If (8) holds, then $(x_0, x, y) = (0, 0, 0)$ solves (7) (i.e., the absence of investment is optimal) and 0 equals the optimal value.

Proof. (2) implies that (7) is equivalent to

$$\left\{ \begin{array}{l} Min\ \theta \\ -\mathbb{E}\left(z \left(x_0 + \sum_{j=1}^m (x_j s_j - y_j S_j) \right) \right) \leq \theta, \forall z \in \Delta_\rho \\ x_0 + \sum_{j=1}^m (Q_j x_j - q_j y_j) \leq 0 \\ x_j, y_j \geq 0,\ j = 1, 2, ..., m, \end{array} \right. \tag{9}$$

[3] Expectiles were introduced in Newey and Powel (1987), though one had to wait for several years until expectiles were also interpreted as coherent risk measures.

[4] In particular, if one deals with robust risk measures, then the conclusions may become model free.

$(\theta, x_0, x, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$ being the decision variable. Bearing in mind (3), and proceeding as Balbás *et al.* (2010) and (2022), the Lagrangian of (9) (see Luenberger, 1969) may simplify to

$$\mathcal{L}(x_0, x, y, \lambda, z) = (\lambda - 1) x_0 + \sum_{j=1}^{m} [\lambda Q_j - \mathbb{E}(s_j z)] x_j + \sum_{j=1}^{m} [\mathbb{E}(S_j z) - \lambda q_j] y_j$$

for every $(x_0, x, y, \lambda, z) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \times \Delta_\rho$ with $\lambda \geq 0$. As in the couple of references above, the dual of (7) has as feasible solutions those $(\lambda, z) \in \mathbb{R} \times \Delta_\rho$ with $\lambda \geq 0$ and such that

$$Inf \ \{\mathcal{L}(x_0, x, y, \lambda, z); \ x_j \geq 0, y_j \geq 0, j = 1, 2, ..., m\} > -\infty, \qquad (10)$$

which is obviously equivalent to (8) plus the equality $\lambda = 1$, in which case the infimum in (10) equals 0. Thus, the dual of (7) becomes (see Luenberger, 1969)

$$\begin{cases} Max \ 0 \\ \lambda = 1 \\ \mathbb{E}(s_j z) \leq Q_j, \quad j = 1, 2, ..., m \\ \mathbb{E}(S_j z) \geq q_j, \quad j = 1, 2, ..., m. \end{cases} \qquad (11)$$

The classical primal-dual relationships imply that (7) is bounded if and only if (11) is feasible, in which case both optimal values coincide. Since the optimal value of (11) equals 0 when this problem is feasible, the lack of investment solves (7) when this problem is bounded. □

Remark 2 *Notice that the necessary and sufficient conditions given in Theorem 1 have a clear financial meaning. Indeed, the unboundedness of (7) requires that there is no* $z \in \Delta_\rho$ *such that, on average, time* T *bid prices scaled by* z, $\{s_1 z, s_2 z, ..., s_n z\}$, *are smaller than time* 0 *ask prices* $\{Q_1, Q_2, ..., Q_m\}$ *and time* T *ask prices scaled by* z, $\{S_1 z, S_2 z, ..., S_n z\}$, *are larger than time* 0 *bid prices* $\{q_1, q_2, ..., q_m\}$. □

Corollary 3 *The statements below are equivalent;*

a) (7) is unbounded.

b) There exists a (7)-feasible $(x_0, x, y) \in \mathbb{R} \times \mathbb{R}_+^m \times \mathbb{R}_+^m$ *such that*

$$\rho \left(x_0 + \sum_{j=1}^{m} (x_j s_j - y_j S_j) \right) < 0.$$

c) There exists a sequence $\left(x_0^{(n}, x^{(n}, y^{(n}\right)_{n \in \mathbb{N}} \subset \mathbb{R} \times \mathbb{R}_+^m \times \mathbb{R}_+^m$ *of (7)-feasible investment strategies such that, under the obvious notation*

$$\begin{cases} x_0^{(n} + \sum_{j=1}^{m} \left(x_j^{(n} Q_j - y_j^{(n} q_j \right) \leq 0, & \forall n \in \mathbb{N} \\ \lim_{n \to \infty} \rho \left(x_0^{(n} + \sum_{j=1}^{m} \left(x_j^{(n} s_j - y_j^{(n} S_j \right) \right) = -\infty \\ \lim_{n \to \infty} \mathbb{E} \left(x_0^{(n} + \sum_{j=1}^{m} \left(x_j^{(n} s_j - y_j^{(n} S_j \right) \right) = +\infty. \end{cases} \qquad (12)$$

Proof. $a) \Longrightarrow b)$ and $c) \Longrightarrow a)$ are obvious. $b) \Longrightarrow a)$ trivially follows from Theorem 1 because the optimal value of (7) equals 0 when this problem is bounded. Thus, let us prove $a) \Longrightarrow c)$. The fulfillment of $a)$ leads to the existence of a sequence $\left(x_0^{(n}, x^{(n}, y^{(n}\right)_{n\in\mathbb{N}}$ of portfolios whose price is never positive and whose risk tends to minus infinity. Moreover, since ρ is mean dominating (Assumption I), under the evident notation

$$\mathbb{E}\left(x_0^{(n} + \sum_{j=1}^{m}\left(x_j^{(n} s_j - y_j^{(n} S_j\right)\right) \geq -\rho\left(x_0^{(n} + \sum_{j=1}^{m}\left(x_j^{(n} s_j - y_j^{(n} S_j\right)\right)$$

for every $n \in \mathbb{N}$. □

Definition 4 *Following Balbás et al. (2019) and (2022),* $(x_0, x, y) \in \mathbb{R} \times \mathbb{R}_+^m \times \mathbb{R}_+^m$ *is said to be a golden strategy if its price is not higher than* 0 *and the strict inequality* $\rho\left(x_0 + \sum_{j=1}^{m}(x_j s_j - y_j S_j)\right) < 0$ *holds.*[5] □

Remark 5 *According to Theorem* 1 *and Corollary* 3*, an unbounded Problem (7), the existence of golden strategies, the failure of (8), and the fulfillment of (12), are equivalent.* □

Remark 6 *(Arbitrage and golden strategies) Recall that* $(x_0, x, y) \in \mathbb{R} \times \mathbb{R}_+^m \times \mathbb{R}_+^m$ *is said to be an arbitrage strategy if*

$$\begin{cases} x_0 + \sum_{j=1}^{m}(Q_j x_j - q_j y_j) \leq 0 \\ \mathbb{P}\left(x_0 + \sum_{j=1}^{m}(x_j s_j - y_j S_j) \geq 0\right) = 1 \\ \mathbb{P}\left(\sum_{j=1}^{m}(x_j s_j - y_j S_j) - \sum_{j=1}^{m}(Q_j x_j - q_j y_j) > 0\right) > 0. \end{cases}$$

In other words, the current price is not positive, the final pay-off will never be negative and (x_0, x, y) *will generate strictly positive profits with strictly positive probability. For instance, if* $j = 1$*,* $q_1 = Q_1 = 0$*,* $s_1 = S_1$ *and* $\mathbb{P}(S_1 = 0) = \mathbb{P}(S_1 = 1) = 0.5$*, then it is easy to see that the purchase of the (unique) risky asset is an arbitrage. However, this purchase is not a golden strategy if* $\rho = CV@R_{0.8}$ *because the strategy risk becomes zero. Conversely, as will be pointed out in Section* 5*, there are also golden strategies which are not arbitrage strategies.*

In spite of the above comments, the notions of "arbitrage" and "golden strategy" are not far apart. The existence of $z \in L^2$ *satisfying (8) and such that* $\mathbb{P}(z > 0) = 1$ *and* $\mathbb{E}(z) = 1$ *is closely related to the absence of arbitrage (Jouini and Kallal, 1995, or see also Luenberger, 2001, if bid prices equal ask prices).*

[5] Actually, Balbás *et al.* (2019) introduced the notion of "golden option" as a European option whose price is strictly higher than the risk provoked by the option sale. Later, Balbás *et al.* (2022) introduced this notion of "golden strategy" and showed that the sale of a golden option plus the investment of the golden option price in a riskless asset is a golden strategy.

Theorem 1 imposes the condition $z \in \Delta_\rho$ rather than the weaker ones $z \in L^2$ with $\mathbb{E}(z) = 1$, and relaxes $\mathbb{P}(z > 0) = 1$ to $\mathbb{P}(z \geq 0) = 1$ (see (3) and (4)). In other words, the absence of arbitrage is related but not equivalent to the existence of finite lower bounds for (7). The absence of arbitrage does not guarantee that Problem (7) is bounded because the absence of arbitrage does not imply the condition $z \in \Delta_\rho$. Conversely, the absence of golden strategy does not guarantee the absence of arbitrage because the condition $z \in \Delta_\rho$ does not imply the equality $\mathbb{P}(z > 0) = 1$. □

4 Incorporating the market depth

Suppose that there are golden strategies in a real market. In other words, suppose that (8) fails and therefore (12) holds. One is facing an anti-intuitive result because it leads to the existence of infinite expected returns with simultaneous minus-infinite risks in an arbitrage-free framework. Nevertheless, Balbás *et al.* (2022) dealt with real market quotations and found golden strategies, although these authors argued that infinite return/risk ratios are not reachable in practice owing to the market depth. Thus, let us incorporate this depth to our approach and suppose the existence of $V_j > 0$ and $W_j > 0$ such that $x_j \leq V_j$ and $y_j \leq W_j$ must hold for $j = 1, 2, ..., m$, *i.e.*, at $t = 0$ investors cannot buy (sell) more than V_j (W_j) units of the $j-th$ asset. Notice that the set of bid prices $\{q_1, q_2, ..., q_m\}$, the set of ask prices $\{Q_1, Q_2, ..., Q_m\}$ and the limits $\{V_1, V_2, ..., V_m\}$ and $\{W_1, W_2, ..., W_m\}$ compose the so called "level I market data". We could consider the "level II market data" too, that is, further levels of bid and ask prices and limits, but Balbás *et al.* (2022) pointed out that models involving the level II may be transformed into equivalent models only involving the level I. Whence, the level I is general enough so as to include every practical situation affecting the order book.

Problem (7) must be replaced by

$$\begin{cases} Min \ \rho\left(x_0 + \sum_{j=1}^{m}(x_j s_j - y_j S_j)\right) & \\ x_0 + \sum_{j=1}^{m}(Q_j x_j - q_j y_j) \leq 0 & \\ x_j \leq V_j, \ y_j \leq W_j, & j = 1, 2, ..., m \\ x_j, y_j \geq 0, & j = 1, 2, ..., m \end{cases} \tag{13}$$

Theorem 7 *a) (13) is feasible, bounded and solvable.*

b) The goal programming problem (14) below is bounded and solvable

$$\begin{cases} Min \ \sum_{j=1}^{m}\left(V_j d_j^+ + W_j d_j^-\right) & \\ \mathbb{E}(s_j z) - d_j^+ \leq Q_j, & j = 1, 2, ..., m \\ \mathbb{E}(S_j z) + d_j^- \geq q_j, & j = 1, 2, ..., m \\ d_j^-, d_j^+ \geq 0, & j = 1, 2, ..., m \end{cases} \tag{14}$$

$\left(z, \left(d_j^-\right)_{j=1}^m, \left(d_j^+\right)_{j=1}^m\right) \in \Delta_\rho \times \mathbb{R}^m \times \mathbb{R}^m$ *being the decision variable. Furthermore, the optimal values of (13) and (14) have the same absolute value and opposite sign.*

c) Suppose that (x_0, x, y) *is (13)-feasible and* $\left(\tilde{z}, \left(d_j^-\right)_{j=1}^m, \left(d_j^+\right)_{j=1}^m\right)$ *is (14)-feasible. Then, they solve the related problem if and only if*

$$\begin{cases} \begin{cases} \mathbb{E}\left(\tilde{z} \sum_{j=1}^m (x_j s_j - y_j S_j)\right) \leq \\ \mathbb{E}\left(z \sum_{j=1}^m (x_j s_j - y_j S_j)\right) \end{cases}, \quad \forall z \in \Delta_\rho \\ x_0 + \sum_{j=1}^m (Q_j x_j - q_j y_j) = 0 \\ d_j^+ (V_j - x_j) = d_j^- (W_j - y_j) = 0, \quad j = 1, 2, ..., m \\ x_j \left(\mathbb{E}(s_j \tilde{z}) - d_j^+ - Q_j\right) = 0, \quad j = 1, 2, ..., m \\ y_j \left(\mathbb{E}(S_j \tilde{z}) + d_j^- - q_j\right) = 0, \quad j = 1, 2, ..., m. \end{cases} \tag{15}$$

d) (7) is bounded (or there are no golden strategies, recall Remark 5*) if and only if the optimal value of (13) equals* 0*. Otherwise, (13) has a strictly negative optimal value and its solution is a golden strategy.*

Proof. *a*) (13) is feasible because $(x_0, x, y) = (0, 0, 0)$ satisfies the constraints. Suppose that (x_0, x, y) also satisfies the constraints of (13), and consider

$$\tilde{x}_0 = -\sum_{j=1}^m (Q_j x_j - q_j y_j) \geq x_0.$$

Obviously, $(\tilde{x}_0, x, y)$ satisfies the constraints of (13). Since ρ is translation invariant (Assumption *I*),

$$\rho\left(\tilde{x}_0 + \sum_{j=1}^m (x_j s_j - y_j S_j)\right) = \rho\left(x_0 + \sum_{j=1}^m (x_j s_j - y_j S_j)\right) - (\tilde{x}_0 - x_0),$$

that is,

$$\begin{cases} \rho\left(\tilde{x}_0 + \sum_{j=1}^m (x_j s_j - y_j S_j)\right) \leq \rho\left(x_0 + \sum_{j=1}^m (x_j s_j - y_j S_j)\right) \\ \rho\left(\tilde{x}_0 + \sum_{j=1}^m (x_j s_j - y_j S_j)\right) = \rho\left(x_0 + \sum_{j=1}^m (x_j s_j - y_j S_j)\right) \Longleftrightarrow \tilde{x}_0 = x_0. \end{cases}$$

Consequently, the inequality in the first constraint of (13) may be replaced by an equality, and the properties of the problem remain the same. In other words, it is sufficient to prove *a*) under an equality in the first constraint. Since ρ is continuous (Assumption *I*), so is the objective function of (13). Besides, since the constraints are given by continuous functions, the feasible set is closed. If one shows that the feasible set is bounded, *a*) will be an evident consequence of the

Weierstrass Theorem. Thus, due to the existence of market depth ($0 \leq x_j \leq V_j$ and $0 \leq y_j \leq W_j,\ j = 1, 2, ..., m$), it is sufficient to show that x_0 is bounded. The first constraint implies that

$$|x_0| = \left| -\sum_{j=1}^{m} (Q_j x_j - q_j y_j) \right| \leq \left| \sum_{j=1}^{m} Q_j x_j \right| + \left| \sum_{j=1}^{m} q_j y_j \right| \leq \sum_{j=1}^{m} |Q_j| \, V_j + \sum_{j=1}^{m} |q_j| \, W_j.$$

b) and c) As in the proof of Theorem 1, (13) is equivalent to

$$\left\{ \begin{array}{ll} Min\ \theta & \\ -\mathbb{E}\left(z\left(x_0 + \sum_{j=1}^{m}(x_j s_j - y_j S_j)\right)\right) \leq \theta, \quad \forall z \in \Delta_\rho & \\ x_0 + \sum_{j=1}^{m}(Q_j x_j - q_j y_j) \leq 0 & \\ x_j \leq V_j,\ y_j \leq W_j, & j = 1, 2, ..., m \\ x_j, y_j \geq 0, & j = 1, 2, ..., m. \end{array} \right. \tag{16}$$

Under the obvious notation

$$\left\{ \begin{array}{l} \mathcal{L}(x_0, x, y, \lambda, z, d^+, d^-) = \\ (\lambda - 1) x_0 + \sum_{j=1}^{m} \left[\lambda Q_j - \mathbb{E}(s_j z) + d_j^+\right] x_j + \sum_{j=1}^{m} \left[\mathbb{E}(S_j z) - \lambda q_j + d_j^-\right] y_j \end{array} \right.$$

for $(x_0, x, y, \lambda, z, d^+, d^-) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \times \Delta_\rho \times \mathbb{R}^m \times \mathbb{R}^m$ with $\lambda \geq 0$, $d^+ \geq 0$ and $d^- \geq 0$. The condition

$$Inf\ \left\{\mathcal{L}\left(x_0, x, y, \lambda, z, d^+, d^-\right);\ x_j \geq 0, y_j \geq 0, j = 1, 2, ..., m\right\} > -\infty,$$

is obviously equivalent to $\lambda = 1$, $\lambda Q_j - \mathbb{E}(s_j z) + d_j^+ \geq 0$ and $\mathbb{E}(S_j z) - \lambda q_j + d_j^- \geq 0$, $j = 1, 2., , , .m$. Therefore, the constraints of (14) are obvious. Furthermore, there is no duality gap between (16) and (14) because the Slater condition (Luenberger, 1969) holds for (16) (recall that $V_j > 0$ and $W_j > 0$ for $j = 1, 2, ..., m$). Lastly, (15) follows from the complementary slackness conditions of linear programming (Anderson and Nash, 1987).

d) Suppose that (7) is bounded and therefore its optimal value equals 0 and is attached at $(x_0, x, y) = (0, 0, 0)$ (Theorem 1). Since this solution is also (13)-feasible, and the (13)-feasible set is obviously included in the (7)-feasible set, one has that $(x_0, x, y) = (0, 0, 0)$ solves (13) and the optimal value of this problem equals 0. Conversely, suppose that (7) is unbounded and therefore (8) has no feasible solutions (Theorem 1). The solution $\left(\tilde{z}, \left(d_j^-\right)_{j=1}^m, \left(d_j^+\right)_{j=1}^m\right)$ of (14) cannot satisfy $\left(\left(d_j^-\right)_{j=1}^m, \left(d_j^+\right)_{j=1}^m\right) = (0, 0)$, since otherwise $\tilde{z}$ would satisfy (8). Hence, the optimal value of (14) will be strictly higher than 0, and therefore the optimal value of (13) will be strictly lower than 0. □

Let us point out that (14) is actually a goal programming problem. See, for instance, Morón *et al.* (1996) for further details about goal programming and compromise programming.

Remark 8 *According to (12) and Remark* 5*, the presence of golden strategies leads to the anti-intuitive equality* $(expected\_wealth, risk) = (+\infty, -\infty)$ *with no net investment of money, and the absence of arbitrage does not affect this finding (Remark* 6*). Nevertheless, Theorem* 7 *shows that (13) is bounded, and things become significantly different when the market depth is considered. In any case, if (7) is unbounded, then the solution* (x_0, x, y) *of (13) is a golden strategy with null price and strictly negative risk, and this is very relevant for every investor. Indeed, consider a portfolio (pay-off)* $u \in L^2$ *composed of securities traded in a different market, and let us compare* u *and*

$$u + x_0 + \sum_{j=1}^{m} (x_j s_j - y_j S_j) . \tag{17}$$

Firstly, both strategies have the same price because the price of (x_0, x, y) *vanishes (second condition in (15)). Secondly, bearing in mind Assumption* I,

$$\rho\left(u + x_0 + \sum_{j=1}^{m} (x_j s_j - y_j S_j)\right) \leq \rho(u) + \rho\left(x_0 + \sum_{j=1}^{m} (x_j s_j - y_j S_j)\right) < \rho(u)$$

and

$$\begin{cases} \mathbb{E}\left(u + x_0 + \sum_{j=1}^{m} (x_j s_j - y_j S_j)\right) = \mathbb{E}(u) + \mathbb{E}\left(x_0 + \sum_{j=1}^{m} (x_j s_j - y_j S_j)\right) \\ \geq \mathbb{E}(u) - \rho\left(x_0 + \sum_{j=1}^{m} (x_j s_j - y_j S_j)\right) > \mathbb{E}(u) . \end{cases}$$

To sum up, (17) beats u *because both strategies have the same price but* u *generates strictly higher risk and strictly lower expected wealth. The golden strategy* (x_0, x, y) *solving (13) allows us to outperform every portfolio by adding this portfolio and* (x_0, x, y)*.* □

Remark 9 *On the one hand, Balbás et al. (2022) already pointed out the potential presence of golden strategies in a buy and hold framework. On the other hand, Remark* 8 *shows the importance of detecting the golden strategies, if they exist. The focus of Balbás et al. (2022) was on "optimal strategies" in a much more complex setting, since they considered three objective functions to be simultaneously optimized. However, if the focus is on golden strategies only, their approach may be significantly simplified from a computational perspective, and the detention of the golden strategies may become much more tractable in practice. Actually, the methods proposed in Balbás et al. (2022) are far more complex than the resolution of the linear goal programming problem (14). Notice that one is dealing with infinite-dimensional linear programming (Anderson and Nash, 1987) because the dual variable* z *belongs to the infinite-dimensional space* L^2 *(see (1)), and it becomes very useful to improve the tractability of problems involving infinitely many dimensions.* □

Algorithm I *Step_I.Solve the linear goal programming problem (14), which, according to Theorem* 7*, is feasible and solvable. Algorithms allowing us to solve infinite-dimensional linear programming problems may be found in Anderson and Nash (1987).*

Step_II. If the solution $\left(z, \left(d_j^-\right)_{j=1}^m, \left(d_j^+\right)_{j=1}^m\right)$ *of (14) satisfies*

$$\left(\left(d_j^-\right)_{j=1}^m, \left(d_j^+\right)_{j=1}^m\right) = 0,$$

then (8) holds and therefore there are no golden strategies (Theorem 1*).*

Step_III. If the solution $\left(z, \left(d_j^-\right)_{j=1}^m, \left(d_j^+\right)_{j=1}^m\right)$ *of (14) satisfies*

$$\left(\left(d_j^-\right)_{j=1}^m, \left(d_j^+\right)_{j=1}^m\right) \neq 0,$$

then *(8) does not hold and there are golden strategies (Theorem* 1*), which are characterized by (15) (Theorem* 7*).* □

5 Numerical experiment involving the Black, Scholes and Merton model

Let us draw on Algorithm I in order to find the golden strategies of some numerical examples. Though real market data in combination with ambiguity could be used in order to prevent the model risk (Balbás *et al.*, 2016*b*), let us choose the BSM model so as to generate the involved data. This choice will show that the most important arbitrage free derivative pricing models may also generate golden strategies in a buy and hold framework, and this assertion remains true after providing these models with order book linked perturbations (frictions). According to the BSM model, consider an international stock index whose drift and volatility will be denoted by μ_0 and σ, respectively. If r is the continuously compounded interest rate and $\mu = \mu_0 - r$ is the index excess return, it is known that the index future contract theoretical quotation $(F_t)_{t\in[0,T]}$ will satisfy the stochastic differential equation

$$dF_t = F_t\left(\mu dt + \sigma dB_t\right), \tag{18}$$

$(B_t)_{t\in[0,T]}$ being a standard Brownian motion. It is also known that the solution of (18) becomes

$$F_T = F_0 Exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}B\right), \tag{19}$$

B being a random variable with a standard normal distribution. Furthermore, the current price of a European future call option with strike k and maturity at T becomes (recall that the riskless rate vanishes, see Section 2),

$$c\left(k\right) = \Phi\left(\frac{F_0/k + \sigma^2 T/2}{\sigma\sqrt{T}}\right)F_0 - \Phi\left(\frac{F_0/k - \sigma^2 T/2}{\sigma\sqrt{T}}\right)k, \tag{20}$$

Φ denoting the cumulative distribution function of the standard normal distribution. We have considered a market composed of a future contract and 12 European future calls with strikes $k = \theta F_0$, with θ taking the values (21) below

$$(0.5; 0.6; 0.65; 0.7; 0.8; 0.85; 0.9; 1; 1.1; 1.15; 1.2) . \tag{21}$$

By means of (20), the 12 call prices $c(\theta F_0)$ may be easily computed as a proportion of F_0, and then one can generate bid and ask prices with the expressions

$$\begin{cases} Bid(\theta F_0) = c(\theta F_0)(1-\alpha) \\ Ask(\theta F_0) = c(\theta F_0)(1+\alpha) \end{cases}$$

for some $0 < \alpha < 1$. In the numerical experiment, the limits $V = W = 10$ and $V_j = W_j = 50$, $j = 1, 2, ..., 12$, have been selected for the future contract and the involved options, respectively. With respect to the payoffs at T, they are the random variables of Table I below,

TABLE I

$Pay - off$ at T

$$\begin{array}{ll} s = F_T - F_0(1+\alpha), & for\ the\ future\ contract \\ S = F_T - F_0(1-\alpha), & for\ the\ future\ contract \\ s_j = Sj = \left(F_T - \xi_j F_0\right)^+, & for\ every\ call \end{array}$$

F_T being given by (19). Table II below reports a brief sample of the obtained results if $\rho = CV@R_{0.8}$.

TABLE II

Risk reduction provoked by the golden strategy under several values for the involved parameters

Row	μ	σ	α	T *(years)*	*Risk*
1	5%	10%	1%	1	23.80%
2	4%	10%	1%	1	23.80%
3	3%	10%	1%	1	25.28%
4	5%	10%	1%	0, 5	33.97%
5	10%	12%	1%	1	94.58%
6	8%	12%	1%	1	26.28%
7	7%	12%	1%	1	38.52%
8	7%	12%	1%	0, 5	27.25%
9	10%	12%	1, 5%	1	87.36%
10	10%	13%	1%	1	72.60%
11	10%	13%	1, 5%	1	61.11%
12	10%	14%	1, 5%	1	46.75%
13	10%	15%	1, 5%	1	48.44%
14	10%	15%	1, 5%	0, 4	33.06%
15	15%	15%	1, 5%	1	230.31%
16	15%	20%	1, 5%	1	83.37%
17	15%	20%	1, 5%	0, 8	43.55%

All the columns on Table II have a clear interpretation except the one on the right hand side. For instance, the first value 23.80% indicates that the optimal value of (13) equals $-0.2380F_0$, that is, the risk of the optimal golden strategy is negative and its absolute value equals 23.80% of the future contract. Obviously, the price of this golden strategy vanishes (recall the second condition of (15)), so for a null price one can build a portfolio whose risk becomes $-0.2380F_0$. The rest of values on the right hand side column have the same interpretation. Moreover, every value of this column has been rounded to the second decimal place.

The most important implication of Table II is that, as already anticipated, the BSM model leads to the existence of buy and hold golden strategies which are not overcome by the existence of market imperfections (frictions). Furthermore, formal expressions connecting the parameters (μ, σ, α, T) and the optimal value of (13) do not seem to exist. In fact, the growth or fall of this optimal value is not clearly related to the growth or fall of the components (μ, σ). At any rate, without mathematical precision, and speaking informally, one might deduce that the optimal value of (13) could increase as μ increases and/or σ decreases, while the relationship with T is more ambiguous. Besides. it is natural and obvious that the right hand side of Table II decreases as α increases.

Table III below yields the golden strategy.

TABLE III

Golden strategy

The parameters are those on the same row of Table I

Row	*Optimal Golden Strategy*
1	$50C(80\%) - 50C(90\%)$
2	$50C(80\%) - 50C(90\%)$
3	$5C(115\%) - 50C(120\%)$
4	$-50C(115\%) - 50C(120\%)$
5	$50C(70\%) - 50C(90\%)$
6	$50C(70\%) - 50C(90\%)$
7	$50C(70\%) + 50C(80\%) - 50C(90\%) - 50C(95\%)$
8	$50C(80\%) - 50C(90\%) + 2C(115\%) - 50C(120\%)$
9	$50C(70\%) + 50C(80\%) - 50C(90\%) - 50C(95\%)$
10	$\begin{cases} 37C(60\%) + 50C(65\%) + 50C(70\%) - 50C(85\%) \\ -50C(90\%) - 50C(95\%) + 14C(100\%) \end{cases}$
11	$50C(70\%) + 37C(80\%) - 50C(90\%) - 50C(95\%) + 14C(100\%)$
12	$50C(65\%) + 50C(70\%) - 50C(85\%) - 50C(90\%)$
13	$50C(65\%) + 50C(70\%) - 50C(85\%) - 50C(90\%))$
14	$7C(115\% - 50C(120\%)$
15	$\begin{cases} 50C(60\%) + 50C(65\%) + 50C(70\%) + 45C(80\%) \\ -50C(85\%) - 50C(90\%) - 50C(95\%) - 50C(100\%) + 5C(110\%) \end{cases}$
16	$\begin{cases} 50C(60\%) + 50C(65\%) + 47C(70\%) - 50C(80\%) \\ -50C(85\%) - 50C(90\%) + 6C(100\%) \end{cases}$
17	$50C(65\%) - 50C(85\%)$

In order to clarify the notation, let us indicate that $C(80\%)$ represents the call option whose strike is $0.8F_0$ and a similar comment applies in the rest of cases. Obviously, every row of Table III is related to the same row of Table II. Moreover, every golden strategy contains the riskless asset, though the amount to lend or borrow has not been reported on Table III because it is easily deduced if one bears in mind that the optimal golden strategy price must vanish (second condition of (15)). Besides, an analysis of Table III indicates that the optimal golden strategy is not easily related to the parameters (μ, σ, α, T), though "in the money calls" are often bought whereas "out of the money calls" are often sold. The future contract is rarely traded. In fact, it is never traded in the selected sample of Tables II and III.

6 Numerical experiment involving real market data

Next let us deal with real market data and the expectile risk measure. The interest of this experiment is threefold. On the one hand, using of real data we will yield further evidence about the existence of golden strategies in the real world. On the other hand, the sub-gradient of the expectile risk measure is more complex than that of the conditional value at risk (see (5) and (6)), and therefore Problem (*14*) and Algorithm I become more complex too. Solving (*14*) for expectiles we will point out the interest and necessity of the efficient and fast procedures proposed in this paper. Finally, as a third noteworthy aspect, close analytical relationships between both the conditional value at risk and the expectile will enable us to prove that the given strategies will remain golden strategies for the conditional value at risk too.

Let us focus on the Spanish derivative market, which provides us with synchronized real quotations delayed 15 minutes.[6] The Spanish index $IBEX-35$ was the underlying asset, and its quotation was 9974 when the existence of golden strategies was analyzed, that is, on January 25^{th} 2024. The riskless rate adopted was 3.90%,[7] and a 30−day negotiation window led to an index historical drift and volatility equaling 9.63% and 12.33%, respectively. The dividend yield was estimated by comparing the index with the index total return, and was 3%. Table IV presents some future contract quotations available on Thursday, January 25^{th} 2024.

TABLE IV

Some available future contracts

Bid	**Depth**	**Ask**	**Depth**	**Maturity**
9920	40	9925	29	Feb_16^{th}
9935	15	9950	1	Mar_15^{th}

[6] See https://www.meff.es/ing/Home

[7] See https://www.euribor-rates.eu/en/current-euribor-rates/

European future options were also available. Up to 250 units of each option could be bought or sold, and all expirations would take place on March 15^{th}, which coincides with the horizon T of this empirical test. Table V below reports bid prices, ask prices and strikes.[8]

TABLE V

Some available European future options

Style	Bid	Ask	Strike	Style	Bid	Ask	Strike
Call	3703	3763	6300	*Put*	15	23	8900
Call	846	906	9100	*Put*	19	27	9000
Call	752	812	9200	*Put*	23	35	9100
Call	710	760	9250	*Put*	28	40	9200
Call	664	714	9300	*Put*	33	45	9250
Call	619	669	9350	*Put*	37	49	9300
Call	574	624	9400	*Put*	42	58	9350
Call	534	584	9450	*Put*	46	55	9400
Call	497	547	9500	*Put*	52	70	9450
Call	185	215	9950	*Put*	59	77	9500
Put	1	6	7800	*Put*	68	86	9550
Put	1	6	7900	*Put*	77	95	9600
Put	2	7	8000	*Put*	208	230	10000
Put	2	7	8100	*Put*	358	398	10250
Put	3	8	8200	*Put*	392	432	10300
Put	4	9	8300	*Put*	427	467	10350
Put	5	10	8400	*Put*	468	518	10400
Put	7	12	8500	*Put*	510	560	10450
Put	8	13	8600	*Put*	553	603	10500
Put	10	18	8700	*Put*	599	649	10550
Put	11	19	8800	*Put*	643	693	10600

Algorithm I reveals the existence of a self-financing golden strategy for the expectile risk measure with parameter $\beta = 0.2$. The global risk equals €$-139,880$, while the expected pay-off becomes €$184,981$. The strategy implies the purchase of 40 and 15 units of the given future contracts (see Table IV) and the investment of €$3,249,472$ in the riskless security. 250 units of the every option above must be sold (see Table V), except those options indicated in Table VI

[8] Many available strikes and maturities are not included because they are not traded in the reported golden strategy.

below, which must be traded according to the reported information.

TABLE VI

Options that do not saturate the sale limit

Style	Strike	Bought units	Sold units
Call	9950	193	0
Put	10000	250	0
Put	10250	0	245

Besides, since there are important analytical relationships between the conditional value at risk and the expectile (see Tadese and Drapeau, 2020, or Balbás *et al.*, 2023), one can also verify that the portfolio above is a golden strategy for many confidence levels of the conditional value at risk. For instance, the inequality

$$CV@R_{1-\beta^*}(u) \leq \mathcal{E}_\beta(u) + \frac{\beta\left(\mathbb{E}(u) + \mathcal{E}_\beta(u)\right)}{\beta^*(1-2\beta)}$$

easily implies that one is facing a $CV@R_{1-\beta^*}-$golden strategy under a confidence level $1-\beta^* \leq 0.8925 = 89.25\%$. Lastly, according to the notion of arbitrage given in Remark 6, it is easy to check that the given portfolio is not an arbitrage strategy.

7 Conclusion

Golden strategies may become very interesting in practice, since every portfolio will be outperformed in a risk/return framework by the golden strategy plus that involved portfolio. This paper has presented a computationally tractable, buy and hold, and potentially model-free approach to detect golden strategies in practice. The setting is very realistic because all the market frictions provoked by the order book constraints have been incorporated. Some numerical experiments have shown how often the golden strategies do exist in derivative markets, and their presence is compatible with the absence of arbitrage. This finding can also shed some light on another problem under discussion, that is, the usefulness of incorporating derivatives in the portfolio of traditional and very risk averse investors. Moreover, the tractable provided methods will be a good starting point in order to implement empirical studies about the practical performance of golden strategies, which will complement and improve the results of previous less general empirical analyses.

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